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On the Eisenstein ideal for imaginary quadratic fields

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Abstract

For certain algebraic Hecke characters χ of an imaginary quadratic field F we define an Eisenstein ideal in a p-adic Hecke algebra acting on cuspidal automorphic forms of $\mathrm{GL}_{2/F}$. By finding congruences between Eisenstein cohomology classes (in the sense of G. Harder) and cuspidal classes we prove a lower bound for the index of the Eisenstein ideal in the Hecke algebra in terms of the special L-value $L(0, \chi)$. We further prove that its index is bounded from above by the p-valuation of the order of the Selmer group of the p-adic Galois character associated to χ^{-1}. This uses the work of R. Taylor *et al.* on attaching Galois representations to cuspforms of $\mathrm{GL}_{2/F}$. Together these results imply a lower bound for the size of the Selmer group in terms of $L(0, \chi)$, coinciding with the value given by the Bloch–Kato conjecture.

1. Introduction

The aim of this work is to demonstrate the use of Eisenstein cohomology, as developed by Harder [Har87], in constructing elements of Selmer groups for Hecke characters of an imaginary quadratic field F. The strategy of first finding congruences between Eisenstein series and cuspforms and then using the Galois representations attached to the cuspforms to prove lower bounds on the size of Selmer groups goes back to Ribet [Rib76], and has been applied and generalized by Wiles [Wil90] in his proof of the Iwasawa main conjecture for characters over totally real fields, and more recently by Skinner and Urban [SU02] and Bellaïche and Chenevier [BC04] amongst others. These all used integral structures coming from algebraic geometry for the congruences. In our case the symmetric space associated to $\mathrm{GL}_{2/F}$ is not hermitian and we therefore use the integral structure arising from Betti cohomology. This alternative, more general approach was outlined for $\mathrm{GL}_{2/\mathbf{Q}}$ in [HP92].

In [Har87] Harder constructed Eisenstein cohomology as a complement to the cuspidal cohomology for the groups GL_2 over number fields and proved that this decomposition respects the rational structure of group cohomology. The case that is interesting for congruences is when this decomposition is not integral, i.e., when there exists an Eisenstein class with integral restriction to the boundary that has a denominator. In [Ber08] such classes were constructed for imaginary quadratic fields and their denominator was bounded from below by the special L-value of a Hecke character. In § 4 we give a general set-up in which cohomological congruences between Eisenstein and cuspidal classes can be proven (Proposition 9) and then apply this to the classes constructed in [Ber08]. The result can be expressed as a lower bound for the index of the Eisenstein ideal of the title in a Hecke algebra in terms of the special L-value. The main obstacle

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to obtaining a congruence from the results in [Ber08] is the occurrence of torsion in higher degree cohomology, which is not very well understood (see the discussion in § 4.4). However, we manage to solve this 'torsion problem' for unramified characters in § 5.

The other main ingredient are the Galois representations associated to cohomological cuspidal automorphic forms, constructed by Taylor *et al.* in [Tay94, HST93] by means of a lifting to the symplectic group. Assuming the existence of congruences between an Eisenstein series and cuspforms we use these representations in § 3 to construct elements in the Selmer group of a Galois character. In fact, we prove that its size is bounded from below by the index of the Eisenstein ideal.

These two results are combined in § 6 to prove a lower bound on the size of Selmer groups of Hecke characters of an imaginary quadratic field in terms of a special L-value, coinciding with the value given by the Bloch–Kato conjecture.

To give a more precise account, let $p>3$ be a prime unramified in the extension $F/\mathbf{Q}$ and let $\mathfrak{p}$ be a prime of F dividing (p). Fix embeddings $\overline{F}\hookrightarrow\overline{F}_{\mathfrak{p}}\hookrightarrow\mathbf{C}$. Let $\phi_1,\phi_2:F^*\backslash\mathbf{A}_F^*\to\mathbf{C}^*$ be two Hecke characters of infinity type z and z^{-1}, respectively, with conductors coprime to (p). Let $\mathcal{R}$ be the ring of integers in a sufficiently large finite extension of $F_{\mathfrak{p}}$. Let $\mathbf{T}$ be the $\mathcal{R}$-algebra generated by Hecke operators acting on cuspidal automorphic forms of $\mathrm{GL}_{2/F}$. For $\phi=(\phi_1,\phi_2)$ we define in § 3 an Eisenstein ideal $\mathbf{I}_\phi$ in $\mathbf{T}$. Following previous work of Wiles [Wil86, Wil90] and Urban [Urb01] we construct elements in the Selmer group of $\chi_{\mathfrak{p}}\epsilon$, where $\chi_{\mathfrak{p}}$ is the p-adic Galois characters associated to $\chi:=\phi_1/\phi_2$ and ϵ is the p-adic cyclotomic character. We obtain a lower bound on the size of the Selmer group in terms of that of the congruence module $\mathbf{T}/\mathbf{I}_\phi$. A complication that arises in the application of Taylor's theorem is that we need to work with cuspforms with cyclotomic central character. This is achieved by a twisting argument (see Lemma 8).

To prove the lower bound on the congruence module in terms of the special L-value (the first step described above), we use the Eisenstein cohomology class $\mathrm{Eis}(\phi)$ constructed in [Ber08] in the cohomology of a symmetric space S associated to $\mathrm{GL}_{2/F}$. The class is an eigenvector for the Hecke operators at almost all places with eigenvalues corresponding to the generators of $\mathbf{I}_\phi$, and its restriction to the boundary of the Borel–Serre compactification of S is integral. The main result of [Ber08], which we recall in § 5, is that the denominator δ of $\mathrm{Eis}(\phi)\in H^1(S,\overline{F}_{\mathfrak{p}})$ is bounded from below by $L^{\mathrm{alg}}(0,\chi)$. As mentioned above, Proposition 9 gives a general set-up for cohomological congruences. It implies the existence of a cuspidal cohomology class congruent to $\delta\cdot\mathrm{Eis}(\phi)$ modulo the L-value supposing that there exists an integral cohomology class with the same restriction to the boundary as $\mathrm{Eis}(\phi)$. The latter can be replaced by the assumption that $H_c^2(S,\mathcal{R})_{\mathrm{torsion}}=0$, and this result is given in Theorem 13. In § 5 we prove that the original hypothesis is satisfied for unramified χ, avoiding the issue of torsion freeness. We achieve this by a careful analysis of the restriction map to the boundary $\partial\overline{S}$ of the Borel–Serre compactification. Starting with a group cohomological result for $\mathrm{SL}_2(\mathcal{O})$ due to Serre [Ser70] (which we extend to all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$), we define an involution on $H^1(\partial\overline{S},\mathcal{R})$ such that the restriction map

$$H^1(S,\mathcal{R})\overset{\mathrm{res}}{\twoheadrightarrow}H^1(\partial\overline{S},\mathcal{R})^-$$

surjects onto the -1-eigenspace. We apply the resulting criterion to $\mathrm{res}(\mathrm{Eis}(\phi))$ to deduce the existence of a lift to $H^1(S,\mathcal{R})$.

Note that the restriction to constant coefficient systems and therefore weight 2 automorphic forms is important only for § 5. It was applied throughout to simplify the exposition. In particular, the results of Theorems 10 and 13 extend (for split p) to characters χ with infinity type $z^{m+2}\overline{z}^{-m}$ for $m \in \mathbf{N}_{\geq 0}$. See [Ber08] for the necessary modifications and results.

Combining the two steps, we obtain in § 6 a lower bound for the size of the Selmer group of $\chi_{\mathfrak{p}}\epsilon$ in terms of $L^{\mathrm{alg}}(0, \chi)$ and relate this result to the Bloch–Kato conjecture. This conjecture has been proven in our case (at least for class number 1) starting from the Main Conjecture of Iwasawa theory for imaginary quadratic fields (see [Han97, Guo93]). Similar results have also been obtained by Hida in [Hid82] for split primes p and $\overline{\chi} = \chi^c$ using congruences of classical elliptic modular forms between CM (complex multiplication) and non-CM forms. We seem to recover base changes of his congruences in this case (but see § 4.6 for a discussion when our congruences do not arise from base change).

However, our method of constructing elements in Selmer groups using cohomological congruences is very different and should be more widely applicable. The analytic theory of Eisenstein cohomology has been developed for many groups, and rationality results are known, for example, for GL_n by the work of Franke and Schwermer [FS98]. Our hope is that the method presented here generalizes to these higher rank groups.

To conclude, we want to mention two related results. In [Fel00] congruences involving degree two Eisenstein cohomology classes for imaginary quadratic fields were constructed but only the L-value of the quadratic character associated to $F/\mathbf{Q}$ was considered. The torsion problem we encounter does not occur for degree two, but the treatment of the denominator of the Eisenstein classes is more difficult. For cases of the Bloch–Kato conjecture when the Selmer groups are infinite see [BC04]. Their method is similar to ours in that they use congruences; however, they work with p-adic families on U(3) and do not use Eisenstein cohomology.

2. Notation and definitions

2.1 General notation

Let $F/\mathbf{Q}$ be an imaginary quadratic extension and d_F its absolute discriminant. Denote the class group by $\mathrm{Cl}(F)$ and the ray class group modulo a fractional ideal $\mathfrak{m}$ by $\mathrm{Cl}_{\mathfrak{m}}(F)$. For a place v of F let F_v be the completion of F at v. We write $\mathcal{O}$ for the ring of integers of F, $\mathcal{O}_v$ for the closure of $\mathcal{O}$ in F_v, $\mathfrak{P}_v$ for the maximal ideal of $\mathcal{O}_v$, π_v for a uniformizer of F_v, and $\hat{\mathcal{O}}$ for $\prod_{v \text{ finite}} \mathcal{O}_v$. We use the notation $\mathbf{A}$, $\mathbf{A}_f$ and $\mathbf{A}_F$, $\mathbf{A}_{F,f}$ for the adeles and finite adeles of $\mathbf{Q}$ and F, respectively, and write $\mathbf{A}^*$ and $\mathbf{A}_F^*$ for the respective group of ideles. We often identify elements a_v of F_v^* for any place v with their image in $\mathbf{A}_F^*$ under the canonical injective homomorphism $a_v \mapsto (1, \dots, 1, a_v, 1 \dots, 1)$. Let $p > 3$ be a prime of $\mathbf{Z}$ that does not ramify in F, and let $\mathfrak{p} \subset \mathcal{O}$ be a prime dividing (p). Let Σ_p be the set of places of F above p.

Denote by G_F the absolute Galois group of F. For Σ a finite set of places of F let G_Σ be the Galois group of the maximal extension of F unramified at all places not in Σ. We fix an embedding $\overline{F} \hookrightarrow \overline{F}_v$ for each place v of F. Denote the corresponding decomposition and inertia groups by G_v and I_v, respectively. Let $g_v = G_v/I_v$ be the Galois group of the maximal unramified extension of F_v. For each finite place v we also fix an embedding $\overline{F}_v \hookrightarrow \mathbf{C}$ that is compatible with the fixed embeddings $i_v : \overline{F} \hookrightarrow \overline{F}_v$ and $i_\infty : \overline{F} \hookrightarrow \mathbf{C}(=\overline{F}_\infty)$. For a topological G_F-module (respectively G_v-module) M write $H^1(F, M)$ for the continuous Galois cohomology group $H^1(G_F, M)$, and $H^1(F_v, M)$ for $H^1(G_v, M)$.

2.2 Hecke characters

A Hecke character of F is a continuous group homomorphism $\lambda : F^*\backslash \mathbf{A}_F^* \to \mathbf{C}^*$. Such a character corresponds uniquely to a character on ideals prime to the conductor, which we also denote by λ. Define the character λ^c by $\lambda^c(x) = \lambda(\overline{x})$.

Lemma 1 [Ber05, Lemma 3.16]. *If λ is an unramified Hecke character then $\lambda^c = \overline{\lambda}$.*

For Hecke characters λ of type (A_0), i.e., with infinity type $\lambda_\infty(z) = z^m\overline{z}^n$ with $m, n \in \mathbf{Z}$, we define (following Weil) a p-adic Galois character

$$\lambda_{\mathfrak{p}} : G_F \to \overline{F}_{\mathfrak{p}}^*$$

associated to λ by the following rule. For a finite place v not dividing p or the conductor of λ, put $\lambda_{\mathfrak{p}}(\mathrm{Frob}_v) = i_{\mathfrak{p}}(i_\infty^{-1}(\lambda(\pi_v)))$ where Frob_v is the *arithmetic* Frobenius at v. It takes values in the integer ring of a finite extension of $F_{\mathfrak{p}}$.

Let $\epsilon : G_F \to \mathbf{Z}_p^*$ be the p-adic cyclotomic character defined by the action of G_F on the p-power roots of unity: $g.\xi = \xi^{\epsilon(g)}$ for ξ with $\xi^{p^m} = 1$ for some m. Our convention is that the Hodge–Tate weight of ϵ at $\mathfrak{p}$ is 1 and we use the arithmetic Frobenius normalization for the Artin reciprocity map rec which implies that $\epsilon(\mathrm{rec}(u)) = \mathrm{Nm}_{\mathcal{O}_{\mathfrak{p}}/\mathbf{Z}_p}(u)^{-1}$ for $u \in \mathcal{O}_{\mathfrak{p}}^*$.

Let λ a Hecke character of infinity type $z^a(z/\overline{z})^b$ with conductor prime to p. Write $L(s, \lambda)$ for the Hecke L-function of λ. Assume $a, b \in \mathbf{Z}$ and $a > 0$ and $b \geq 0$. Put

$$L^{\mathrm{alg}}(0, \lambda) := \Omega^{-a-2b}\left(\frac{2\pi}{\sqrt{d_F}}\right)^b \Gamma(a+b) \cdot L(0, \lambda),$$

where Ω is a complex period. In most cases, this normalization is integral, i.e., lies in the integer ring of a finite extension of $F_{\mathfrak{p}}$. See [Ber08, Theorem 3] for the exact statement. Put

$$L^{\mathrm{int}}(0, \lambda) = \begin{cases} L^{\mathrm{alg}}(0, \lambda) & \text{if } \mathrm{val}_p(L^{\mathrm{alg}}(0, \lambda)) \geq 0, \\ 1 & \text{otherwise.} \end{cases}$$

2.3 Selmer groups

Let $\rho : G_F \to \mathcal{R}^*$ be a continuous Galois character taking values in the ring of integers $\mathcal{R}$ of a finite extension L of $F_{\mathfrak{p}}$. Write $\mathfrak{m}_{\mathcal{R}}$ for its maximal ideal and put $\mathcal{R}^\vee = L/\mathcal{R}$. Let $\mathcal{R}_\rho$, L_ρ, and $W_\rho = L_\rho/\mathcal{R}_\rho = \mathcal{R}_\rho \otimes_{\mathcal{R}} \mathcal{R}^\vee$ be the free rank one modules on which G_F acts via ρ.

Following Bloch and Kato [BK90] we define the following Selmer groups. Let

$$H^1_f(F_v, L_\rho) = \begin{cases} \ker(H^1(F_v, L_\rho) \to H^1(I_v, L_\rho)) & \text{for } v \nmid p, \\ \ker(H^1(F_v, L_\rho) \to H^1(F_v, B_{\mathrm{cris}} \otimes L_\rho)) & \text{for } v \mid p, \end{cases}$$

where B_{cris} denotes Fontaine's ring of p-adic periods. Put

$$H^1_f(F_v, W_\rho) = \mathrm{im}(H^1_f(F_v, L_\rho) \to H^1(F_v, W_\rho)).$$

For a finite set of places Σ of F define

$$\mathrm{Sel}^\Sigma(F, \rho) = \ker\left(H^1(F, W_\rho) \to \prod_{v \notin \Sigma} \frac{H^1(F_v, W_\rho)}{H^1_f(F_v, W_\rho)}\right).$$

We write $\mathrm{Sel}(F, \rho)$ for $\mathrm{Sel}^\emptyset(F, \rho)$.

If p splits in $F/\mathbf{Q}$ and $\rho=\lambda_{\mathfrak{p}}$ for a Hecke character λ of infinity type $z^a\overline{z}^b$ with $a,b\in\mathbf{Z}$ then one can show (see [Hid82, Equation (1.6a)]) that

$$\rho(\mathrm{rec}(u_1,u_2))=u_1^a\overline{u}_2^{\,b} \text{ for } (u_1,u_2)\in\mathcal{O}_{\mathfrak{p}}^*\times\mathcal{O}_{\overline{\mathfrak{p}}}^* \quad \text{with } (u_1,u_2)\equiv(1,1) \bmod \mathfrak{f}_\lambda\mathcal{O}_{\mathfrak{p}}\times\mathfrak{f}_\lambda\mathcal{O}_{\overline{\mathfrak{p}}}, \tag{1}$$

where $\mathfrak{f}_\lambda$ is the conductor of λ. The character ρ is therefore 'locally algebraic' in the sense of [Ser68], and a theorem of Tate [Ser68, ch. III, Appendix A7, Theorem 3] implies that the local Galois representations $\rho|_{G_{\mathfrak{p}}}$ and $\rho|_{G_{\overline{\mathfrak{p}}}}$ are of Hodge–Tate type with weights $-a$ and $-b$, respectively. Following Greenberg [Gre89], we define in this 'ordinary case'

$$F_{\mathfrak{p}}^+L_\rho=\begin{cases} L_\rho & \text{if } a<0 \text{ (i.e., Hodge–Tate weight of } \rho>0),\\ \{0\} & \text{if } a\geq 0 \text{ (i.e., Hodge–Tate weight of } \rho\leq 0)\end{cases}$$

and

$$F_{\overline{\mathfrak{p}}}^+L_\rho=\begin{cases} L_\rho & \text{if } b<0,\\ \{0\} & \text{if } b\geq 0.\end{cases}$$

In the ordinary case we have $H^1_f(F_v,L_\rho)=\mathrm{im}(H^1(F_v,F_v^+L_\rho)\to H^1(F_v,L_\rho))$ for $v\mid p$ (see [Guo96, p. 361], [Fla90, Lemma 2]).

Lemma 2. *Let ρ be unramified at $v\nmid p$. If $\rho(\mathrm{Frob}_v)\not\equiv\epsilon(\mathrm{Frob}_v) \bmod p$ then*

$$\mathrm{Sel}^{\Sigma}(F,\rho)=\mathrm{Sel}^{\Sigma\setminus\{v\}}(F,\rho).$$

Proof. By definition $\mathrm{Sel}^{\Sigma\setminus\{v\}}(F,\rho)\subset\mathrm{Sel}^{\Sigma}(F,\rho)$ for any v. For places v as in the lemma we have

$$H^1_f(F_v,W_\rho)=\ker(H^1(F_v,W_\rho)\to H^1(I_v,W_\rho)^{g_v}).$$

It is clear that $H^1(I_v,W_\rho)^{g_v}=\mathrm{Hom}_{g_v}(I_v^{\mathrm{tame}},W_\rho)=\mathrm{Hom}_{g_v}(I_v^{\mathrm{tame}},W_\rho[\mathfrak{m}_{\mathcal{R}}^n])$ for some n. By our assumption, therefore, $H^1(I_v,W_\rho)^{g_v}=0$ since Frob_v acts on I_v^{tame} by $\epsilon(\mathrm{Frob}_v)$. □

2.4 Cuspidal automorphic representations

We refer to [Urb95, § 3.1] as a reference for the following. For $K_f=\prod_v K_v\subset\mathrm{GL}_2(\mathbf{A}_{F,f})$ a compact open subgroup, denote by $S_2(K_f)$ the space of cuspidal automorphic forms of $\mathrm{GL}_2(F)$ of weight 2, right-invariant under K_f. For ω a finite order Hecke character write $S_2(K_f,\omega)$ for the forms with central character ω. This is isomorphic as a $\mathrm{GL}_2(\mathbf{A}_{F,f})$-module to $\bigoplus\pi_f^{K_f}$ for automorphic representations π of a certain infinity type (see Theorem 3 below) with central character ω. For $g\in\mathrm{GL}_2(\mathbf{A}_{F,f})$ we have the Hecke action of $[K_fgK_f]$ on $S_2(K_f)$ and $S_2(K_f,\omega)$. For places v with $K_v=\mathrm{GL}_2(\mathcal{O}_v)$ we define $T_v=\left[K_f\left(\begin{smallmatrix}\pi_v&0\\0&1\end{smallmatrix}\right)K_f\right]$.

2.5 Cohomology of symmetric space

Let $G=\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2/F}$ and let B be the restriction of scalars of the Borel subgroup of upper triangular matrices. For any $\mathbf{Q}$-algebra R we consider a pair of characters $\phi=(\phi_1,\phi_2)$ of $R^*\times R^*$ as characters of $B(R)$ by defining $\phi\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)=\phi_1(a)\phi_2(b)$. Put $K_\infty=U(2)\cdot\mathbf{C}^*\subset G(\mathbf{R})$. For an open compact subgroup $K_f\subset G(\mathbf{A}_f)$ we define the adelic symmetric space

$$S_{K_f}=G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K_f.$$

Note that S_{K_f} has several connected components. In fact, strong approximation implies that the fibers of the determinant map

$$S_{K_f}\twoheadrightarrow\pi_0(K_f):=\mathbf{A}_{F,f}^*/\det(K_f)F^*$$

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are connected. Any $\gamma \in G(\mathbf{A}_f)$ gives rise to an injection

$$G_\infty \hookrightarrow G(\mathbf{A}),$$
$$g_\infty \mapsto (g_\infty, \gamma)$$

and, after taking quotients, to a component $\Gamma_\gamma \backslash G_\infty / K_\infty \to S_{K_f}$, where

$$\Gamma_\gamma := G(\mathbf{Q}) \cap \gamma K_f \gamma^{-1}.$$

This component is the fiber over $\det(\gamma)$. Choosing a system of representatives for $\pi_0(K_f)$ we therefore have

$$S_{K_f} \cong \coprod_{[\det(\gamma)] \in \pi_0(K_f)} \Gamma_\gamma \backslash \mathbf{H}_3,$$

where G_∞/K_∞ has been identified with the three-dimensional hyperbolic space $\mathbf{H}_3 = \mathbf{R}_{>0} \times \mathbf{C}$.

We denote the Borel–Serre compactification of S_{K_f} by $\overline{S}_{K_f}$ and write $\partial \overline{S}_{K_f}$ for its boundary. The Borel–Serre compactification $\overline{S}_{K_f}$ is given by the union of the compactifications of its connected components. For any arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$, the boundary of the Borel–Serre compactification of $\Gamma \backslash \mathbf{H}_3$, denoted by $\partial(\Gamma \backslash \overline{\mathbf{H}}_3)$, is homotopy equivalent to

$$\coprod_{[\eta] \in \mathbf{P}^1(F)/\Gamma} \Gamma_{B^\eta} \backslash \mathbf{H}_3, \tag{2}$$

where we identify $\mathbf{P}^1(F) = B(\mathbf{Q}) \backslash G(\mathbf{Q})$, take $\eta \in G(\mathbf{Q})$, and put $\Gamma_{B^\eta} = \Gamma \cap \eta^{-1} B(\mathbf{Q}) \eta$.

For $X \subset \overline{S}_{K_f}$ and R an $\mathcal{O}$-algebra we denote by $H^i(X, R)$ (respectively $H^i_c(X, R)$) the ith (Betti) cohomology group (respectively with compact support), and the interior cohomology, i.e., the image of $H^i_c(X, R)$ in $H^i(X, R)$, by $H^i_!(X, R)$.

There is a Hecke action of double cosets $[K_f g K_f]$ for $g \in G(\mathbf{A}_f)$ on these cohomology groups (see [Urb98, § 1.4.4] for the definition). We put $T_{\pi_v} = \left[K_f \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) K_f\right]$ and $S_{\pi_v} = \left[K_f \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & \pi_v \end{smallmatrix}\right) K_f\right]$.

The connection between cohomology and cuspidal automorphic forms is given by the Eichler–Shimura–Harder isomorphism (in this special case see [Urb98, Theorem 1.5.1]). For any compact open subgroup $K_f \subset G(\mathbf{A}_f)$ we have

$$S_2(K_f) \xrightarrow{\sim} H^1_!(S_{K_f}, \mathbf{C}) \tag{3}$$

and the isomorphism is Hecke-equivariant.

One knows (see, for example, [Ber08, Proposition 4]) that for any $\mathcal{O}[1/6]$-algebra R there is a natural R-functorial isomorphism

$$H^1(\Gamma \backslash \mathbf{H}_3, R) \cong H^1(\Gamma, R), \tag{4}$$

where the group cohomology $H^1(\Gamma, R)$ is just given by $\mathrm{Hom}(\Gamma, R)$.

2.6 Galois representations associated to cuspforms for imaginary quadratic fields

Combining the work of Taylor, Harris, and Soudry [HST93, Tay94] with results of Bump, Friedberg and Hoffstein [BFH90] and Laumon [Lau97, Lau05] or Weissauer [Wei05], one can show the following (see [BH07]).

Theorem 3. *Given a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ with π_∞ isomorphic to the principal series representation corresponding to*

$$\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \left(\frac{t_1}{|t_1|}\right)\left(\frac{|t_2|}{t_2}\right)$$

and cyclotomic central character ω (i.e. $\omega^c = \omega$), let Σ_π denote the set of places above p, the primes where π or π^c is ramified, and primes ramified in $F/\mathbf{Q}$.

Then there exists a continuous Galois representation

$$\rho_\pi : G_F \to \mathrm{GL}_2(\overline{F}_{\mathfrak{p}})$$

such that if $v \notin \Sigma_\pi$, then ρ_π is unramified at v and the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_v)$ is $x^2 - a_v(\pi)x + \omega(\mathfrak{P}_v)\mathrm{Nm}_{F/\mathbf{Q}}(\mathfrak{P}_v)$, where $a_v(\pi)$ is the Hecke eigenvalue corresponding to T_v. The image of the Galois representation lies in $\mathrm{GL}_2(L)$ for a finite extension L of $F_{\mathfrak{p}}$ and the representation is absolutely irreducible. □

Remarks.

(i) Taylor relates π to a low weight Siegel modular form via a theta lift and uses the Galois representation attached to this form (via pseudorepresentations and the Galois representations of cohomological Siegel modular forms) to find ρ_π.

(ii) Taylor had some additional technical assumption in [Tay94] and only showed the equality of Hecke and Frobenius polynomial outside a set of places of zero density. For this strengthening of Taylor's result see [BH07].

(iii) Conjecture 3.2 in [CD06] describes a conjectural extension of Taylor's theorem for cuspforms of general weight.

Urban studied in [Urb98] the case of ordinary automorphic representations π, and together with results in [Urb05] on the Galois representations attached to ordinary Siegel modular forms, shows the following theorem.

THEOREM 4 [Urb05, Corollaire 2]. *If π is unramified at $\mathfrak{p}$ and ordinary at $\mathfrak{p}$, i.e., $|a_{\mathfrak{p}}(\pi)|_p = 1$, then the Galois representation ρ_π is ordinary at $\mathfrak{p}$, i.e.,*

$$\rho_\pi|_{G_{\mathfrak{p}}} \cong \begin{pmatrix} \Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix},$$

where $\Psi_2|_{I_{\mathfrak{p}}} = 1$, and $\Psi_1|_{I_{\mathfrak{p}}} = \det(\rho_\pi)|_{I_{\mathfrak{p}}} = \epsilon$.

For p inert we will need a stronger statement (we refer the reader to [DFG04, § 1.1.2] for the definition of a short crystalline Galois representation, and note that we assume $p > 3$).

CONJECTURE 5. If π is unramified at $\mathfrak{p}$ then $\rho_\pi|_{G_{\mathfrak{p}}}$ is crystalline and short.

3. Selmer group and Eisenstein ideal

In this section we define an Eisenstein ideal in a Hecke algebra acting on cuspidal automorphic forms of $\mathrm{GL}_{2/F}$ and show that its index gives a lower bound on the size of the Selmer group of a Galois character.

Let ϕ_1 and ϕ_2 be two Hecke characters with infinity type z and z^{-1}, respectively. Assume that their conductors are coprime to (p). Let $\mathcal{R}$ be the ring of integers in the finite extension L of $F_{\mathfrak{p}}$ containing the values of the finite parts of ϕ_i and $L^{\mathrm{alg}}(0, \phi_1/\phi_2)$. Denote its maximal ideal by $\mathfrak{m}_{\mathcal{R}}$. Let Σ_ϕ be the finite set of places dividing the conductors of the characters ϕ_i and their complex conjugates and the places dividing pd_F. Let $K_f = \prod_v K_v \subset G(\mathbf{A}_f)$ be a compact open subgroup such that $K_v = \mathrm{GL}_2(\mathcal{O}_v)$ if $v \notin \Sigma_\phi \backslash \Sigma_p$. In § 4.2 we will specify K_v at the other 'bad' places $v \in \Sigma_\phi \backslash \Sigma_p$. We will leave this open for now since we focus on the places dividing p in this section.

Because of the condition on the central character in Theorem 3 we assume that there exists a finite order Hecke character η unramified outside Σ_ϕ such that

$$(\phi_1\phi_2\eta^2)^c = \phi_1\phi_2\eta^2. \tag{5}$$

Denote by $\mathbf{T}$ the $\mathcal{R}$-algebra generated by the Hecke operators $T_v, v \notin \Sigma_\phi \backslash \Sigma_p$, acting on $S_2(K_f, \phi_1\phi_2)$. Call the ideal $\mathbf{I}_\phi \subseteq \mathbf{T}$ generated by

$$\{T_v - \phi_1(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v) - \phi_2(\mathfrak{P}_v) \mid v \notin \Sigma_\phi \backslash \Sigma_p\}$$

the *Eisenstein ideal associated to* $\phi = (\phi_1, \phi_2)$.

Using the notation of § 2.2, we define Galois characters

$$\rho_1 = \phi_{1,\mathfrak{p}}\epsilon, \quad \rho_2 = \phi_{2,\mathfrak{p}}, \quad \rho = \rho_1 \otimes \rho_2^{-1}.$$

Note that ρ depends only on the quotient ϕ_1/ϕ_2. Our first main result is the following inequality.

Theorem 6. *Assuming Conjecture 5 if p is inert in $F/\mathbf{Q}$, we have*

$$\mathrm{val}_p(\#\mathrm{Sel}^{\Sigma_\phi \backslash \Sigma_p}(F, \rho)) \geq \mathrm{val}_p(\#(\mathbf{T}/\mathbf{I}_\phi)).$$

Proof. We can assume that

$$\mathbf{T}/\mathbf{I}_\phi \neq 0.$$

Let $\mathfrak{m} \subset \mathbf{T}$ be a maximal ideal containing $\mathbf{I}_\phi$. Localizing at $\mathfrak{m}$ we write

$$S_2(K_f, \phi_1\phi_2)_\mathfrak{m} = \bigoplus_{i=1}^n V_{\pi_{i,f}}^{K_f},$$

where V_{π_f} denotes the representation space of the (finite part) of a cuspidal automorphic representation π.

By twisting the cuspforms by the finite order character η of (5) we can ensure that their central character is cyclotomic. Hence we can apply Theorem 3 to associate Galois representations $\rho_{\pi_i \otimes \eta} : G_{\Sigma_\phi} \to \mathrm{GL}_2(L_i)$ to each $\pi_i \otimes \eta, i = 1, \ldots, n$, for some finite extensions $L_i/F_\mathfrak{p}$. Taking all of them together (and untwisting by η) we obtain a continuous, absolutely irreducible Galois representation

$$\rho_T := \bigoplus_{i=1}^n \rho_{\pi_i \otimes \eta} \otimes \eta^{-1} : G_{\Sigma_\phi} \to \mathrm{GL}_2(\mathbf{T}_\mathfrak{m} \otimes_\mathcal{R} L).$$

Here we use the fact that $\mathbf{T}_\mathfrak{m} \otimes_\mathcal{R} L = \prod_{i=1}^n L_i$, which follows from the strong multiplicity one theorem. We have an embedding

$$\mathbf{T}_\mathfrak{m} \hookrightarrow \prod_{i=1}^n L_i,$$
$$T_v \mapsto ((a_v(\pi_i)),$$

where $a_v(\pi_i)$ is the T_v-eigenvalue of π_i. The coefficients of the characteristic polynomial $\mathrm{char}(\rho_T)$ lie in $\mathbf{T}_\mathfrak{m}$ and by the Chebotarev density theorem

$$\mathrm{char}(\rho_T) \equiv \mathrm{char}(\rho_1 \oplus \rho_2) \mod \mathbf{I}_\phi.$$

For any finite free $\mathbf{T}_\mathfrak{m} \otimes L$-module $\mathcal{M}$, any $\mathbf{T}_\mathfrak{m}$-submodule $\mathcal{L} \subset \mathcal{M}$ that is finite over $\mathbf{T}_\mathfrak{m}$ and such that $\mathcal{L} \otimes L = \mathcal{M}$ is called a $\mathbf{T}_\mathfrak{m}$-*lattice*. Specializing to our situation [Urb01, Theorem 1.1] we get the following.

Theorem 7 (Urban). *Given a Galois representation ρ_T as above there exists a G_{Σ_ϕ}-stable $\mathbf{T}_\mathfrak{m}$-lattice $\mathcal{L} \subset (\mathbf{T}_\mathfrak{m} \otimes L)^2$ such that G_{Σ_ϕ} acts on $\mathcal{L}/\mathbf{I}_\phi\mathcal{L}$ via the short exact sequence*

$$0 \to \mathcal{R}_{\rho_1} \otimes_\mathcal{R} (\mathcal{N}/\mathbf{I}_\phi) \to \mathcal{L}/\mathbf{I}_\phi\mathcal{L} \to \mathcal{R}_{\rho_2} \otimes_\mathcal{R} (\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi) \to 0,$$

where $\mathcal{N} \subset \mathbf{T}_\mathfrak{m} \otimes L$ is a $\mathbf{T}_\mathfrak{m}$-lattice with $\mathrm{val}_p(\#\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi) \leq \mathrm{val}_p(\#\mathcal{N}/\mathbf{I}_\phi\mathcal{N}) < \infty$ and no quotient of $\mathcal{L}$ is isomorphic to $\overline{\rho}_1 := \rho_1 \bmod \mathfrak{m}_\mathcal{R}$.

Proof. Note that the $\mathcal{R}$-algebra map $\mathcal{R} \twoheadrightarrow \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$ is surjective and that $\mathcal{L}/\mathbf{I}_\phi\mathcal{L} \cong \mathcal{L} \otimes_\mathcal{R} \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$. Hence this short exact sequence recovers the one in the statement of [Urb01, Theorem 1.1]. For the statement about $\mathrm{val}_p(\#\mathcal{N}/\mathbf{I}_\phi\mathcal{N})$ see [Urb01, p. 519] and use the fact that any $\mathcal{R}$-submodule of $\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$ or $\mathcal{N}/\mathbf{I}_\phi\mathcal{N}$ is a $\mathbf{T}_\mathfrak{m}$-submodule.

See [Ber05, § 7.3.2] for an alternative construction of such a lattice using arguments of Wiles [Wil86, Wil90]. □

Using the properties of the Galois representations attached to cuspforms listed in § 2.6 we can now conclude the proof of Theorem 6 by similar arguments as in [Urb01]. To ease notation we put $\mathcal{T} := \mathcal{N}/\mathbf{I}_\phi$ and $\Sigma := \Sigma_\phi$.

Identifying $\mathcal{R}_\rho$ with $\mathrm{Hom}_\mathcal{R}(\mathcal{R}_{\rho_2}, \mathcal{R}_{\rho_1})$ and writing $s : \mathcal{R}_{\rho_2} \otimes \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi \to \mathcal{L} \otimes \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$ for the section as $\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$-modules, we define a 1-cocycle $c : G_F \to G_\Sigma \to \mathcal{R}_\rho \otimes \mathcal{T}$ by

$$c(g)(m) = \text{the image of } s(m) - g.s(g^{-1}.m) \text{ in } \mathcal{R}_{\rho_1} \otimes \mathcal{T}.$$

Consider the $\mathcal{R}$-homomorphism

$$\varphi : \mathrm{Hom}_\mathcal{R}(\mathcal{T}, \mathcal{R}^\vee) \to H^1(F, W_\rho), \quad \varphi(f) = \text{the class of } (1 \otimes f) \circ c.$$

We will show that:

(i) $\mathrm{im}(\varphi) \subset \mathrm{Sel}^{\Sigma\setminus\Sigma_p}(F, \rho)$;

(ii) $\ker(\varphi) = 0$.

From statement (i) it follows that

$$\mathrm{val}_p(\#\mathrm{Sel}^{\Sigma\setminus\Sigma_p}(F, \rho)) \geq \mathrm{val}_p(\#\mathrm{im}(\varphi)).$$

From statement (ii) it follows that

$$\begin{aligned} \mathrm{val}_p(\#\mathrm{im}(\varphi)) &\geq \mathrm{val}_p(\#\mathrm{Hom}_\mathcal{R}(\mathcal{T}, \mathcal{R}^\vee)) \\ &= \mathrm{val}_p(\#\mathcal{T}) \\ &\geq \mathrm{val}_p(\#\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi). \end{aligned}$$

To prove statement (ii) we use an argument explained to us by Chris Skinner (personal communication). We first observe that, for any $f \in \mathrm{Hom}_\mathcal{R}(\mathcal{T}, \mathcal{R}^\vee)$, $\ker(f)$ has finite index in $\mathcal{T}$ since $\mathcal{T}$ is a finite $\mathcal{R}$-module, and so $f \in \mathrm{Hom}_\mathcal{R}(\mathcal{T}, \mathcal{R}^\vee[\mathfrak{m}_\mathcal{R}^n])$ for some n. Suppose now that $f \in \ker(\varphi)$. We claim that the class of c in $H^1(G_\Sigma, \mathcal{R}_\rho \otimes_\mathcal{R} \mathcal{T}/\ker(f))$ is zero. To see this, let $X = \mathcal{R}^\vee/\mathrm{im}(f)$ and observe that there is an exact sequence

$$H^0(G_\Sigma, \mathcal{R}_\rho \otimes_\mathcal{R} X) \to H^1(G_\Sigma, \mathcal{R}_\rho \otimes_\mathcal{R} \mathcal{T}/\ker(f)) \to H^1(G_\Sigma, \mathcal{R}_\rho \otimes_\mathcal{R} \mathcal{R}^\vee).$$

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Since $f \in \ker(\varphi)$ and the second arrow in the sequence comes from the inclusion $\mathcal{T}/\ker(f) \hookrightarrow \mathcal{R}^\vee$ induced by f, the image in the right module of the class of c in the middle is zero. Our claim follows therefore if the module on the left is trivial. But the dual of this module is a subquotient of $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}_\rho, \mathcal{R})$ on which G_Σ acts trivially. By assumption, however, $\mathcal{R}_\rho$ is a rank one module on which G_Σ acts non-trivially.

Suppose in addition that f is non-trivial, i.e., $\ker(f) \subsetneq \mathcal{T}$. Note that any $\mathcal{R}$-submodule of $\mathcal{T}$ is actually a $\mathbf{T}_\mathfrak{m}$-submodule since $\mathcal{R} \twoheadrightarrow \mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi$. Hence there exists a $\mathbf{T}_\mathfrak{m}$-module A with $\ker(f) \subset A \subset \mathcal{T}$ such that $\mathcal{T}/A \cong \mathcal{R}/\mathfrak{m}_\mathcal{R}$. Since c represents the trivial class in $H^1(G_\Sigma, \mathcal{R}_\rho \otimes_\mathcal{R} \mathcal{T}/\ker(f))$ it follows that the $\mathbf{T}_\mathfrak{m}[G_\Sigma]$-extension

$$0 \to \mathcal{R}_{\rho_1} \otimes_\mathcal{R} \mathcal{R}/\mathfrak{m}_\mathcal{R} \cong \mathcal{R}_{\rho_1} \otimes_\mathcal{R} \mathcal{T}/A \to (\mathcal{L}/\mathbf{I}_\phi\mathcal{L})/(\mathcal{R}_{\rho_1} \otimes_\mathcal{R} A) \to \mathcal{R}_{\rho_2} \otimes_\mathcal{R} (\mathbf{T}_\mathfrak{m}/\mathbf{I}_\phi) \to 0$$

is split. But this would give a $\mathbf{T}_\mathfrak{m}[G_\Sigma]$-quotient of $\mathcal{L}$ isomorphic to $\overline{\rho}_1$, which contradicts one of the properties of the lattice constructed by Urban. Hence $\ker(\varphi)$ is trivial.

For statement (i) we have to show that the local conditions of the Selmer group are satisfied. Firstly, if $v \notin \Sigma$ then $v \nmid p$ and ρ is unramified at v, so we have

$$H^1_f(F_v, W_\rho) = \ker(H^1(F_v, W_\rho) \to H^1(I_v, W_\rho))$$

by [Rub00, Lemma 1.3.5(iv)]. Since the extension in Theorem 7 is unramified outside Σ the image of φ maps to zero in $H^1(I_v, W_\rho)$.

For the places $v \mid p$ we divide them into the split and inert cases. For split p we claim that we only have a non-trivial condition at $\mathfrak{p}$ since $H^1_f(F_{\overline{\mathfrak{p}}}, W_\rho) = H^1(F_{\overline{\mathfrak{p}}}, W_\rho)_{\mathrm{div}} = H^1(F_{\overline{\mathfrak{p}}}, W_\rho)$. For this we use the fact that $\rho = \epsilon$ on $I_{\overline{\mathfrak{p}}}$ by (1). Then we know by [NSW08, Proposition 7.3.10] that $H^1(F_{\overline{\mathfrak{p}}}, W_\epsilon)$ is divisible and $W_\epsilon^{I_{\overline{\mathfrak{p}}}} = \{1\}$ since $p > 3$, so by applying the inflation-restriction sequence for both W_ρ and W_ϵ we get

$$H^1(F_{\overline{\mathfrak{p}}}, W_\rho) \cong H^1(I_{\overline{\mathfrak{p}}}, W_\rho)^{g_{\overline{\mathfrak{p}}}} \cong H^1(F_{\overline{\mathfrak{p}}}, W_\epsilon).$$

At $\mathfrak{p}$ it suffices to prove that the extension in Theorem 7 is split when considered as an extension of $\mathbf{T}_\mathfrak{m}[G_\mathfrak{p}]$-modules, because then the class in $H^1(F_\mathfrak{p}, \mathcal{R}_\rho \otimes \mathcal{T})$ determined by c is the zero class. In this case the Hecke eigenvalues $a_\mathfrak{p}(\pi_i) \equiv p \cdot \phi_1(\mathfrak{p}) + \phi_2(\mathfrak{p}) \not\equiv 0 \bmod \mathfrak{m}_R$; hence the cuspforms $\pi_i \otimes \eta$ are ordinary at $\mathfrak{p}$, so Theorem 4 applies and ρ_T is ordinary. Observing that the Hodge–Tate weights at $\mathfrak{p}$ of ρ_1 and ρ_2 are 0 and 1, respectively, the splitting of the extension as $\mathbf{T}_\mathfrak{m}[G_\mathfrak{p}]$-modules follows from comparing the basis given by Theorem 7 with the one coming from ordinarity.

For inert p we use the observation from the proof of statement (ii) that $\mathrm{im}(\varphi) \subset H^1(F, W_\rho[\mathfrak{m}^n_\mathcal{R}])$ for some n. Following [DFG04, p. 697] we define $H^1_f(F_\mathfrak{p}, W_\rho[\mathfrak{m}^n_\mathcal{R}]) \subset H^1(F_\mathfrak{p}, W_\rho[\mathfrak{m}^n_\mathcal{R}])$ to be the subset consisting of those cohomology classes which correspond to extensions of $\mathcal{R}/\mathfrak{m}^n_\mathcal{R}[G_\mathfrak{p}]$-modules

$$0 \to \mathcal{R}^\vee_{\rho_1}[\mathfrak{m}^n_\mathcal{R}] \to \mathcal{E} \to \mathcal{R}^\vee_{\rho_2}[\mathfrak{m}^n_\mathcal{R}] \to 0$$

such that $\mathcal{E}$ is in the essential image of the functor $\mathbf{V}$ defined in [DFG04, § 1.1.2] (extended using the Tate twist [DFG04, p. 711]). We will not need the precise definition of $\mathbf{V}$, just that its essential image is closed under taking subobjects, quotients and finite direct sums, and contains all short crystalline $G_\mathfrak{p}$-representations. Conjecture 5 therefore implies that $\mathrm{im}(\varphi) \subset H^1_f(F_\mathfrak{p}, W_\rho[\mathfrak{m}^n_\mathcal{R}])$

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(see [Klo09, Lemma 9.19]). Since by [DFG04, Proposition 2.2] and [Klo09, Proposition 9.20]

$$\varinjlim_j H^1_f(F_{\mathfrak{p}}, W_\rho[\mathfrak{m}^j_{\mathcal{R}}]) \cong H^1_f(F_{\mathfrak{p}}, W_\rho),$$

this proves that $\mathrm{im}(\varphi)$ satisfies the required local condition in this case, too. $\square$

The following lemma will later provide us with the finite order character η of (5) used in the twisting above.

Lemma 8. *If $\chi = \phi_1/\phi_2$ satisfies $\chi^c = \overline{\chi}$ then there exists a finite order character η unramified outside Σ_ϕ such that $(\phi_1\phi_2\eta^2)^c = \phi_1\phi_2\eta^2$.*

Proof. In the lemma in [Gre83, p. 81], Greenberg defines a Hecke character $\mu_G : F^*\backslash\mathbf{A}^*_F \to \mathbf{C}^*$ of infinity type z^{-1} such that $\mu^c_G = \overline{\mu}_G$ and μ_G is ramified exactly at the primes ramified in $F/\mathbf{Q}$. We claim that there exists a finite order Hecke character μ unramified outside Σ_ϕ such that

$$\chi' := \chi\mu^2_G = \mu/\mu^c.$$

Given such a character μ we then define $\eta = \mu_G/\mu\phi_2$ and one can check that $\phi_1\phi_2\eta^2 = (\mu\mu^c)^{-1}$.

Since $\chi'^{-1} = \overline{\chi}' = \chi'^c$ we have that

$$\chi' \equiv 1 \quad \text{on } \mathrm{Nm}_{F/\mathbf{Q}}(\mathbf{A}^*_F) \subset \mathbf{A}^*_{\mathbf{Q}} \subset \mathbf{A}^*_F.$$

Thus χ' restricted to $\mathbf{Q}^*\backslash\mathbf{A}^*_{\mathbf{Q}}$ is either the quadratic character of $F/\mathbf{Q}$ or trivial. Since our finite order character has trivial infinite component, χ' has to be trivial on $\mathbf{Q}^*\backslash\mathbf{A}^*_{\mathbf{Q}}$.

Looking at the exact sequence

$$1 \to F^* \to \mathbf{A}^*_F \to F^*\backslash\mathbf{A}^*_F \to 1,$$

by Hilbert's Theorem 90 (see, e.g., [NSW08, Theorem 6.2.1]) applied to F^* and $\mathrm{Gal}(F/\mathbf{Q})$ we find that $H^0(\mathrm{Gal}(F/\mathbf{Q}), F^*\backslash\mathbf{A}^*_F) = \mathbf{Q}^*\backslash\mathbf{A}^*_{\mathbf{Q}}$. Thus the kernel of $x \mapsto x/x^c$ is given by $\mathbf{Q}^*\backslash\mathbf{A}^*_{\mathbf{Q}}$. Since we showed that χ' vanishes on $\mathbf{Q}^*\backslash\mathbf{A}^*_{\mathbf{Q}}$ it therefore factors through $\mathbf{A}^*_F \to A$, where A is the subset of $\mathbf{A}^*_F$ of elements of the form x/x^c and the map is $x \mapsto x/x^c$. If $y \in A \cap F^*$ then y has trivial norm and so, by Hilbert's Theorem 90, $y = x/x^c$ for some $x \in F^*$. Thus the induced character $A \to \mathbf{C}^*$ vanishes on $A \cap F^*$. This implies that there is a continuous finite order character $\mu : F^*\backslash\mathbf{A}^*_F \to \mathbf{C}^*$ which restricts to this character on A and $\chi' = \mu/\mu^c$ (this argument is taken from the proof of [Tay94, Lemma 1]).

By the following argument we can further conclude that the induced character vanishes on $A \cap \prod_{v\notin\Sigma_\phi} \mathcal{O}^*_v$ (recall our identification $F^*_v \hookrightarrow \mathbf{A}^*_F$ from § 1.1) and therefore find μ on $F^*\backslash\mathbf{A}^*_F/\prod_{v\notin\Sigma_\phi}\mathcal{O}^*_v$ restricting to the character $A \to \mathbf{C}^*$. Writing $U_{F,\ell} = \prod_{v|\ell}\mathcal{O}^*_v$ for a prime ℓ in $\mathbf{Q}$ we have an injection

$$H^1\Bigg(\mathrm{Gal}(F/\mathbf{Q}), \prod_{v\notin\Sigma_\phi}\mathcal{O}^*_v\Bigg) \hookrightarrow \prod_{\ell\notin\Sigma_\phi} H^1(\mathrm{Gal}(F/\mathbf{Q}), U_{F,\ell}),$$

where '$\ell \notin \Sigma_\phi$' denotes those $\ell \in \mathbf{Z}$ such that $v \mid \ell \Rightarrow v \notin \Sigma_\phi$. Here we use the fact that by our definition of Σ_ϕ we know that $v \in \Sigma_\phi \Rightarrow \overline{v} \in \Sigma_\phi$. In fact, all these groups are trivial since all $\ell \notin \Sigma_\phi$ are unramified in $F/\mathbf{Q}$ and so

$$H^1(\mathrm{Gal}(F/\mathbf{Q}), U_{F,\ell}) \cong H^1(G_v, \mathcal{O}^*_v) = 1.$$

If $y \in A \cap \prod_{v\notin\Sigma_\phi}\mathcal{O}^*_v$ then y has trivial norm in $\prod_{v\notin\Sigma_\phi}\mathcal{O}^*_v$. But as shown, $H^1(\mathrm{Gal}(F/\mathbf{Q}), \prod_{v\notin\Sigma_\phi}\mathcal{O}^*_v)$ is trivial so there exists $x \in \prod_{v\notin\Sigma_\phi}\mathcal{O}^*_v \cap \mathbf{A}^*_F$ such that $y = x/x^c$. Since χ' is unramified

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outside Σ_ϕ the image of y under the induced character therefore equals $\chi'(x) = 1$, as claimed above. $\square$

4. Bounding the Eisenstein ideal

In [Ber08] we constructed a class $\mathrm{Eis}(\phi)$ in the cohomology of a symmetric space associated to $\mathrm{GL}_{2/F}$ that has integral non-zero restriction to the boundary of the Borel–Serre compactification, and is an eigenvector for the Hecke operators at almost all places. By a result of Harder [Har87, Corollary 4.2.1] one knows that $\mathrm{Eis}(\phi)$ is rational. The main result of [Ber08] is a lower bound on its denominator (defined in (6) below) in terms of the L-value of a Hecke character. In this section we show that if there exists an integral cohomology class with the same restriction to the boundary as $\mathrm{Eis}(\phi)$ then there exists a congruence modulo the L-value between $\mathrm{Eis}(\phi)$, multiplied by its denominator, and a cuspidal cohomology class.

4.1 The Eisenstein cohomology set-up

Recall the notation and definitions introduced in § 2.5. Let $\mathcal{R}$ denote the ring of integers in the finite extension L of $F_\mathfrak{p}$ obtained by adjoining the values of the finite part of both ϕ_i and $L^{\mathrm{alg}}(0, \phi_1/\phi_2)$. We write

$$\widetilde{H}^1(X, \mathcal{R}) := H^1(X, \mathcal{R})_{\mathrm{free}} = \mathrm{im}(H^1(X, \mathcal{R}) \to H^1(X, L))$$

for $X = S_{K_f}$ or $\partial \overline{S}_{K_f}$. For $c \in H^1(S_{K_f}, L)$ define the denominator (ideal) by

$$\delta(c) := \{a \in \mathcal{R} : a \cdot c \in \widetilde{H}^1(S_{K_f}, \mathcal{R})\}. \tag{6}$$

We have the long exact sequence of relative cohomology (see, e.g., [Bre97, ch. II, § 12, Equation (22)])

$$\cdots \to H^1_c(S_{K_f}, R) \to H^1(S_{K_f}, R) \stackrel{\mathrm{res}}{\to} H^1(\partial \overline{S}_{K_f}, R) \to H^2_c(S_{K_f}, R) \to \cdots$$

for any $\mathcal{R}$-algebra R.

4.1.1 *The set-up.* Suppose we are given a pair of Hecke characters $\phi = (\phi_1, \phi_2)$ as in § 3 and a class $\mathrm{Eis}(\phi) \in H^1(S_{K_f}, L)$ satisfying the following properties.

(E1) The image of $\mathrm{Eis}(\phi)$ under res lies in $\widetilde{H}^1(\partial \overline{S}_{K_f}, \mathcal{R})$.

(E2) For all places v outside the conductors of the ϕ_i the class $\mathrm{Eis}(\phi)$ is an eigenvector for the Hecke operator

$$T_{\pi_v} = \left[K_f \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} K_f\right]$$

with eigenvalue

$$\phi_2(\mathfrak{P}_v) + \mathrm{Nm}(\mathfrak{P}_v)\phi_1(\mathfrak{P}_v).$$

(E3) The central character of $\mathrm{Eis}(\phi)$ is given by $\phi_1\phi_2$, i.e., the Hecke operators

$$S_{\pi_v} = \left[K_f \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} K_f\right]$$

act on it by multiplication by $(\phi_1\phi_2)(\pi_v)$.

(E4) The denominator of $\mathrm{Eis}(\phi)$ is bounded below by $L^{\mathrm{int}}(0, \phi_1/\phi_2)$, i.e.,

$$\delta(\mathrm{Eis}(\phi)) \subseteq (L^{\mathrm{int}}(0, \phi_1/\phi_2)).$$

Suppose we are also given:

(H1) There exists $c_\phi \in \widetilde{H}^1(S_{K_f}, \mathcal{R})$ with

$$\mathrm{res}(c_\phi) = \mathrm{res}(\mathrm{Eis}(\phi)) \in \widetilde{H}^1(\partial \overline{S}_{K_f}, \mathcal{R}).$$

(H2) There exists an idempotent e_ω acting on $H^1(S_{K_f}, \mathbf{C})$ such that $S_{\pi_v} e_\omega = (\phi_1\phi_2)(\pi_v) e_\omega$ for v not dividing the conductors of the ϕ_i.

The following provides a bound on the congruence module introduced in the previous section.

Proposition 9. *Given the above set-up there is an $\mathcal{R}$-algebra surjection*

$$\mathbf{T}/\mathbf{I}_\phi \twoheadrightarrow \mathcal{R}/(L^{\mathrm{int}}(0, \phi_1/\phi_2)).$$

Proof. Put $\widetilde{H}^1_!(S_{K_f}, \mathcal{R}) = H^1_!(S_{K_f}, L) \cap \widetilde{H}^1(S_{K_f}, \mathcal{R})$ and $\omega = \phi_1\phi_2$. Under the Eichler–Shimura–Harder isomorphism (see (3)) we have

$$e_\omega H^1_!(S_{K_f}, \mathbf{C}) \cong S_2(K_f, \omega).$$

Hence the Hecke algebra $\mathbf{T}$ from § 3 is isomorphic to the $\mathcal{R}$-subalgebra of

$$\mathrm{End}_{\mathcal{R}}(e_\omega \widetilde{H}^1_!(S_{K_f}, \mathcal{R}))$$

generated by the Hecke operators T_{π_v} for all primes $v \notin \Sigma_\phi$, and we will identify the two.

Note that for $c_\phi \in \widetilde{H}^1(S_{K_f}, \mathcal{R})$ given by (H1) we have

$$\mathrm{res}(e_\omega c_\phi) = e_\omega \mathrm{res}(c_\phi) = e_\omega \mathrm{res}(\mathrm{Eis}(\phi)) = \mathrm{res}(\mathrm{Eis}(\phi))$$

since $S_v(\mathrm{Eis}(\phi)) = \omega(\pi_v)\mathrm{Eis}(\phi)$ by (E3).

Without loss of generality, we can assume that $\delta(\mathrm{Eis}(\phi)) \neq \mathcal{R}$; there is nothing to prove otherwise by (E4). Let δ be a generator of $\delta(\mathrm{Eis}(\phi))$. Then $\delta \cdot \mathrm{Eis}(\phi)$ is an element of an $\mathcal{R}$-basis of $e_\omega \widetilde{H}^1(S_{K_f}, \mathcal{R})$. By construction, $c_0 := \delta \cdot (e_\omega c_\phi - \mathrm{Eis}(\phi)) \in e_\omega H^1_!(S_{K_f}, L)$ is a nontrivial element of an $\mathcal{R}$-basis of $e_\omega \widetilde{H}^1_!(S_{K_f}, \mathcal{R})$. Extend c_0 to an $\mathcal{R}$-basis $c_0, c_1, \ldots c_d$ of $e_\omega \widetilde{H}^1_!(S_{K_f}, \mathcal{R})$. For each $t \in \mathbf{T}$ write

$$t(c_0) = \sum_{i=0}^{d} a_i(t) c_i, \; a_i(t) \in \mathcal{R}.$$

Then

$$\mathbf{T} \to \mathcal{R}/(\delta), \quad t \mapsto a_0(t) \bmod \delta \tag{7}$$

is an $\mathcal{R}$-module surjection. We claim that it is a homomorphism of $\mathcal{R}$-algebras with the Eisenstein ideal $\mathbf{I}_\phi$ contained in its kernel. To prove this it suffices to check that each $a_0(T_{\pi_v} - \phi_2(\mathfrak{P}_v) - \mathrm{Nm}(\mathfrak{P}_v)\phi_1(\mathfrak{P}_v))$, $v \notin \Sigma_\phi$ is contained in $\delta\mathcal{R}$. This is an easy calculation using (E2). Since $\mathcal{R}/(\delta) \twoheadrightarrow \mathcal{R}/(L^{\mathrm{int}}(0, \chi))$ by (E4), this concludes the proof. □

In the following sections, we will indicate how to produce the elements in the set-up of the proposition. Under certain conditions on the characters ϕ_i, to be reviewed in § 4.2, we constructed in [Ber08] a class $\mathrm{Eis}(\phi)$ satisfying (E1)–(E4) using Harder's Eisenstein cohomology. Assumption (H2) is of a technical nature and will be discussed in § 4.3. We are interested in controlling the central character via (H2) because of the restriction in Theorem 3. The most difficult ingredient to procure is (H1), see § § 4.4 and 5.

Remarks.

(i) As already remarked in the introduction, the constant cohomology coefficients above can be replaced by coefficient systems arising from finite dimensional representations of $\mathrm{GL}_{2/F}$. See [Ber08, § 2.4 and 3.1] for the necessary modifications.

(ii) Note also that except for the explicit Hecke operators we did not use any information specific to $\mathrm{GL}_{2/F}$, i.e., S_{K_f} could be replaced by a symmetric space associated to a different group G and ϕ by a tuple of automorphic forms on the Levi part of a parabolic subgroup of G. Since the analytic theory of Eisenstein cohomology has been developed for a wide variety of groups, and rationality results are known, for example, for GL_n by the work of Franke and Schwermer [FS98], we hope that these techniques generalize to these groups.

4.2 Construction of Eisenstein class

Following Harder we constructed in [Ber08] Eisenstein cohomology classes in the Betti cohomology group $H^1(S_{K_f}, \mathbf{C})$. Given a pair of Hecke characters $\phi = (\phi_1, \phi_2)$ with $\phi_{1,\infty}(z) = z$ and $\phi_{2,\infty}(z) = z^{-1}$ these depend on a choice of a function Ψ_{ϕ_f} in the induced representation

$$V_{\phi_f,\mathbf{C}}^{K_f} = \{\Psi : G(\mathbf{A}_f) \to \mathbf{C} \mid \Psi(bg) = \phi_f(b)\Psi(g)\ \forall b \in B(\mathbf{A}_f),\ \Psi(gk) = \Psi(g)\ \forall k \in K_f\}.$$

In the notation of [Ber08] we take $K_f = K_f^S$ and $\Psi_{\phi_f} = \Psi_\phi^0$. We recall the definition of the compact open K_f. Denote by S the finite set of places where both ϕ_i are ramified, but ϕ_1/ϕ_2 is unramified. Write $\mathfrak{M}_i$ for the conductor of ϕ_i. For an ideal $\mathfrak{N}$ in $\mathcal{O}$ and a finite place v of F put $\mathfrak{N}_v = \mathfrak{N}\mathcal{O}_v$. We define

$$K^1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}),\ a-1, c \equiv 0 \mod \mathfrak{N} \right\},$$
$$K^1(\mathfrak{N}_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v),\ a-1, c \equiv 0 \mod \mathfrak{N}_v \right\},$$

and

$$U^1(\mathfrak{N}_v) = \{k \in \mathrm{GL}_2(\mathcal{O}_v) : \det(k) \equiv 1 \bmod \mathfrak{N}_v\}.$$

Now put

$$K_f := \prod_{v \in S} U^1(\mathfrak{M}_{1,v}) \prod_{v \notin S} K^1((\mathfrak{M}_1\mathfrak{M}_2)_v).$$

The exact definition of Ψ_{ϕ_f} will not be required in the following; we refer the interested reader to [Ber08, § 3.2]. Let $\mathrm{Eis}(\phi)$ be the cohomology class denoted by $[\mathrm{Eis}(\Psi^0_{(\phi_1,\phi_2)_f})]$ in [Ber08].

The rationality of $\mathrm{Eis}(\phi)$, i.e., the fact that $\mathrm{Eis}(\phi) \in H^1(S_{K_f}, L)$, was proven by Harder, see [Ber08, Proposition 13]. Properties (E2) and (E3) are satisfied by construction, see [Ber08, Lemma 9]. The integrality of the constant term (E1) is analyzed in [Ber08, Proposition 16]. The main result of [Ber08] is the bound on the denominator (E4). The latter two require certain conditions on the characters ϕ_1 and ϕ_2. However, since in the combination of Theorem 6 and Proposition 9 the main object of interest is the character $\chi = \phi_1/\phi_2$, we will from now on focus on χ and view ϕ_1 and ϕ_2 as being auxiliary.

Theorem 10 [Ber08, Proposition 16, Theorem 29]. *Let χ be a Hecke character of infinity type z^2 with conductor $\mathfrak{M}$ coprime to (p). Assume in addition that either:*

(i) *p splits in F, χ has split conductor, and $L(0, \overline{\chi})/L(0, \chi) \in \mathcal{R}$; or*

(ii) *$\chi^c = \overline{\chi}$, no ramified primes divide $\mathfrak{M}$ and no inert primes congruent to $-1 \bmod p$ divide $\mathfrak{M}$*

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with multiplicity one, and

$$\omega_{F/\mathbf{Q}}(\mathfrak{M})\frac{\tau(\tilde{\chi})}{\sqrt{\mathrm{Nm}(\mathfrak{M})}}=1,$$

where $\omega_{F/\mathbf{Q}}$ is the quadratic Hecke character associated to the extension $F/\mathbf{Q}$ and $\tau(\tilde{\chi})$ the Gauss sum of the unitary character $\tilde{\chi}:=\chi/|\chi|$.

Then there exists a factorization $\chi=\phi_1/\phi_2$ such that $\mathrm{Eis}(\phi)$ satisfies (E1)–(E4). □

Remarks.

(i) Berger [Ber08, Proposition 16] shows that $L(0,\overline{\chi})/L(0,\chi)\in L$.

(ii) By considering non-constant coefficient systems, Berger [Ber08] in fact proves this for characters χ of infinity type $z^{m+2}\overline{z}^{-m}$ for $m\in\mathbf{N}_{\geq 0}$ if p splits in $F/\mathbf{Q}$.

4.3 Existence of an idempotent (H2)

LEMMA 11. *Let K_f be the compact open defined in § 4.2. If $p\nmid\#\mathrm{Cl}_{\mathfrak{M}_1\mathfrak{M}_2}(F)$ then (H2) is satisfied.*

Proof. By [Urb98, § § 1.2 and 1.4.5] the action of the diamond operators S_{π_v}, $v\nmid\mathfrak{M}_1\mathfrak{M}_2$ on $H^1(S_{K_f},\mathbf{C})$ is determined by the class in $\mathrm{Cl}_{\mathfrak{M}_1\mathfrak{M}_2}(F)$ of the ideal determined by π_v and induces an $\mathcal{R}$-linear action of $\mathrm{Cl}_{\mathfrak{M}_1\mathfrak{M}_2}(F)$ on $H^1(S_{K_f},\mathbf{C})$. Here we use the fact that

$$K_f\supset K(\mathfrak{M}_1\mathfrak{M}_2):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\hat{\mathcal{O}}):\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\bmod\mathfrak{M}_1\mathfrak{M}_2\right\}.$$

By assumption the ray class group has order prime to p, so $\mathcal{R}[\mathrm{Cl}_{\mathfrak{M}_1\mathfrak{M}_2}(F)]$ is semisimple. For $\omega:=\phi_1\phi_2$, which can be viewed as a character of $\mathrm{Cl}_{\mathfrak{M}_1\mathfrak{M}_2}(F)$, let e_ω be the idempotent associated to ω, so that $S_{\pi_v}e_\omega=\omega(\pi_v)e_\omega$. □

Remark. By enlarging K_f the condition $p\nmid\#(\mathcal{O}/\mathfrak{M}_1\mathfrak{M}_2)^*$ can be weakened to the order of $\phi_i|_{\hat{\mathcal{O}}^*}$ being coprime to p, see [Ber05, § 6.1].

4.4 Torsion problem (H1)

Hypothesis (H1) is related to the question of the occurrence of torsion classes in $H^2_c(S_{K_f},\mathcal{R})$ as follows. There exists $c_\phi\in\widetilde{H}^1(S_{K_f},\mathcal{R})$ with $\mathrm{res}(c_\phi)=\mathrm{res}(\mathrm{Eis}(\phi))\in\widetilde{H}^1(\partial\overline{S}_{K_f},\mathcal{R})$ if and only if $\mathrm{res}(\mathrm{Eis}(\phi))$ maps to zero in $\widetilde{H}^2_c(S_{K_f},\mathcal{R})$. Let $d_\phi\in\widetilde{H}^2_c(S_{K_f},\mathcal{R})$ be the image of $\mathrm{res}(\mathrm{Eis}(\phi))$ and write δ for a generator of the denominator $\delta(\mathrm{Eis}(\phi))$. Since $\delta\mathrm{Eis}(\phi)\in\widetilde{H}^1(S_{K_f},\mathcal{R})$ we know that δd_ϕ is trivial in $\widetilde{H}^2_c(S_{K_f},\mathcal{R})$. So if we knew that $H^2_c(S_{K_f},\mathcal{R})_{\mathrm{torsion}}=0$ then hypothesis (H1) would be satisfied.

This problem does not arise for $\mathrm{GL}_{2/\mathbf{Q}}$ because no such torsion classes exist with the Hecke eigenvalues under consideration (see [HP92]). In our situation, we know by Lefschetz duality (see [Gre67, Equation (28.18)] or [Mau80, Theorem 5.4.13]) that

$$H^2_c(S_{K_f},\mathcal{R})\cong H_1(S_{K_f},\mathcal{R}),$$

so this question reduces to the problem of torsion in Γ^{ab} for arithmetic subgroups $\Gamma\subset G(\mathbf{Q})$. This has been studied in [EGM82, SV83], and [GS93] (see also [EGM98, § 7.5]). An arithmetic interpretation or explanation for the torsion has not been found yet in general (but see [EGM82]

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for examples in the case of $\mathbf{Q}(\sqrt{-1})$). Based on computer calculations [GS93, Equation (2) on p. 62] suggests that for $\Gamma \subset \mathrm{PSL}_2(\mathcal{O})$, apart from 2 and 3, only primes less than or equal to $\frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}) : \Gamma]$ occur in the torsion of Γ^{ab}. Even restricting to the ordinary part there can be torsion, see [Tay88, § 4]. In all cases calculated so far, $\mathrm{PSL}_2(\mathcal{O})^{\mathrm{ab}}$ has only 2- or 3-torsion (see also [Swa71, Ber06]) but this is not known in general; hence our different approach in the following section, where we will prove the following.

PROPOSITION 12. *Let $\chi = \phi_1/\phi_2$ be an unramified Hecke character of infinity type z^2. Assume that 1 is the only unit in $\mathcal{O}^*$ congruent to 1 modulo the conductor of ϕ_1. If (E1) holds for K_f and $\mathrm{Eis}(\phi)$ as defined in § 4.2 then (H1) is satisfied.*

4.5 Congruence results

We will summarize in this section the conditions under which we can procure the ingredients for Proposition 9 and hence prove the existence of cohomological congruences.

THEOREM 13. *Assume p splits in $F/\mathbf{Q}$. Let χ be a Hecke character of infinity type z^2 with split conductor $\mathfrak{M}$ coprime to (p). Assume $L(0, \overline{\chi})/L(0, \chi) \in \mathcal{R}$ and*

$$p \nmid \#(\mathcal{O}/\mathfrak{M})^* \cdot \#\mathrm{Cl}(F).$$

Let $q > \#(\mathcal{O}^*)$ be any rational prime coprime to $(p)\mathfrak{M}$ and split in F such that $p \nmid q-1$ and $\mathfrak{q}$ is a prime of F dividing (q). If $H_c^2(S_{\tilde{K}_f}, \mathbf{Z}_p)_{\mathrm{torsion}} = 0$, where

$$\tilde{K}_f = \{k \in K^1(\mathfrak{M}) \mid \det(k) \equiv 1 \bmod \mathfrak{q}\},$$

then there exists a pair of characters $\phi = (\phi_1, \phi_2)$ such that $\chi = \phi_1/\phi_2$ and there is an $\mathcal{R}$-algebra surjection

$$\mathbf{T}/\mathbf{I}_\phi \twoheadrightarrow \mathcal{R}/(L^{\mathrm{int}}(0, \chi)).$$

Remark. As noted before, this result is true, in fact, for characters χ of infinity type $z^{m+2}\overline{z}^{-m}$ for $m \in \mathbf{N}_{\geq 0}$.

Proof. Let ϕ_1 be a Hecke character with conductor $\mathfrak{q}$ of infinity type z (for existence see, e.g., [Ber08, Lemma 24]). This is the character used in the proof of [Ber08, Theorem 29] and $K_f = \tilde{K}_f$ for this pair $(\phi_1, \phi_1/\chi)$, so the theorem follows from Proposition 9, Theorem 10, and Lemma 11, together with the comments at the beginning of § 4.4. □

A similar result can be deduced for characters χ satisfying $\overline{\chi} = \chi^c$ by taking as ϕ_1 the character used in the proof of [Ber08, Theorem 29]. Its construction is more involved and we refer the reader to the account in [Ber08, §§ 5.2 and 5.3]. The conductor $\mathfrak{M}_1$ of ϕ_1 in this case is given by $r\mathcal{D}$ for $\mathcal{D}$ the different of F and $r \in \mathbf{Z}$ any integer coprime to $(p)\mathfrak{M}$, but such that no inert prime congruent to -1 modulo p divides r with multiplicity one.

To be able to apply Lemma 2 in § 6 and to satisfy the assumption in Proposition 12 we want to impose the following extra condition on the conductor $\mathfrak{M}_1$:

(ϕ) 1 is the only unit in $\mathcal{O}^*$ congruent to 1 modulo $\mathfrak{M}_1$, and

$$v \mid \mathfrak{M}_1 \Rightarrow v = \overline{v} \quad \text{and} \quad \#\mathcal{O}_v/\mathfrak{P}_v \not\equiv \pm 1 \mod p.$$

We therefore assume in addition that $p \nmid \#\mathrm{Cl}(F)$ and that $\ell \not\equiv \pm 1 \bmod p$ for $\ell \mid d_F$. Also we choose r appropriately such that $p \nmid (\mathcal{O}/r)^*$ and that (ϕ) holds.

We leave the counterpart of Theorem 13 for characters χ satisfying $\overline{\chi} = \chi^c$ to the interested reader and instead give the following result, which does not require torsion freeness. By Lemma 1, unramified characters χ satisfy $\chi^c = \overline{\chi}$, so we deduce the following theorem from Proposition 9 together with Lemma 11, Theorem 10, and Proposition 12.

Theorem 14. *Assume in addition that $p \nmid \#\mathrm{Cl}(F)$ and that $\ell \not\equiv \pm 1 \bmod p$ for $\ell \mid d_F$. Let χ be an unramified Hecke character of infinity type z^2. Then there exists a pair of characters $\phi = (\phi_1, \phi_2)$ satisfying (ϕ) such that $\chi = \phi_1/\phi_2$ and there is an $\mathcal{R}$-algebra surjection*

$$\mathbf{T}/\mathbf{I}_\phi \twoheadrightarrow \mathcal{R}/(L^{\mathrm{int}}(0, \chi)). \qquad \square$$

4.6 Discussion of results

These are the first such cohomological congruences between Eisenstein series and cuspforms for GL_2 over an imaginary quadratic field, except for the results for degree two Eisenstein classes associated to unramified characters in [Fel00]. There are two options: either these congruences first arise for $\mathrm{GL}_{2/F}$ or they show that congruences over $\mathbf{Q}$ can be lifted, in accordance with Langlands functoriality. The congruences constructed in [Fel00] turn out to be base changes of congruences over $\mathbf{Q}$ (see [Fel00, Satz 3.3]).

Recall from [GL79, Theorem 2] and [Cre92, p. 413] that a cuspform over F is a base change if and only if its Hecke eigenvalues at complex conjugate places coincide. Observe that if $\overline{\chi} \neq \chi^c$ then the Hecke eigenvalues of our Eisenstein cohomology class $\mathrm{Eis}(\phi)$ (and all twists by a character) are distinct at complex conjugate places (see (E2) for the definition of the eigenvalues). Therefore in this case our congruences are new, i.e., are not base changed.

If $\overline{\chi} = \chi^c$ then the proof of Lemma 8 implies that there exists a twist of the Eisenstein class such that its eigenvalues at conjugate places coincide. However, we cannot determine if the congruences are base changed, as for cohomology in degree one the arguments of [HLR86] do not apply. We plan to investigate this question further. We refer the reader to [BK, Remark 4.6], where we exhibit conditions under which this question can be answered. Consider split p and let $\rho_0 : G_{\Sigma_\phi} \to \mathrm{GL}_2(\mathcal{R}/\mathfrak{m}_\mathcal{R})$ be a continuous representation of the form

$$\begin{pmatrix} 1 & * \\ 0 & \chi_\mathfrak{p}\epsilon \bmod \mathfrak{m}_\mathcal{R} \end{pmatrix}$$

with scalar centralizer. Under assumptions ensuring the uniqueness of ρ_0 up to isomorphism we then prove that no character twist of the congruent cuspforms of § 4.5 arises from base change.

5. The case of unramified characters

In this section we will prove Proposition 12, i.e., show the existence of an integral lift of the constant term of the Eisenstein cohomology class $\mathrm{Eis}(\phi)$, as defined in § 4.2. Our strategy is to find an involution on the boundary cohomology such that the restriction map surjects onto the -1-eigenspace of this involution, i.e., such that (for each connected component of $\overline{S}_{K_f}$)

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, \mathcal{R}) \overset{\mathrm{res}}{\twoheadrightarrow} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{R})^- \subset H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{R}),$$

where the superscript '$-$' indicates the -1-eigenspace. We prove the existence of such an involution for all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$, extending a result of Serre for $\mathrm{SL}_2(\mathcal{O})$. Proposition 12 is then proven by showing that $\mathrm{res}(\mathrm{Eis}(\phi))$ lies in this -1-eigenspace.

5.1 Involutions and the image of the restriction map

Let $\Gamma \subset G(\mathbf{Q})$ be an arithmetic subgroup. Given an involution ι on $X = \Gamma\backslash\overline{\mathbf{H}}_3$ or $\partial(\Gamma\backslash\overline{\mathbf{H}}_3)$ we define an involution on $H^1(X, R)$ via the pullback of ι on the level of singular cocycles. Assuming that we have an orientation-reversing involution on $\Gamma\backslash\overline{\mathbf{H}}_3$ such that

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, \mathcal{R}) \stackrel{\text{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{R})^- \subset H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{R})$$

we show that the map is, in fact, surjective. The existence of such an involution will be shown for maximal arithmetic subgroups in the following sections. We first recall the following.

Theorem 15 (Poincaré and Lefschetz duality). *Let R be a Dedekind domain in which 2 and 3 are invertible. Let ι be an orientation-reversing involution on $\Gamma\backslash\overline{\mathbf{H}}_3$. Denoting by a superscript $+$ (respectively $-$) the $+1$-(respectively -1-) eigenspaces for the induced involutions on cohomology groups, we have perfect pairings*

$$H_c^r(\Gamma\backslash\overline{\mathbf{H}}_3, R)^{\pm} \times H^{3-r}(\Gamma\backslash\overline{\mathbf{H}}_3, R)^{\mp} \to R \quad \text{for } 0 \leq r \leq 3$$

and

$$H^r(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^{\pm} \times H^{2-r}(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^{\mp} \to R \quad \text{for } 0 \leq r \leq 2.$$

Furthermore, the maps in the exact sequence

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \xrightarrow{\text{res}} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \xrightarrow{\partial} H_c^2(\Gamma\backslash\overline{\mathbf{H}}_3, R)$$

are adjoint, i.e.,

$$\langle \text{res}(x), y\rangle = \langle x, \partial(y)\rangle.$$

References. Serre states this in the proof of [Ser70, Lemma 11] for field coefficients, Ash and Stevens [AS86, Lemma 1.4.3] prove the perfectness for fields R and Urban [Urb95, Theorem 1.6] for Dedekind domains as above. Other references for this Lefschetz or 'relative' Poincaré duality for oriented manifolds with boundary are [May99, ch. 21, § 4] and [Gre67, Theorem (28.18)]. The pairings are given by the cup product and evaluation on the respective fundamental classes. We use the fact that $\overline{\mathbf{H}}_3$ is an oriented manifold with boundary and that Γ acts on it properly discontinuously and without reversing orientation. The lemma in [Fel00, § 1.1] shows that the order of any finite subgroup of $G(\mathbf{Q})$ is divisible only by 2 or 3. See also [Ber05, Theorem 5.1 and Lemma 5.2].

Lemma 16. *Suppose in addition to the conditions of the previous theorem that R is a complete discrete valuation ring with finite residue field of characteristic $p > 2$. Suppose that we have an involution ι as in the theorem such that*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \stackrel{\text{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^{\epsilon},$$

where $\epsilon = +1$ or -1. Then, in fact, the restriction map is surjective.

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R. Since the cohomology modules are finitely generated (so the Mittag–Leffler condition is satisfied for $\varprojlim H^1(\cdot, R/\mathfrak{m}^r)$), it suffices to prove the surjectivity for each $r \in \mathbf{N}$ of

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R/\mathfrak{m}^r) \twoheadrightarrow H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^{\epsilon}.$$

For these coefficient systems we are dealing with finite groups and can count the number of elements in the image and the eigenspace of the involution; they turn out to be the same.

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We observe that $H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r) = H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^+ \oplus H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^-$ and that, by the last lemma,

$$\#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^+ = \#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^-.$$

Similarly, we deduce from the adjointness of res and ∂ and the perfectness of the pairings that $\mathrm{im}(\mathrm{res})^\perp = \mathrm{im}(\mathrm{res})$ and so

$$\#\mathrm{im}(\mathrm{res}) = \tfrac{1}{2}\#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r). \qquad \square$$

5.2 Involutions for maximal arithmetic subgroups of $\mathrm{SL}_2(F)$

For $\eta \in G(\mathbf{Q})$ let B^η be the parabolic subgroup defined by $B^\eta(\mathbf{Q}) = \eta^{-1}B(\mathbf{Q})\eta$. Let $\Gamma \subset G(\mathbf{Q})$ be an arithmetic subgroup. The set $\{B^\eta : [\eta] \in B(\mathbf{Q})\backslash G(\mathbf{Q})/\Gamma\}$ is a set of representatives for the Γ-conjugacy classes of Borel subgroups. Let U^η be the unipotent radical of B^η. For $D \in \mathbf{P}^1(F)$ let $\Gamma_D = \Gamma \cap U_D$, where U_D is the unipotent subgroup of $\mathrm{SL}_2(F)$ fixing D. Note that if $D_\eta \in \mathbf{P}^1(F)$ corresponds to $[\eta] \in B(\mathbf{Q})\backslash G(\mathbf{Q})$ under the isomorphism of $B(\mathbf{Q})\backslash G(\mathbf{Q}) \cong \mathbf{P}^1(F)$ given by right action on $[0:1] \in \mathbf{P}^1(F)$ then we have that $U_{D_\eta} = U^\eta(\mathbf{Q})$ and $\Gamma_{D_\eta} = \Gamma \cap U^\eta(\mathbf{Q}) =: \Gamma_{U^\eta}$.

Let $U(\Gamma)$ be the direct sum $\bigoplus_{[D]\in\mathbf{P}^1(F)/\Gamma}\Gamma_D$. Up to canonical isomorphism this is independent of the choice of representatives $[D] \in \mathbf{P}^1(F)/\Gamma$. Since $U(\Gamma)$ is abelian we use additive notation in the following. The inclusions $\Gamma_D \hookrightarrow \Gamma$ define a homomorphism

$$\alpha : U(\Gamma) \to \Gamma^{\mathrm{ab}}.$$

For $\Gamma = \mathrm{SL}_2(\mathcal{O})$, Serre [Ser70] shows that there is a well-defined action of complex conjugation on $U(\mathrm{SL}_2(\mathcal{O}))$ induced by the complex conjugation action on the matrix entries of $G_\infty = \mathrm{GL}_2(\mathbf{C})$. Denoting by U^+ the set of elements of $U(\mathrm{SL}_2(\mathcal{O}))$ invariant under the involution and by U' the set of elements $u + \overline{u}$ for $u \in U(\mathrm{SL}_2(\mathcal{O}))$, Serre proves the following.

Theorem 17 (Serre [Ser70, Théorème 9]). *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ the kernel of the homomorphism $\alpha : U(\mathrm{SL}_2(\mathcal{O})) \to \mathrm{SL}_2(\mathcal{O})^{\mathrm{ab}}$ satisfies the inclusions*

$$6U' \subseteq \ker(\alpha) \subseteq U^+.$$

In the following we generalize this theorem to all maximal arithmetic subgroups. After we had discovered this generalization we found out that it had already been stated in [Blu92], but for our application we need more detail than is provided there.

For $\mathfrak{b}$ a fractional ideal let

$$H(\mathfrak{b}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \;\middle|\; a, d \in \mathcal{O}, b \in \mathfrak{b}, c \in \mathfrak{b}^{-1} \right\}.$$

This is a maximal arithmetic subgroup of $\mathrm{SL}_2(F)$ and any maximal arithmetic subgroup is conjugate to $H(\mathfrak{b})$ (see [EGM98, Proposition 7.4.5]). In order to study the structure of $U(H(\mathfrak{b}))$ we define $j : \mathbf{P}^1(F) \to \mathrm{Cl}(F)$ to be the map

$$j([z_1 : z_2]) = [z_1\mathfrak{b} + z_2\mathcal{O}].$$

Theorem 18. *For $\Gamma = H(\mathfrak{b})$, the induced map*

$$j : \mathbf{P}^1(F)/\Gamma \to \mathrm{Cl}(F)$$

is a bijection.

Proof. Let (x_1, x_2), $(y_1, y_2) \in F \times F$. It is easy to check (see [EGM98, Theorem VII 2.4] for $\mathrm{SL}_2(\mathcal{O})$, [Ber05, Lemma 5.10] for the general case) that the following are equivalent.

(1) $x_1\mathfrak{b} + x_2\mathcal{O} = y_1\mathfrak{b} + y_2\mathcal{O}$.

(2) There exists $\sigma \in H(\mathfrak{b})$ such that $(x_1, x_2) = (y_1, y_2)\sigma$.

It remains to show the surjectivity of j. Given a class in $\mathrm{Cl}(F)$ take $\mathfrak{a} \subset \mathcal{O}$ representing it. By the Chinese Remainder theorem one can choose $z_2 \in \mathcal{O}$ such that:

- $\mathrm{ord}_\wp(z_2) = \mathrm{ord}_\wp(\mathfrak{a})$ if $\wp|\mathfrak{a}$;
- $\mathrm{ord}_\wp(z_2) = 0$ if $\wp \nmid \mathfrak{a}$, $\mathrm{ord}_\wp(\mathfrak{b}) \neq 0$.

Then one chooses z_1 such that:

- $\mathrm{ord}_\wp(z_1\mathfrak{b}) > \mathrm{ord}_\wp(z_2)$ if $\wp|\mathfrak{a}$ or $\mathrm{ord}_\wp(\mathfrak{b}) \neq 0$;
- $\mathrm{ord}_\wp(z_1\mathfrak{b}) = 0$ if $\wp|z_2$, $\wp \nmid \mathfrak{a}$, and $\mathrm{ord}_\wp(\mathfrak{b}) = 0$.

These choices ensure that $\mathrm{ord}_\wp(z_1\mathfrak{b} + z_2\mathcal{O}) = \mathrm{ord}_\wp(\mathfrak{a})$ for all prime ideals $\wp$. □

Following Serre [Ser70] we now calculate explicitly $\Gamma_{[z_1:z_2]}$ for $\Gamma = H(\mathfrak{b})$ and $[z_1 : z_2] \in \mathbf{P}^1(F)$.

Lemma 19. *For $\Gamma = H(\mathfrak{b})$, $\Gamma_{[z_1:z_2]}$ is conjugate in $H(\mathfrak{b})$ to*

$$\left\{\theta \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \theta^{-1} : t \in \mathfrak{a}^{-2}\mathfrak{b}\right\},$$

where $\mathfrak{a} = z_1\mathfrak{b} + z_2\mathcal{O}$ and θ is an isomorphism $\mathcal{O} \oplus \mathfrak{b} \xrightarrow{\sim} \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1, i.e., such that its second exterior power

$$\Lambda^2\theta : \Lambda^2(\mathcal{O} \oplus \mathfrak{b}) = \mathfrak{b} \to \Lambda^2(\mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}) = \mathfrak{a} \otimes \mathfrak{a}^{-1}\mathfrak{b} = \mathfrak{b}$$

is the identity.

Proof. The main change to [Ser70, § 3.6] is that we consider the lattice $L := \mathcal{O} \oplus \mathfrak{b}$ instead of $\mathcal{O}^2$. We claim there exists a projective rank one submodule E of L containing a multiple of (z_1, z_2). Let E be the kernel of the $\mathcal{O}$-homomorphism $L = \mathcal{O} \oplus \mathfrak{b} \to F$ given by $(x, y) \mapsto yz_1 - xz_2$. Since the image is $\mathfrak{a} = z_1\mathfrak{b} + z_2\mathcal{O}$, we get $L/E \cong \mathfrak{a}$, so L/E is projective of rank one and L decomposes as $E \oplus L/E$.

By definition $\Gamma_{[z_1:z_2]}$ fixes $L \cap \{\lambda(z_1, z_2), \lambda \in F\}$, but this is exactly E. Since $\Gamma_{[z_1:z_2]}$ is unipotent it can therefore be identified with $\mathrm{Hom}_{\mathcal{O}}(L/E, E)$. For any fractional ideal $\mathfrak{a}$, $\Lambda^2(\mathfrak{a}) = 0$ and so $\mathfrak{b} = \Lambda^2(L) = \Lambda^2(E \oplus L/E) = E \otimes_{\mathcal{O}} L/E$; therefore E is isomorphic to $(L/E)^{-1} \otimes \mathfrak{b}$. This implies an isomorphism $\mathrm{Hom}_{\mathcal{O}}(L/E, E) = (L/E)^{-1} \otimes E \cong (L/E)^{-1} \otimes (L/E)^{-1} \otimes \mathfrak{b} \cong \mathfrak{a}^{-2}\mathfrak{b}$. Choosing an isomorphism $\theta : L \to L/E \oplus E \cong \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1 we can represent $\Gamma_{[z_1:z_2]}$ as stated above. □

Note that since $H(\mathfrak{b})$ is the stabilizer of any lattice $\mathfrak{m} \oplus \mathfrak{n}$ with $\mathfrak{m}$ and $\mathfrak{n}$ fractional ideals of F such that $\mathfrak{m}^{-1}\mathfrak{n} = \mathfrak{b}$, one can deduce the following.

Lemma 20. *Let $\mathfrak{a}, \mathfrak{b}$ be two fractional ideals of F. If $[\mathfrak{a}] = [\mathfrak{b}]$ in $\mathrm{Cl}(F)/\mathrm{Cl}(F)^2$, then $H(\mathfrak{a}) = H(\mathfrak{b})^\gamma$ with $\gamma \in \mathrm{GL}_2(F)$. If the fractional ideals differ by the square of an $\mathcal{O}$-ideal, then γ can be taken to be in $\mathrm{SL}_2(F)$.*

If the class of $\mathfrak{b}$ in $\mathrm{Cl}(F)$ is a square, $H(\mathfrak{b})$ is isomorphic to $\mathrm{SL}_2(\mathcal{O})$ by Lemma 20, and the involution on $U(\mathrm{SL}_2(\mathcal{O}))$ induced by complex conjugation and Serre's Théorème 9 [Ser70] can easily be transferred to $U(H(\mathfrak{b}))$. We therefore turn our attention to the case when

$$[\mathfrak{b}] \text{ is not a square in } \mathrm{Cl}(F).$$

Note that this implies that $[\mathfrak{b}]$ has even order, since any odd order class can be written as a square.

Define an involution on $H(\mathfrak{b})$ to be the composition of complex conjugation with an Atkin–Lehner involution, i.e., by

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto A\overline{H}A^{-1} = \begin{pmatrix} \overline{d} & -\mathrm{Nm}(\mathfrak{b})\overline{c} \\ -\overline{b}\mathrm{Nm}(\mathfrak{b})^{-1} & \overline{a} \end{pmatrix},$$

where $A = \left(\begin{smallmatrix} 0 & 1 \\ -\mathrm{Nm}(\mathfrak{b})^{-1} & 0 \end{smallmatrix}\right)$.

Like Serre, we will choose a set of representatives for the cusps $\mathbf{P}^1(F)/H(\mathfrak{b})$ on which this involution acts. For this we observe that if $\Gamma_{[z_1:z_2]}$ fixes $[z_1 : z_2]$ then $A\overline{\Gamma}_{[z_1:z_2]}A^{-1}$ fixes $[\overline{z}_1 : \overline{z}_2]A^{-1} = [\overline{z}_2 : -\mathrm{Nm}(\mathfrak{b})\overline{z}_1]$. We use the isomorphism $j : \mathbf{P}^1(F)/H(\mathfrak{b}) \to \mathrm{Cl}(F)$ to show that this action on the cusps is fixpoint-free. We observe that if $j([z_1 : z_2]) = \mathfrak{a}$ then $j([\overline{z}_1 : \overline{z}_2]A^{-1}) = [\overline{z}_2\mathfrak{b} + \mathrm{Nm}(\mathfrak{b})\overline{z}_1\mathcal{O}] = [\overline{\mathfrak{a}}\mathfrak{b}]$. Note that $[\mathfrak{a}] \neq [\overline{\mathfrak{a}}\mathfrak{b}]$ in $\mathrm{Cl}(F)$ since otherwise $[\mathfrak{a}^2] = [\mathrm{Nm}(\mathfrak{a})\mathfrak{b}] = [\mathfrak{b}]$, i.e., $[\mathfrak{b}]$ is a square, contradicting our hypothesis. So $\mathrm{Cl}(F)$ can be partitioned into pairs $(\mathfrak{a}_i, \overline{\mathfrak{a}_i}\mathfrak{b})$.

Choosing $[z_1^i : z_2^i] \in \mathbf{P}^1(F)$ such that $\mathfrak{a}_i = z_1^i\mathfrak{b} + z_2^i\mathcal{O}$ we obtain

$$U(H(\mathfrak{b})) = \bigoplus_{(\mathfrak{a}_i, \overline{\mathfrak{a}_i}\mathfrak{b})} (\Gamma_{[z_1^i:z_2^i]} \oplus A\overline{\Gamma}_{[z_1^i:z_2^i]}A^{-1}).$$

Our choice of representatives of $\mathbf{P}^1(F)/H(\mathfrak{b})$ shows that the involution operates on $U(H(\mathfrak{b}))$ and, in fact, by identifying $\Gamma_{[z_1^i:z_2^i]}$ with $\{\theta\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)\theta^{-1} : s \in \mathfrak{a}_i^{-2}\mathfrak{b}\}$ for $\theta : \mathcal{O} \oplus \mathfrak{b} \to \mathfrak{a}_i \oplus \mathfrak{a}_i^{-1}\mathfrak{b}$ and $A\overline{\Gamma}_{[z_1^i:z_2^i]}A^{-1}$ with $\{\theta'\left(\begin{smallmatrix} 1 & 0 \\ -t & 1 \end{smallmatrix}\right)\theta'^{-1} : t \in \overline{\mathfrak{a}_i}^{-2}\mathfrak{b}^{-1}\}$ for $\theta' = A\overline{\theta}A^{-1} : \mathcal{O} \oplus \mathfrak{b} \to \overline{\mathfrak{a}_i}^{-1} \oplus \overline{\mathfrak{a}_i}\mathfrak{b}$, we can describe the involution on each of the pairs as

$$(s, t) \in \mathfrak{a}_i^{-2}\mathfrak{b} \oplus \overline{\mathfrak{a}_i}^{-2}\mathfrak{b}^{-1} \mapsto (\overline{t}\mathrm{Nm}(\mathfrak{b}), \overline{s}\mathrm{Nm}(\mathfrak{b})^{-1}).$$

Now denote by U^+ the set of elements of $U(H(\mathfrak{b}))$ invariant under the involution $H \mapsto A\overline{H}A^{-1}$, and by U' the set of elements $u + A\overline{u}A^{-1}$ for $u \in U(H(\mathfrak{b}))$.

Theorem 21. *For $\Gamma = H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$, the kernel N of the homomorphism*

$$\alpha : U(\Gamma) \to \Gamma^{\mathrm{ab}}$$

coming from the inclusion $\Gamma_D \hookrightarrow \Gamma$ for $D \in \mathbf{P}^1(F)$ satisfies $6U' \subset N \subset U^+$.

Proof. With small modifications, we follow Serre's proof of his Théorème 9. As in Serre's case, it suffices to prove the inclusion $6U' \subset N$, i.e., that $6(u + A\overline{u}A^{-1})$ maps to an element of the commutator $[H(\mathfrak{b}), H(\mathfrak{b})]$.

Suppose that we have $6U' \subset N$, but that there exists an element $u \in N$ not contained in U^+. Then the subgroup of N generated by $6U'$ and u has rank $\#\mathrm{Cl}(F) + 1$. This contradicts the fact that the kernel of α has rank $\#\mathrm{Cl}(F)$ (see [Ser70, Théorème 7]). (The latter is proven by showing dually that the rank of the image of the restriction map $H^1(H(\mathfrak{b})\backslash\overline{\mathbf{H}}_3, R) \to H^1(\partial(H(\mathfrak{b})\backslash\overline{\mathbf{H}}_3), R)$ has half the rank of that of the boundary cohomology. This we have shown in the proof of Lemma 16.)

To prove $6U' \subset N$ we make use of Serre's Proposition 6.

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Proposition 22 [Ser70, Proposition 6]. *Let $\mathfrak{q}$ be a fractional ideal of F and let $t \in \mathfrak{q}$ and $t' = \overline{t}/\mathrm{Nm}(\mathfrak{q})$ so that $t' \in \mathfrak{q}^{-1}$. Put $x_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ and $y_t = \left(\begin{smallmatrix} 1 & 0 \\ -t' & 1 \end{smallmatrix}\right)$. Then $(x_t y_t)^6$ lies in the commutator subgroup of $H(\mathfrak{q})$.*

Put $\mathfrak{a} := z_1\mathfrak{b} + z_2\mathcal{O}$. If $u \in \Gamma_{[z_1:z_2]}$, identify it with $\theta^{-1}\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\theta$ for some $t \in \mathfrak{a}^{-2}\mathfrak{b}$ and $\theta : \mathcal{O} \oplus \mathfrak{b} \to \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1. One can easily check that $A\overline{u}A^{-1}$ then corresponds to $(A\overline{\theta}A^{-1})\left(\begin{smallmatrix} 1 & 0 \\ -\overline{t}\mathrm{Nm}(\mathfrak{b})^{-1} & 1 \end{smallmatrix}\right)(A\overline{\theta}A^{-1})$. Like Serre, we use the fact that since $[\overline{\mathfrak{a}}] = [\mathfrak{a}^{-1}]$, $A\overline{u}A^{-1}$ is also given by Theorem 18 by $B^{-1}\theta^{-1}\left(\begin{smallmatrix} 1 & 0 \\ -t' & 1 \end{smallmatrix}\right)\theta B$ for $t' = \overline{t}\mathrm{Nm}(\mathfrak{b})^{-1}\mathrm{Nm}(\mathfrak{a})^2$ and $B \in H(\mathfrak{b})$ taking $\left(\begin{smallmatrix} \mathrm{Nm}(\mathfrak{b})\overline{z}_2 \\ \overline{z}_1 \end{smallmatrix}\right)$ to $\mathrm{Nm}(\mathfrak{a})^{-1}\left(\begin{smallmatrix} \mathrm{Nm}(\mathfrak{b})\overline{z}_2 \\ \overline{z}_1 \end{smallmatrix}\right)$.

Since $\theta^{-1}x_t y_t\theta$ is a representative of $u + BA\overline{u}A^{-1}B^{-1}$, we deduce from the above proposition with $\mathfrak{q} = \mathfrak{a}^{-2}\mathfrak{b}$ that $6(u + BA\overline{u}A^{-1}B^{-1})$ and therefore $6(u + A\overline{u}A^{-1})$ lie in $[H(\mathfrak{b}), H(\mathfrak{b})]$. □

The following observation links $U(\Gamma)$ to the cohomology of the boundary components.

Lemma 23. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma \subset \mathrm{SL}_2(F)$ an arithmetic subgroup, P a parabolic subgroup of $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SL}_{2/F})$ with unipotent radical U_P, and R a ring in which 2 is invertible, we have*

$$H^1(\Gamma_P, R) \cong H^1(\Gamma_{U_P}, R),$$

where $\Gamma_P = \Gamma \cap P(\mathbf{Q})$ and $\Gamma_{U_P} = \Gamma \cap U_P(\mathbf{Q})$.

Proof. Serre shows in [Ser70, Lemme 7] that $\Gamma_{U_P} \lhd \Gamma_P$ and that the quotient $W_P = \Gamma_P/\Gamma_{U_P}$ can be identified with a subgroup of the roots of unity of F, i.e., of $\{\pm 1\}$ since $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$. The lemma follows from the Inflation–Restriction sequence. See also [Tay88, p. 110]. □

By (2), (4) (in § 2.5), and Lemma 23 we have

$$H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \cong \coprod_{[\eta] \in \mathbf{P}^1(F)/\Gamma} H^1(\Gamma_{U^\eta}, R) = H^1(U(\Gamma), R). \tag{8}$$

We now reinterpret Serre's theorem (Theorem 17) and its generalization (Theorem 21) as follows.

Proposition 24. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ and R a ring in which 2 and 3 is invertible, the image of the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \stackrel{\mathrm{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)$$

is contained in the -1-eigenspace of the involution induced by

- *$\iota : \mathbf{H}_3 \to \mathbf{H}_3 : (z, t) \mapsto (\overline{z}, t)$ if $\Gamma = \mathrm{SL}_2(\mathcal{O})$,*
- *$\iota : \mathbf{H}_3 \to \mathbf{H}_3 : (z, t) \mapsto A.(\overline{z}, t)$ for $A = \left(\begin{smallmatrix} 0 & 1 \\ -\mathrm{Nm}(\mathfrak{b})^{-1} & 0 \end{smallmatrix}\right)$ if $\Gamma = H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$,*

and these involutions are orientation-reversing.

By Lemma 16 this immediately implies the following.

Corollary 25. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma = \mathrm{SL}_2(\mathcal{O})$ or $H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$, and R a complete discrete valuation ring in which 2 and 3 are invertible and with finite residue field of characteristic $p > 2$, the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \stackrel{\mathrm{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^-$$

surjects onto the -1-eigenspace of the involutions defined in the proposition.

Proof of Proposition 24. Write $I:\Gamma\to\Gamma$ for the involution

$$\begin{cases}\gamma\mapsto\overline{\gamma} & \text{if } \Gamma=\mathrm{SL}_2(\mathcal{O}),\\ \gamma\mapsto A\overline{\gamma}A^{-1} & \text{if } \Gamma=H(\mathfrak{b}).\end{cases}$$

The involutions ι extend canonically to $\overline{\mathbf{H}}_3$. One can check that for $\gamma\in\Gamma$ we have

$$\iota(\gamma.(z,t))=I(\gamma)\iota(z,t). \tag{9}$$

This implies that the involutions operate on $\Gamma\backslash\mathbf{H}_3$ and $\Gamma\backslash\overline{\mathbf{H}}_3$, and hence on $\partial(\Gamma\backslash\overline{\mathbf{H}}_3)$. To show that they act by reversing the orientation note that complex conjugation corresponds to reflection in a half-plane of $\mathbf{H}_3$ and therefore reverses the orientation. Furthermore, $\mathrm{GL}_2(\mathbf{C})$ acts on $\mathbf{H}_3$ via $A'=(\det(A)^{-1/2})A\in\mathrm{SL}_2(\mathbf{C})$ and $\mathrm{SL}_2(\mathbf{C})$ acts without reversing orientation, as can be seen from the geometric definition of its action via the Poincaré extension of the action on $\mathbf{P}^1(\mathbf{C})$ (see [EGM98, pp. 2–3]).

Using (9) one can show that under the isomorphism

$$H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3),R)\overset{(8)}{\cong}H^1(U(\Gamma),R)$$

ι corresponds to the involution on $H^1(U(\Gamma),R)=\mathrm{Hom}(U(\Gamma),R)$ given by $\varphi\mapsto I(\varphi)$, where $I(\varphi)(u):=\varphi(I(u))$.

We can therefore check that the image of the restriction maps is contained in the -1-eigenspace on the level of group cohomology: the restriction map is given by

$$\mathrm{Hom}(\Gamma^{\mathrm{ab}},R)\to\mathrm{Hom}(U(\Gamma),R):\varphi\mapsto\varphi\circ\alpha.$$

By Serre's theorem (Theorem 17) and Theorem 21, $0=\varphi(\alpha(uI(u)))=\varphi(\alpha(u))+\varphi(\alpha(I(u)))$, so $I(\varphi\circ\alpha)(u)=\varphi(\alpha(I(u)))=-\varphi(\alpha(u))$ for any $u\in U(\Gamma)$. □

5.3 Integral lift of constant term

Recall the statement of Proposition 12.

Proposition 26 (=Proposition 12). *Let $\chi=\phi_1/\phi_2$ be an unramified Hecke character of infinity type z^2. Assume that 1 is the only unit in $\mathcal{O}^*$ congruent to 1 modulo the conductor of ϕ_1. Let K_f and $\mathrm{Eis}(\phi)$ be defined as in § 4.2. Assume $\mathrm{res}(\mathrm{Eis}(\phi))\in\widetilde{H}^1(\partial\overline{S}_{K_f},\mathcal{R})$. Then there exists $c_\phi\in\widetilde{H}^1(S_{K_f},\mathcal{R})$ with*

$$\mathrm{res}(c_\phi)=\mathrm{res}(\mathrm{Eis}(\phi))\in\widetilde{H}^1(\partial\overline{S}_{K_f},\mathcal{R}).$$

First observe that everywhere unramified characters with infinity type z^2 exist only for $F\neq\mathbf{Q}(\sqrt{-1}),\mathbf{Q}(\sqrt{-3})$. For unramified χ we have

$$K_f=\prod_{v\mid\mathfrak{M}_1}U^1(\mathfrak{M}_{1,v})\prod_{v\nmid\mathfrak{M}_1}\mathrm{GL}_2(\mathcal{O}_v).$$

Recall that $U^1(\mathfrak{M}_{1,v})=\{k\in\mathrm{GL}_2(\mathcal{O}_v):\det(k)\equiv 1 \bmod \mathfrak{M}_{1,v}\}$. By assumption, 1 is the only unit in $\mathcal{O}^*$ congruent to 1 modulo $\mathfrak{M}_1$ so we get $K_f\cap\mathrm{GL}_2(F)=\mathrm{SL}_2(\mathcal{O})$.

Recall from § 2.5 the decomposition of S_{K_f} into its connected components. The above implies that we can write S_{K_f} as a disjoint union of $\Gamma\backslash\mathbf{H}_3$ with $\Gamma=H(\mathfrak{b})$ for suitable fractional ideals $\mathfrak{b}$.

For a finite idele a, denote by (a) the corresponding fractional ideal. Write

$$S_{K_f} \cong \coprod_{i=1}^{\#\pi_0(K_f)} \Gamma_{t_i}\backslash \mathbf{H}_3,$$

where $\Gamma_{t_i} = G(\mathbf{Q}) \cap t_i K_f t_i^{-1}$ and the $t_i \in G(\mathbf{A}_f)$ are given by $t_i = \left(\begin{smallmatrix} \gamma_j a_k b_m & 0 \\ 0 & b_m \end{smallmatrix}\right)$, with:

- $\{\gamma_j\}$ a system of representatives of

$$\ker(\pi_0(K_f) \to \mathrm{Cl}(F)) \cong \mathcal{O}^* \backslash \prod_v \mathcal{O}_v^* / \det(K_f);$$

- $\{a_k\}$ a set of representatives of $\mathrm{Cl}(F)/(\mathrm{Cl}(F))^2$ in $\mathbf{A}^*_{F,f}$ (and we represent the principal ideals by (1));
- $\{b_m^2\}$ a set representing $\mathrm{Cl}(F)^2$.

Note that for these choices $\Gamma_{t_i} = H((a_k))$ and either $a_k = 1$ or $[(a_k)]$ is not a square in $\mathrm{Cl}(F)$. This allows us to apply our results for maximal arithmetic subgroups from the previous section by considering the restriction maps to the boundary separately for each connected component.

Proposition 27. *We have*

$$[\mathrm{res}(\mathrm{Eis}(\phi))] \in (H^1(\partial \overline{S}_{K_f}, \mathcal{R})^-)_{\mathrm{free}},$$

where $H^1(\partial \overline{S}_{K_f}, \mathcal{R})^-$ is defined via the isomorphism to

$$\bigoplus_{i=1}^{\#\pi_0(K_f)} H^1(\partial(\Gamma_{t_i}\backslash \overline{\mathbf{H}}_3), \mathcal{R})^-$$

where '$-$' indicates the -1-eigenspace of the involutions defined in Proposition 24.

Remark. Together with Corollary 25 this shows the existence of an integral lift of the constant term and proves Proposition 12.

Proof. We will consider the restriction maps to the boundary separately for each connected component $\Gamma_{t_i}\backslash \mathbf{H}_3$:

$$H^1(\Gamma_{t_i}\backslash \mathbf{H}_3, \mathcal{R}) \xrightarrow{\mathrm{res}} H^1(\partial(\Gamma_{t_i}\backslash \overline{\mathbf{H}}_3), \mathcal{R}) \overset{(2)}{\cong} \bigoplus_{[\eta] \in \mathbf{P}^1(F)/\Gamma_{t_i}} H^1(\Gamma_{t_i, B^\eta}\backslash \mathbf{H}_3, \mathcal{R}),$$

where $\Gamma_{t_i, B^\eta} = \Gamma_{t_i} \cap \eta^{-1} B(\mathbf{Q}) \eta$. By (4) and Lemma 23 we have $H^1(\Gamma_{t_i, B^\eta}\backslash \mathbf{H}_3, \mathcal{R}) \cong H^1(\Gamma_{t_i, U^\eta}, \mathcal{R})$. We recall from [Ber08, Proposition 10, Lemma 11, and Proof of Proposition 16] that $\mathrm{res}(\mathrm{Eis}(\phi))$ restricted to this boundary component is represented by

$$\eta_\infty^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \eta_\infty \mapsto x \Psi_\phi(\eta_f t_i) - \frac{L(0, \overline{\chi})}{L(0, \chi)} W(\chi) \cdot \overline{x} \Psi_{w_0.\phi}(\eta_f t_i), \tag{10}$$

where $W(\chi)$ is the root number of χ, η_f and η_∞ denote the images of $\eta \in G(\mathbf{Q})$ in $G(\mathbf{A}_f)$ and $G(\mathbf{R})$, respectively, $w_0.(\phi_1, \phi_2) = (\phi_2 \cdot |\cdot|, \phi_1 \cdot |\cdot|^{-1})$, and $\Psi_\phi : G(\mathbf{A}_f) \to \mathbf{C}$ satisfies

$$\Psi_\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k\right) = \phi_1(a)\phi_2(d) \quad \text{for } \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbf{A}_f), k \in \prod_v \mathrm{SL}_2(\mathcal{O}_v) \subset K_f.$$

Note that, in particular, $\Psi_\phi(bg) = \phi_\infty^{-1}(b)\Psi_\phi(g)$ for $b \in B(F) \subset G(\mathbf{A}_f)$, where we use the convention introduced in § 2.5 for considering ϕ as a character of $B(\mathbf{A})$. By Lemma 1, $\chi^c = \overline{\chi}$, so

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$L(0,\chi)=L(0,\overline{\chi})$. Furthermore, $W(\chi)=i^2(\chi/|\chi|)(\delta_F^{-1})$ for δ_F the different ideal of $F/\mathbf{Q}$ (see, e.g., the proof of [AH06, Proposition 2.1.7]). Since $\delta_F=g'(\alpha)\mathcal{O}$, where $\mathcal{O}=\mathbf{Z}[\alpha]$ and $g\in\mathbf{Z}[x]$ is the minimal polynomial of α (see [Neu99, Proposition III.2.4]) one can check that for all imaginary quadratic fields there exists a generator δ of δ_F satisfying $\overline{\delta}=-\delta$. We deduce therefore that in our case $W(\chi)=1$.

We need to prove that (10) lies in the -1-eigenspace of the involution induced by $u\mapsto\overline{u}$ for $\Gamma_{t_i}=\mathrm{SL}_2(\mathcal{O})$ and by $u\mapsto A\overline{u}A^{-1}$ for $\Gamma_{t_i}=H(\mathfrak{b})$, where $A=\left(\begin{smallmatrix}0&1\\-N^{-1}&0\end{smallmatrix}\right)$ with $N=\mathrm{Nm}(\mathfrak{b})$.

(1) *Case* $\Gamma_{t_i}=\mathrm{SL}_2(\mathcal{O})$: Recall that in this case $t_i=\left(\begin{smallmatrix}\gamma_i b_i&0\\0&b_i\end{smallmatrix}\right)$ for some $\gamma_i\in\hat{\mathcal{O}}^*$ and $b_i\in\mathbf{A}_{F,f}^*$.

It suffices to prove that $\Psi_\phi(\eta_f t_i)=\Psi_{w_0.\phi}(\overline{\eta}_f t_i)$. For this we use the Bruhat decomposition of matrices in $\mathrm{GL}_2(F)$ given by

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{cases}\begin{pmatrix}1&b/d\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&d\end{pmatrix} & \text{if } c=0,\\ \begin{pmatrix}1&a/c\\0&1\end{pmatrix}\begin{pmatrix}(ad-bc)/c&0\\0&-c\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&d/c\\0&1\end{pmatrix} & \text{otherwise.}\end{cases}$$

Since $\Psi_\phi(\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)g)=\Psi_\phi(\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right))\Psi_\phi(g)$ we can consider separately the cases:

(a) $\eta=\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)$ for $a,b,d\in F$; and

(b) $\eta=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&e\\0&1\end{smallmatrix}\right)$ for $e\in F$.

We check that, for case (a),

$$\Psi_\phi\left(\eta_f\begin{pmatrix}\gamma_i b_i&0\\0&b_i\end{pmatrix}\right)=\phi_1(\gamma_i b_i)\phi_2(b_i)\Psi_\phi(\eta_f)$$

and

$$\Psi_{w_0.\phi}\left(\overline{\eta}_f\begin{pmatrix}\gamma_i b_i&0\\0&b_i\end{pmatrix}\right)=\phi_2(\gamma_i)|\gamma_i|\phi_1(b_i)\phi_2(b_i)\Psi_{w_0.\phi}(\overline{\eta}_f).$$

Since $\gamma_i\in\hat{\mathcal{O}}^*$ and $\chi=\phi_1/\phi_2$ is unramified, it suffices to show that $\Psi_\phi(\eta_f)=\Psi_{w_0.\phi}(\overline{\eta}_f)$. In case (b) we similarly reduce to this assertion.

In case (a) we get $\Psi_\phi(\eta_f)=\phi_{1,\infty}^{-1}(a)\phi_{2,\infty}^{-1}(d)=d/a$. Since $w_0.\phi$ has infinity type $(\overline{z},\overline{z}^{-1})$ this equals $\Psi_{w_0.\phi}(\overline{\eta}_f)$. In case (b) we need to calculate the Iwasawa decomposition of η in $\mathrm{GL}_2(F_v)$ if $e\notin\mathcal{O}_v$ (at all other places $\Psi_\phi(\eta_v)=\Psi_{w_0.\phi}(\overline{\eta}_{\overline{v}})=1$). It is given by

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&e\\0&1\end{pmatrix}=\begin{pmatrix}e^{-1}&0\\0&e\end{pmatrix}\begin{pmatrix}-1&0\\-e^{-1}&-1\end{pmatrix}.$$

So, if $e\notin\mathcal{O}_v$ then $\Psi_\phi(\eta_v)=(\phi_2/\phi_1)_v(e)=\chi_v^{-1}(e)$, which matches $\Psi_{w_0.\phi}(\overline{\eta}_{\overline{v}})=(\phi_1/\phi_2)_{\overline{v}}(\overline{e})|\overline{e}|_{\overline{v}}^{-2}$ because $\chi^c=\overline{\chi}$ and $\chi\overline{\chi}=|\cdot|^2$.

(2) *Case* $\Gamma_{t_i}=H(\mathfrak{b})$: The involution maps the cusp corresponding to B^η to $B^{\overline{\eta}A^{-1}}$. We therefore have to prove that

$$\Psi_\phi(\eta_f t_i)=\Psi_{w_0.\phi}(\overline{\eta}_f A^{-1}t_i). \tag{11}$$

Recall that $t_i=\left(\begin{smallmatrix}x_i b_i&0\\0&b_i\end{smallmatrix}\right)$ for some $x_i,b_i\in\mathbf{A}_{F,f}^*$. Again making use of the Bruhat decomposition, we need to only consider η as in cases (a) and (b) above. Following the arguments used for case (1) above, case(a) reduces immediately to showing that $\Psi_\phi(t_i)=\Psi_{w_0.\phi}(A^{-1}t_i)$. The left-hand

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side equals $\phi_{1,f}(x_ib_i)\phi_{2,f}(b_i)$, the right-hand side is

$$\Psi_{w_0.\phi}\left(\begin{pmatrix}N & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x_ib_i & 0\\ 0 & b_i\end{pmatrix}\right) = N^{-1}\Psi_{w_0.\phi}\left(\begin{pmatrix}b_i & 0\\ 0 & x_ib_i\end{pmatrix}\right)$$
$$= N^{-1}\phi_{1,f}(x_ib_i)\phi_{2,f}(b_i)|x_i|_f^{-1}.$$

Equality follows from $|x_i|_f^{-1} = \mathrm{Nm}(\mathfrak{b})$.

For case (b), one can quickly check that for $\eta = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ the two sides in (11) agree. For general $\eta = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & e\\ 0 & 1\end{smallmatrix}\right)$ one can show that, on the one hand,

$$\eta_f\begin{pmatrix}x_ib_i & 0\\ 0 & b_i\end{pmatrix} = \begin{pmatrix}b_i & 0\\ 0 & x_ib_i\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & ex_i\\ 0 & 1\end{pmatrix},$$

and, on the other hand,

$$\overline{\eta}_f A^{-1}\begin{pmatrix}x_ib_i & 0\\ 0 & b_i\end{pmatrix} = \begin{pmatrix}x_ib_i & 0\\ 0 & b_iN\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & \overline{e}x_i/N\\ 0 & 1\end{pmatrix}.$$

Since $(x_i\overline{x}_i) = (N)$ the valuations of $\overline{e}x_i/N$ agree with that of $\overline{ex_i}$. Repeating the calculation for $\eta = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ and then applying the argument from case 1(b) (since χ is unramified we are only concerned about the valuation of the upper right-hand entry) we also obtain equality. □

6. Bloch–Kato Conjecture for Hecke characters

Combining Theorem 6 with Theorem 13 or 14 we get lower bounds on the size of the Selmer group of Hecke characters. We want to demonstrate this application under the particular conditions of Theorem 14 and relate it to the Bloch–Kato conjecture.

Theorem 28. *Assume that $p > 3$, $p \nmid d_F\#\mathrm{Cl}(F)$ and that $\ell \not\equiv \pm 1 \bmod p$ for $\ell \mid d_F$. If p is inert in $F/\mathbf{Q}$ then assume Conjecture 5. Let χ be an unramified Hecke character of infinity type z^2. Then we have*

$$\mathrm{val}_p\#\mathrm{Sel}(F, \chi_{\mathfrak{p}}\epsilon) \geq \mathrm{val}_p(\#\mathcal{R}/L^{\mathrm{int}}(0, \chi)).$$

Proof. Put $\rho := \chi_{\mathfrak{p}}\epsilon$. Theorem 6 together with Lemma 8 and Theorem 14 imply that

$$\mathrm{val}_p\#\mathrm{Sel}^{\Sigma_\phi\backslash\Sigma_p}(F, \chi_{\mathfrak{p}}\epsilon) \geq \mathrm{val}_p(\#\mathcal{R}/L^{\mathrm{int}}(0, \chi)),$$

where $\phi = (\phi_1, \phi_2)$ is given by Theorem 14. For the definition of Σ_ϕ see the start of § 3. Recall that by (ϕ) the set $\Sigma_\phi\backslash\Sigma_p$ contains only places v such that $\overline{v} = v$ and $\#\mathcal{O}_v/\mathfrak{P}_v \not\equiv \pm 1 \bmod p$. By Lemma 1 we have $\chi^c = \overline{\chi}$, which implies that ρ is anticyclotomic, and so we get $\rho(\mathrm{Frob}_v) = \rho(\mathrm{Frob}_v^c) = \rho^{-1}(\mathrm{Frob}_v)$, or $\rho(\mathrm{Frob}_v) = \pm 1$. Hence we have ensured that

$$\rho(\mathrm{Frob}_v) \not\equiv \epsilon(\mathrm{Frob}_v) \bmod p$$

for all $v \in \Sigma_\phi\backslash\Sigma_p$, so the theorem follows from applying Lemma 2. □

Example 29. A numerical example in which the conditions of our theorem are satisfied and a non-trivial lower bound on a Selmer group is obtained is given by the following. Let $F = \mathbf{Q}(\sqrt{-67})$ and $p = 19$. One can check that 19 splits in F. Since the class number is 1, there is only one unramified Hecke character of infinity type z^2. Up to p-adic units $L^{\mathrm{alg}}(0, \chi)$ is given by $L(0, \chi)/\Omega^2$ where Ω is the Neron period of the elliptic curve $y^2 + y = x^3 - 7370x + 243582$, which

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has conductor 67^2 and complex multiplication by $\mathcal{O}$. Using MAGMA and ComputeL [Dok04] one calculates that $L^{\mathrm{alg}}(0,\chi) \in \mathbf{Z}_{19}$ and

$$\mathrm{val}_{19}(L^{\mathrm{alg}}(0,\chi)) = 1.$$

6.1 Comparison with other results

Assume from now on that $\#\mathrm{Cl}(F) = 1$. Let $\Psi : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ be a Hecke character of infinity type z^{-1} which satisfies $\Psi^c = \overline{\Psi}$. Then there exists an elliptic curve E over $\mathbf{Q}$ with complex multiplication by $\mathcal{O}$ and associated Grössencharacter Ψ. Consider

$$\rho = (\Psi^k \overline{\Psi^{-j}})_{\mathfrak{p}} \quad \text{for } k > 0, j \geq 0.$$

We now have the following proposition from [Dee99, Proposition 4.4.3 and § 5.3].

Proposition 30 (Dee). *The group* $\mathrm{Sel}(F,\rho)$ *is finite if and only if* $\mathrm{Sel}(F,\rho^{-1}\epsilon)$ *is finite. If this is the case then*

$$\#\mathrm{Sel}(F,\rho) = \#\mathrm{Sel}(F,\rho^{-1}\epsilon) < \infty.$$

By considering $\chi = \Psi^{-2}$ for some Ψ as above and $(k,j) = (2,0)$, therefore compare Theorem 28 with the following.

Theorem 31 (Han [Han97]). *Suppose* $k > j+1$. *For inert* p *also assume that* ρ *is non-trivial when restricted to* $\mathrm{Gal}(F(E_{\mathfrak{p}})/F)$. *Then* $\mathrm{Sel}(F,\rho)$ *is finite and*

$$\mathrm{val}_p \#\mathrm{Sel}(F,\rho) = \mathrm{val}_p(\#\mathcal{R}/L^{\mathrm{alg}}(0,\Psi^{-k}\overline{\Psi}^j)).$$

Previously, Kato proved this in the case $k > 0$ and $j = 0$; cf. [Kat93]. For a similar result in the case of split p see [Guo93]. Han claims that his method extends to general class numbers. All proofs take as input the proof of the Main Conjecture of Iwasawa theory by Rubin [Rub91].

We refer to [Guo96, § 3] for the proof that this statement on the size of the Selmer group is equivalent to the (critical case of the) p-part of the Bloch–Kato Tamagawa number conjecture for the motives associated to the Hecke characters.

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References

AH06 A. Agboola and B. Howard, *Anticyclotomic Iwasawa theory of CM elliptic curves*, Ann. Inst. Fourier (Grenoble) **56** (2006), 1001–1048.

AS86 A. Ash and G. Stevens, *Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues*, J. Reine Angew. Math. **365** (1986), 192–220.

BC04 J. Bellaïche and G. Chenevier, *Formes non tempérées pour* U(3) *et conjectures de Bloch-Kato*, Ann. Sci. École Norm. Sup. (4) **37** (2004), 611–662.

Ber05 T. Berger, *An Eisenstein ideal for imaginary quadratic fields*, PhD thesis, University of Michigan, Ann Arbor, 2005.

Ber08 T. Berger, *Denominators of Eisenstein cohomology classes for* GL_2 *over imaginary quadratic fields*, Manuscripta Math. **125** (2008), 427–470.

BH07 T. Berger and G. Harcos, *l-adic representations associated to modular forms over imaginary quadratic fields*, Int. Math. Res. Not. IMRN (2007), Art. ID rnm113, 16

BK T. Berger and K. Klosin, *A deformation problem for Galois representations over imaginary quadratic fields*, Journal de l'Institut de Math. de Jussieu, to appear.

Ber06 E. Berkove, *The integral cohomology of the Bianchi groups*, Trans. Amer. Math. Soc. **358** (2006), 1033–1049, electronic.

BK90 S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, in *The Grothendieck Festschrift,* Vol. I, Progress in Mathematics, vol. 86 (Birkhäuser, Boston, MA, 1990), 333–400.

Blu92 J. Blume-Nienhaus, *Lefschetzzahlen für Galois-Operationen auf der Kohomologie arithmetischer Gruppen*, Bonner Mathematische Schriften, vol. 230 (Universität Bonn Mathematisches Institut, Bonn, 1992).

Bre97 G. E. Bredon, *Sheaf theory*, Graduate Texts in Mathematics, vol. 170, second edition (Springer, New York, 1997).

BFH90 D. Bump, S. Friedberg and J. Hoffstein, *Nonvanishing theorems for L-functions of modular forms and their derivatives*, Invent. Math. **102** (1990), 543–618.

CD06 F. Calegari and N. M. Dunfield, *Automorphic forms and rational homology 3-spheres*, Geometry and Topology **10** (2006), 295–329.

Cre92 J. E. Cremona, *Abelian varieties with extra twist, cusp forms, and elliptic curves over imaginary quadratic fields*, J. London Math. Soc. **45** (1992), 404–416.

Dee99 J. Dee, *Selmer groups of Hecke characters and Chow groups of self products of CM elliptic curves*, Preprint (1999), arXiv:math.NT/9901155.

DFG04 F. Diamond, M. Flach and L. Guo, *The Tamagawa number conjecture of adjoint motives of modular forms*, Ann. Sci. École Norm. Sup. (4) **37** (2004), 663–727.

Dok04 T. Dokchitser, *Computing special values of motivic L-functions*, Experiment. Math. **13** (2004), 137–149.

EGM82 J. Elstrodt, F. Grunewald and J. Mennicke, *On the group* $\mathrm{PSL}_2(\mathbf{Z}[i])$, Number theory days, Exeter, 1980, London Mathematical Society Lecture Note Series, vol. 56 (Cambridge University Press, Cambridge, 1982), 255–283.

EGM98 J. Elstrodt, F. Grunewald and J. Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics (Springer, Berlin, 1998).

Fel00 D. Feldhusen, *Nenner der Eisensteinkohomologie der* GL(2) *über imaginär quadratischen Zahlkörpern*, Bonner Mathematische Schriften, vol. 330 (Universität Bonn Mathematisches Institut, Bonn, 2000).

Fla90 M. Flach, *A generalisation of the Cassels-Tate pairing*, J. Reine Angew. Math. **412** (1990), 113–127.

FS98 J. Franke and J. Schwermer, *A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups*, Math. Ann. **311** (1998), 765–790.

GL79 P. Gérardin and J.-P. Labesse, *The solution of a base change problem for* GL(2) *(following Langlands, Saito, Shintani), Automorphic forms, representations and L-functions*, Oregon State University, Corvallis, OR, 1977, Part 2, Proceedings of Symposia in Pure Mathematics, XXXIII (American Mathematical Society, Providence, RI, 1979), 115–133.

Gre67 M. Greenberg, *Lectures on algebraic topology* (W. A. Benjamin, New York, 1967).

Gre83 R. Greenberg, *On the Birch and Swinnerton-Dyer conjecture*, Invent. Math. **72** (1983), 241–265.

Gre89 R. Greenberg, *Iwasawa theory for p-adic representations*, Algebraic number theory, Advanced Studies in Pure Mathematics, vol. 17 (Academic Press, Boston, MA, 1989), 97–137.

GS93 F. Grunewald and J. Schwermer, *Subgroups of Bianchi groups and arithmetic quotients of hyperbolic 3-space*, Trans. Amer. Math. Soc. **335** (1993), 47–78.

Guo93 Li Guo, *General Selmer groups and critical values of Hecke L-functions*, Math. Ann. **297** (1993), 221–233.

Guo96 Li Guo, *On the Bloch-Kato conjecture for Hecke L-functions*, J. Number Theory **57** (1996), 340–365.

Han97 B. Han, *On Bloch-Kato's Tamagawa number conjecture for Hecke characters of imaginary quadratic fields*, PhD thesis, University of Illinois, Chicago, 1997, http://front.math.ucdavis.edu/ANT/0070.

Har87 G. Harder, *Eisenstein cohomology of arithmetic groups. The case* GL_2, Invent. Math. **89** (1987), 37–118.

HST93 M. Harris, D. Soudry and R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields. I. Lifting to* $\mathrm{GSp}_4(\mathbf{Q})$, Invent. Math. **112** (1993), 377–411.

Hid82 H. Hida, *Kummer's criterion for the special values of Hecke L-functions of imaginary quadratic fields and congruences among cusp forms*, Invent. Math. **66** (1982), 415–459.

HLR86 G. Harder, R. Langlands and M. Rapoport, *Algebraische Zyklen auf Hilbert-Blumenthal-Flächen*, J. Reine Angew. Math. **366** (1986), 53–120.

HP92 G. Harder and R. Pink, *Modular konstruierte unverzweigte abelsche p-Erweiterungen von* $\mathbf{Q}(\zeta_p)$ *und die Struktur ihrer Galoisgruppen*, Math. Nachr. **159** (1992), 83–99.

Kat93 K. Kato, *Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via* B_{dR}*. I*, in *Arithmetic algebraic geometry*, Trento, 1991, Lecture Notes in Mathematics, vol. 1553 (Springer, Berlin, 1993), 50–163.

Klo09 K. Klosin, *Congruences among modular forms on* $U(2,2)$ *and the Bloch-Kato conjecture*, Ann. Inst. Fourier **59** (2009), 81–166.

Lau97 G. Laumon, *Sur la cohomologie à supports compacts des variétés de Shimura pour* $\mathrm{GSp}(4)_{\mathbf{Q}}$, Compositio Math. **105** (1997), 267–359.

Lau05 G. Laumon, *Fonctions zêtas des variétés de Siegel de dimension trois*, Astérisque **302** (2005), 1–66 (Formes automorphes. II. Le cas du groupe GSp(4)).

Mau80 C. R. F. Maunder, *Algebraic topology* (Cambridge University Press, Cambridge, 1980).

May99 J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics (University of Chicago Press, Chicago, IL, 1999).

Neu99 J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322 (Springer, Berlin, 1999). (Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.)

NSW08 J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, second edition (Springer, Berlin, 2008).

Rib76 K. A. Ribet, *A modular construction of unramified p-extensions of* $\mathbf{Q}(\mu_p)$, Invent. Math. **34** (1976), 151–162.

Rub91 K. Rubin, *The main conjectures of Iwasawa theory for imaginary quadratic fields*, Invent. Math. **103** (1991), 25–68.

Rub00 K. Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147 (Princeton University Press, Princeton, NJ, 2000). (Hermann Weyl Lectures. The Institute for Advanced Study.)

Ser68 J.-P. Serre, *Abelian l-adic representations and elliptic curves*, McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute (W. A. Benjamin, Inc., New York-Amsterdam, 1968).

https://doi.org/10.1112/S0010437X09003984 Published online by Cambridge University Press

Ser70 J.-P. Serre, *Le probleme de groupes de congruence pour* SL_2, Ann. of Math. **92** (1970).

SU02 C. Skinner and E. Urban, *Sur les déformations p-adiques des formes de Saito-Kurokawa*, C. R. Math. Acad. Sci. Paris **335** (2002), 581–586.

SV83 J. Schwermer and K. Vogtmann, *The integral homology of* SL_2 *and* PSL_2 *of Euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), 573–598.

Swa71 R. G. Swan, *Generators and relations for certain special linear groups*, Adv. Math. **6** (1971), 1–77 1971

Tay88 R. Taylor, *Congruences of modular forms*, PhD thesis, Harvard, 1988.

Tay94 R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields. II*, Invent. Math. **116** (1994), 619–643.

Urb95 E. Urban, *Formes automorphes cuspidales pour* GL_2 *sur un corps quadratique imaginaire. Valeurs spéciales de fonctions L et congruences*, Compositio Math. **99** (1995), 283–324.

Urb98 E. Urban, *Module de congruences pour* GL(2) *d'un corps imaginaire quadratique et théorie d'Iwasawa d'un corps CM biquadratique*, Duke Math. J. **92** (1998), 179–220.

Urb01 E. Urban, *Selmer groups and the Eisenstein-Klingen ideal*, Duke Math. J. **106** (2001), 485–525.

Urb05 E. Urban, *Sur les représentation p-adiques associées aux représentations cuspidales de* $\mathrm{GSp}_{4/\mathbf{Q}}$, Astérisque **302** (2005), 151–176.

Wei05 R. Weissauer, *Four dimensional Galois representations*, Astérisque **302** (2005), 67–150 (Formes automorphes. II. Le cas du groupe GSp(4)).

Wil86 A. Wiles, *On p-adic representations for totally real fields*, Ann. of Math. (2) **123** (1986), 407–456.

Wil90 A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. of Math. (2) **131** (1990), 493–540.

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