Can. J. Math., Vol. XXXII, No. 6, 1980, pp. 1361-1371

ARITHMETICAL SEMIGROUP RINGS

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1. Introduction. Throughout this paper the ring R and the semigroup S are commutative with identity; moreover, it is assumed that S is cancellative, i.e., that S can be embedded in a group. The aim of this note is to determine necessary and sufficient conditions on R and S that the semigroup ring $R[S]$ should be one of the following types of rings: principal ideal ring (PIR), ZPI-ring, Bezout, semihereditary or arithmetical. These results shed some light on the structure of semigroup rings and provide a source of examples of the rings listed above. They also play a key role in the determination of all commutative reduced arithmetical semigroup rings (without the cancellative hypothesis on S) which will appear in a forthcoming paper by Leo Chouinard and the authors [**4**].

Our results are motivated in large part by the paper [**11**] of R. Gilmer and T. Parker. In particular, Theorem 1.1 of [**11**] asserts that if R and S are as above and, moreover, if S is torsion-free, then the following are equivalent conditions: (1) $R[S]$ is a Bezout ring; (2) $R[S]$ is a Prüfer ring; (3) R is a (von Neumann) regular ring and S is isomorphic to either a subgroup of the additive rationals or the positive cone of such a subgroup. One could very naturally include a fourth condition, namely: (4) $R[S]$ is arithmetical. L. Fuchs [**7**] defines an *arithmetical ring* as a commutative ring with identity for which the ideals form a distributive lattice. Since a Prüfer ring is one for which $(A + B) \cap C = (A \cap C) + (B \cap C)$ whenever at least one of the ideals A, B or C contains a regular element (see [**18**]), arithmetical rings are certainly Prüfer. On the other hand, it is well known that every Bezout ring is arithmetical, so that (4) is indeed equivalent to (1)–(3) in Theorem 1.1.

In Theorem 3.6 of this paper we drop the requirement that S be torsion-free and determine necessary and sufficient conditions for the semigroup ring of a cancellative semigroup to be arithmetical. Examples are included to show that for these more general semigroup rings, the equivalences of the torsion-free case are no longer true. Theorems 4.1 and 4.2 provide characterizations of semigroup rings that are ZPI-rings and PIR's. Again, the corresponding results in [**18**] for torsion-free semigroups fail to hold in the more general case.

We would like to thank Leo Chouinard for showing us how to remove

Received February 7, 1979 and in revised form July 25, 1979.

the requirement that the ring R be reduced when S is torsion in our original statement of Theorem 3.6.

2. Terminology and notations. As in [**11**], elements of the semigroup ring $R[S]$ will be written in Northcott's "polynomial" notation: S is written additively with identity 0 and a typical element of $R[S]$ is $r_1X^{s_1} + r_2X^{s_2} + \ldots + r_nX^{s_n}$, $r_i \in R$ and $s_i \in S$, $i = 1, \ldots, n$. For basic properties of semigroups, we refer to [**5**] and for group rings, to [**16**]. For this note a *local ring* is a ring with unique maximal ideal, a *valuation ring* is a ring whose ideal lattice is totally ordered and a *regular ring* is understood to be a von Neumann regular ring. A regular element means a nonzero-divisor and the *total quotient ring* of R, denoted by $\mathrm{tot}(R)$, is the localization of R at the set of regular elements of R. The localization of R at a prime ideal P is denoted by R_P and the localization of R at a multiplicative system T of R is generally denoted by $T^{-1}R$.

We say that the semigroup S has a *group of quotients* G if S embeds in the group G and G is (group) generated by the elements of S. Such a G exists exactly when S is cancellative. In this case if T is a subsemigroup of S, then $T^{-1}S$ is the subsemigroup of G generated by S and inverses of elements of T.

If T is a multiplicative system in the semigroup ring $R[S]$, the following items are easily verified and will be used freely:

1. If $T \subseteq R$, then $(T^{-1}R)[S] \cong T^{-1}(R[S])$.
2. If $T \subseteq S$ and S is cancellative, then T consists of regular elements of the ring $R[S]$ and

$$T^{-1}(R[S]) \cong R[T^{-1}S].$$

If G is a group (always assumed to be abelian in this paper), we denote the torsion subgroup of G by tG. In the case that G is torsion and p is a prime integer, then G_p is the p-primary component of G. If G is an ordered group, then the *positive cone* of G is the set $G^+ = \{x \in G | x \geqq 0\}$. (Note that $0 \in G^+$.)

Finally, we denote the rational integers and rational numbers by $\mathbf{Z}$ and $\mathbf{Q}$, respectively.

3. The main result. First we require a slight extension of Lemma 2.1 of [**11**] concerning group rings of direct sums of groups; since the construction of the isomorphism required in Lemma 3.1 (and its inverse) are obvious, we omit the proof. When S is a semigroup with 0, the notation "$S = A \oplus B$" is understood to mean that S is generated by the union of subsemigroups A and B, and $A \cap B = 0$.

LEMMA 3.1. *Let R be a ring and S a (commutative, cancellative with* 0) *semigroup such that $S = G \oplus T$ where G is a group. Then* $R[S] \cong (R[G])[T]$.

Next, it is necessary to identify local semigroup rings.

LEMMA 3.2. *The semigroup ring $R[S]$ is a local ring if and only if R is a local ring with maximal ideal M and S is a p-group, where $p = \operatorname{char}(R/M)$.*

Proof. The sufficiency of the above conditions appears in [**12**] and [**19**]. In case S is a group, the necessity has been proved in [**3**]. We complete the proof by showing that if $R[S]$ is local, then S must be a group.

Let I be a proper ideal of S (i.e., $I + S \subseteq I$). It suffices to show that $I = \emptyset$[**5**, p. 6]. If not, let S/I be the semigroup of equivalence classes of the congruence $\sim$ defined on S by $s \sim t$ if and only if $s = t$ or $s, t \in I$ (i.e., S/I is the Rees factor semigroup of I). The natural map of S onto S/I induces a map of $R[S]$ onto $R[S/I]$. So $R[S/I]$ is local. But if ∞ is the equivalence class I, then X^∞ is an idempotent of the ring $R[S/I]$ which is not 0 or 1. Thus $I = \emptyset$, as desired.

Next, we need the following characterization of semigroup rings which are valuation rings, due to I. Kuzhukhov [**15**], which we state without proof.

LEMMA 3.3. *The semigroup ring $R[S]$ is a valuation ring if and only if R is a field of characteristic p and S is a cyclic or quasicyclic p-group.*

Before proving the next lemma, let us recall the following important description of arithmetical rings due to C. Jensen [**14**, p. 115]: The ring R is arithmetical if and only if every localization of R at a maximal ideal is a valuation ring. It follows rather easily that any localization of an arithmetical ring at a multiplicative system yields an arithmetical ring.

LEMMA 3.4. *The direct limit of arithmetical rings is arithmetical.*

Proof. Let (R_i, μ_{ij}) be a directed system of arithmetical rings and let M be a maximal ideal of $R = \text{proj. lim } R_i$. Let $M_i = M \cap R_i$, the contraction of M relative to the direct limit map $\mu_i : R_i \rightarrow R$. Then M_i is a prime ideal of R_i and for $i \leqq j$, $\mu_{ij}(M_i) \subseteq M_j$. By a standard result from ([**10**], p. 130) R_M is the direct limit of the system $((R_i)_{M_i}, \mu_{ij}^*)$ where the μ_{ij}^* are the natural maps induced by μ_{ij}. But each $(R_i)_{M_i}$ is a valuation ring, whence each localization R_M of R is also a valuation ring. Therefore, R is arithmetical, as desired.

Finally, we require a technical lemma about a congruence which is needed for the statement of our main result. The proof is obvious and therefore we omit it.

LEMMA 3.5. *Let S be a (cancellative) semigroup with group of quotients G. The relation ρ defined on S by $x\rho y$ if and only if $x = y + f$ for some f in $tS = \{s \in S | ns = 0$ for some positive integer $n\}$ is a congruence on S.*

Moreover, if $tG \subseteq S$, then the semigroup S/ρ of congruence classes of ρ is torsion-free and cancellative with group of quotients G/tG.

We can now prove our main result, which characterizes the arithmetical semigroup rings of cancellative, commutative semigroups with zero in terms of the constituent ring and semigroup. We remind the reader that the condition that a group ring $R[G]$ be a regular ring is well known [**16**, p. 155] to be equivalent to the following conditions on R and G: R is a regular ring and G is a torsion group such that the order of every element of G is a unit in R.

THEOREM 3.6. *Let R be a commutative ring and S a commutative cancellative semigroup with group of quotients G. Then $R[S]$ is arithmetical if and only if one of the following holds:*

(1) *tG is a proper subsemigroup of S, $R[tG]$ is regular and S/ρ is isomorphic to an additive subgroup of $\mathbf{Q}$ or the positive cone of such a group.*

(2) *R is arithmetical and $S = G$ is a torsion group such that if $G_p \neq 0$ for some prime $p = \operatorname{char}(R/\mathfrak{m})$, where $\mathfrak{m}$ is a maximal ideal of R, then G_p is cyclic or quasicyclic and $R_\mathfrak{m}$ is a field.*

Proof. We first assume that S contains elements of infinite order and show that the arithmeticality of $R[S]$ is equivalent to (1). Suppose that $R[S]$ is arithmetical. If u is a nonzero element of G of finite order, then $X^u \in \operatorname{tot}(R[S])$ and X^u is integral over $R[S]$. But $R[S]$ is integrally closed by Theorem 10.18 of [**18**, p. 237]. Therefore $X^u \in R[S]$ and $u \in S$. It follows that if ρ is the congruence of Lemma 3.5, then S/ρ is a torsion-free cancellative semigroup with 0 (here $S/\rho = S$ if S is itself torsion-free). The natural map $S \to S/\rho$ induces a map of $R[S]$ onto $R[S/\rho]$; so $R[S/\rho]$ is arithmetical. By Theorem 3.1 of [**11**], R is regular and S/ρ is described as in (1). It remains to show that orders of elements of tG are units in R. By way of contradiction, suppose that for some prime integer p and maximal ideal $\mathfrak{m}$ of R, $p \cdot 1 \in \mathfrak{m}$ and $(tG)_p \neq 0$. Since G is rank one, there is a free cyclic subgroup F of G such that G/F is torsion. The group epimorphisms $G \to G/F \to (G/F)_p$ induce a map of $R[G]$ onto $R[(G/F)_p]$; therefore, $R[(G/F)_p]$ and its ring of quotients $T = R_\mathfrak{m}[(G/F)_p]$ are arithmetical. But T is already local by Lemma 3.2; hence T is a valuation ring. Thus, $(G/F)_p$ is cyclic or quasicyclic by Lemma 3.3. But tG embeds in G/F since $F \cap tG = 0$, so $(tG)_p$ is cyclic or quasicyclic. In the latter case $(tG)_p$ is an injective group and hence a direct summand of G; in the former case $(tG)_p$ is a bounded pure subgroup of G, hence by Theorem 27.5 of [**8**, p. 118], $(tG)_p$ is again a summand of G. Thus

$$R[G] \cong (R[(tG)_p])[H]$$

by Lemma 3.1. Taking $R[(tG)_p]$ in place of R in the first part of the proof, we get that $R[(tG)_p]$ is regular. Since orders of nonzero elements of $(tG)_p$

are not invertible in R, this is a contradiction unless $(tG)_p = 0$, as desired.

Conversely, suppose that the conditions of (1) hold. Write $\mathbf{Q}$ as the ascending union of cyclic subgroups $C_n = (n!^{-1})$ and for each n let

$$(S/\rho)_n = (S/\rho) \cap C_n.$$

By condition (1) we have that each $(S/\rho)_n$ is either a cyclic torsion-free group or the positive cone of such a group. So choose $z_n \in S \backslash tS$ such that $(S/\rho)_n$ is generated by $\rho(z_n)$, either as a group or semigroup, according as $(S/\rho)_n$ is a group or not. Now set

$$S_n = \rho^{-1}((S/\rho)_n),$$

so that $S = \cup S_n$ and $R[S] = \cup R[S_n]$. In light of Lemma 3.4, it suffices to show that each $R[S_n]$ is arithmetical. If $(S/\rho)_n$ is not a group, then for each $x \in S_n$, for some nonnegative integer k we have $\rho(x) = k\rho(z_n) = \rho(kz_n)$. Thus $x = kz_n + f$ for some $f \in tS \subseteq S_n$. But this representation is unique, whence $S_n = tS \oplus (z_n)$. On the other hand, if $(S/\rho)_n$ is a group and $x \in S_n$, then $\rho(x) = k\rho(z_n)$ for some integer k. For a given z_n there is an element $w \in (S/\rho)_n$ such that $\rho(z_n + w) = 0$. Hence $z_n + w = f \in tG \subseteq S$. Therefore, if we set $z_n' = w - f \in S$, the semigroup generated by z_n and z_n' is isomorphic to the group $\mathbf{Z}$. Moreover, it is easily checked that $S = tS \oplus (z_n, z_n')$. We thus see from Lemma 3.1 that $K[S_n] \cong R[tG][T]$, where T is one of the semigroups (z_n) or (z_n, z_n'). But $R[tG]$ is regular by (1), so that each $R[S_n]$ is indeed Bezout by Corollary 3.1 of [**11**], so certainly arithmetical.

To complete the proof we show that when $S = G$, a torsion group, then $R[G]$ is arithmetical if and only if condition (2) holds. First suppose that $R[G]$ is arithmetical. Then the ring R, a homomorphic image of $R[G]$, is also arithmetical. Let $\mathfrak{m}$ be a maximal ideal of R with $\mathrm{char}(R/\mathfrak{m}) = p$. If $G_p \neq 0$, then $G = G_p \oplus H$ and the projection map $G \rightarrow G_p$ induces a map from $R[G]$ onto $R[G_p]$. Therefore, $R[G_p]$ and $R_\mathfrak{m}[G_p]$ are arithmetical rings. But $R_\mathfrak{m}$ is a local ring of characteristic p; hence, $R_\mathfrak{m}[G_p]$ is a local ring by Lemma 3.2. It follows that this ring is a valuation ring and Lemma 3.3 produces the remaining conditions of (2).

To prove the converse, suppose that $R[S]$ satisfies the conditions of (2). We must show that $R[G]$ is arithmetical. Since $S = G$ is the direct limit of its finitely generated (hence finite) subgroups we may, in view of Lemma 3.4, reduce to the case of finite G. Moreover, by induction on the order of G, the identity $R[K \oplus H] \cong (R[K])[H]$ and the Fundamental Theorem for Finite Abelian Groups, we may further reduce to the case in which G is a prime power cyclic group, say of order p^n, for a prime p. Let M be a maximal ideal of $R[G]$ and set $\mathfrak{m} = M \cap R$, so that $R[G]_M$ is a localization of the group ring $R_\mathfrak{m}[G]$. If $R_\mathfrak{m}[G]$ is arithmetical, then

$R[G]_{\mathfrak{m}}$ is a valuation ring and we are done. To simplify notation assume that $R = R_{\mathfrak{m}}$; so R is a valuation ring with unique maximal ideal and, let us say, unique minimal prime ideal $\mathfrak{p}$ (the prime ideals of a valuation ring are linearly ordered). By hypothesis, if $\operatorname{char}(R/\mathfrak{m}) = p$, then R is a field whence $R[G]$ is a valuation ring by Lemma 3.3. So we may suppose that $\operatorname{char}(R/\mathfrak{m}) \neq p$. Set

$$A = R[G]/\mathfrak{p}[G] \cong (R/\mathfrak{p})[G].$$

Then A is a torsion-free module over the valuation domain $R/\mathfrak{p}$; now any finitely generated ideal of A is also a finitely generated torsion-free $R/\mathfrak{p}$-module and therefore $R/\mathfrak{p}$-projective (see e.g. Proposition 8 of [**16**, p. 85]). Hence, finitely generated ideals of A are A-projective by Lemma 3 of [**16**, p. 154]. It follows that A is semihereditary and therefore a reduced arithmetical ring by a result of [**6**]. Now the minimal primes of A are finite in number, since they correspond to primes of the finite integral extension $(R/\mathfrak{p})[G]$ of $R/\mathfrak{p}$ which contract to zero in $R/\mathfrak{p}$. Therefore $R[G]$ is a finite direct product of rings with unique minimal prime ideal which contracts to $\mathfrak{p}$ in R (see e.g. Theorem 2.2 of [**17**]). It suffices to show that each of these factors is arithmetical.

Suppose that T is such a summand of $R[G]$ and that P is the unique minimal prime of T. Then P corresponds to a minimal prime of $R[G]$ and in fact $\mathfrak{p}[G]$ is the intersection of such minimal primes since A is reduced. It follows that $\mathfrak{p}T = P$. Moreover, T/P is a summand of A and therefore T/P is arithmetical. Now let u be the image of a generator of G in T, so that $T = R[u]$ is a finite integral extension of the valuation ring R. Suppose $f \in T \backslash P$. The ring $\operatorname{tot}(R/\mathfrak{p})$ embeds in the field $\operatorname{tot}(T/P)$ and since u is integral over R, the latter field is generated by the images of u and $\operatorname{tot}(R/\mathfrak{p})$. Hence, there exists an element $g \in T$ and $x \in R\backslash\mathfrak{p}$ such that

$$fg \equiv x (\operatorname{mod} \mathfrak{p}[u]),$$

i.e. $fg - x = p_0 + \ldots + p_k u^k$ for suitable $p_i \in \mathfrak{p}$. But each p_i is a multiple of x (R is a valuation ring), so $fg = x(1 - y)$ for some nilpotent $y \in R$. Therefore $fgT = xT$, which implies that $\mathfrak{p}[u] = P \subseteq fT$ and so elements of P are comparable to elements outside P.

Let N be a maximal ideal of T and f, g arbitrary elements of T. We claim that the images of f and g in T_N are comparable, which will show that T_N is a valuation ring. If one of the elements is in P while the other is not, the elements are even comparable in T by the preceding paragraph. If both are not in P, then since T/P is arithmetical, there is an inclusion, say $fT_N \subseteq gT_N \bmod P_N$ which lifts to an inclusion $fT_N \subseteq gT_N$ in T_N since $P \subseteq gT$.

There remains the case in which f, $g \in P$. Observe that T, being a summand of $R[G]$, is a projective and therefore free R-module. Also note that $T/\mathfrak{p}T$ is an integral extension of $R/\mathfrak{p}$ by the image of u. Therefore, there is an integer k such that the images of $1, u, \ldots, u^k$ form an $R/\mathfrak{p}$-basis for $T/\mathfrak{p}T$. But then the elements $1, u, \ldots, u^k$ form an R-basis for T. Now think of f and g as polynomials in u of degree at most k and with coefficients in $\mathfrak{p}$. Let d be a greatest common divisor of the coefficients of f and g, so that $f = df'$ and $g = dg'$ for suitable $f', g' \in T$. Then at least one of f', g' does not belong to $\mathfrak{p}[u] = P$. Since f' and g' are therefore comparable in T_N by the preceding cases, f and g are also comparable and the proof is complete.

The following corollary is an extension of Théorème 3.2 of [**13**].

COROLLARY 3.7. *The semigroup ring $R[S]$ is semihereditary if and only if $R[S]$ is arithmetical, R is semihereditary and the order of every torsion element of S is a unit in R.*

Proof. Assume that $R[S]$ is semihereditary. As we have noted in the preceding proof, $R[S]$ is reduced arithmetical. If S is not a torsion group, then Theorem 3.6 yields the remaining conditions. So suppose that $S = G$ is a torsion group and let I be a finitely generated ideal of R. Then the ideal $I[G]$ of $R[G]$ is also finitely generated, hence projective; say $I[G] \oplus M \cong \oplus R[G]$. But as R-modules, $I[G] \cong \oplus I$ and $R[G] \cong \oplus R$. Therefore, I is a direct summand of a free R-module; hence I is projective and R must be semihereditary. Finally, suppose $0 \neq g \in G$, $0(g) = p^k$ and R has a maximal ideal $\mathfrak{m}$ with $\operatorname{char}(R/\mathfrak{m}) = p$. By Theorem 3.6 (2), R_N is a field. But then $R_{\mathfrak{m}}[G]$ is semihereditary and yet this ring contains the nonzero nilpotent element $1 - X^g$. This contradiction shows that the order of every element of G is a unit in R.

For the converse first assume that $S = G$ is a torsion group. It is known that a ring A is semihereditary if and only if A is Prüfer and $\operatorname{tot}(A)$ is regular [**9**, p. 66]; therefore, $\operatorname{tot}(R)$ is regular. But then so is $(\operatorname{tot}(R))[G]$. It follows that

$$(\operatorname{tot}(R))[G] = \operatorname{tot}(R[G]).$$

Hence $R[G]$ is semihereditary. On the other hand, if S is a group but not torsion, the desired result follows from Théorème 3.2 of [**13**, p. 211].

Finally, if S is not a group, its group of quotients G is one of the groups above; therefore, $R[G]$ is semihereditary. But then $R[S]$ is a Prüfer ring with regular total quotient ring $(= \operatorname{tot}(R[G]))$; i.e. $R[S]$ is semihereditary.

Example 3.8. Recall that the classes of Prüfer, arithmetical and Bezout rings coincide for semigroup rings of torsion-free semigroups. Such is not the case for more general types of nontrivial semigroup rings, as the following examples show. Let F be a field of characteristic $p \neq 0$ and G a finite (abelian) p-group that is not cyclic. Then one obtains from Lemmas 3.2 and 3.3 that the group ring $F[G]$ is a local ring but not a valuation ring; therefore $F[G]$ is not arithmetical. However, $F[G]$ is trivially a Prüfer ring since it has no regular nonunits.

Next, let $R = \mathbf{Z}[\sqrt{-5}]$, $T = \{11^n | n \text{ a nonnegative integer}\}$ and G a cyclic group of order 11. Then R, being the ring of integers of a number field, is certainly a Dedekind domain. Indeed, R is a classical example of a Dedekind domain which is not a PIR. Now the prime 11 remains prime in R. Consequently, it is easily seen that R and $T^{-1}R$ have isomorphic ideal class groups. Therefore the group ring $(T^{-1}R)[G]$ provides an example of a nontrivial group ring which is arithmetical (by Theorem 3.6) but not Bezout.

4. ZPI-rings and PIR's. A Noetherian arithmetical (Prüfer) domain is a Dedekind domain, i.e., a domain in which every ideal is a product of prime ideals. W. Krull called a ring with the latter property a (general) ZPI-ring. For semigroup rings of torsion-free semigroups, Gilmer and Parker used their characterization of Prüfer semigroup rings to show that a semigroup ring $R[S]$ is a ZPI-ring if and only if R is a finite direct product of fields and S is isomorphic to $\mathbf{Z}$ or $\mathbf{Z}^+$. For details see [**11**, p. 229]. Furthermore, these ZPI-rings are principal ideal rings.

The ZPI-rings are known [**1**] to be exactly those rings that are a finite direct product of Dedekind domains and special primary rings (artinian local PIR's). Also R. B. Warfield showed in [**20**, p. 170] that such rings are precisely the Noetherian arithmetical rings. We use these facts in the following result.

THEOREM 4.1. *Let R be a commutative ring and S a commutative cancellative semigroup with group of quotients G. Then $R[S]$ is a* ZPI-*ring if and only if either* (1) *R is a* ZPI-*ring, $S = G$ is a finite group and $R[S]$ is arithmetical; or* (2) *R is semisimple, $S \cong \mathbf{Z} \oplus F$ or $S \cong \mathbf{Z}^+ \oplus F$ where F is a finite group whose order is a unit in R. Furthermore, in case* (2) *$R[S]$ is a* PIR.

Proof. In the case that S is a group the Noetherian hypothesis for $R[S]$ is equivalent to the condition that R be Noetherian and S finitely generated (see [**16**, p. 153]). Hence, the necessity and sufficiency of (1) follows from Theorem 3.6.

Now assume that S is not a torsion group. If $R[S]$ satisfies (2), then $R[S] \cong (R[F])[H]$ where H is $\mathbf{Z}$ or $\mathbf{Z}^+$. But $R[F]$ is a Noetherian regular ring, so a finite product of fields. Hence $R[S]$ is a finite product of rings of the form $k[\mathbf{Z}^+]$ or $k[\mathbf{Z}]$ where k is a field. Such rings are PIR's, whence $R[S]$ is a PIR. Conversely, suppose that $R[S]$ is a ZPI-ring. Let G be the quotient group of S, so that the localization $R[G]$ of $R[S]$ is also ZPI and hence G is finitely generated by the preceding remarks. The ring R is a homomorphic image of $R[G]$, so R is a Noetherian regular ring (Theorem 3.6) and therefore semisimple. The proof of Theorem 3.6 shows that G, being finitely generated, must have the form $F \oplus \mathbf{Z}$, where F is the torsion subgroup of G. But F is finite since G is finitely generated. Furthermore, even if S is not a group we have that $F \subseteq S$ and that the group generator of $\mathbf{Z}$ can be taken to belong to S. But then it follows easily that $S = F \oplus \mathbf{Z}^+$, as required. The order condition of (2) is immediate from Theorem 3.6, (1), so the proof is complete.

Not all nontrivial ZPI semigroup rings are PIR's, for if D is a Dedekind domain that is not a PID and if G is a finite group whose order is a unit in D, then $D[G]$ is a ZPI-ring that is not a PIR. The ring $T^{-1}R$ of Example 3.8 is one such ring.

Our last result describes the rings $R[G]$ that are PIR's when G is a torsion group. Together with Theorem 4.1, the PIR's which occur as semigroup rings of cancellative semigroups are completely characterized. Recall that an abelian group G is said to have *exponent* N if N is the least positive integer for which $NG = 0$.

THEOREM 4.2. *Let R be a commutative ring and assume G is an abelian torsion group. Then $R[G]$ is a* PIR *if and only if G is a finite group and R is a* PIR *such that $R = R_1 \oplus \ldots \oplus R_n$ where the R_i are* PID's *or special primary rings subject to the conditions:*

1) *The order of G is a unit in any R_i that is not a field.*
2) *G_p is cyclic if any R_i is a field of characteristic p.*
3) *For any component R_i that is not a special primary ring, if m divides the exponent of G and u is a primitive m^{th}-root of unity over the quotient field of R_i, then $R_i[u]$ is a* PID.

Proof. Assume that $R[G]$ is a PIR. Then R is also a PIR; hence, $R \cong R_1 \oplus \ldots \oplus R_n$ where the R_i's are PID's or special primary rings [**18**, p. 207], and $R[G]$ is a finite direct sum of the PIR's $R_i[G]$. Since domains and local rings that are not themselves fields are not fields locally, conditions (1) and (2) follow from Theorem 3.6. Let R_i, m and u be as described in (3) and let H be a subgroup of G such that $G/H = K$ is cyclic of order m. Then $R_i[K]$ is a PIR. Furthermore, if $g + H$ is a generator of K, then $R[K]$ can be mapped onto $R[u]$ by defining $\theta(X^{g+H}) = u$ and extending linearly. Condition (3) follows.

For the converse, assume that R and G are as described. It suffices to prove that $R_i[G]$ is a PIR for $i = 1, 2, \ldots, n$. If R_i is a field or a special primary ring, then $R_i[G]$ is an artinian arithmetical ring (Theorem 3.6), hence a PIR.

It remains to consider the case where R_i is a Dedekind domain that is not a field. We write R for R_i. Also fix an algebraic closure of the quotient field of R and consider all algebraic extensions in the following to be subfields of this closure. From (1) and Theorem 4.1, $R[G]$ is a ZPI-ring which is, in fact, reduced. Thus $R[G]$ is a direct sum of Dedekind domains

$D_j = R[G]/P_j$ where the P_j's are the minimal primes of $R[G]$. Since $R[G]$ is integral over R, $P_j \cap R$ is a minimal prime of R[**18**, p. 84]; i.e., $P_j \cap R = 0$ for all j and R embeds in each D_j.

Now, if $G = \langle g \rangle$ is cyclic, the projection maps $\rho_j : R[G] \to D_j$ are determined by $\rho_j(X^g)$ since

$$\rho_j\left(\sum_k r_k X^{dg}\right) = \sum_k \rho_j(X^g)^k.$$

Therefore, $D_j = R[u_j]$ where $u_j = \rho_j(X^g)$ is an n^{th}-root of unity over F, the quotient field of R. If d is the multiplicative order of u_j in D_j, then u_j is a primitive d^{th}-root of unity over F. By (3) each D_j is a PID; thus, $R[G]$ is a PIR.

In general G is a finite direct sum of cyclic groups C_k of prime power order n_k, $k = 1, \ldots, m$, then

$$R[G] \cong (((R[C_1])[C_2]) \ldots)[C_m].$$

From the preceding case, $R[C_1]$ is a direct sum of PID's $R[u_{1l}]$ where the u_{1l} are primitive $k_{1l}{}^{\text{th}}$-roots of unity over F and k_{1l} divides n_l. Thus

$$R[C_1 \oplus C_2] \cong (R[C_1])[C_2] \cong (\oplus\, R[u_{1l}])[C_2] \cong \oplus\, ((R[u_{1l}])[C_2])$$

and the order of C_2 is a unit in each $R[u_{1l}]$. Apply the cyclic group case to each of the latter summands to get $(R[u_{1l}])[C_2]$ isomorphic to a finite sum of Dedekind domains $(R[u_{1l}])[u_{2r}] = R[v_{lr}]$ for some primitive k^{th}-root of unity v_{lr} over F, where k divides the exponent of G. By (3) each $R[v_{lr}]$ is a PID. Continue in a similar fashion to find a direct sum of PID's isomorphic to $R[G]$, and the proof is complete.

Examples. Let F be a finite abelian group of order n. Then $\mathbf{Z}[F]$ is not arithmetical since the localizations $\mathbf{Z}_{(p)}$ for prime divisors p of n are not fields (Theorem 3.6). However, these rings come close to being arithmetical; for if we pass to $T^{-1}\mathbf{Z}$, the ring of quotients of $\mathbf{Z}$ with respect to the multiplicative system T consisting of the nonnegative powers of n, then $T^{-1}\mathbf{Z}[F]$ is arithmetical. In fact, $T^{-1}\mathbf{Z}[F]$ is a ZPI-ring. Is it also a PIR? By Theorem 4.2 this ring is a PIR only if $T^{-1}\mathbf{Z}[u]$ is a PIR for certain roots of unity u. This requirement is quite restrictive; for if $n = p$ is a prime, to verify condition (3) of Theorem 4.2 we must determine when $T^{-1}\mathbf{Z}[u]$ is a PID, where u is a primitive p^{th}-root of unity. Clearly, if $\mathbf{Z}[u]$ is a PID, then so is $T^{-1}\mathbf{Z}[u]$. The converse is also true. Let $S = \{p^i | i \geqq 0\}$ and set $B = \mathbf{Z}[u]$. Then $pB = (u-1)^{p-1}B$ and $(u-1)B$ is a prime ideal of B [**21**, p. 314] thus, $p = (u-1)^{p-1} \cdot f$ for some unit f in B. Let

$$W = \{(u-1)^i | i \geqq 0\}.$$

One can easily show that $S^{-1}B = W^{-1}B$; thus if $S^{-1}B$ is a PIR, so is B

https://doi.org/10.4153/CJM-1980-106-x Published online by Cambridge University Press

[**18**, p. 193]. In particular, for F a group of order p, $T^{-1}\mathbf{Z}[F]$ is a PID if and only if $\mathbf{Z}[u]$ is a PID, for u a primitive p^{th}-root of unity. Certainly $\mathbf{Z}[u]$ is not always a PID; in fact it is well known that 2, 3, 5, 7, 11, 13, 17 and 19 are the only primes less than 100 for which $\mathbf{Z}[u]$ is a PID.

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