MAPS WITH DISCRETE BRANCH SETS BETWEEN MANIFOLDS OF CODIMENSION ONE

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1. Introduction. Let M^n and N^p be separable manifolds of dimensions n and p, respectively, with $n \geqq p$, and without boundary unless otherwise indicated. A map $f: M \to N$ is *proper* if, for each compact set $K \subset N$, $f^{-1}(K)$ is compact. It is *topologically equivalent* to $g: X \to Y$ if there exist homeomorphisms α of M onto X and β of N onto Y such that $\beta f \alpha^{-1} = g$. At $x \in M$, f is *locally topologically equivalent* to g if, for every neighbourhood $W \subset M$ of x, there exist neighbourhoods $U \subset W$ of x and V of $f(x)$ such that $f|U: U \to V$ is topologically equivalent to g.

Definition 1.1. Let f be a map of M^n (possibly with boundary) into N^p. The *branch set* $B_f \subset M^n$ is defined in (**12**) by: x is an element of $M^n - B_f$ if and only if f at x is locally topologically equivalent to the natural product projection map of E^n or $E_+{}^n$ onto E^p, where $E_+{}^n$ is a (closed) euclidean half-space.

THEOREM 1.2. *If $f: M^{p+1} \to N^p$ is proper and B_f is discrete, then either*

(a) *B_f is empty and f is a fibre bundle map, or*

(b) $p = 1$ or 3, *and for each $q \in B_f$ the map f at q is locally topologically equivalent to θ, where*

(i) *$\theta: E^2 \to E^1$ by $\theta(z) = |z|$,*

(ii) *$\theta: E^2 \to E^1$ by $\theta(z) =$ Re z^d, d an integer greater than one, or*

(iii) *$\theta: E^4 \to E^3$ by the natural extension of the Hopf map from S^3 onto S^2 to a map of the open cone of S^3 onto the open cone of S^2.*

If f is said to be differentiable of order m, it is understood that M^n and N^p are differentiable manifolds of order m. The *critical set* of f is the collection of points in M^n at which the map has rank less than p.

COROLLARY 1.3. *If $f: M^{p+1} \to N^p$ is a C^m proper map with critical set discrete, then either* (a) *or* (b) *of Theorem* 1.2 *is satisfied.*

Proof. The set B_f is contained in the set of critical points by the Rank Theorem (**6**, p. 273, Theorem 10.3.1).

Remark 1.4. If $f: M^n \to N^p$ and $\dim f^{-1}(y) \leqq 0$ for each $y \in f(B_f)$, then for

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any $q \in B_f$ there are connected open neighbourhoods $U \subset M^n$ of q and $V \subset N^p$ of $f(q)$ such that $f|U$: $U \to V$ is proper. (Same proof as in (**5**, p. 74, Lemma 1.14)).

COROLLARY 1.5. *Let* f: $M^{p+1} \to N^p$ *be a* C^{p+1} *map with* $B_f \neq \emptyset$ *and* $\dim f^{-1}(y) \leqq 0$ *for each* $y \in f(B_f)$.

(a) *Then there is a closed set* $Y \subset B_f$ *such that* $\dim Y < \dim B_f$, *and at each point* $x \in B_f - Y$ *the map* f *is locally topologically equivalent to* $\theta \times \iota_k$, *where* θ *is the map in* (i) *or* (iii) *of Theorem* 1.2 *and* ι_k *is the identity map on* E^k.

(b) *If* f *is proper and onto, only* θ *in* (iii) *occurs.*

Proof. By Remark 1.4 there exist neighbourhoods U of $q \in B_f$ and V of $f(q)$ such that f: $U \to V$ is a proper map. By Theorem 1.2, f restricted to $U - B_f$ is a bundle map, and hence $B_f = A_f$ in (**5**, p. 72, Definition 1.4). It follows from (**5**, p. 83, Lemma 4.1) that f has the desired local structure.

Remark 1.6. Although f is assumed differentiable in Corollaries 1.3 and 1.5, the local structure of f about $q \in B_f$ is given by a topological equivalence, and this cannot be improved to a differential equivalence (**5**, p. 72, Remark 1.6).

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2. Preliminary lemmas. If f: $M^p \to N^p$, then the branch set B_f coincides with the set of points at which f fails to be a local homeomorphism, a set which has been very significant in studies by Church, Hemmingsen, and others; see, for example, (**1**–**4**). If f is proper and is a local homeomorphism, then it is a covering map (**9**, p. 128, Theorem 4.2).

PROPOSITION 2.1. *Let* M^{p+1} (*possibly with boundary*) *and* N^p *be connected manifolds. If* f: $M^{p+1} \to N^p$ *is proper and* $B_f = \emptyset$, *then* f *is a fibre bundle map.*

Proof. By (**12**, p. 63, Lemma 2.3) there is a factorization of f into hg, where g is a monotone map onto the p-manifold K^p and h is a covering map. Since h is a local homeomorphism, B_g is empty, and it suffices to show that g is locally trivial.

If y is an element of K^p, then $g^{-1}(y)$ is homeomorphic to S^1 or a closed interval. If $g^{-1}(y)$ is homeomorphic to S^1, let $\{U_i: i = 0, 1, \ldots, n-1\}$ be a minimal cover of $g^{-1}(y)$ by open sets of M^{p+1}, ordered so that $U_i \cap U_j \cap g^{-1}(y)$ is empty if $j \not\equiv (i-1, i, \text{or } i+1) \bmod (n)$, and selected so that there exist homeomorphisms α_i: $U_i \to E^p \times E^1$ and β: $g(U_i) \to E^p$ (where β and $g(U_i)$ are the same for each i) with $\beta g \alpha_i^{-1}$ the product projection map of $E^p \times E^1$ onto E^p. For each i and $j \equiv (i-1) \bmod (n)$ choose a point q_i in $U_i \cap U_j \cap g^{-1}(y)$. Let W_i be a neighbourhood of q_i contained in $U_i \cap U_j$ such that $W_i \cap W_k = \emptyset$ for $i \not\equiv k \bmod (n)$, $g(W_i) = g(W_k)$, β restricted to $g(W_i)$ is a homeomorphism onto an open p-disk D contained in E^p, and α_i restricted to W_i is a homeomorphism onto $D \times \{t_i\}$ for some t_i in E^1. Without loss of generality, assume that $t_i = 0$ for each i and assume that α_i was selected so that its second coordinate function is positive on q_k, where $k \equiv (i+1) \bmod (n)$.

Define the homeomorphism ϕ_i: $D \times E^1 \to D \times E^1$ by $\phi_i(x, t) = (x, s_i(x, t))$ where $s_i(x, t) = (nt/2\pi - i)u_i(x)$, and $u_i(x)$ is the point in E^1 such that $(x, u_i(x)) \in \alpha_i(W_k)$, $k \equiv (i+1) \bmod (n)$. The sets W_i and W_k separate $g^{-1}(\beta^{-1}(D))$; therefore, if L_i is the union of $W_i \cup W_k$ and the component of $g^{-1}(\beta^{-1}(D)) - (W_i \cup W_k)$ contained in U_i, then

$$\alpha_i(L_i) = \phi_i(D \times [2\pi i/n, 2\pi k/n]).$$

Since $\bigcup_i L_i = g^{-1}(\beta^{-1}(D))$ and $\bigcup_i (D \times [2\pi i/n, 2\pi k/n])$ can be considered as $D \times S^1$, the map γ: $g^{-1}(\beta^{-1}(D)) \to D \times S^1$ defined by $\gamma(z) = \phi_i^{-1}\alpha_i(z)$, where $z \in L_i$, is a homeomorphism such that $\beta g \gamma^{-1}$ is the natural product projection of $D \times S^1$ onto D. Hence, f and g are fibre bundle maps.

When M^n is a manifold with non-empty boundary, a similar proof yields that f and g are fibre bundle maps, where the fibre for g is a closed interval.

LEMMA 2.2. *If f: $M^n \to N^p$ is a proper map with B_f consisting of isolated points, then*

(a) *$f(B_f)$ consists of isolated points;*

(b) *if N^p is connected and $p > 1$, then f is open and onto;*

(c) *$f^{-1}(y)$ has a finite number of components for each point $y \in N^p$;*

Proof. The proof of (a) will be omitted.

The map f restricted to $M' = M^n - f^{-1}(f(B_f))$ is a proper map into $N^p - f(B_f)$, which implies that $f(M')$ is a closed subset of $N^p - f(B_f)$. Since M' is open and $f|M'$ is interior, $f(M')$ is an open subset of $N^p - f(B_f)$. If N^p is connected and $p > 1$, then $N^p - f(B_f)$ is connected, and thus f is an onto map. The map f is interior except possibly at points in B_f. If $X \subset B_f$ is the set of all elements in M^n at which f is not interior, suppose that $q \in X$ and let U be an open neighbourhood of q such that $U \cap B_f = \{q\}$. If Q is the component of $U \cap f^{-1}(f(q))$ containing q and $Q - \{q\}$ is not empty, then $f(q)$ is an interior point of the image of any open neighbourhood of q contained in U. Thus, f is interior at q, which is a contradiction; hence, each $q \in X$ is a point component of $f^{-1}(f(q))$. Since $X \cap \mathrm{Cl}(M^n - X) = X$, $f(q)$ must locally separate N^p (**14**, p. 149, Theorem 7.81). If $p > 1$, however, then $f(q)$ cannot locally separate N^p; thus, X is empty and f is an open map.

Statement (c) is true for each $y \in N^p - f(B_f)$, since $f^{-1}(y)$ is either empty or a compact $(n-p)$-manifold. Consider $y \in N^p$ such that there exists at least one element q in $B_f \cap f^{-1}(y)$. If $p = 1$ and q is a point component of $f^{-1}(y)$, then there exists a neighbourhood V of $f(q)$ such that if U is the component of $f^{-1}(V)$ containing q, then $U \cap B_f = \{q\}$. By (**12**, p. 63, Lemma 2.3), there exists a positive integer k such that $U \cap f^{-1}(x)$ has k components for each x in a component of $V - \{y\}$. Suppose that $U \cap f^{-1}(y)$ has at least $k+2$ components, and let W be a neighbourhood of y contained in V such that $U \cap f^{-1}(W)$ has at least $k+2$ components $Q, Q_1, \ldots, Q_{k+1}$, where Q contains q. Since $f(Q_i) = W$, for each x in $W - \{y\}$ there are at least

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$k + 1$ components in $U \cap f^{-1}(x)$, which is a contradiction to the choice of k. Hence, $U \cap f^{-1}(y)$ has a finite number of components if q is a component of $f^{-1}(y)$. If q is not a point component of $f^{-1}(y)$, suppose that $\{q_i\}$ is a sequence of point components of $f^{-1}(y)$ contained in B_f and converging to $q' \in f^{-1}(y)$. Then q' is also an element of B_f, which is a contradiction to B_f consisting of isolated points; hence, there exists an open neighbourhood V of y with $V \cap f(B_f) = \{y\}$ and with the property that if U is the component of $f^{-1}(V)$ containing q, then no component of $U \cap f^{-1}(y)$ is a single point. By a process similar to the one just discussed, $U \cap f^{-1}(y)$ has a finite number of components, and since $f^{-1}(y)$ is compact, it has a finite number of components. For $p > 1$, the desired conclusion follows from (**12**, p. 64, Lemma 2.5).

LEMMA 2.3. *Let M^n and N^p be connected manifolds, $n \geqq p \geqq 2$, and let $f: M^n \to N^p$ be a proper map. If $f(B_f)$ consists of isolated points and* $\dim(f^{-1}(y)) \leqq n - 2$ *for each $y \in f(B_f)$, then there exists a unique factorization of f into hg, where*

(a) *$g: M^n \to K^p$ is a monotone map onto the p-manifold K^p, and either*

(b) *$p = 2$ and $h: K^2 \to N^2$ is locally topologically equivalent at $x \in K^2$ to an analytic function $\theta(z) = z^d$, d a positive integer, or*

(c) *$p \geqq 3$ and $h: K^p \to N^p$ is a k*-to-1 *covering map.*

In particular, if B_f is discrete and $p \geqq 2$, then f has such a factorization.

Proof. If V is an open connected set in N^p and U is a component of $f^{-1}(V)$, then $f(U) = V$. The manifold $N^p = \bigcup_{i=1}^{\infty} X_i$, where X_i is compact and $X_i \subset X_{i+1}$. Since f is proper, f restricted to $f^{-1}(X_i)$ has a unique monotone-light factorization (**14**, p. 141, Theorem 4.1); it follows that f is also equal to hg, where g is a monotone map onto an intermediate space K and h is light.

The space $M' = M^n - f^{-1}(f(B_f))$ is connected (**7**, p. 48, Theorem IV 4), and by the proof of (**12**, p. 75, Lemma 2.3), $f|M'$ can be factored into the restriction of g to a monotone map of M' onto the p-manifold $K' = K - h^{-1}(f(B_f))$ followed by the restriction of h to a k-to-1 covering map of K' onto $N^p - f(B_f)$.

Let $y \in f(B_f)$ and let D be a closed euclidean neighbourhood of y such that $D \cap f(B_f) = \{y\}$. Each component of $f^{-1}(D)$ is mapped by f onto D; hence, $h^{-1}(y)$ contains at most k points. The components of $h^{-1}(D - \{y\})$ are homeomorphic to $D - \{y\}$, and thus the components of $h^{-1}(D)$ are each topologically the union of disjoint closed euclidean p-disks identified at their centre points. If L is a component of $h^{-1}(D)$ that is separated by an element z in $h^{-1}(y)$, then $g^{-1}(z)$ must separate $g^{-1}(L)$. However, $g^{-1}(L)$ is a component of $f^{-1}(D)$ and is an n-manifold with boundary, while dim $(g^{-1}(z)) \leqq n - 2$, thus by (**7**, p. 48, Theorem IV 4), $g^{-1}(z)$ cannot separate $g^{-1}(L)$. Hence, no component of $h^{-1}(D)$ is separated by a point in $h^{-1}(y)$, each component of $h^{-1}(D)$ is homeomorphic to D, and K is a p-manifold K^p.

The set B_h is contained in $h^{-1}(f(B_f))$. If B_h is empty, then (**9**, p. 128, Theorem 4.2) implies (c); if B_h is not empty, then by (**3**, p. 535, Theorem 5.9),

$p = 2$, and (**11**) implies (b). If B_f is discrete, then $f(B_f)$ is discrete by Lemma 2.2 and $\dim(f^{-1}(y)) \leqq n - p$ for each y in $f(B_f)$; therefore, f has the required factorization.

Remark 2.4. Suppose that $f: L \to S^n$ is a locally trivial fibre map with fibre homeomorphic to an open, half-open, or closed interval. Then f is trivial (topologically equivalent to a product projection) if $n > 1$, or if $n = 1$ and there exist at least two disjoint cross-sections.

3. Proof of Theorem 1.2.

LEMMA 3.1. *If $f: M^{n+1} \to N^n$ is a proper map with B_f a non-empty discrete set, then for each $q \in B_f$ either*

(a) *q is a point component of $f^{-1}(f(q))$, or*

(b) *$n = 1$, q is not a point component of $f^{-1}(f(q))$, and $f^{-1}(f(q))$ fails to be locally euclidean at q.*

Proof. Let Y be the component of $f^{-1}(f(q))$ containing q, and suppose that Y is not equal to $\{q\}$. The main body of this proof will consist of three parts: (I), (II), and (III). In (I), there is an open neighbourhood of q in Y which is homeomorphic to a half-open interval with q as the end point; in (II) there is an open neighbourhood of q in Y which is homeomorphic to an open interval, and in (III) there is an open neighbourhood of q in Y which is homeomorphic to a collection of three or more half-open or closed arcs with end points identified at q.

By Lemma 2.2, $f(B_f)$ consists of isolated points and $f^{-1}(f(q))$ has a finite number of components; therefore, there exists a euclidean neighbourhood V of $f(q)$ such that $V \cap f(B_f) = f(q)$ and if U is the component of $f^{-1}(V)$ containing q, Y the component of $f^{-1}(f(q))$ containing q, then $U \cap f^{-1}(f(B_f)) = U \cap Y$. If $\lambda = f|U$, then λ is proper and $B_\lambda = Y \cap B_f$. By Lemma 2.2, λ is open and onto if $n > 1$; however, since no element of B_λ is a point component of $\lambda^{-1}(\lambda(q))$, the same proof shows that λ is also open and onto when $n = 1$.

If $n = 1$, then by Proposition 2.1, λ restricted to each component of $U - Y$ is a locally trivial fibre map onto a component of $V - f(q)$ homeomorphic to an open interval. Hence, λ restricted to each component of $U - Y$ is trivial (**10**, p. 53, Corollary 11.6) and for each $y \in V - f(q)$, $\lambda^{-1}(y)$ is homeomorphic to a finite disjoint union of copies of S^1. If $n \geqq 2$, then Lemma 2.3 indicates that there exists a factorization of λ into hg. Since V is simply connected, if h is a covering map, it actually is a homeomorphism. If $n = 2$, let $x \in Y - B_f$ and let W be a neighbourhood of x contained in U such that λ restricted to W is topologically equivalent to the natural projection map of E^3 onto E^2. There exists a point y in $\lambda(W)$ such that $(h^{-1}(y)) \cap g(W)$ has d components $C_1, C_2, \ldots, C_d$, where $g^{-1}(C_i)$ is not empty for each $i = 1, 2, \ldots, d$. Thus, $W \cap \lambda^{-1}(y)$ is not connected if $d > 1$, which is a contradiction. Hence, for $n \geqq 2$, h is a homeomorphism, λ is monotone, and $\lambda^{-1}(y)$ is homeomorphic to S^1 for each y in $V - f(q)$.

The set Y is compact, connected, and, except for a discrete set, is a 1-manifold. In addition, Y is locally connected, since if it were not it would fail to be locally connected on a set contained in B_f that was not discrete (**14**, p. 18, (12.1)). Thus, there exists an open connected neighbourhood $W(q)$ of q with compact closure W' such that $Y \cap W(q)$ is connected, $W' \subset U$, and $W' \cap B_\lambda = \{q\}$. Since each point x in $Y \cap (\text{bdy } W')$ is disjoint from B_λ, there exists an open euclidean neighbourhood $W(x)$ of x with compact closure C such that $C \subset U$, $Y \cap C(x)$ is homeomorphic to a closed interval disjoint from B_λ, and $Y \cap (\text{bdy } C(x))$ consists of exactly two points. Since $Y \cap (\text{bdy } W')$ is compact, there is a finite collection of points $x_1, x_2, \ldots, x_m$ such that $\{W(x_i)\}$ is an open cover. Let X be the closure of the component containing q of the open set $W(q) - \bigcup_{i=1}^{m} C(x_i)$. Then X is contained in U, $X \cap Y$ is connected and $X \cap B_\lambda = \{q\}$. In addition,

$$\text{bdy}(X \cap (Y - \{q\})) = (\text{bdy}(X)) \cap Y.$$

Case I. If $W(q)$ can be selected so that $W(q) \cap Y$ is homeomorphic to a half-open interval and has end point q, then $X \cap Y$ is homeomorphic to a closed interval. The set $X - Y$ is connected; thus, there exists a component U' of $U - Y$ containing $X - Y$ on which λ is a monotone map (in fact, if $n > 1$, U' is all of $U - Y$). Let y be an element of $\lambda(X - Y)$, and define $S(y)$ to be $\lambda^{-1}(y) \cap U'$ (which is homeomorphic to S^1). The single point in $(\text{bdy}(X)) \cap Y$ is not an element of B_λ, hence there exists an open neighbourhood W_1 of it contained in U such that $\lambda|W_1$ is topologically equivalent to the projection of E^{n+1} onto E^n. Let $W = W_1 \cup (\text{int}(X))$ and suppose that $(X - W) \cap S(y_i)$ is not empty for an infinite sequence of points $\{y_i\}$ contained in $\lambda(W_1 - Y)$ with limit point $f(q)$. Since λ is a proper map, $\lim\sup((X - W) \cap S(y_i))$ is not empty and is contained in $(X - W) \cap Y$. However, by the selection of X and W', $(X - W) \cap Y$ is empty. Hence, there exists an integer η such that for $k > \eta$, $X \cap S(y_k) \subset W$. If $y \in \{y_k\}$ such that $S(y) - W_1$ is homeomorphic to a closed interval with one end point in $W - W_1$ and the other end point in $U - X$, then $X \cap S(y)$ is not contained in W, which is a contradiction. Thus, q is not an end point of Y and Case I cannot occur.

Case II. If $W(q)$ can be selected so that $W(q) \cap Y$ is homeomorphic to an open interval, then $X \cap Y$ is homeomorphic to a closed interval.

Let σ: $[-1, 1] \to X \cap Y$ be a homeomorphism into Y such that $\sigma([-1, 1])$ is contained in the interior of X and $\sigma(0) = q$. Since $\sigma(1)$ and $\sigma(-1)$ are not elements of B_λ, there exist open neighbourhoods $W(1)$ of $\sigma(1)$ and $W(-1)$ of $\sigma(-1)$ contained in X such that λ restricted to $W(i)$ $(i = +1, -1)$ is topologically equivalent to the natural projection map of E^{n+1} onto E^n. In addition, $W(1)$ and $W(-1)$ can be selected so that $W(1) \cap W(-1) = \emptyset$ and $\lambda(W(1)) = \lambda(W(-1)) = D$, where D is a euclidean neighbourhood of $f(q)$. Let $D(i)$ $(i = +1, -1)$ be a cross-sectional n-disk (for λ) contained in $W(i)$ which contains $\sigma(i)$ and which maps onto D.

For $n = 1$, a component of $D(1) - Y$ and one of $D(-1) - Y$ is contained in a component P of $\lambda^{-1}(D - f(q))$, while the other components of $D(i) - Y$ are contained in a component P' of $\lambda^{-1}(D - f(q))$. If $D(1) \cup D(-1)$ does not separate a component of $\lambda^{-1}(D)$, then it does not separate P or P'. However, by the choice of $D(1)$ and $D(-1)$, their union does separate P and P'; hence, $D(1) \cup D(-1)$ separates a component of $\lambda^{-1}(D)$. For $n > 1$, $\dim(\lambda^{-1}(D) - (D(1) \cup D(-1))) \geqq 3$ and $\dim(Y) = 1$; hence, if $\lambda^{-1}(D) - (D(1) \cup D(-1)) = C'$ is connected, then $C' - Y$ is also connected (**7**, p. 48, Theorem IV 4). By choice of $\{D(i)\}$, $C' - Y$ is not connected. Hence, for all $n \geqq 1$, $D(1) \cup D(-1)$ separates $\lambda^{-1}(D)$. Define Γ to be the union of $D(1) \cup D(-1)$ and the component of C' containing q.

Let k be a non-zero integer and let $m = |k|$. Since $\sigma(k^{-1})$ is not an element of B_λ, there exists for each k with $m \geqq 2$ an open euclidean neighbourhood $W(k)$ of $\sigma(k^{-1})$ contained in Γ, such that $\lambda | W(k)$ is topologically equivalent to the natural projection map of E^{n+1} onto E^n. The $W(k)$'s can be selected to be pairwise disjoint and disjoint from $D(1) \cup D(-1)$. Select in $W(k)$ a pair of partial cross-sectional open n-disks $(D(k), D'(k))$ through $\sigma(k^{-1})$ such that $D'(k) \subset D(k)$, $\lambda(D(k), D'(k)) = \lambda(D(-k), D'(-k))$, and so that there exists a homeomorphism $\mu: S^{n-1} \times (0, 1) \to D - f(q)$ with

$$\mu\left(S^{n-1} \times \left(0, \frac{1}{m}\right)\right) = \lambda(D(k) - y)$$

and

$$\mu\left(S^{n-1} \times \left(0, \frac{1}{m+1}\right)\right) = \lambda(D'(k)) - Y).$$

The map $\mu^{-1}\lambda$ restricted to $\Gamma - Y$ has empty branch set and is proper; thus, by Proposition 2.1 it is a fibre bundle map with fibre homeomorphic to $[-1, 1]$. By (**10**; p. 53, Theorem 11.4), this map is bundle equivalent to

$$\xi: (\Gamma \cap \lambda^{-1}(\mu(S^{n-1} \times \{t\}))) \times (0, 1) \to S^{n-1} \times (0, 1)$$

defined by $\xi(x, u) = (\mu^{-1}\lambda(x), u)$, where t and u are elements of $(0, 1)$ and $x \in \Gamma \cap \lambda^{-1}(\mu(S^{n-1} \times \{t\}))$. It follows from Remark 2.4 that ξ is trivial, and therefore there exists a homeomorphism $\alpha: \Gamma - Y \to S^{n-1} \times (0, 1) \times [-1, 1]$ such that $\mu^{-1}\lambda\alpha^{-1} = \pi$, where π is the product projection map onto $S^{n-1} \times (0, 1)$.

For $z \in S^{n-1} \times (0, 1)$, let $v_k(z) = \pi^{-1}(z) \cap \alpha(D(k) - Y)$. Let

$$u_m: (0, 1) \to [0, 1]$$

be defined by

$$u_m(t) = 0 \quad \text{if } t \in \left(0, \frac{1}{m+1}\right),$$

$$u_m(t) = m^2t + mt - m \quad \text{if } t \in \left[\frac{1}{m+1}, \frac{1}{m}\right),$$

and

$$u_m(t) = 1 \quad \text{if } t \in \left[\frac{1}{m}, 1\right).$$

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Define s_k: $S^n \times (0, 1) \to [-1, 1]$ for $m = 1$ by $s_k(z) = v_k(z)$, and for $m \geqq 2$ by

$$s_k(z) = \tfrac{1}{2}u_m(t)(s_{m-1}(z) + s_{-m+1}(z)) + (1 - u_m(t))v_k(z),$$

where $z \in S^{n-1} \times \{t\}$.

If $\rho_k(z) = (z, s_k(z))$, then $\alpha^{-1}\rho_k\mu^{-1}$ is a cross-section for $\lambda|\Gamma - Y$. Let $E(k) = \sigma(k^{-1}) \cup (\alpha^{-1}\rho_k\mu^{-1}(D - f(q)))$. Then $E(k)$ is a cross-sectional n-disk which is an extension of the partial cross-section $D'(k)$. In addition, $E(k)$ and $E(-k)$ are equal over $\mu(S^{n-1} \times [m^{-1}, 1])$ while they are otherwise disjoint.

For $m \geqq 2$, the union of $E(k)$ and the component of $\Gamma - E(k)$ disjoint from $E(-k)$ will be called $L(k)$. Since $B_\lambda \cap L(k) = \emptyset$, $L(k)$ is the total space of a product bundle over D with suitable restriction of λ as map and fibre homeomorphic to a closed interval (Proposition 2.1 and (**10**, p. 53, Corollary 11.6)). In addition, $L(m) - L(m - 1)$ and $L(-m) - L(-m + 1)$ are total spaces of product bundles over $\mu(S^{n-1} \times (0, m^{-1})) \cup f(q)$ with suitable restriction of λ as map and fibre a half-open interval. Let $S_j(r)$ be the open euclidean j-ball of radius r centred at the origin $\{0\}$ in E^j. Define β: $D \to S_n(1)$ to be a homeomorphism with $\beta\mu(S^{n-1} \times (0, t)) = S_n(t) - \{0\}$ for each $t \in (0, 1)$. Consider $S_n(1) \times [-1, 1]$ as a subset of $E^n \times E^1$ and let δ: $S_n(1) \times [-1, 1] \to S_n(1)$ be a product projection. For each $m \geqq 3$ there exists a homeomorphism

$$h_m: L(m) \cup L(-m) \to (S_n(1) \times [-1, 1]) - S_{n+1}(m^{-1})$$

so that $\beta\lambda h_m^{-1}|S_n(1) \times [-1, 1] - S_{n+1}(0, m^{-1})$ is the appropriate restriction of δ, and $h_m|L(m - 1) \cup L(-m + 1) = h_{m-1}$. If $x \in L - \{q\}$, then there exists an integer k such that $x \in L(k)$; thus define $\gamma(x) = h_m(x)$, and let $\gamma(q)$ be the origin in E^{n+1}. Then γ is a homeomorphism of Γ onto $S_n(1) \times [-1, 1]$, and $\beta\lambda\gamma^{-1} = \delta$. However, $q \notin B_\lambda$, which is a contradiction; hence, Y is not locally euclidean at q and Case II cannot occur.

Case III. Suppose that $W(q)$ is selected so that $W(q) \cap Y$ is homeomorphic to a collection of three or more half-open or closed arcs with end points identified at q. Then $X \cap ((Y) - \{q\})$ has at least three components. Let x_1, x_2, and x_3 be three points in the interior of X which are elements of different components of $X \cap (Y - \{q\})$, and let $W(i)$, $i = 1, 2, 3$, be a neighbourhood of x_i contained in X such that $\lambda|W(i)$ is topologically equivalent to the product projection of E^{n+1} onto E^n. Choose $\{W(i)\}$ so that their closures are pairwise disjoint and disjoint from q. Let $D(i)$ be a partial cross-sectional n-disk in $W(i)$ through x_i such that $\lambda(D(i))$ is the same euclidean neighbourhood D of $f(q)$ for each i. The map $\lambda|\lambda^{-1}(D) - Y$ is a bundle map onto $D - f(q)$ with fibre S^1 (by Proposition 2.1) and cross-sections $D(i) - x_i$. Let E be an open neighbourhood of q contained in $\lambda^{-1}(D)$ and disjoint from each cross-section $D(i)$. If $n > 1$, then $E - Y$ is connected (**7**, p. 48, Theorem IV 4); therefore, there exists a component L of $\lambda^{-1}(D) - ((\bigcup_{i=1}^{3} D(i)) \cup Y)$ which contains $E - Y$, and $\lambda|L$ is a bundle map onto $D - f(q)$ with fibre homeomorphic to an open interval. By (**10**, p. 53, Theorem 11.4) and Remark 2.4, $\lambda|L$ is topologically equivalent to the projection map of $(D - f(q)) \times (0, 1)$ onto $(D - f(q))$. However, the closure of L has non-empty intersection with each $D(i)$, which is a contradiction. Hence, either (a) q is a point component of $f^{-1}(f(q))$, or (b) $n = 1$, q is not a point component of $f^{-1}(f(q))$, and $f^{-1}(f(q))$ fails to be locally euclidean at q.

Now, for the *proof of Theorem* 1.2, observe that if $B_f = \emptyset$, then f is a locally trivial fibre map by Proposition 2.1; if q is an element of B_f and is a point component of $f^{-1}(f(q))$, then by Lemma 2.2, q is an isolated point in $f^{-1}(f(B_f))$, and thus Proposition 2.1 implies that q is an isolated point in the set A_f defined in (**13**; **12**; or **5**). It follows from (**12**, p. 62, Theorem 1.6) that (i) or (ii) must occur. If $n = 1$, then q is not a point component of $f^{-1}(f(q))$, and $f^{-1}(f(q))$ fails to be locally euclidean at q, then f is an interior map at q and Nathan (**8**) has shown that (ii) must hold.

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