5
The observables of a reaction

Interesting spin effects are seen in many hadronic reactions, such as $pp \to pp$, $np \to np$, $pp \to n\Delta$, $\Lambda p \to \Lambda p$, $pp \to \pi X$ etc. And recently more complete measurements have been made on $\Lambda p \to \Lambda p$ and the related reaction $\bar{p}p \to \bar{\Lambda}\Lambda$, especially at LEAR at CERN. In addition experiments using polarized deuteron beams and targets are becoming relatively commonplace.

Given the interest and variety of reactions that are or will be studied it seems worthwhile to set up a general description for an arbitrary $2 \to 2$ reaction with particles of any spin. Indeed we shall set up a general scheme which is, surprisingly, simpler to work with than the usual one for $NN \to NN$ and from which the relevant information for a specific reaction can be easily read off.

Our emphasis here will be upon those quantities, the *observables* that can be measured and upon how they are related to the helicity amplitudes.

We begin with total cross-section measurements, which yield information about the forward amplitudes, and then consider more general observables. For the latter we work first in the CM and then relate the CM observables to the Lab frames where the measurements are actually made.

A comprehensive list of linearly independent measurable reaction parameters and their relation to the helicity amplitudes, for various reactions, is given in Appendix 10.

5.1 The generalized optical theorem

For spinless particles, in our normalization, the usual optical theorem (see e.g. Messiah, 1958) relates the imaginary part of the forward helicity amplitude H to the total cross-section as follows:

$$\mathrm{Im}\, H(\theta = 0) = \frac{1}{4\sqrt{\pi}}\sigma_{\mathrm{tot}}. \tag{5.1.1}$$

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For particles with spin, the direct generalization of (5.1.1) is

$$\mathrm{Im}\, H_{\lambda_A\lambda_B;\lambda_A\lambda_B}(\theta=0) = \frac{1}{4\sqrt{\pi}}\sigma_{\mathrm{tot}}(\lambda_A,\lambda_B) \tag{5.1.2}$$

where $\sigma_{\mathrm{tot}}(\lambda_A,\lambda_B)$ means the total cross-section measured with the initial particles A and B in the unique helicity states λ_A, λ_B respectively, a situation that can sometimes be realized using a polarized beam and target.

The *unpolarized* total cross-section is defined as

$$\sigma_{\mathrm{tot}} = \frac{1}{(2s_A+1)(2s_B+1)}\sum_{\lambda_A,\lambda_B}\sigma_{\mathrm{tot}}(\lambda_A,\lambda_B) \tag{5.1.3}$$

so that from (5.1.2)

$$\sum_{\lambda_A,\lambda_B}\mathrm{Im}\, H_{\lambda_A\lambda_B;\lambda_A\lambda_B} = \frac{(2s_A+1)(2s_B+1)}{4\sqrt{\pi}}\sigma_{\mathrm{tot}}, \tag{5.1.4}$$

where $H_{\lambda_A\lambda_B;\lambda_A\lambda_B}$ is evaluated at $\theta=0$. For photons the factor $2s+1$ is replaced by 2 in (5.1.3) and (5.1.4).

Relations (5.1.4) and (5.1.2) are very valuable. Equation (5.1.4), which is easy to use in practice, allows a determination of the imaginary part of the forward 'spin-averaged' amplitude whereas (5.1.2), which may be difficult in practice, gives the imaginary parts of the individual amplitudes $H_{\lambda_A\lambda_B;\lambda_A\lambda_B}$ at $\theta=0$.

However, (5.1.2) is not the most general form of the optical theorem. There are other amplitudes, not of the form $\lambda_A\lambda_B\to\lambda_A\lambda_B$, which need not vanish in the forward direction (see Section 4.3), namely those of the form $\lambda_A\lambda_B\to\lambda'_A\lambda'_B$ where $\lambda'_A-\lambda'_B=\lambda_A-\lambda_B$; *all* these can be measured by suitably preparing the initial states of beam and target.

Let $\rho_{\mathrm{i}}(A,B)$ be the joint helicity density matrix for the initial particles. Then (Bialkowski, 1970) the generalization of (5.1.2) is

$$\sum_{\lambda'_A-\lambda'_B=\lambda_A-\lambda_B}\rho_{\mathrm{i}\,\lambda'_A\lambda'_B;\lambda_A\lambda_B}(A,B)\,\mathrm{Im}\, H_{\lambda'_A\lambda'_B;\lambda_A\lambda_B}(\theta=0) = \frac{1}{4\sqrt{\pi}}\sigma_{\mathrm{tot}}(\rho_{\mathrm{i}}) \tag{5.1.5}$$

where $\sigma_{\mathrm{tot}}(\rho_{\mathrm{i}})$ is the total cross-section measured with the beam and target described by ρ_{i}.

Usually the beam and target are uncorrelated, so that

$$\rho_{\mathrm{i}}(A,B) = \rho_{\mathrm{i}}(A)\otimes\rho_{\mathrm{i}}(B). \tag{5.1.6}$$

We shall illustrate the use of (5.1.5) in nucleon–nucleon scattering and then consider a more general reaction.

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5.1.1 Nucleon–nucleon scattering

Let the spin-polarization vectors for the beam ($\boldsymbol{\mathcal{P}}^A$) and for the target ($\boldsymbol{\mathcal{P}}^B$) *both* be specified in the Lab frame, as is commonly done in experiments. Then the CM helicity density matrices for particles A and B will be

$$\begin{aligned} \rho(A) &= \tfrac{1}{2}\left(I + \boldsymbol{\mathcal{P}}^A \cdot \boldsymbol{\sigma}\right) \\ \rho(B) &= \tfrac{1}{2}\left(I + \tilde{\boldsymbol{\mathcal{P}}}^B \cdot \boldsymbol{\sigma}\right) \end{aligned} \tag{5.1.7}$$

where, because the Lab frame is rotated from the helicity rest frame for B (see Fig. 3.1 and discussion thereafter)

$$\tilde{\boldsymbol{\mathcal{P}}}^B \equiv (\mathscr{P}^B_x, -\mathscr{P}^B_y, -\mathscr{P}^B_z). \tag{5.1.8}$$

Substituting in (5.1.5) yields

$$\begin{aligned} &\mathrm{Im}\ H_{++;++}(1 - \mathscr{P}^A_z \mathscr{P}^B_z) \\ &+\ \mathrm{Im}\ H_{+-;+-}(1 + \mathscr{P}^A_z \mathscr{P}^B_z) \\ &+\ \mathrm{Im}\ H_{++;--}(\mathscr{P}^A_x \mathscr{P}^B_x + \mathscr{P}^A_y \mathscr{P}^B_y) = \frac{1}{2\sqrt{\pi}} \sigma_{\mathrm{tot}}(\boldsymbol{\mathcal{P}}^A, \boldsymbol{\mathcal{P}}^B) \end{aligned} \tag{5.1.9}$$

where $(\pm)$ is short for $(\pm 1/2)$.

The connection between our helicity amplitudes and the notation commonly used in nucleon–nucleon (NN) physics (Goldberger *et al.*, 1960) is, aside from normalization,

$$\begin{array}{ccc} H_{++;++} = \phi_1 & H_{++;--} = \phi_2 & H_{+-;+-} = \phi_3 \\ & H_{+-;-+} = \phi_4 \qquad H_{++;+-} = \phi_5 & \end{array} \tag{5.1.10}$$

If $\overset{\leftarrow}{\rightarrow}$ indicates complete polarization along or opposed to the incoming beam direction and $\uparrow\downarrow$ indicates polarizations transverse to the beam then (5.1.9) gives the now familiar results

$$\Delta\sigma_L \equiv \sigma_{\mathrm{tot}\underset{\leftarrow}{\rightarrow}} - \sigma_{\mathrm{tot}\underset{\rightarrow}{\rightarrow}} = 4\sqrt{\pi}\ \mathrm{Im}\ \left(H_{++;++} - H_{+-;+-}\right), \tag{5.1.11}$$

where the top arrow refers to the beam polarization and the bottom arrow to the target polarization, and

$$\Delta\sigma_T \equiv \sigma_{\mathrm{tot}\uparrow\downarrow} - \sigma_{\mathrm{tot}\uparrow\uparrow} = -4\sqrt{\pi}\ \mathrm{Im}\ H_{++;--}, \tag{5.1.12}$$

where the first arrow refers to the beam polarization and the second arrow to the target polarization.

Measurements of $\Delta\sigma_L$ and $\Delta\sigma_T$ have produced rather interesting results, as will be discussed in Chapter 14.

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5.1.2 Particles of arbitrary spin

It is now simplest to specify the initial CM helicity density matrix in terms of the multipole parameters $t^l_m(A)$ and $t^L_M(B)$ (see eqn (3.1.32)). Then (5.1.5) becomes

$$\sum_m \sum_{l,L} t^l_m(A) t^L_m(B) \text{ Im } h_{lL}(m) = \frac{1}{4\sqrt{\pi}} \sigma_{\text{tot}}(\rho_{\text{i}}),$$

where $h_{lL}(m)$ is a linear combination of forward amplitudes:

$$h_{lL}(m) \equiv \frac{(2l+1)(2L+1)}{(2s_A+1)(2s_B+1)} \text{ Tr } \left[T^l_m T^L_m H(\theta = 0) \right]. \tag{5.1.13}$$

Parity invariance gives

$$h_{lL}(-m) = (-1)^{l+L} h_{lL}(m) \tag{5.1.14}$$

and time-reversal invariance yields

$$h_{lL}(m) = (-1)^{l+L} h_{lL}(m). \tag{5.1.15}$$

Thus only even values of $l + L$ can occur and we end up with the result

$$\frac{1}{4\sqrt{\pi}} \sigma_{\text{tot}}(\rho_{\text{i}}) = \sum_{\substack{l,L \\ l+L \text{ even}}} \sum_{m \geq 0} (2 - \delta_{m0}) \text{ Im } h_{lL}(m) \text{ Re } \left[t^l_m(A) t^L_m(B) \right]. \tag{5.1.16}$$

Notice that there is no interference between even and odd ranks of polarization.

For identical particles one also has

$$h_{lL}(m) = h_{Ll}(m). \tag{5.1.17}$$

By suitably choosing the $t^l_m(A)$ and $t^L_m(B)$ one can measure the linear combinations of forward amplitudes $h_{lL}(m)$.

Note that since $\left(T^l_m \right)_{ij} = 0$ unless $i = j + m$, all amplitudes in the sum (5.1.16) are of the form

$$H_{\lambda_A+m, \lambda_B+m; \lambda_A \lambda_B}$$

with, of course $|m| \leq \min\{2s_A, 2s_B\}$. Thus we have an important result: *The determination of the imaginary part of a forward amplitude of the form $H_{\lambda_A+m, \lambda_B+m; \lambda_A \lambda_B}$ requires polarization of rank $l \geq |m|$ in* both *beam and target.*

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Once the $h_{lL}(m)$ are determined, the individual helicity amplitudes can be obtained via

$$\begin{aligned} H_{\lambda_A+m,\lambda_B+m;\lambda_A\lambda_B} = &(-1)^{s_A+s_B-\lambda_A-\lambda_B}\,[(2s_A+1)(2s_B+1)]^{1/2} \\ &\times \sum_{lL} \frac{h_{lL}(m)}{[(2l+1)(2L+1)]^{1/2}} \langle l,m|s_A,\lambda_A+m;s_A,-\lambda_A\rangle \\ &\times \langle L,m|s_B,\lambda_B+m;s_B,-\lambda_B\rangle . \end{aligned} \tag{5.1.18}$$

5.1.3 Application to deuteron–nucleon and deuteron–deuteron scattering

Consider a magnetically prepared beam and target with axes of quantization in the Lab frame specified by polar angles $\theta = \beta_A, \phi = \gamma_A$ and $\theta = \beta_B, \phi = 0$ respectively, as shown in Fig. 5.1.

Let $\hat{t}^l_0$ and $\hat{t}^L_0$ be the multipole parameters of beam and target when referred to the frames in which their quantization axes are along OZ. Then the CM multipole parameters needed are found from (3.3.1) and (3.3.2), and (5.1.16) becomes

$$\begin{aligned} \frac{1}{4\sqrt{\pi}}\sigma_{\text{tot}}(\rho_{\text{i}}) = &\sum_{\substack{l,L \\ l+L \text{ even}}} \hat{t}^l_0 \hat{t}^L_0 \sum_{m\geq 0} \cos(\gamma_A m)\, d^l_{m0}(\beta_A) \\ &\times d^L_{m0}(\pi-\beta_B)(2-\delta_{m0}) \operatorname{Im} h_{lL}(m). \end{aligned} \tag{5.1.19}$$

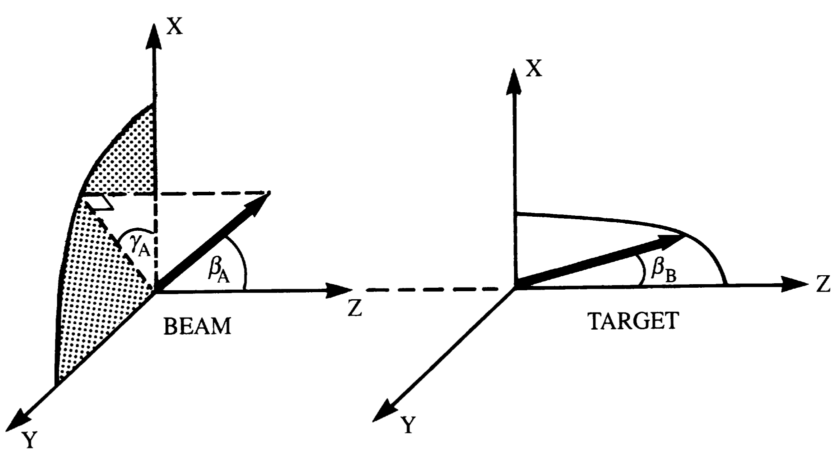

Fig. 5.1. Angles specifying quantization axes of beam and target.

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In terms of the polarization and the alignment[1] of beam and target, (5.1.19) gives the following.

For $d + N \to d + N$

$$\begin{aligned}\frac{1}{4\sqrt{\pi}}\sigma_{\text{tot}}(\boldsymbol{\mathcal{P}}^d, \mathscr{A}; \boldsymbol{\mathcal{P}}^N) = {} & \operatorname{Im} h_{00}(0) \\ & + \tfrac{1}{\sqrt{6}}\left[\mathscr{P}_x^d \mathscr{P}_x^N \operatorname{Im} h_{11}(1) - \mathscr{P}_z^d \mathscr{P}_z^N \operatorname{Im} h_{11}(0)\right] \\ & + \tfrac{3}{\sqrt{10}}\mathscr{A}\left(3\cos^2\beta - 1\right) \operatorname{Im} h_{20}(0) \qquad (5.1.20)\end{aligned}$$

where

$$\begin{aligned} h_{00}(0) &= \tfrac{1}{3}\left(H_{11/2;11/2} + H_{01/2;01/2} + H_{-11/2;-11/2}\right) \\ h_{11}(1) &= \sqrt{3}H_{11/2;0-1/2} \\ h_{11}(0) &= \sqrt{\tfrac{3}{2}}\left(H_{11/2;11/2} - H_{-11/2;-11/2}\right) \\ h_{20}(0) &= \tfrac{1}{3}\sqrt{\tfrac{5}{2}}\left(H_{11/2;11/2} + H_{-11/2;-11/2} - 2H_{01/2;01/2}\right) \end{aligned} \qquad (5.1.21)$$

and where β is the angle between $\boldsymbol{\mathcal{P}}^d$ and the beam direction. Note that four measurements are needed to find all the amplitudes, and only polarizations along and transverse to the beam are required.

For $d + d \to d + d$, labelling the beam and target deuterons by A and B respectively, one gets[2]

$$\begin{aligned}\frac{1}{4\sqrt{\pi}}&\sigma_{\text{tot}}(\boldsymbol{\mathcal{P}}^B, \mathscr{A}^B; \boldsymbol{\mathcal{P}}^A, \mathscr{A}^A) \\ = {} & \operatorname{Im} h_{00}(0) + \tfrac{1}{2}\left[\mathscr{P}_x^A \mathscr{P}_x^B \operatorname{Im} h_{11}(1) - \mathscr{P}_z^A \mathscr{P}_z^B \operatorname{Im} h_{11}(0)\right] \\ & + \tfrac{1}{2\sqrt{10}}\left[\mathscr{A}^A\left(3\cos^2\beta_A - 1\right) + \mathscr{A}^B\left(3\cos^2\beta_B - 1\right)\right] \operatorname{Im} h_{20}(0) \\ & + \tfrac{1}{40}\mathscr{A}^A\mathscr{A}^B\Big[\left(3\cos^2\beta_A - 1\right)\left(3\cos^2\beta_B - 1\right) \operatorname{Im} h_{22}(0) \\ & \qquad - 12\cos\gamma_A \sin\beta_A \cos\beta_A \sin\beta_B \cos\beta_B \operatorname{Im} h_{22}(1) \\ & \qquad + 3\cos 2\gamma_A \sin^2\beta_A \sin^2\beta_B \operatorname{Im} h_{22}(2)\Big]. \qquad (5.1.22)\end{aligned}$$

[1] See subsection 3.1.12.

[2] This can be written in simpler form using the T_{ij} of eqn (3.1.59). We have not done so because experimentally it is easier to think in terms of the alignment.

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Finally, the amplitude combinations measured by seven experiments are:

$$
\begin{aligned}
h_{00}(0) &= \tfrac{2}{9}\left(H_{11;11} + 2H_{10;01} + H_{1-1;1-1} + \tfrac{1}{2}H_{00;00}\right) \\
h_{11}(1) &= H_{11;00} + H_{10;0-1} \\
h_{11}(0) &= H_{11;11} - H_{1-1;1-1} \\
h_{20}(0) &= \tfrac{\sqrt{10}}{9}\left(H_{11;11} - H_{10;10} + H_{1-1;1-1} - H_{00;00}\right) \qquad (5.1.23)\\
h_{22}(0) &= \tfrac{5}{9}\left(H_{11;11} - 4H_{10;10} + H_{1-1;1-1} + 2H_{00;00}\right) \\
h_{22}(1) &= \tfrac{5}{3}\left(H_{11;00} - H_{10;0-1}\right) \\
h_{22}(2) &= \tfrac{5}{3}H_{11;-1-1}.
\end{aligned}
$$

Note that now the polarizations of both beam and target have to be set at some angle other than along or transverse to the beam for at least one measurement. For example, one could choose $\gamma_A = 0$, $\beta_A = \beta_B = 45°$.

5.2 The final state helicity density matrix

We consider now the definition, and some important properties, of the helicity density matrix of the final particles produced in a reaction. Initially we deal with $2 \to 2$ reactions, but this will be generalized in Section 5.8.

5.2.1 Definition

We consider an arbitrary reaction $A + B \to C + D$. For given initial helicities a, b, the helicity amplitudes $H_{cd;ab}$ are a measure of the probability amplitude for finding the final helicities c, d. Thus, in analogy with eqn (3.2.2) the joint CM helicity density matrix for the final state is

$$\rho'_{cd;c'd'}(C,D) = \sum_{\substack{a,b\\a',b'}} H_{cd;ab}\ \rho_{\mathrm{i}\,ab;a'b'}(A,B)\ H^*_{c'd';a'b'} \qquad (5.2.1)$$

where $\rho_\mathrm{i}(A,B)$ is the initial state helicity density matrix. To avoid the profusion of indices we write (5.2.1) in matrix form:

$$\rho'(C,D) = \mathbf{H}\rho_\mathrm{i}(A,B)\mathbf{H}^\dagger. \qquad (5.2.2)$$

If $\rho_\mathrm{i}(A,B)$ is correctly normalized, so that $\mathrm{Tr}\ \rho_\mathrm{i}(A,B) = 1$, it will be found that $\rho'(C,D)$ is not normalized to trace 1, so for computing expectation values of observables in the final state we must always use

$$\rho(C,D) \equiv \frac{\rho'(C,D)}{\mathrm{Tr}\ \rho'(C,D)}.$$

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With our normalization for $H_{\{\lambda\}}$, eqn (4.1.4),

$$\text{Tr}\ \rho'(C,D) = 2\pi \frac{d^2\sigma}{dtd\phi}(\rho_{\rm i}), \tag{5.2.3}$$

where $(d^2\sigma/dtd\phi)(\rho_{\rm i})$ is the differential cross-section into the momentum transfer range $t \to t + dt$ and the azimuthal range $\phi \to \phi + d\phi$ for an initial state specified by $\rho_{\rm i}$.

5.2.2 Rank conditions

Since the rank (see subsection (3.1.2)) of a product of matrices must be less than or equal to the rank of any matrix in the product, (5.2.2) implies that

$$r_f \leq r_{\rm i} \tag{5.2.4}$$

where r_f and $r_{\rm i}$ are the ranks of the final and initial state density matrices.

This condition can be a very stringent one. For example, in

$$\pi + N \to \pi + N^*(J)$$

where $N^*(J)$ is a high-spin resonance, $r_{\rm i}$ cannot be greater than 2 ($\rho_{\rm i}$ is a 2×2 matrix) and therefore ρ_f, which is $2J \times 2J$ and could thus be a huge matrix, must have rank ≤ 2.

If it happens that only the even part ρ_+ of the final state density matrix can be measured (see subsection 3.1.7), then the weaker rank condition

$$\text{rank}\ \rho_+ \leq 2r_{\rm i}$$

holds. In our $N^*(J)$ example above, if $J = 3/2$ we end up with rank $\rho_+ \leq 4$ which is no restriction at all, bearing in mind that ρ_+ is a 4×4 matrix! If both C and D have non-zero spin and we consider the effective density matrix of, say, C, then its rank must satisfy a much weaker bound than (5.2.4), namely,

$$\text{rank}\ \rho(C) \leq (2s_D + 1)\ \text{rank}\ \rho_{\rm i} \tag{5.2.5}$$

with analogous constraint for D.

Generally a large number of relations may exist amongst the elements of ρ_f and they must be taken into account experimentally.

5.2.3 Angular momentum constraints near $\theta = 0, \pi$

The behaviour of the $H_{\{\lambda\}}$ near $\theta = 0$ and π (Section 4.3) imposes constraints on $\rho(C,D)$ near the forward and backward regions. These depend upon $\rho_{\rm i}(A,B)$.

The strongest conditions apply when the initial state is *unpolarized.* Then at $\theta = 0$ or π

$$\rho_{cd;c'd'}(C,D) = 0$$

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unless both

$$c - d = c' - d' \quad \text{and} \quad |c - d| \leq s_A + s_B.$$

Near these points, the behaviour is

$$\rho_{cd;c'd'} \propto (\sin\theta/2)^{s_1}(\cos\theta/2)^{c_1} \tag{5.2.6}$$

where

$$s_1 \equiv |c - d - c' + d'| + (1+\epsilon)M$$
$$c_1 \equiv |c - d + c' - d'| + (1-\epsilon)M$$

with

$$\epsilon \equiv \text{sign}\,\{(c-d)(c'-d')\}$$

and

$$M = \begin{cases} 0 & \text{when either } |c-d| \text{ or } |c'-d'| \leq s_A + s_B \\ \min\left\{\Big||c-d| - s_A - s_B\Big|; \Big||c'-d'| - s_A - s_B\Big|\right\} & \text{otherwise.} \end{cases}$$

For the effective single-particle density matrix, say of particle C, we have

$$\rho_{c'c}(C) = 0, \tag{5.2.7}$$

unless both

$$c = c' \quad \text{and} \quad |c| \leq s_A + s_B + s_D,$$

and

$$\rho_{c'c}(C) \propto (\sin\theta/2)^{\bar{s}_1}(\cos\theta/2)^{\bar{c}_1} \tag{5.2.8}$$

where

$$\bar{s}_1 \equiv |c - c'| + (1+\bar{\epsilon})\overline{M}$$
$$\bar{c}_1 \equiv |c + c'| + (1-\bar{\epsilon})\overline{M}$$

with

$$\bar{\epsilon} \equiv \text{sign}\,\{cc'\}$$

and

$$\overline{M} = \begin{cases} 0 & \text{when either } |c| \text{ or } |c'| \leq s_A + s_B + s_D \\ \min\left\{\Big||c| - s_A - s_B - s_D\Big|; \Big||c'| - s_A - s_B - s_D\Big|\right\} & \text{otherwise.} \end{cases}$$

The above constraints must be respected in any data analysis. It will be seen in Section 5.4 that the multipole parameters have a much simpler behaviour than ρ at $\theta \simeq 0, \pi$.

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5.3 The CM observables and the dynamical reaction parameters

Several discussions of the variables valid for relativistic scattering have been given in the literature for nucleon–nucleon scattering. Detailed references are given in Bourrely, Leader and Soffer, (1980).

Our treatment is more general, applying to any reaction, and is actually simpler. We expand the initial and correctly normalized final density matrices in terms of joint multipole parameters $t^{lL}_{mM}(A,B)$, $t^{l'L'}_{m'M'}(C,D)$, according to eqn (3.1.31) as generalized to combined systems of particles, and substitute in (5.2.2). There results a relation between the initial and final multipole parameters of the reaction:

$$\begin{aligned} t^{l'L'}_{m'M'}(C,D)\frac{d^2\sigma}{dtd\phi}\,(\rho_{\mathrm{i}}) &= \left(\frac{2}{3}\right)^{n_\gamma}\frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{\substack{lL\\mM}}(2l+1)(2L+1) \\ &\quad\times (l,m;L,M|l',m';L',M')_\phi\, t^l_m(A)t^L_M(B) \end{aligned} \tag{5.3.1}$$

where n_γ is the number of photons in the initial state. We have assumed that the beam and target are uncorrelated. Equation (5.3.1) gives the value of $t^{l'L'}_{m'M'}(C,D)$ when C's direction is at polar angles θ,ϕ in the CM. The outcome of the experiment is controlled by the fundamental *CM dynamical reaction parameters* (we shall simply call them 'reaction parameters'),

$$\begin{aligned} &(l,m;L,M|l',m';L',M')_\phi \\ &\equiv \left(\frac{d\sigma}{dt}\right)^{-1}\frac{1}{(2s_A+1)(2s_B+1)} \\ &\quad\times\ \mathrm{Tr}\,[H\,T^{l\dagger}_m(s_A)T^{L\dagger}_M(s_B)H^\dagger\,T^{l'}_{m'}(s_C)T^{L'}_{M'}(s_D)], \end{aligned} \tag{5.3.2}$$

where H is the matrix whose elements are $H_{cd;ab}(\theta,\phi)$. The use of matrix notation is compact and efficient, but to avoid any confusion we write out the trace in (5.3.2) in full detail:

$$\begin{aligned} &\mathrm{Tr}\,[H\,T^{l\dagger}_m(s_A)T^{L\dagger}_M(s_B)H^\dagger\,T^{l'}_{m'}(s_C)T^{L'}_{M'}(s_D)] \\ &\qquad = H_{cd;ab}\left(T^{l\dagger}_m(s_A)\right)_{aa'}\left(T^{L\dagger}_M(s_B)\right)_{bb'} \\ &\qquad\quad \times H^*_{c'd';a'b'}\left(T^{l'}_{m'}(s_C)\right)_{c'c}\left(T^{L'}_{M'}(s_D)\right)_{d'd}. \end{aligned}$$

The reaction parameters (5.3.2) are a direct generalization of the Wolfenstein parameters. All the dynamics is contained in these parameters, which can be evaluated in terms of the helicity amplitudes. They depend on both θ (or t) and ϕ, but the ϕ-dependence is trivial:

$$(l,m;L,M|l',m';L',M')_\phi = e^{i\phi(M-m)}\,(l,m;L,M|l',m';L',M')\,, \tag{5.3.3}$$

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where the right-hand side parameters are at $\phi = 0$. When no ϕ-label is shown we shall always mean $\phi = 0$ in the reaction parameters.

It should be noted that the order of symbols is

$$(\text{beam};\ \text{target}|\text{scattered};\ \text{recoil})$$

and the normalization is such that

$$(0,0;0,0|0,0;0,0) = 1. \tag{5.3.4}$$

Note that for some colliding-beam experiments the spin measurements are carried out in the CM, so that (5.3.1) will apply *directly* to the measured quantities.

In using (5.3.1) and various special cases to be derived from it, it must be remembered that for all *photons*, whether polarized or not, because of the absence of states with helicity $\lambda = 0$ one has $t_0^2 = 1/\sqrt{10}$, as is explained in subsection 3.1.12. Also of use in this case is the result

$$(2,0;0,0|0,0;0,0) = 1/\sqrt{10},$$

which follows from (5.3.2) and the properties of T_0^2 as given in (3.1.26).

5.3.1 Properties of the CM reaction parameters

The reaction parameters are not all independent as a consequence of the symmetry properties of the helicity amplitudes and of the T_m^l matrices.

(*i*) *Reality*. From $T_m^{l\dagger} = (-1)^m T_{-m}^l$ follows

$$\begin{aligned}&(l,m;L,M|l',m';L',M')^* \\ &\qquad = (-1)^{m+M+m'+M'} \times (l,-m;L,-M|l',-m';L',-M').\end{aligned} \tag{5.3.5}$$

(*ii*) *Parity*. Using $\left(T_m^l\right)_{-m_1-m_2} = (-1)^l \left(T_{-m}^l\right)_{m_1m_2}$ and the space inversion properties eqn (4.2.1) in both the $H_{\{\lambda\}}$ in (5.3.2) yields

$$\begin{aligned}&(l,m;L,M|l',m';L',M') \\ &\quad = (-1)^{m+M+m'+M'}(-1)^{l+L+l'+L'} \\ &\qquad \times (l,-m;L,-M|l',-m';L',-M').\end{aligned} \tag{5.3.6}$$

Thus

$$(l,0;L,0|l',0;L',0) = 0 \tag{5.3.7}$$

if $l + L + l' + L'$ is odd.

When this is combined with (5.3.5) we have the important result

$$\begin{aligned}&(l,m;L,M|l',m';L',M') \quad \text{is} \left\{\begin{matrix}\text{real}\\ \text{imaginary}\end{matrix}\right\} \\ &\text{as } l+L+l'+L' \quad \text{is} \left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}.\end{aligned} \tag{5.3.8}$$

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(*iii*) *Time-reversal.* Using eqn (4.2.4) in both the $H_{\{\lambda\}}$ in (5.3.2) and also the fact that the T^l_m are real gives

$$(l,m;L,M|l',m';L',M')^{AB\to CD} = (l',m';L',M'|l,m;L,M)^{CD\to AB}_{\phi=\pi}. \tag{5.3.9}$$

For *elastic* reactions eqn (4.2.5) yields

$$(l,m;L,M|l',m';L',M') = (-1)^{m+M+m'+M'}\,(l',m';L',M'|l,m;L,M)\,. \tag{5.3.10}$$

(*iv*) *Identical particles.* Using (4.2.11a,b) we find the following.
If $A=B$,

$$(l,m;L,M|l',m';L',M')^{\theta} = (-1)^{m'+M'}(L,M;l,m|l',m';L',M')^{\pi-\theta}. \tag{5.3.11}$$

Thus at $\theta=\pi/2$

$$(l,m;l,m|l',m';L',M') = 0 \qquad \text{if } m'+M' \text{ is odd.} \tag{5.3.12}$$

If $C=D$,

$$(l,m;L,M|L',M';l',m')^{\theta} = (-1)^{m+M}(l,m;L,M|l',m';L',M')^{\pi-\theta}. \tag{5.3.13}$$

Thus at $\theta=\pi/2$

$$(l,m;L,M|l',m';l',m') = 0 \qquad \text{if } m+M \text{ is odd.} \tag{5.3.14}$$

Equations (5.3.13), (5.3.14) also hold for reactions of the type

$$A+A\to D+\overline{D}$$

provided the reaction is invariant under charge conjugation.

Finally, if $A=B$ and $C=D$ then

$$(L,M;l,m|L',M';l',m') = (-1)^{m+M+m'+M'}(l,m;L,M|l',m';L',M'). \tag{5.3.15}$$

This also holds for reactions of the type

$$A+\overline{A}\to D+\overline{D}$$

if charge conjugation is a good symmetry.

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(*v*) *Additional parity and time-reversal constraints.* The application of the above symmetry results will not, in general, reduce the number of independent reaction parameters to the expected N^2 in the case where there are N independent helicity amplitudes. The additional relations can be obtained by applying the symmetry concerned to just *one* $H_{\{\lambda\}}$ in (5.3.2). The results are as follows.

Parity:

$$\begin{aligned}(l,m;L,M|l',m';L',M') \\ = \eta \sum_{\substack{\text{repeated}\\ \text{indices}}} \mathscr{A}_{l_1m_1}(lm)\mathscr{A}_{L_1M_1}(LM)\mathscr{A}_{l'_1m'_1}(l'm')\mathscr{A}_{L'_1M'_1}(L'M') \\ \times (l_1,m_1;L_1,M_1|l'_1,m'_1;L'_1,M'_1)\end{aligned} \tag{5.3.16}$$

where η is defined in eqn (4.2.1). The $\mathscr{A}(lm)$ are given in terms of vector addition coefficients and are tabulated for $s = 1/2, 1$ and $3/2$ in Appendix 6.

Time reversal: (for elastic reactions):

$$\begin{aligned}(l,m;L,M|l',m';L',M') = \sum_{\substack{\text{repeated}\\ \text{indices}}} \mathscr{C}^{lm;l'm'}_{l_1m_1;l'_1m'_1}\mathscr{C}^{LM;L'M'}_{L_1M_1;L'_1M'_1} \\ \times (l_1,m_1;L_1,M_1|l'_1,m'_1;L'_1,M'_1)\end{aligned} \tag{5.3.17}$$

The coefficients $\mathscr{C}$ are explained in Appendix 7 and are tabulated for $s = 1/2$. In Appendix 10 we give a comprehensive list of linearly independent reaction parameters for various reactions and their relation to the helicity amplitudes.

(*vi*) *Behaviour near* $\theta = 0$ *or* π. In the forward and backward scattering regions we find

$$(l,m;L,M|l',m';L',M')^{\theta\to 0} \propto (\sin\theta/2)^{|m-M-m'+M'|} \tag{5.3.18a}$$

and

$$(l,m;L,M|l',m';L',M')^{\theta\to \pi} \propto (\cos\theta/2)^{|m-M+m'-M'|} \tag{5.3.18b}$$

The phenomenological consequences of these properties will emerge in the following sections.

5.4 Experimental determination of the CM reaction parameters

In this section we assume that we are given the CM multipole parameters for an arbitrarily prepared initial state and that we are able to measure the joint CM multipole parameters of the final state. The connection with measurements carried out in the Lab and the question of *how* one

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measures the multipole parameters will be dealt with in Sections 5.5 and 5.6.

From (5.3.1), exhibiting explicitly the ϕ-dependence, the outcome of an experiment is controlled by

$$t^{l'L'}_{m'M'}(C,D)\frac{d^2\sigma}{dtd\phi} = \left(\frac{2}{3}\right)^{n_\gamma}\frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{lL}(2l+1)(2L+1) \times \sum_{mM} e^{i\phi(M-m)} t^l_m(A) t^L_M(B) \times (l,m;L,M|l',m',L',M') \tag{5.4.1}$$

where, of course, the left-hand side is measured for an initial state specified by $t^l_m(A)$ and $t^L_M(B)$. (For photons we recall the discussion after eqn (5.3.4).)

There are, in general, two ways to utilize (5.4.1) experimentally in order to learn about the reaction parameters. The first way takes advantage of the simple ϕ-dependence to study asymmetries such as 'up–down' or 'left–right'. The most sophisticated example would involve measuring over the whole range of ϕ at fixed θ and then taking experimental averages of $e^{i\mu\phi}$ over the data at fixed θ, the μ being integers.

The second way looks at the changes induced in a measured observable when the density matrix of the initial state is altered, e.g. by reversal of the ordinary (rank-1) polarization of beam or target. For spin $> 1/2$ the method is less efficacious than for $s = 1/2$, where one can maximize the effect by fully reversing the sign of the polarization. It is not generally possible to reverse the sign of an arbitrary t^l_m when $l \geq 2$. We shall discuss an example where the t^l_m are altered by the passage through a magnetic field.

5.4.1 Unpolarized initial state

(i) Measurements of the generalized polarizing power and the final state polarization correlation parameters

Since all $t^l_m(A)$ and $t^L_M(B)$ are zero except $t^0_0(A) = t^0_0(B) = 1$, there is no ϕ-dependence left in (5.4.1) and, remembering that by definition

$$\int_0^{2\pi} d\phi \frac{d^2\sigma}{dtd\phi}(\text{unpol. initial state}) = \frac{d\sigma}{dt},$$

one obtains

$$t^{l'L'}_{m'M'}(C,D;\text{unpol. initial state}) = (0,0;0,0|l',m';L',M'). \tag{5.4.2}$$

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For an elastic reaction time reversal, eqn (5.3.10), then gives the parameters:

$$(l', m'; L', M'|0,0;0,0) = (-1)^{m'+M'}(0,0;0,0|l', m'; L', M').$$

Note that the parameters $(0,0;0,0|l', m';0,0)$ and $(0,0;0,0|0,0;L', M')$ are analogous to the usual polarizing power of the reaction but are here generalized to specify the rank of the polarization produced. We shall refer to them as the 'lm polarizing power'. The parameters with both l' and L' non-zero are generalizations of the final state polarization correlation parameters C_{ij} used in nucleon–nucleon scattering. All these parameters can be determined, in principle, using an unpolarized initial state.

(ii) Properties of the final state multipole parameters

From (5.4.2) and the properties (5.3.5)–(5.3.17) of the reaction parameters we learn the following properties of the final state CM joint (or effective) helicity-basis multipole parameters for a *parity-conserving* reaction with *unpolarized* initial state:

(a) $t^{lL}_{mM}(C, D)$ is independent of ϕ.

(b) As always $t^{lL}_{-m-M}(C, D) = (-1)^{m+M} t^{lL}_{mM}(C, D)^*$.

(c) $$t^{lL}_{mM} \text{ is } \left\{ \begin{matrix} \text{real} \\ \text{or} \\ \text{imaginary} \end{matrix} \right\} \quad \text{for } l + L \left\{ \begin{matrix} \text{even} \\ \text{or} \\ \text{odd} \end{matrix} \right\}. \qquad (5.4.3)$$

(d) Hence, $t^{lL}_{-m-M}(C, D) = (-1)^{l+L+m+M} t^{lL}_{mM}(C, D)$
and $t^{lL}_{00} = 0 \quad$ if $l + L$ is odd.

As an example, consider the famous result that the spin-polarization vector $\mathcal{P}$ of the final particles in a parity-conserving two-body reaction with unpolarized initial state must be perpendicular to the reaction plane. The properties (c) and (d) imply that $t^1_0 = 0$ and $t^1_{\pm1}$ is pure imaginary respectively. The result then follows from eqn (3.1.35).

In reverse, we note that a non-zero value of, say, the longitudinal component of $\mathcal{P}$ (i.e. the component along the particle's momentum) signals a parity violation. Some of the most beautiful electroweak experiments play upon just this feature.

(e) If particles C and D are identical then

$$t^{lL}_{mM}(\theta) = t^{Ll}_{Mm}(\pi - \theta). \qquad (5.4.4)$$

(f) If, in addition, particles A and B are identical then

$$t^{lL}_{mM}(\theta) = (-1)^{m+M} t^{Ll}_{Mm}(\theta) \qquad (5.4.5)$$

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and it follows that

$$t^{lL}_{mM}(\pi/2) = 0 \qquad \text{if } m + M \text{ is odd.} \tag{5.4.6}$$

As an example, in $pp \to pp$ we have $t^1_{\pm 1} = 0$ at $\theta = \pi/2$. Thus the spin-polarization vector $\mathcal{P}$ has magnitude 0 at $\theta = \pi/2$. Equivalently one can say that the polarizing power vanishes at $\theta = \pi/2$.

(g)
$$\begin{aligned} &\text{As } \theta \to 0 \qquad t^{lL}_{mM} \propto (\sin\theta/2)^{|m-M|} \\ &\text{As } \theta \to \pi \qquad t^{lL}_{mM} \propto (\cos\theta/2)^{|m-M|} \end{aligned} \tag{5.4.7}$$

Most of the above properties have obvious consequences for the helicity density matrix itself. The most interesting result follows from (d), namely

$$\rho_{c'd';cd} = (-1)^{c-c'+d-d'}\rho_{-c'-d';-c-d}. \tag{5.4.8}$$

Note that from eqn (3.1.43) the even- and odd-polarization parts of the final state density matrix are, in this case, simply the real and imaginary parts of ρ, i.e.

$$\begin{aligned} \rho_+ &= \text{ Re } \rho \\ \rho_- &= i \text{ Im } \rho. \end{aligned} \tag{5.4.9}$$

In the transversity basis the analogue for the effective density matrix of either of the final particles is

$$\rho^{\mathrm{T}}_{c'c} = 0 \quad \text{if} \quad c' - c \text{ is odd,} \tag{5.4.10}$$

thus giving ρ^{T} a 'chequerboard' pattern and forcing $[t^l_m]_{S^{\mathrm{T}}} = 0$ if m is odd.

5.4.2 Polarized beam, unpolarized target

We consider the measurement of the cross-section and the final state multipole parameters for an arbitrarily polarized beam. We also give some results for specific types of initial polarization.

(i) Measurement of cross-section asymmetries
– the generalized analysing power

From (5.4.1) we have, in general (for photons we recall the discussion after eqn (5.3.4))

$$\frac{d^2\sigma}{dtd\phi} = \left(\frac{2}{3}\right)^{n_\gamma} \frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{l,m}(2l+1)t^l_m(A)(l,m;0,0|0,0;0,0)e^{-im\phi}. \tag{5.4.11}$$

The parameters $(l,m;0,0|0,0;0,0)$ play the rôle of the 'analysing power' of the reaction for lm-type initial polarization, since they govern the magnitude of the asymmetry or ϕ-dependence in $d^2\sigma/dtd\phi$. From eqn

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(5.3.10) we see that for an elastic reaction the magnitudes of the lm polarizing power and lm analysing power are equal.[1]

In a typical polarized-beam experiment let the quantization axis for the beam have polar angles $\theta = \beta$, $\phi = \gamma$ in the rest frame of the beam (see Fig. 5.1). Let $\hat{t}^l_0$ be the (known) helicity multipole parameters in the frame whose Z-axis is along the quantization axis. Then the CM multipole parameters $t^l_m(A)$ needed for (5.4.11) are, from (3.3.1),

$$t^l_m(A) = e^{i\gamma m} d^l_{m0}(\beta)\hat{t}^l_0. \tag{5.4.12}$$

We refer to the plane $\phi = \gamma$, i.e. the plane containing the beam and the quantization axis, as the *quantization plane*. For this discussion there is no loss of generality in choosing $\gamma = 0$, so that the quantization plane is the XZ-plane. In detail (5.4.11) now becomes (recall that $(l, m; 0, 0|0, 0; 0, 0)$ is pure imaginary when l is odd)

$$\begin{aligned}\frac{d^2\sigma}{dt d\phi} = \left(\frac{2}{3}\right)^{n_\gamma} \frac{1}{2\pi}\frac{d\sigma}{dt}\Bigg(1 + \tfrac{1}{2}\sum_{l\geq 1}(2l+1)\hat{t}^l_0 \sum_{m\geq 0}(2-\delta_{m0})d^l_{m0}(\beta) \\ \times \left\{[1+(-1)^l]\cos m\phi - i[1-(-1)^l]\sin m\phi\right\} \\ \times (l, m; 0, 0|0, 0; 0, 0)\Bigg)\end{aligned} \tag{5.4.13}$$

where ϕ is the azimuthal angle measured from the quantization plane.

The asymmetries with respect to the quantization plane, or the detailed ϕ-dependence itself, can be used to isolate the combinations such as

$$\mathscr{C}_m = \sum_{l\geq m}(2l+1)\hat{t}^l_0 d^l_{m0}(\beta)(l, m; 0, 0|0, 0; 0, 0) \tag{5.4.14}$$

for each $m \geq 0$.

To measure the individual $(l, m; 0, 0|0, 0; 0, 0)$ one must be able to vary the $t^l_m(A)$ of the beam for each m. One way to do this is to deflect the polarized beam in a magnetic field, between the production reaction and the main reaction. We shall discuss one simple example.

[1] Because of this and eqn (5.4.2), an analogue of (5.4.11) appears in the non-relativistic literature with $(l, m; 0, 0|0, 0; 0, 0)$ replaced by $(-1)^m t^l_m(\theta)$, the latter being the CM final state multipole parameters for A when produced from an unpolarized initial state. We avoid this in practice since it confuses properties of the beams in special situations with properties of the reaction. Moreover in *relativistic double-scattering experiments* the $t^l_m(A)$ to be inserted into (5.4.11) are NOT the final state CM multipole parameters of the first reaction but, rather, are the $[t^l_m(\theta)]_{S_{LC}}$ discussed in subsection 3.3.2 (see eqn (3.3.14)).

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(ii) Use of a magnetic field to vary the initial state density matrix

Let a polarized beam having multipole parameters t'^{l}_{m} emerge in the XZ-plane and pass through a uniform magnetic field $\mathbf{B}$ oriented along OY. The particles are deflected around OY through an angle θ_{cyc} (the *cyclotron angle*) as measured in the Lab. (θ_{cyc} is zero for neutral particles.)

The helicity density matrix of the beam, considered, as usual, to be arriving along the Z-direction of the main reaction, will then be described by the CM helicity multipole parameters t^{l}_{m} given by

$$t^{l}_{m}(\delta) = \sum_{m'} d^{l}_{mm'}(\delta) t'^{l}_{m'} \tag{5.4.15}$$

where δ is the *angle of precession* of the spin vector of the particle during the passage through the magnetic field.

For a particle of mass m, charge Q, arbitrary spin s and total magnetic moment μ, the g-factor is defined by

$$\boldsymbol{\mu} = g\frac{Q}{2mc}\mathbf{s}. \tag{5.4.16}$$

Then the precession angle δ is given, in terms of the cyclotron angle, by

$$\delta = \left(\frac{g}{2} - 1\right)\frac{E_{\text{L}}}{mc^2}\theta_{\text{cyc}}. \tag{5.4.17}$$

where E_{L} is the Lab energy of the beam particles. For protons and deuterons one has

$$\left(\frac{g}{2} - 1\right)_{\text{proton}} = 1.79 \qquad \left(\frac{g}{2} - 1\right)_{\text{deuteron}} = -0.14.$$

It is thus difficult to cause a sizeable alteration of the t'^{l}_{m} for deuterons. Nevertheless a successful experiment of this type, using 410 MeV deuterons, was carried out by Button and Mermod (1960), and the idea seems to stem from Lakin (1955).

For neutral particles

$$\delta = -2\mu\left(\frac{E_{\text{L}}}{p_{\text{L}}c}\right)\frac{e_p}{2m_pc}Bd \tag{5.4.18}$$

where μ is the magnetic moment in units of the proton magneton, e_p and m_p are the charge and mass of the proton and d is the distance through the magnetic field traversed by the particle.

If B is measured in gauss and d in metres then

$$\delta \approx -3.2 \times 10^{-5}\mu\left(\frac{E_{\text{L}}}{p_{\text{L}}c}\right)Bd. \tag{5.4.19}$$

We have, for example,

$$\mu_{\text{neutron}} = -1.91 \qquad \mu_{\Lambda} = -0.61.$$

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Clearly one must utilize as many different values δ_i of δ as there are l-values appearing in the sum (5.4.14) and measure $C_m(\delta_i)$ for each. The individual $(l,m;0,0|0,0;0,0)$ are then obtained by solving a set of simultaneous equations.

(iii) Measurement of the generalized depolarization and polarization-transfer parameters

Consider the case where we measure the effective multipole parameters of particle C. From (5.4.1) and (5.4.2) we have

$$\left[t^{l'}_{m'}(C)\frac{d^2\sigma}{dtd\phi}\right]_{\text{pol.bm.}} - \left[t^{l'}_{m'}(C)\frac{d^2\sigma}{dtd\phi}\right]_{\text{unpol.}}$$
$$= \left(\frac{2}{3}\right)^{n_\gamma}\frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{\substack{l\geq 1\\ m}}(2l+1)t^l_m(A)$$
$$\times (l,m;0,0|l',m';0,0)e^{-im\phi} \tag{5.4.20}$$

where $d\sigma/dt$ is, of course, the unpolarized cross-section.

Equation (5.4.20) indicates the significance of the generalized *depolarization parameters* $(l,m;0,0|l',m';0,0)$ which can be measured by studying the asymmetry in ϕ of the left-hand side for several values of l' and m', bearing in mind the ϕ-independence of the second term on the left-hand side. As in (ii) above, the isolation of individual parameters will be possible only if the initial $t^l_m(A)$ can be varied.

If it is the density matrix of D that is measured, completely analogous equations hold and one determines thereby the generalized $A \to D$ *polarization-transfer parameters* $(l,m;0,0|0,0;L',M')$.

If the joint multipole parameters for C and D can be measured, one learns analogously about the 'three-spin' parameters $(l,m;0,0|l',m';L',M')$.

(iv) Properties of the final state

From (5.4.13), (5.4.20) and the properties (5.3.5)–(5.3.17) of the reaction parameters we find that the special properties of $d^2\sigma/dtd\phi$ and $t^{lL}_{mM}(C,D)$ for our main reaction, as listed below, hold for any of the following situations.

(s_1) The magnetically prepared beam has $\beta = \pi/2$, i.e. the quantization axis is perpendicular to the beam.

(s_2) The beam is a secondary beam emerging from a previous parity conserving reaction $R_1 : E + F \to A + G$, with unpolarized initial state, and our Y-axis is along $\mathbf{p}_E \times \mathbf{p}_A$.

(s_3) As in (s_2), but R_1 can have a polarized beam E, a polarized target F or both, provided that the quantization axes are normal to the scattering plane of R_1.

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The properties are:

(a) $d^2\sigma/dtd\phi$ is symmetric under reflection in the beam-containing plane that is perpendicular to the quantization plane, i.e. under $\phi \to -\phi$. (Exceptionally, if particles A have spin 1/2, this holds also for any angle β of the quantization axis; furthermore, Re t^{lL}_{mM} for $l+L$ even and Im t^{lL}_{mM} for $l+L$ odd are symmetric under $\phi \to -\phi$.)

(b) $t^{lL}_{mM}(C,D)$ now depends on ϕ in general.

(c) As always

$$t^{lL}_{-m-M} = (-1)^{m+M} t^{lL*}_{mM}. \tag{5.4.21}$$

(d)

$$t^{lL*}_{mM}(\phi) = (-1)^{l+L} t^{lL}_{-m-M}(-\phi). \tag{5.4.22}$$

Thus at $\phi = 0$

$$t^{lL}_{mM} \text{ is } \left\{ \begin{array}{c} \text{real} \\ \text{or} \\ \text{imaginary} \end{array} \right\} \text{ as } l+L \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{or} \\ \text{odd} \end{array} \right\}. \tag{5.4.23}$$

(e) Hence $t^{lL}_{-m-M}(\phi) = (-1)^{m+M+l+L} t^{lL}_{mM}(-\phi)$, and

$$t^{lL}_{00}(\phi = 0) = 0 \qquad \text{if } l+L \text{ is odd.} \tag{5.4.24}$$

As an example, an incoming beam with its spin-polarization vector $\boldsymbol{\mathcal{P}}^A$ perpendicular to the scattering plane satisfies the condition (s_1). Then use of (5.4.22) together with (3.1.35) tells us that the spin-polarization vectors $\boldsymbol{\mathcal{P}}^C$ and $\boldsymbol{\mathcal{P}}^D$ must also be perpendicular to the scattering plane.

(f) For an arbitrary initial polarization, if $C = D$ one has

$$t^{lL}_{mM}(\theta, \phi) = t^{Ll}_{Mm}(\pi - \theta, \phi + \pi). \tag{5.4.25}$$

(g)

$$\begin{array}{ll} \text{As } \theta \to 0 & t^{lL}_{mM} \propto (\sin\theta/2)^{\Lambda} \\ \text{As } \theta \to \pi & t^{lL}_{mM} \propto (\cos\theta/2)^{\Lambda} \end{array} \tag{5.4.26}$$

where $\Lambda = \max\{0,\ |m - M| - \overline{m}'\}$ and $\overline{m}'$ is the largest value of $|m'|$ that occurs in the $t^{l'}_{m'}(A)$ of the polarized beam.

For the density matrix itself, the results given in subsection 5.4.1 hold at $\phi = 0$. For $\phi \neq 0$ one has

$$\rho_{c'd';cd}(\phi) = (-1)^{c-c'+d-d'} \rho_{-c'-d';-c-d}(-\phi). \tag{5.4.27}$$

In particular ρ satisfies (5.4.8) at $\phi = 0$ under the experimental conditions (s_1)–(s_3).

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There are other results that hold for rather special circumstances. For example, if the quantization axis lies *in* the scattering plane and if the beam possesses *only* even-rank or *only* odd-rank polarization, then

$$\left(\text{Re } t^{lL}_{mM}\right)_{\text{pol.bm.}} \text{ for } \left\{\begin{array}{c}\text{even-rank}\\ \text{odd-rank}\end{array}\right\} \text{ polarization}$$

$$= \left(\text{Re } t^{lL}_{mM}\right)_{\text{unpol.}} \text{ for } l+L \left\{\begin{array}{c}\text{odd}\\ \text{even}\end{array}\right\} \tag{5.4.28}$$

and

$$\left(\text{ Im } t^{lL}_{mM}\right)_{\text{pol.bm.}} \text{ for } \left\{\begin{array}{c}\text{even-rank}\\ \text{odd-rank}\end{array}\right\} \text{ polarization}$$

$$= \left(\text{ Im } t^{lL}_{mM}\right)_{\text{unpol.}} \text{ for } l+L \left\{\begin{array}{c}\text{even}\\ \text{odd}\end{array}\right\}. \tag{5.4.29}$$

These are particularly powerful when the beam consists of spin-1/2 particles, since in this case only rank-1 polarization is possible. As an example, if the spin-polarization vector of the beam, $\mathcal{P}^A$, lies *in* the scattering plane then the spin-polarization vector $\mathcal{P}^C$ can have components both in $\left(\mathcal{P}^C_{\parallel}\right)$ and perpendicular to $\left(\mathcal{P}^C_{\perp}\right)$ to the scattering plane. Equations (5.4.28), (5.4.29) together with (3.1.35) tell us that $\mathcal{P}^C_{\perp}$ is independent of the vector $\mathcal{P}^A$, i.e. it is the same as it would have been if the beam were unpolarized.

5.4.3 Polarized target, unpolarized beam

The transcription of the results of subsection 5.4.2 to the situation where the target is polarized and the beam is unpolarized is absolutely straightforward. Only one point requires mention.

If the experiment involves a stationary target in the laboratory and if the target quantization axis is specified by polar angles $\theta = \beta'$, $\phi = \gamma'$ in the Lab frame, then in place of (5.4.12) one must have (see eqn (3.3.2))

$$t^L_M(B) = e^{-i\gamma' M} d^L_{M0}(\pi - \beta')\hat{t}^L_0. \tag{5.4.30}$$

If, however, the experiment involves colliding beams and if β', γ' refer to the quantization axis for B in its helicity rest frame S_B (see Fig. 3.1) then (5.4.12) should be used to calculate $t^L_M(B)$.

5.4.4 Polarized beam and target

For either the differential cross-section or the final state multipole parameters, the general result when the beam and target are both polarized is, from (5.4.1), of the form (for photons recall the discussion after eqn

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(5.3.4))

$$\left[t^{l'L'}_{m'M'}(C,D)\frac{d^2\sigma}{dtd\phi}\right]_{\substack{\text{pol.bm.}\\ \text{pol.targ.}}} - \left[t^{l'L'}_{m'M'}\frac{d^2\sigma}{dtd\phi}\right]_{\text{pol.bm.}}$$

$$- \left[t^{l'L'}_{m'M'}\frac{d^2\sigma}{dtd\phi}\right]_{\text{pol.targ.}} + \left[t^{l'L'}_{m'M'}\frac{d^2\sigma}{dtd\phi}\right]_{\text{unpol.}}$$

$$= \left(\frac{2}{3}\right)^{n_\gamma}\frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{\substack{l,L\geq 1\\ m,M}}(2l+1)(2L+1)t^l_m(A)t^L_M(B)$$

$$\times (l,m;L,M|l',m';L',M')e^{-i(M-m)\phi}. \tag{5.4.31}$$

Here, obviously, the state of polarization of beam (or target) must, where labelled, be the same on both sides of the equation.

The generalized *initial state polarization correlation parameters*,

$$(l,m;L,M|0,0;0,0),$$

which are the analogues of A_{ij} in nucleon–nucleon scattering, can be studied from the ϕ-dependence of the differential cross-section. Other three- and four-spin tensors require measurements of the final state multipole parameters.

For arbitrarily polarized beam and target the final state parameters t^{lL}_{mM} do not possess any special symmetry properties. If, however, the following experimental condition holds,

(s$_4$) the quantization axes of beam and target are parallel

then the properties (a)–(f) listed in subsection 5.4.2(iv) continue to hold *in the situations* (s$_1$)–(s$_3$).

The behaviour near $\theta = 0, \pi$ is now as follows.

$$\begin{aligned} &\text{For } \theta \to 0 \qquad t^{lL}_{mM} \propto (\sin\theta/2)^{\Lambda'} \\ &\text{For } \theta \to \pi \qquad t^{lL}_{mM} \propto (\cos\theta/2)^{\Lambda'} \end{aligned} \tag{5.4.32}$$

where $\Lambda' = \max\{0, \ |m-M| - \overline{\mu}\}$ and $\overline{\mu}$ is the largest value of $|m'-M'|$ that occurs in the $t^{l'}_{m'}(A)$ and $t^{L'}_{M'}(B)$ of the polarized beam and target.

5.5 The laboratory reaction parameters

For some colliding-beam experiments the measurements are carried out in the CM so that the multipole parameters that appear in (5.4.1) are the ones measured. For fixed targets in the laboratory what one actually measures are the multipole parameters in the Lab natural analysing frames (see subsection 3.3.2, especially Fig. 3.5). It is straightforward to translate these

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measurements into statements about the CM multipole parameters so that (5.4.1) again applies. However, for psychological reasons, experimentalists prefer to utilize the analogue of (5.4.1), which connects directly what goes into the experiment with what comes out in the Lab.

With all quantities measured respectively in the Lab analysing frames S_{LA}, S_{LB}, S_{LC}, S_{LD} one has

$$\left[t^{l'L'}_{m'M'}(C,D)\ \frac{d^2\sigma}{dt d\phi}\right]_{S_{LC},S_{LD}} = \left(\frac{2}{3}\right)^{n_\gamma} \frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{\substack{lL\\ mM}}(2l+1)(2L+1) \times \left[t^l_m(A)\right]_{S_{LA}}\left[t^L_M(B)\right]_{S_{LB}} \times (l,m;L,M|l',m';L',M')_{\mathrm{Lab}}\, e^{i(M-m)\phi} \tag{5.5.1}$$

where, from (3.3.14), the Lab *reaction parameters* are

$$(l,m;L,M|l',m';L',M')_{\mathrm{Lab}} = \sum_{m'',M''}(l,m;L,M|l',m'';L',M'') \times d^{l'}_{m''m'}(-\alpha_C)d^{L'}_{M''M'}(-\alpha_D) \tag{5.5.2}$$

with the angles α_C, α_D being given by (2.2.13).

Note that the Lab reaction parameters enjoy the same reality property (5.3.8) as do the CM ones.

Clearly the entire analysis of measurements in the CM can be taken over unchanged to discuss the extraction of the Lab reaction parameters from the Lab experimental data.

The symmetry properties that relate many of the CM parameters to each other will give rise, via (5.5.2), to similar, though more complicated-looking, relations amongst the Lab parameters.

Only the parity result looks simple:

$$(l,-m;L,-M|l',-m';L',-M')_{\mathrm{Lab}} = (-1)^{m+M+m'+M'}(-1)^{l+L+l'+L'} \times (l,m;L,M|l',m';L',M')_{\mathrm{Lab}} \tag{5.5.3}$$

from which one gets

$$(l,0;L,0|l',0;L',0)_{\mathrm{Lab}} = 0 \qquad \text{if } l+L+l'+L' \text{ is odd.} \tag{5.5.4}$$

For the other symmetries there is no point in writing down the general results. In a specific reaction it is best to write them out explicitly for the CM parameters and then to substitute the inverse of (5.5.2) to get the relations amongst the Lab reaction parameters.

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The properties of the CM final state multipole parameters $t^{lL}_{mM}(C,D)$ listed in subsections 5.4.1(ii) and 5.4.2(iv) hold also for the Lab multipole parameters provided that they are measured in the respective Lab analysing frames S_{LC} and S_{LD}.

5.6 Applications: Cartesian formalism for initial particles with spin 1/2

For particles of spin ≥ 1 the above formalism is the simplest and most compact. For spin-1/2 particles, however, one is accustomed to working with the Cartesian components of the spin-polarization vectors $\boldsymbol{\mathcal{P}} = \langle \boldsymbol{\sigma} \rangle$. Moreover only the values $l = 0, 1$ occur in the reaction parameters so that a simpler notation is possible.

The transformation between multipole parameters and components of the spin-polarization vector for spin 1/2 is

$$\sqrt{2l+1}\; t^l_m = \sum_\mu U_{lm;\mu} \mathscr{P}^\mu \tag{5.6.1}$$

where lm ($= 00, 11, 10, 1-1$) labels the rows, and μ ($= 0, X, Y, Z$) labels the columns, with $\mathscr{P}^0 \equiv 1$. The matrix U is given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix} \tag{5.6.2}$$

with $U^\dagger U = 1$.

5.6.1 The reaction spin 1/2 + spin 1/2 → spin 1/2 + spin 1/2

The Cartesian analogue of the CM relation (5.4.1) is then

$$\begin{aligned} \langle \sigma_{\alpha'}(C)\sigma_{\beta'}(D)\rangle \frac{d^2\sigma}{dt d\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt} \sum_{\alpha,\beta} \langle \sigma_\alpha(A)\rangle \langle \sigma_\beta(B)\rangle \\ \times (\alpha\beta|\alpha'\beta')_\phi, \end{aligned} \tag{5.6.3}$$

where $\alpha, \beta, \alpha', \beta'$ take on the values $0, X, Y, Z$, corresponding to the usual three Pauli matrices $\boldsymbol{\sigma}$ supplemented by a fourth matrix $\sigma_0 \equiv I$, the unit 2×2 matrix.[1]

Equation (5.6.3) relates the final state spin expectation values to those of the initial state in the CM. We shall make much use of this result when studying electroweak and QCD reactions.

It must be remembered that the directions X, Y, Z refer to the CM frame but that the physical interpretation of each $\langle \boldsymbol{\sigma}(K) \rangle$ is that it is the

[1] According to convention we use upper-case X, Y, Z here.

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mean spin vector for particle K in its helicity rest frame S_K reached from the CM.

The CM *Cartesian reaction parameters* are given by

$$
\begin{aligned}
(\alpha\beta|\alpha'\beta')_\phi = & \sum_{\substack{\text{repeated}\\ \text{indices}}} \sqrt{2l+1}\sqrt{2L+1}U_{lm;\alpha}U_{LM;\beta} \\
& \times (l,m;L,M|l',m';L',M')_\phi \\
& \times U^*_{l'm';\alpha'}U^*_{L'M';\beta'}\sqrt{2l'+1}\sqrt{2L'+1} \\
= & \frac{1}{4}\left(\frac{d\sigma}{dt}\right)^{-1} \text{Tr}\,(\sigma_\alpha\sigma_\beta H^\dagger\sigma_{\alpha'}\sigma_{\beta'}H)
\end{aligned} \tag{5.6.4}
$$

in complete analogy to (5.3.2).

The explicit ϕ-dependence of $(\alpha\beta|\alpha'\beta')_\phi$ can be found from (5.6.4) and (5.3.3). One gets

$$
(\alpha\beta|\alpha'\beta')_\phi = \mathscr{R}^z_{\alpha\alpha''}(\phi)\mathscr{R}^z_{\beta\beta''}(-\phi)(\alpha''\beta''|\alpha'\beta') \tag{5.6.5}
$$

with

$$
\mathscr{R}^z(\phi) = \begin{pmatrix} 1 & 0 \quad 0 \quad 0 \\ 0 & \\ 0 & R[r_z(\phi)] \\ 0 & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.6.6}
$$

In (5.6.5), as usual, absence of a ϕ-label implies $\phi = 0$.

In a similar fashion the Lab spin expectation values in the final state are related to those of the initial state, provided each is measured in the natural analysing frame $S_{\text{L}K}$, by

$$
\begin{aligned}
& \left[\langle \sigma_{\alpha'}(C)\sigma_{\beta'}(D)\rangle \frac{d^2\sigma}{dt d\phi} \right]_{S_{\text{L}C},S_{\text{L}D}} \\
& = \frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{\alpha,\beta} \left[\langle\sigma_\alpha(A)\rangle\,\langle\sigma_\beta(B)\rangle\right]_{S_{\text{L}A},S_{\text{L}B}} (\alpha\beta|\alpha'\beta')_{\substack{\phi\\ \text{Lab}}}
\end{aligned} \tag{5.6.7}
$$

where now the directions X, Y, Z for, say, particle K refer to the spin projections along the X-, Y-, Z-axes of the Lab frame $S_{\text{L}K}$.

From (5.6.4) and (5.5.2) we find

$$
\begin{aligned}
(\alpha\beta|\alpha'\beta')_{\text{Lab}} = & \sum_{\alpha'',\beta''} (\alpha\beta|\alpha''\beta'') \\
& \times \mathscr{R}^y_{\alpha''\alpha'}(-\alpha_C)\mathscr{R}^y_{\beta''\beta'}(-\alpha_D)
\end{aligned} \tag{5.6.8}
$$

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where, for any angle ω,

$$\mathscr{R}^y(\omega) = \begin{pmatrix} 1 & 0 \quad 0 \quad 0 \\ 0 & \\ 0 & R[r_y(\omega)] \\ 0 & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & 0 & \sin\omega \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\omega & 0 & \cos\omega \end{pmatrix}. \tag{5.6.9}$$

It follows that the azimuthal ϕ-dependence of the Lab reaction parameters is also given by (5.6.5).

It should be remembered that for $NN \to NN$ one has $\alpha_C = \theta_{\rm L}$ and $\alpha_D = \theta_{\rm R}$, where $\theta_{\rm L}$ is the Lab scattering angle and $\theta_{\rm R}$ is the Lab recoil angle; however for a reaction like $\Lambda p \to \Lambda p$ one will have $\alpha_D = \theta_{\rm R}$ but $\alpha_C \neq \theta_{\rm L}$ (see subsection 2.2.4).

Some of the results of the most exciting experiments on spin dependence in NN scattering carried out at Argonne have been reported using a slightly different choice of Lab reference frame for each particle. The Argonne Lab frames $S_{{\rm L}K}^{\rm ARG}$ are

$$\begin{aligned} S_{{\rm L}A}^{\rm ARG} &= S_{{\rm L}A} \\ S_{{\rm L}C}^{\rm ARG} &= S_{{\rm L}C} \\ S_{{\rm L}B}^{\rm ARG} &= S_{\rm L} = r_z(-\pi)r_y(-\pi)S_{{\rm L}B} \\ S_{{\rm L}D}^{\rm ARG} &= r_z(-\pi)S_{{\rm L}D} \end{aligned} \tag{5.6.10}$$

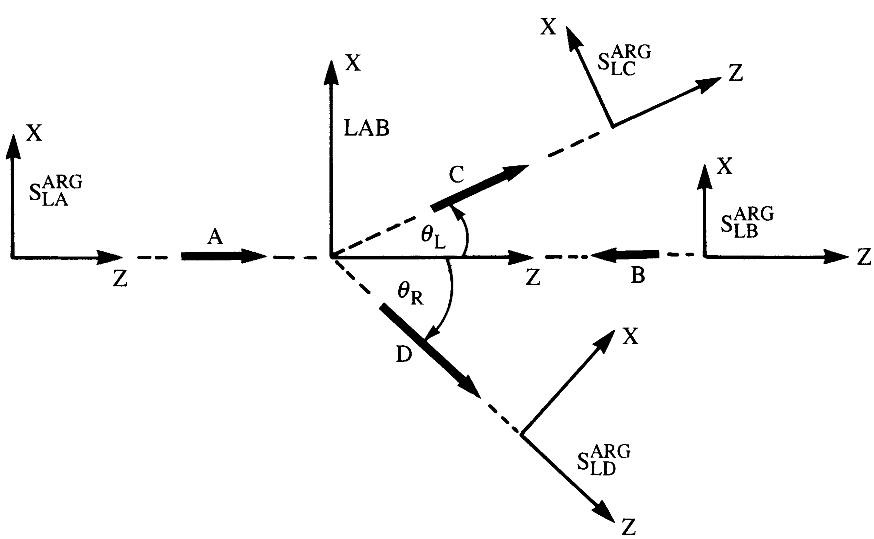

Fig. 5.2 The Argonne Lab frames for $A + B \to C + D$ as used in the reporting of several experiments.

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and, as shown in Fig. 5.2, all have their Y-axes in the same direction. The Argonne frames *seem* to be a simple, sensible choice. Their drawback is that they ignore the fact that both B and D have azimuthal angle $\phi = \pi$.

By convention the directions in *each* Argonne Lab frame are not referred to as X, Y, Z but by the symbols

$$\begin{aligned} L &= \text{longitudinal} = \text{along } OZ \\ N &= \text{normal} = \text{along } OY \\ S &= \text{sideways} = \text{along } OX \end{aligned}$$

and the Argonne final state measurements are related to their initial state ones by an equation almost identical to (5.6.7) but involving the Argonne reaction parameters

$$(\alpha\beta|\alpha'\beta')^{\text{ARG}}_{\text{Lab}} = \epsilon_\beta(\alpha\beta|\alpha'\beta')_{\text{Lab}}\nu_{\beta'} \tag{5.6.11}$$

where

$$\begin{aligned} \epsilon_0 = \epsilon_X = \nu_0 = \nu_Z = +1 \\ \epsilon_Y = \epsilon_Z = \nu_X = \nu_Y = -1 \end{aligned}$$

The Argonne reaction parameters are connected to the CM parameters by an obvious change in (5.6.8). Clearly they have essentially the same ϕ-dependence as the Lab reaction parameters.

The detailed symmetry properties of the CM Cartesian reaction parameters and of the Argonne Lab reaction parameters are given in Appendix 8, both for $NN \to NN$ and for the more general case of reactions like $\Lambda N \to \Lambda N$.

In Appendix 9 we list the 'shorthand' notation and the nomenclature commonly used for the Argonne Lab parameters, both for $NN \to NN$ and for $\Lambda N \to \Lambda N$.

Let us look at an example of the use of (5.6.3) or (5.6.7) for a parity-conserving reaction $A + B \to A + B$ where both beam and target may be polarized and one measures the differential cross-section. Let

$$\begin{aligned} \boldsymbol{\mathcal{P}}^A &= \left(\mathscr{P}^A_x, \mathscr{P}^A_y, \mathscr{P}^A_z\right) \\ \boldsymbol{\mathcal{P}}^B &= \left(\mathscr{P}^B_x, \mathscr{P}^B_y, \mathscr{P}^B_z\right) \end{aligned}$$

be the components of the spin-polarization vectors relative to the CM or Lab frames, in which A moves along OZ.

Then, according to subsection 3.3.1 (see also eqn (5.1.7)) the spin-polarization vector that must be used for B in (5.6.3) is

$$\tilde{\boldsymbol{\mathcal{P}}}^B = \left(\mathscr{P}^B_x, -\mathscr{P}^B_y, -\mathscr{P}^B_z\right).$$

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Thus, in either the CM or the Lab, using (5.6.5) in (5.6.3) or (5.6.7), one has

$$\begin{aligned}\frac{d^2\sigma}{dtd\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\Big\{1 &+ A^{(A)}\left(\mathscr{P}_y^A\cos\varphi - \mathscr{P}_x^A\sin\phi\right)\\ &- A^{(B)}\left(\mathscr{P}_y^B\cos\varphi + \mathscr{P}_x^B\sin\phi\right)\\ &+ A_{xx}\Big[\cos^2\varphi\ \mathscr{P}_x^A\mathscr{P}_x^B - \sin^2\phi\ \mathscr{P}_y^A\mathscr{P}_y^B\\ &\qquad + \cos\phi\sin\phi\left(-\mathscr{P}_x^A\mathscr{P}_y^B + \mathscr{P}_y^A\mathscr{P}_x^B\right)\Big]\\ &+ A_{yy}\Big[\sin^2\varphi\ \mathscr{P}_x^A\mathscr{P}_x^B - \cos^2\phi\ \mathscr{P}_y^A\mathscr{P}_y^B\\ &\qquad + \cos\phi\sin\phi\left(\mathscr{P}_x^A\mathscr{P}_y^B - \mathscr{P}_y^A\mathscr{P}_x^B\right)\Big]\\ &- A_{zz}\mathscr{P}_z^A\mathscr{P}_z^B + A_{zx}\mathscr{P}_z^A\left(\cos\varphi\ \mathscr{P}_x^B - \sin\varphi\ \mathscr{P}_y^B\right)\\ &- A_{xz}\mathscr{P}_z^B\left(\cos\phi\ \mathscr{P}_x^A + \sin\phi\ \mathscr{P}_y^A\right)\Big\}\end{aligned} \tag{5.6.12}$$

where we have used the following abbreviations for the various analysing powers:

$$\begin{aligned}A^{(A)} &\equiv (Y0|00)_{\mathrm{CM}} = (Y0|00)_{\mathrm{Lab}}\\ A^{(B)} &\equiv (0Y|00)_{\mathrm{CM}} = (0Y|00)_{\mathrm{Lab}}\\ A_{ij} &\equiv (ij|00)_{\mathrm{CM}} = (ij|00)_{\mathrm{Lab}}.\end{aligned} \tag{5.6.13}$$

The equality of the Lab and CM generalized analysing powers follows from (5.6.8).

Note that for identical fermions, e.g. for $pp \to pp$, $A_{zx} = -A_{xz}$ and $A^{(A)} = -A^{(B)}$. Conventionally one writes $A^{(A)} = -A^{(B)} = A_N$.

Equation (5.6.12) indicates how the analysing powers could be measured from a study of the azimuthal dependence or by comparing 'left' ($\phi = 0$) and 'right' ($\phi = \pi$) scattering, with various settings of the spin-polarization vectors.

Alternatively, if the analysing powers are known, the ϕ-dependence can be used to get some information about the spin-polarization vector of beam and target, an important issue in 'polarimetry'.

If the spin-polarization vector refers to the Argonne choice of reference frames then in the Argonne notation (5.6.12) will hold with the following substitutions:

$$\begin{array}{ccc} A_y^{(A)} \to A^{(A)} & A_y^{(B)} \to -A^{(B)} & \\ A_{xx} \to A_{SS} & A_{yy} \to -A_{NN} & A_{zz} \to -A_{LL}\\ A_{xz} \to -A_{SL} & A_{zx} \to A_{LS} & A_{yx} \to A_{NS}\end{array} \tag{5.6.14}$$

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5.6.2 *The reactions* spin $0+$ spin $1/2 \rightarrow$ spin $0+$ spin $1/2$ *and* spin $1/2+$ spin $1/2 \rightarrow$ spin $0+$ spin 0

We have in mind here processes like $\pi N \rightarrow \pi N$ and $\overline{N}N \rightarrow \pi\pi$.

(*i*) $0+1/2 \rightarrow 0+1/2$. The formulae (5.6.3), (5.6.4) and (5.6.7) apply provided the following simplifications are made.

(a) Suppress completely the labels α and α'.
(b) Replace the factor $1/4$ by $1/2$ in the relation (5.6.4) for $(\beta|\beta')_\phi$ in terms of the trace over H.

(*ii*) $1/2+1/2 \rightarrow 0+0$. Again (5.6.3), (5.6.4) and (5.6.7) apply provided that one suppresses the labels α' and β' everywhere.

5.6.3 *The reactions* spin $1/2+$ spin $1/2 \rightarrow$ arbitrary-spin particles

Undoubtedly many of the most interesting experiments in the next decade will consist of the production of high-spin particles from collisions of spin-1/2 particles. We therefore recast our general results (5.3.1) and (5.5.1) into a hybrid form that takes advantage of the Cartesian formalism for the initial particles but retains the multipole description for the final particles. We get in the CM

$$t^{lL}_{mM}(C,D)\frac{d^2\sigma}{dtd\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{\alpha,\beta}\langle\sigma_\alpha(A)\rangle\,\langle\sigma_\beta(B)\rangle \times (\alpha\beta|l,m;L,M)_\phi, \tag{5.6.15}$$

where the *hybrid CM reaction parameters* are given by

$$(\alpha\beta|l,m;L,M)_\phi = \frac{1}{4}\left(\frac{d\sigma}{dt}\right)^{-1}\ \mathrm{Tr}\ \left(\sigma_\alpha\sigma_\beta H^\dagger T^l_m T^L_M H\right). \tag{5.6.16}$$

For $\phi = 0$ one finds that the parameters are real or imaginary according as $l+L+\delta_{\alpha 0}+\delta_{\beta 0}$ is even or odd.

The Lab version of (5.6.15), using the natural Lab analysing frames, is

$$\left[t^{lL}_{mM}(C,D)\frac{d^2\sigma}{dtd\phi}\right]_{S_{\mathrm{L}C}S_{\mathrm{L}D}} = \frac{1}{2\pi}\frac{d\boldsymbol{\sigma}}{dt}\sum_{\alpha,\beta}\langle\sigma_\alpha(A)\rangle_{S_{\mathrm{L}A}}\langle\sigma_\beta(B)\rangle_{S_{\mathrm{L}B}} \times (\alpha\beta|l,m;L,M)^\phi_{\mathrm{Lab}} \tag{5.6.17}$$

where the directions α,β for A and B refer to the frames $S_{\mathrm{L}A}$, $S_{\mathrm{L}B}$, with

$$(\alpha,\beta|l,m;L,M)_{\mathrm{Lab}} = \sum_{m',M'}(\alpha\beta|l,m';L,M') \times d^l_{m'm}(-\alpha_C)d^l_{M'M}(-\alpha_D). \tag{5.6.18}$$

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If the Argonne Lab frames are used, the analogue of (5.6.17) has in it

$$(\alpha\beta|l,m;L,M)^{\text{ARG}}_{\text{Lab}} = \epsilon_\beta(\alpha\beta|l,m;L,M)_{\text{Lab}}(-1)^M \tag{5.6.19}$$

where, as earlier,

$$\epsilon_0 = \epsilon_X = 1 \qquad \epsilon_Y = \epsilon_Z = -1.$$

The ϕ-dependence of the hybrid CM or Lab reaction parameters is still given by (5.6.5) and the ϕ-dependence of the Argonne parameters then follows from (5.6.19).

Consider now the most general possible experiment for a $2 \to 2$ reaction with polarized spin-1/2 beam and target. Let $\boldsymbol{\mathcal{P}}^{(A)} = \left(\mathscr{P}^A_x, \mathscr{P}^A_y, \mathscr{P}^A_z\right)$ and $\tilde{\boldsymbol{\mathcal{P}}}^{(B)} = \left(\tilde{\mathscr{P}}^B_x, \tilde{\mathscr{P}}^B_y, \tilde{\mathscr{P}}^B_z\right)$ be the components of the spin-polarization vectors of beam and target specified in the correct helicity frames for A and B (see subsection 3.3.1) using either the CM or natural Lab analysing frames. Recall (see eqn (5.1.7)) that if one specifies the components of the initial spin-polarization vector in the CM or Lab frames where A moves along OZ then, for B, $\tilde{\boldsymbol{\mathcal{P}}}^{(B)} = (\mathscr{P}^B_x, -\mathscr{P}^B_y, -\mathscr{P}^B_z)$. Then, *with f standing for the final state labels $l,m;L,M$*, one has for $AB \to CD$

$$\begin{aligned}
t^{lL}_{mM}(C,D)\frac{d^2\sigma}{dtd\phi}
&= \frac{1}{2\pi}\left[t^{lL}_{mM}(C,D)\frac{d\sigma}{dt}\right]_{\text{unpol.}} \\
&+ \frac{1}{2\pi}\frac{d\sigma}{dt}\Big\{\mathscr{P}^A_x\left[\cos\phi\ (X0|f) - \sin\phi\ (Y0|f)\right] \\
&+ \tilde{\mathscr{P}}^B_x\left[\cos\phi\ (0X|f) - \sin\phi\ (0Y|f)\right] \\
&+ \mathscr{P}^A_y\left[\cos\phi\ (Y0|f) + \sin\phi\ (X0|f)\right] \\
&+ \tilde{\mathscr{P}}^B_y\left[\cos\phi\ (0Y|f) + \sin\phi\ (0X|f)\right] \\
&+ \mathscr{P}^A_z(Z0|f) + \tilde{\mathscr{P}}^B_z(0Z|f) + \mathscr{P}^A_z\tilde{\mathscr{P}}^B_z(ZZ|f) \\
&+ \mathscr{P}^A_x\tilde{\mathscr{P}}^B_x\Big[\cos^2\phi\ (XX|f) + \sin^2\phi\ (YY|f) \\
&\qquad\qquad - \cos\phi\sin\phi\ \langle(XY|f) + (YX|f)\rangle\Big] \\
&+ \mathscr{P}^A_y\tilde{\mathscr{P}}^B_y\Big[\sin^2\phi\ (XX|f) + \cos^2\phi\ (YY|f) \\
&\qquad\qquad + \cos\phi\sin\phi\ \langle(XY|f) + (YX|f)\rangle\Big] \\
&+ \mathscr{P}^A_x\tilde{\mathscr{P}}^B_y\Big[\cos^2\phi\ (XY|f) - \sin^2\phi\ (YX|f) \\
&\qquad\qquad + \cos\phi\sin\phi\,\langle(XX|f) - (YY|f)\rangle\Big] +
\end{aligned}$$

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$$+\mathscr{P}_y^A\tilde{\mathscr{P}}_x^B\Big[\cos^2\phi\ (YX|f)-\sin^2\phi\ (XY|f)$$
$$+\cos\phi\sin\phi\ \langle(XX|f)-(YY|f)\rangle\Big]$$
$$+\mathscr{P}_x^A\tilde{\mathscr{P}}_z^B\left[\cos\phi\ (XZ|f)-\sin\phi\ (YZ|f)\right]$$
$$+\mathscr{P}_z^A\tilde{\mathscr{P}}_x^B\left[\cos\phi\ (ZX|f)-\sin\phi\ (ZY|f)\right]$$
$$+\mathscr{P}_y^A\tilde{\mathscr{P}}_z^B\left[\cos\phi\ (YZ|f)+\sin\phi\ (XZ|f)\right]$$
$$+\mathscr{P}_z^A\tilde{\mathscr{P}}_y^B\left[\cos\phi\ (ZY|f)+\sin\phi\ (ZX|f)\right]\Big\} \tag{5.6.20}$$

where the reaction parameters and the final state multipole parameters should carry an appropriate label to indicate which set of reference frames is implied.

The above is completely general in the sense that no discrete symmetries have been assumed.

We mention some of the simpler properties of the hybrid parameters.

(i) *Reality*. Using $T_m^{l\dagger}=T_{-m}^l(-1)^m$ and $\sigma_\alpha^\dagger=\sigma_\alpha$ in (5.6.16), one finds

$$(\alpha\beta|l,m;L,M)^*=(-1)^{m+M}(\alpha\beta|l,-m;L,-M). \tag{5.6.21}$$

(ii) *Parity*. For *any* one set of the above reference frames one has

$$(\alpha\beta|l,m;L,M)=\xi_\alpha^{\mathscr{P}}\xi_\beta^{\mathscr{P}}(-1)^{l+L+m+M}(\alpha\beta|l,-m;L,-M) \tag{5.6.22}$$

where $\xi_0^{\mathscr{P}}=\xi_Y^{\mathscr{P}}=+1$ and $\xi_X^{\mathscr{P}}=\xi_Z^{\mathscr{P}}=-1$. Thus

$$(\alpha\beta|l,0;L,0)=0 \quad \text{if } \xi_\alpha^{\mathscr{P}}\xi_\beta^{\mathscr{P}}(-1)^{l+L}=-1. \tag{5.6.23}$$

Combining (5.6.22) and (5.6.21) we have

$$(\alpha\beta|l,m;L,M) \text{ is } \left\{\begin{matrix}\text{real}\\ \text{imaginary}\end{matrix}\right\} \text{ as } \xi_\alpha^{\mathscr{P}}\xi_\beta^{\mathscr{P}}(-1)^{l+L}=\pm 1. \tag{5.6.24}$$

(iii) *Identical particles*. If $A=B$, then for the CM reaction parameters

$$(\alpha\beta|l,m;L,M)^\theta=(-1)^{m+M}(\beta\alpha|l,m;L,M)^{\pi-\theta}. \tag{5.6.25}$$

As regards the properties of the final state multipole parameters they are of course no different from those discussed in subsections 5.4.1(ii) and 5.4.2(iv), provided that they are measured in the correct reference frames of the set being used.

Equations (5.6.20), (5.6.23), and (5.6.24) give a complete description of the states of polarization of the final particles that are possible for various choices of the initial state polarizations with and without the imposition of parity invariance.

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5.6.4 Connection between photon and spin-1/2 *induced reactions*

We saw in subsection 3.1.12 that because of the absence of states $|m = 0\rangle$ for photons, the helicity density matrix for a photon is essentially a 2×2 matrix and can therefore be expressed in terms of the Pauli matrices.

Upon comparing (3.1.85) and (3.1.88) with the result (3.1.23) for spin-1/2 particles, it is clear that we can map any formulae for the observables in the reaction

$$\text{spin } 1/2 + B \to X,$$

where the spin-1/2 particles have a spin-polarization vector $\boldsymbol{\mathcal{P}}$, into the corresponding formula for the observable for the photon-induced reaction

$$\gamma + B \to X.$$

This is done by making the following replacements:

(a)
$$\mathscr{P}_x \to -\cos 2\gamma\ \mathscr{P}_{\text{lin}} \qquad \mathscr{P}_y \to -\sin 2\gamma\ \mathscr{P}_{\text{lin}}$$

corresponding to a photon linearly polarized in the XY-plane at angle γ to OX, with degree of linear polarization $\mathscr{P}_{\text{lin}}$ (see subsection 3.1.12(ii));

(b)
$$\mathscr{P}_z \to \mathscr{P}_{\text{circ}}$$

corresponding to a circularly polarized photon with circular polarization $\mathscr{P}_{\text{circ}}$ (see subsection 3.1.12(ii));

(c)
$$H_{X;\lambda\lambda_B} \to H_{X;\lambda_\gamma=2\lambda,\lambda_B}$$

for the helicity amplitudes, where λ refers to the spin-1/2 particle.

5.7 Non-linear relations amongst the observables

Consider a reaction $A + B \to C + D$ for which, after application of all the symmetries of the situation, there are found to exist n independent helicity amplitudes. Let us label these H_j with $j = 1, \ldots, n$. Since all the observables are quadratic in the H_j there will clearly exist n^2 linearly independent observables O_α of the form

$$O_\alpha = \sum_{j,k} a^\alpha_{jk} H_j H^*_k \tag{5.7.1}$$

with known coefficients a^α_{jk}. Knowing the value of the n^2 observables O_α is tantamount to knowing the value of the n^2 quantities

$$O_{jk} \equiv H_j H^*_k. \tag{5.7.2}$$

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However, the number of experiments that can be carried out, i.e. the number of reaction parameters that exist, is

$$N = (2s_A + 1)(2s_B + 1)(2s_C + 1)(2s_D + 1),$$

a number that is generally much larger than n^2. For example, in elastic nucleon–nucleon scattering $n^2 = 25$ whereas $N = 256$. The symmetry properties of the reaction parameters given in subsection 5.3.1 yield linear relations amongst them just such as to reduce their number to n^2 independent parameters. The case of nucleon–nucleon scattering is displayed in some detail in Appendix 8.

To start with, though, there are only $2n$ independent real functions, the real and imaginary parts of the H_j. Moreover because the observables are quadratic functions of the H_j, one overall phase is irrelevant and can never be determined experimentally. Thus in fact all experiments must be describable in terms of $2n - 1$ real functions. This implies that there must exist $n^2 - (2n - 1) = (n - 1)^2$ relations amongst the n^2 observables O_α. As will be seen, they are *non-linear* relations. The method for finding them is due to Klepikov, Kogan and Shamanin (1967) and Bourrely and Soffer (1975).

Consider the matrix O whose elements are the O_{jk} of (5.7.2). It is an $n \times n$ hermitian, positive matrix of rank 1. It is clear from (5.7.2) that

$$O_{jk}O_{lm} = O_{jm}O_{lk} \tag{5.7.3}$$

from which it follows that

$$O^2 = O \;\; \mathrm{Tr} \;\; O. \tag{5.7.4}$$

Conversely one can show that if a given square matrix has elements O_{jk} such that (with no summation over repeated indices)

$$(O_{ik})^2 = O_{ii}O_{kk} \tag{5.7.5}$$

for all $i \neq k$ and

$$O_{ij}O_{jk} = O_{ik}O_{jj} \tag{5.7.6}$$

for all $i \neq k$ and any *one* value of j ($\neq i$ or k), then (5.7.3) and (5.7.4) follow. Equations (5.7.5) and (5.7.6) are the necessary and sufficient conditions for O_{jk} to be of the form (5.7.2). They constitute the desired non-linear relations amongst the observables.

The best-known example occurs in $\pi N \to \pi N$ where the reaction parameters

$$P \equiv (00|0N)^{\mathrm{ARG}}_{\mathrm{Lab}}$$

$$A \equiv (0L|0S)^{\mathrm{ARG}}_{\mathrm{Lab}}$$

$$R \equiv (0S|0S)^{\mathrm{ARG}}_{\mathrm{Lab}}$$

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(the nomenclature P, A, R is historical) satisfy

$$P^2 + R^2 + A^2 = 1. \tag{5.7.7}$$

For nucleon–nucleon scattering the 16 non-linear relations can be found in Bourrely and Soffer (1975).

5.8 Multiparticle and inclusive reactions

We consider the simplest kind of multiparticle production process

$$A + B \to C + D_1 + D_2 + D_3 + \cdots$$

where all the variables specifying the multiparticle

$$X = D_1 + D_2 + D_3 + \cdots$$

are integrated over (except its mass M_X), i.e. we consider the *single-particle inclusive reaction*

$$A + B \to C + X$$

where A, B, C can have arbitrary spins. It does not matter, in what follows, whether X contains a fixed number of particles or whether we sum over different numbers of particles.

5.8.1 CM reaction parameters and final state density matrix

For each fixed number of particles, X can be considered as a composite 'particle', with many internal degrees of freedom and with a definite momentum $P = p_A + p_B - p_C$. It has a variable spin S_X and helicity Λ. In summing over all possible configurations of the particles that make up X we also sum over all the values of Λ incoherently.

It is then clear that, in so far as helicity dependence is concerned, the unnormalized final state density matrix for C, $\rho'(C)$, is given, in analogy with (5.2.1), by

$$\rho'_{cc'}(C; s, t, M_X^2, \phi) = \sum_{\substack{\text{internal}\\ \text{variables}}} \sum_{\Lambda} \sum_{\substack{ab\\ a'b'}} H_{c\Lambda;ab}(s, t, M_X^2, \phi) \times \rho_{\mathrm{iab;a'b'}}(A, B) H^*_{c'\Lambda;a'b'}(s, t, M_X^2, \phi) \tag{5.8.1}$$

where the H are generalized helicity amplitudes the knowledge of whose detailed properties is not necessary for our discussion.

We normalize the H in such a way that

$$\mathrm{Tr}\ \rho'(C) = 2\pi s \frac{d^3\sigma}{dt d\phi dM_X^2}. \tag{5.8.2}$$

From now on $\rho(C)$ will mean the properly normalized density matrix. It is clear that those symmetry properties of $\rho(C)$ in the $2 \to 2$ process

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$A+B \to C+D$ which arise from parity conservation (for example (5.4.27)) and in which the spin s_X and the intrinsic parity η_X do not appear explicitly will continue to hold for $\rho(C)$ in $A+B \to C+X$.

In complete analogy with (5.3.1) for $2 \to 2$ scattering, we now have for the final state multipole parameter of C, in the CM,

$$\frac{d^3\sigma}{dtd\phi dM_X^2} t^{l'}_{m'}(C) = \frac{1}{2\pi}\frac{d^2\sigma}{dtdM_X^2}\sum_{\substack{lm\\LM}}(2l+1)(2L+1)t^l_m(A)t^L_M(B) \times (l,m;L,M|l',m')^{\text{inc}}_{\phi} \tag{5.8.3}$$

where

$$(l,m;L,M|l',m')^{\text{inc}}_{\phi} \equiv \sum_{\substack{\text{internal}\\\text{variables}}} (l,m;L,M|l',m';L'=0,M'=0)^{AB\to CX}_{\phi} \tag{5.8.4}$$

and now depends on ϕ, t and M_X^2 as well, and where L', M' refer to 'particle' X.

Because the sum is incoherent, $(l,m;L,M|l',m')^{\text{inc}}_{\phi}$ has the same properties (5.3.3), (5.3.5), (5.3.6), (5.3.7), (5.3.8) and (5.3.11) as it would have had if X were a single spinless particle. It does not enjoy those properties like (5.3.16) that depend upon the intrinsic parity of 'X'.

As a consequence the properties of $t^l_m(C)$ exactly mimic those given in subsections 5.4.1(ii) and 5.4.2(iv) for $A+B \to C+D$, if s_D is put equal to zero.

In particular, for an unpolarized initial state and a parity-conserving reaction, the polarization vector of C must be perpendicular to the plane defined by $\mathbf{p}_A$ and $\mathbf{p}_C$.

If A and B have spins 1/2 then (5.8.3) can be re-cast in an obvious way (Section 5.6) into a pure Cartesian or a hybrid Cartesian–spherical form. (See (5.6.3), (5.6.12) in which s_D would be put equal to zero.)

The CM reaction parameters can, in the present case, be related to discontinuities across the cut in M_X^2 of the forward $3 \to 3$ amplitudes for the process $A+B+\overline{C} \to A+B+\overline{C}$ (see Fig. 5.3). The original work, relating only to the unpolarized case, is due to Mueller (1970). For the generalization to spin-dependent terms see the work of Goldstein and Owens (1976). In the notation of the latter, our reaction parameter is

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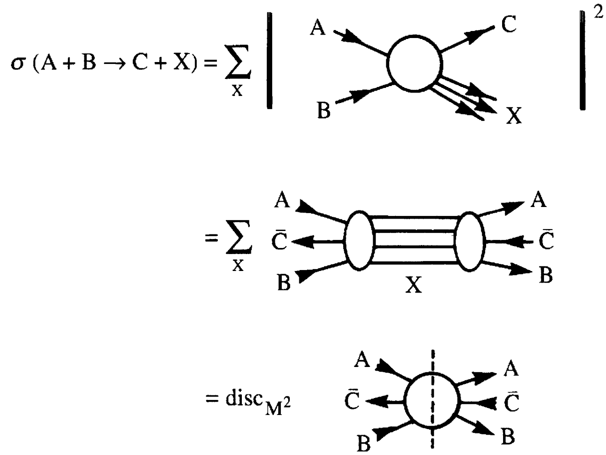

Fig. 5.3. Schematic form of the Mueller formula for inclusive cross-sections: see the text for a discussion of the quantities.

given by

$$(l, m; L, M|l', m')^{\text{inc}}_{\phi} = \sum_{\substack{ab\bar{c} \\ a'b'\bar{c}'}} (T^{l'\dagger}_{m'})_{\bar{c}'\bar{c}} (T^{l}_{m})_{a'a} (T^{L}_{M})_{b'b} \times \frac{\operatorname{disc}_{M_X^2}\, g_{a'b'\bar{c}';ab\bar{c}}}{\sum_{ab\bar{c}} \operatorname{disc}_{M_X^2}\, g_{ab\bar{c};ab\bar{c}}} \tag{5.8.5}$$

where $g_{a'b'\bar{c}';ab\bar{c}}$ is the forward $3 \to 3$ amplitude and 'disc' refers to the discontinuities across the cut in M_X^2.

Near the forward and backward directions for C, i.e. $\theta = 0, \pi$, the $3 \to 3$ amplitude has the behaviour

$$g_{a'b'\bar{c}';ab\bar{c}} \propto (\sin\theta/2)^{\zeta_1} (\cos\theta/2)^{\zeta_2} \tag{5.8.6}$$

where $\zeta_1 = |a - b - \bar{c} - a' + b' + \bar{c}'|$ and $\zeta_2 = |a - b + \bar{c} - a' + b' - \bar{c}'|$, which implies for the reaction parameters the behaviour

$$(l, m; L, M|l', m')^{\text{inc}} \propto (\sin\theta/2)^{|m-M+m'|} (\cos\theta/2)^{|m-M-m'|}. \tag{5.8.7}$$

We end this section with a brief comment on *two-particle semi-inclusive reactions* of the type

$$A + B \to C + D + E_1 + E_2 + \cdots$$

where all variables specifying the multiparticle state $X = E_1 + E_2 + \cdots$ are integrated over, except its mass M_X.

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The properties of the joint density matrix for C and D will then be analogous to those of $C + D$ in the $2 \to 3$ process

$$A + B \to C + D + X$$

where X is considered to be a particle of spin zero, but of indefinite parity.

The outcome of an experiment will be controlled by the reaction parameters

$$(l, m; L, M | l', m'; L', M')^{\text{inc}} = \sum_{\substack{\text{internal}\\ \text{variables}\\ \text{of } X}} (l, m; L, M | l', m'; L', M'; 0, 0)^{AB \to CDX}. \tag{5.8.8}$$

The most important new element is that the polarization vectors of C and D need not be perpendicular to the reaction planes ABC or ABD respectively, even for unpolarized initial beam and target.

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