Adv. Appl. Prob. (SGSA) **42,** 631–658 (2010)
Printed in Northern Ireland

CRITICALITY OF THE EXPONENTIAL RATE OF DECAY FOR THE LARGEST NEAREST-NEIGHBOR LINK IN RANDOM GEOMETRIC GRAPHS

BHUPENDER GUPTA,* *Indian Institute of Information Technology*
SRIKANTH K. IYER,** *Indian Institute of Science*

Abstract

Let n points be placed independently in d-dimensional space according to the density $f(x) = A_d \mathrm{e}^{-\lambda\|x\|^\alpha}$, $\lambda, \alpha > 0$, $x \in \mathbb{R}^d$, $d \geq 2$. Let d_n be the longest edge length of the nearest-neighbor graph on these points. We show that $(\lambda^{-1}\log n)^{1-1/\alpha}\,\mathrm{d}_n - b_n$ converges weakly to the Gumbel distribution, where $b_n \sim ((d-1)/\lambda\alpha)\log\log n$. We also prove the following strong law for the normalized nearest-neighbor distance $\tilde{d}_n = (\lambda^{-1}\log n)^{1-1/\alpha}\,\mathrm{d}_n/\log\log n$: $(d-1)/\alpha\lambda \leq \liminf_{n\to\infty} \tilde{d}_n \leq \limsup_{n\to\infty} \tilde{d}_n \leq d/\alpha\lambda$ almost surely. Thus, the exponential rate of decay $\alpha = 1$ is critical, in the sense that, for $\alpha > 1$, $d_n \to 0$, whereas, for $\alpha \leq 1$, $d_n \to \infty$ almost surely as $n \to \infty$.

Keywords: Random geometric graph; nearest-neighbor graph; Poisson point process; largest nearest-neighbor link; vertex degree

2010 Mathematics Subject Classification: Primary 60D05; 60G70
Secondary 05C05; 90C27

1. Introduction and main results

In this paper we prove weak and strong laws for the largest nearest-neighbor distance of points distributed according to the probability density function

$$f(x) = A_d \mathrm{e}^{-\lambda\|x\|^\alpha}, \qquad \lambda > 0,\ \alpha > 0,\ x \in \mathbb{R}^d,\ d \geq 2, \tag{1.1}$$

where $\|\cdot\|$ is the Euclidean (ℓ_2) norm on $\mathbb{R}^d$ and

$$A_d = \frac{\alpha\lambda^{d/\alpha}\Gamma(d/2+1)}{d\pi^{d/2}\Gamma(d/\alpha)}. \tag{1.2}$$

If X has density given by (1.1) then $R = \|X\|$ has density,

$$f_R(r) = A'_d r^{d-1}\mathrm{e}^{-\lambda r^\alpha}, \qquad 0 < r < \infty,\ d \geq 2, \tag{1.3}$$

where $A'_d = \alpha\lambda^{d/\alpha}/\Gamma(d/\alpha)$. The basic object of study will be the graphs G_n with vertex set $\mathcal{X}_n = \{X_1, X_2, \ldots, X_n\}$, $n = 1, 2, \ldots$, where the vertices are independently distributed according to f. Edges of G_n are formed by connecting each of the vertices in $\mathcal{X}_n$ to its nearest

Received 17 June 2006; revision received 18 March 2010.
* Postal address: Department of Computer Science and Engineering, Indian Institute of Information Technology, Jabalpur 482011, India.
** Postal address: Department of Mathematics, Indian Institute of Science, Bangalore 560012, India.
Email address: skiyer@math.iisc.ernet.in
Research supported in part by UGC SAP IV and a grant from the DRDO-IISc program on Mathematical Engineering.

neighbor. The longest edge of the graph G_n is denoted by d_n. We will refer to G_n as the nearest-neighbor graph (NNG) on $\mathcal{X}_n$ and to d_n as the largest nearest-neighbor distance (LNND). For any finite subset $\mathcal{X} \subset \mathbb{R}^d$, let $G(\mathcal{X}, r)$ denote the graph with vertex set $\mathcal{X}$ and edges between all pairs of vertices that are at distances less than r. Thus, d_n is the minimum r_n required so that the graph $G(\mathcal{X}_n, r_n)$ has no isolated nodes.

The largest nearest-neighbor link has been studied in the context of computational geometry (see Dette and Henze (1989) and Steele and Tierney (1986)) and has applications in statistics, computer science, biology, and the physical sciences. For a detailed description of random geometric graphs, their properties, and applications, we refer the reader to Penrose (2003) and the references therein.

The asymptotic distribution of d_n was derived in Penrose (1997), assuming that f is uniform on the unit cube. It is shown that if the metric is assumed to be the toroidal, and if θ is the volume of the unit ball, then $n\theta d_n^d - b_n$ converges weakly to the Gumbel distribution, where $b_n \sim \log n$. Penrose (1998) showed that, for normally distributed points ($\alpha = 2$), $\sqrt{(2\log n)}\,\mathrm{d}_n - b_n$ converges weakly to the Gumbel distribution, where $b_n \sim (d-1)\log\log n$. The above result is also shown to be true for the longest edge of the minimal spanning tree. The notation $a_n \sim b_n$ implies that a_n/b_n converges to 1 as $n \to \infty$. Hsing and Rootzén (2005) derived the asymptotic distribution for d_n in the case $d = 2$, for a large class of densities, including elliptically contoured distributions, distributions with independent Weibull-like marginals, and distributions with parallel level curves (which includes the densities defined by (1.1)). Appel and Russo (1997) proved strong laws for d_n for graphs on uniform points in the d-dimensional unit cube. Penrose (1999) extended this to general densities having compact support Ω for which $\min_{x\in\Omega} f(x) > 0$.

Our aim in this paper is to show that when the tail of the density decays like an exponential or slower ($\alpha \le 1$), d_n diverges, whereas, for superexponential decay of the tail, $d_n \to 0$, almost surely (a.s.) as $n \to \infty$. Properties of one-dimensional exponential random geometric graphs have been studied in Gupta *et al.* (2005). In this case, spacings between the ordered nodes are independent and exponentially distributed. This allows for explicit computations of many characteristics for the graph and both strong and weak laws can be established.

It is often easier to study the graph G_n via the NNG P_n on the set $\mathcal{P}_n = \{X_1, X_2, \dots, X_{N_n}\}$, $n \ge 1$, where $\{N_n\}_{n\ge 1}$ is a sequence of Poisson random variables that are independent of the sequence $\{X_n\}_{n\ge 1}$ with $\mathrm{E}[N_n] = n$. Here $\mathcal{P}_n$ is an inhomogeneous Poisson point process with intensity function $nf(\cdot)$ (see Penrose (2003, Proposition 1.5)). Note that the graphs G_n and P_n are coupled, since the first $\min(n, N_n)$ vertices of the two graphs are identical. We also assume that the random variables N_n are nondecreasing, so that $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset \cdots$.

Let $W_n(r_n)$ and $W_n'(r_n)$ be the numbers of vertices of degree 0 (isolated nodes) in $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$, respectively. Let θ_d denote the volume of the d-dimensional unit ball in $\mathbb{R}^d$, and let $\mathrm{Po}(\lambda)$ denote a Poisson distribution with mean $\lambda > 0$. In what follows we will write $\log_2 n$ for $\log\log n$ and $\log_3 n$ for $\log\log\log n$, etc.

For any $\beta \in \mathbb{R}$, let $\{r_n\}_{n\ge 1}$ be a sequence of edge distances that satisfies

$$r_n(\lambda^{-1}\log n)^{1-1/\alpha} - \frac{d-1}{\lambda\alpha}\log_2 n + \frac{d-1}{2\lambda\alpha}\log_3 n \to \frac{\beta}{\lambda\alpha} \tag{1.4}$$

as $n \to \infty$. We now state our main results.

Theorem 1.1. *Let $\{r_n\}_{n\ge 1}$ satisfy (1.4) as $n \to \infty$. Then*

$$W_n(r_n) \to \mathrm{Po}\left(\frac{\mathrm{e}^{-\beta}}{C_d}\right) \tag{1.5}$$

in distribution, where

$$C_d = \frac{\alpha^{1-d}\theta_{d-1}(d-1)!}{2}\left(\frac{d-1}{2\pi}\right)^{(d-1)/2}. \tag{1.6}$$

An easy consequence of the above result is the following limiting distribution for d_n.

Theorem 1.2. *Let $f(\cdot)$ be the d-dimensional density defined in (1.1). Let d_n be the largest nearest-neighbor link of the graph G_n of n independent and identically distributed points $\mathcal{X}_n = \{X_1, X_2, \ldots, X_n\}$ distributed according to f. Then, for any $\gamma \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathrm{P}\left[\lambda\alpha(\lambda^{-1}\log n)^{1-1/\alpha}\,\mathrm{d}_n - (d-1)\log_2 n + \frac{d-1}{2}\log_3 n \le \gamma - \log(C_d)\right] \to \exp(-\mathrm{e}^{-\gamma}).$$

The above result for the case $\alpha = 2$ was derived in Penrose (1998). In dimension $d = 2$, Theorem 1.2 follows from Theorem 7 of Hsing and Rootzén (2005) (see also Example 3 therein). Their method is based on spatial blocking and uses a locally orthogonal coordinate system with respect to the level curves. We follow the approach in Penrose (1998) and use the Chen–Stein method.

Strong laws exist in the literature only for densities that do not vanish and whose support is bounded. Suppose that $d \ge 2$, the density f is continuous, has support Ω, and that the boundary $\partial\Omega$ is a compact $(d-1)$-dimensional C^2 submanifold of $\mathbb{R}^d$. Let $f_0 > 0$ be the essential infimum of f restricted to Ω, and let $f_1 = \inf_{\partial\Omega} f$. Then (see Theorem 7.2 of Penrose (2003)), $\lim_{n\to\infty} n d_n^d/\log n = \max\{c_0 f_0^{-1}, c_1 f_1^{-1}\}$ a.s. Thus, the asymptotic behavior of the LNND depends on the (reciprocal of the) infimum of the density, since it is in the vicinity of this infimum that points will be sparse and, hence, be farthest from each other. If f_0 or f_1 is 0, then the right-hand side is infinite, implying that the scaling on the left is not the appropriate one. We now state a strong law for the LNND in our case.

Theorem 1.3. *Let d_n be the LNND of the NNG G_n defined on the collection $\mathcal{X}_n$ of n points distributed independently and identically according to the density $f(\cdot)$ as defined in (1.1). Then, a.s., for any $d \ge 2$,*

$$\liminf_{n\to\infty} \frac{(\lambda^{-1}\log n)^{1-1/\alpha} d_n}{\log_2 n} \ge \frac{d-1}{\alpha\lambda}, \tag{1.7}$$

$$\limsup_{n\to\infty} \frac{(\lambda^{-1}\log n)^{1-1/\alpha} d_n}{\log_2 n} \le \frac{d}{\alpha\lambda}.$$

Note that it follows from Theorem 1.2 that the opposite inequality holds as well in (1.7). Thus, the inequality in (1.7) can be replaced by an equality.

2. Supporting results and proofs of Theorems 1.1 and 1.2

In what follows, C, C_1, C_2, C', c_1, etc. will denote constants whose values might change from place to place. For any $x \in \mathbb{R}^d$, let $B(x, r)$ denote the open ball of radius r centered at x. Let

$$I(x, r) = \int_{B(x,r)} f(y)\,\mathrm{d}y.$$

For $\rho > 0$, define $I(\rho, r) = I(\rho e, r)$, where e is the d-dimensional unit vector $(1, 0, 0, \ldots, 0)$. Due to the radial symmetry of f, $I(x, r) = I(\|x\|, r)$. The following lemma, which provides a large ρ asymptotic for $I(\rho, r)$, will be crucial in subsequent calculations.

Lemma 2.1. *Let $d \geq 2$, and let $\{\rho_n\}_{n\geq 1}$ and $\{r_n\}_{n\geq 1}$ be sequences of positive numbers satisfying $\rho_n \to \infty$, $r_n/\rho_n \to 0$, $r_n^2\rho_n^{\alpha-2} \to 0$, and $r_n\rho_n^{\alpha-1} \to \infty$. Then*

$$K_d \mathrm{e}^{-\lambda w_1(n)} \frac{(\Gamma((d+1)/2) + E_n)}{\Gamma((d+1)/2)} g(\rho_n, r_n) \leq I(\rho_n, r_n) \leq K_d \mathrm{e}^{-\lambda w_2(n)} g(\rho_n, r_n),$$

where

$$K_d = A_d \theta_{d-1} 2^{(d-1)/2} \Gamma\left(\frac{d+1}{2}\right), \tag{2.1}$$

$$w_1(n) = \begin{cases} \frac{\alpha}{2} r_n^2 (\rho_n^2 - 2r_n\rho_n)^{\alpha/2-1}, & \alpha \leq 2, \\ \frac{\alpha}{2} r_n^2 (\rho_n^2 + 2r_n\rho_n)^{\alpha/2-1} [1 + (\alpha-2)\rho_n^2(\rho_n^2 - 2r_n\rho_n)^{-1}], & \alpha > 2, \end{cases} \tag{2.2}$$

$$w_2(n) = \begin{cases} \frac{\alpha(\alpha-2)}{2} (r_n\rho_n)^2 (\rho_n^2 - 2r_n\rho_n)^{\alpha/2-2}, & 0 < \alpha \leq 2, \\ 0, & \alpha > 2, \end{cases}$$

$$|E_n| \leq \frac{C_1}{r_n\rho_n^{\alpha-1}}, \tag{2.3}$$

$$g(\rho, r) = r^d \mathrm{e}^{-\lambda(\rho^\alpha - \alpha r \rho^{\alpha-1})} (\lambda\alpha r\rho^{\alpha-1})^{-(d+1)/2}, \qquad r, \rho \geq 0, \tag{2.4}$$

A_d is as defined in (1.2), θ_{d-1} is the volume of the $(d-1)$-dimensional unit ball, and C_1 is some constant. As $n \to \infty$, $E_n \to 0$, and $w_i(n) \to 0$, $i = 1, 2$.

The proof of Lemma 2.1 is given in Appendix A. We first prove a Poissonized version of Theorem 1.1 for the number of isolated nodes, i.e. (1.5) with $W_n(r_n)$ replaced by $W_n'(r_n)$. To this end, we first find a sequence r_n for which $\mathrm{E}[W_n'(r_n)]$ converges. From the Palm theory for Poisson processes (see Equation (8.45) of Penrose (2003)), we obtain

$$\mathrm{E}[W_n'(r_n)] = n \int_{R^d} \exp(-nI(x, r_n)) f(x)\, \mathrm{d}x.$$

Changing to polar coordinates gives

$$\mathrm{E}[W_n'(r_n)] = n \int_0^\infty \exp(-nI(s, r_n)) f_R(s)\, \mathrm{d}s, \tag{2.5}$$

where f_R is defined in (1.3). Define the sequence of functions $\{\rho_n\}_{n\geq 1}$ by

$$\rho_n(t)^\alpha = \frac{t + a_n}{\lambda}, \qquad t \geq -a_n, \tag{2.6}$$

where

$$a_n = \log n + \left(\frac{d}{\alpha} - 1\right) \log_2 n - \log\left(\Gamma\left(\frac{d}{\alpha}\right)\right). \tag{2.7}$$

Choose r_n so that the remaining factor in (2.5) also converges. Making the change of variable $t = \rho_n^{-1}(s)$ in (2.5) we obtain

$$\mathrm{E}[W_n'(r_n)] = \int_{-a_n}^\infty \exp(-nI(\rho_n(t), r_n)) g_n(t)\, \mathrm{d}t, \tag{2.8}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

where

$$\begin{aligned} g_n(t) &= nf_R(\rho_n(t))\rho_n'(t) \\ &= \frac{n\lambda^{d/\alpha-1}}{\Gamma(d/\alpha)}\left(\frac{t+a_n}{\lambda}\right)^{d/\alpha-1} \mathrm{e}^{-(t+a_n)} \\ &= \left(\frac{t+a_n}{\log n}\right)^{d/\alpha-1} \mathrm{e}^{-t} \\ &= \left(\frac{t+\log n+(d/\alpha-1)\log_2 n-\log(\Gamma(d/\alpha))}{\log n}\right)^{d/\alpha-1} \mathrm{e}^{-t} \\ &\to \mathrm{e}^{-t} \quad \text{as } n\to\infty \text{ for all } t\in\mathbb{R}. \end{aligned} \tag{2.9}$$

Lemma 2.2. *Suppose that the sequence* $\{r_n\}_{n\geq 1}$ *satisfies (1.4). Let* $t\in\mathbb{R}$, *and set* $\rho_n(t)^\alpha = ((t+a_n)/\lambda)\,\mathbf{1}_{\{t\geq -a_n\}}$, *where* a_n *is as defined in (2.7). Then*

$$\lim_{n\to\infty} nI(\rho_n(t), r_n) = C_d \mathrm{e}^{\beta-t}, \tag{2.10}$$

where C_d *is as defined in (1.6).*

Proof. It is easy to verify that, for each fixed $t\in\mathbb{R}$, $\rho_n=\rho_n(t)$ and r_n satisfy the conditions of Lemma 2.1, and so we have $nI(\rho_n, r_n)\sim K_d n g(\rho_n, r_n)$. Substituting for $\lambda\rho_n^\alpha(t)$ from (2.6) and (2.7), we obtain

$$nI(\rho_n, r_n) \sim \frac{nK_d\Gamma(d/\alpha)\mathrm{e}^{-t}}{n(\log n)^{d/\alpha-1}} r_n^d \exp(\lambda\alpha r_n\rho_n^{\alpha-1})(\lambda\alpha r_n\rho_n^{\alpha-1})^{-(d+1)/2}. \tag{2.11}$$

From (1.4) we can write

$$\begin{aligned} r_n &= \frac{d-1}{\lambda\alpha}\frac{\log_2 n}{(\lambda^{-1}\log n)^{1-1/\alpha}} - \frac{d-1}{2\lambda\alpha}\frac{\log_3 n}{(\lambda^{-1}\log n)^{1-1/\alpha}} + \frac{\beta+o(1)}{\lambda\alpha(\lambda^{-1}\log n)^{1-1/\alpha}} \\ &= \frac{d-1}{\lambda\alpha}\frac{\log_2 n}{(\lambda^{-1}\log n)^{1-1/\alpha}}(1+o(1)), \end{aligned} \tag{2.12}$$

and, hence,

$$\begin{aligned} \lambda\alpha r_n\rho_n^{\alpha-1} &= \left(\frac{(d-1)\log_2 n}{(\lambda^{-1}\log n)^{1-1/\alpha}} - \frac{d-1}{2}\frac{\log_3 n}{(\lambda^{-1}\log n)^{1-1/\alpha}} + \frac{\beta+o(1)}{(\lambda^{-1}\log n)^{1-1/\alpha}}\right) \\ &\quad\times\left(\frac{1}{\lambda}\left(t+\log n+\left(\frac{d}{\alpha}-1\right)\log_2 n-\log\left(\Gamma\left(\frac{d}{\alpha}\right)\right)\right)\right)^{(\alpha-1)/\alpha} \\ &= \left((d-1)\log_2 n - \frac{d-1}{2}\log_3 n+\beta+o(1)\right) \\ &\quad\times\left(1+\frac{t}{\log n}+\left(\frac{d}{\alpha}-1\right)\frac{\log_2 n}{\log n}-\frac{\log(\Gamma(d/\alpha))}{\log n}\right)^{(\alpha-1)/\alpha} \end{aligned} \tag{2.13}$$

$$= (d-1)\log_2 n - \frac{d-1}{2}\log_3 n + \beta + o(1). \tag{2.14}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

Substituting (2.12) and (2.14) into (2.11), we obtain

$$\begin{aligned} nI(\rho_n, r_n) &\sim \frac{K_d\Gamma(d/\alpha)\mathrm{e}^{-t}}{(\log n)^{d/\alpha-1}}\left(\frac{d-1}{\lambda\alpha}\frac{\log_2 n}{(\lambda^{-1}\log n)^{1-1/\alpha}}(1+o(1))\right)^d \\ &\quad\times\left(\frac{\exp((d-1)\log_2 n - \log_3 n(d-1)/2+\beta+o(1))}{((d-1)\log_2 n - \log_3 n(d-1)/2+\beta+o(1))^{(d+1)/2}}\right) \\ &\to C_d\mathrm{e}^{\beta-t}, \end{aligned} \tag{2.15}$$

where $C_d = K_d\Gamma(d/\alpha)(d-1)^{(d-1)/2}\lambda^{-d/\alpha}\alpha^{-d}$. Substituting for K_d from (2.1) (and for A_d from (1.2)) and simplifying, we obtain

$$C_d = \alpha^{1-d}\theta_{d-1}2^{(d-1)/2}(d-1)^{(d-1)/2}\frac{\Gamma((d+1)/2)\Gamma(d/2+1)}{d\pi^{d/2}}.$$

Since $\Gamma((d+1)/2)\Gamma(d/2+1) = 2^{-d}\Gamma(d+1)\sqrt{\pi}$, we obtain the expression for C_d given in (1.6).

Lemma 2.3. *For any $t \in \mathbb{R}$ and sufficiently large n, let $g_n(t)$ be as defined in (2.9). There exists a constant M depending on α, d, and λ such that the following inequalities hold for all large enough n.*

1. *Suppose that $d/\alpha \geq 1$ and $\lambda r_n^\alpha - a_n \leq t \leq 0$ or that $d/\alpha < 1$ and $-\log n/\log_2 n \leq t \leq 0$. Then $g_n(t) \leq M\mathrm{e}^{-t}$.*
2. *For $d/\alpha < 1$ and $\lambda r_n^\alpha - a_n \leq t \leq -\log n/\log_2 n$, $g_n(t) \leq M(\log_2 n/\log n)^{d-\alpha}\mathrm{e}^{-t}$.*

Remark 2.1. From (2.7), it follows that, for large n,

$$\tfrac{1}{2}\log n \leq a_n \leq 2\log n. \tag{2.16}$$

Hence, from (2.12) and (2.16), we have, for large n, $\lambda r_n^\alpha - a_n < -\log n/\log_2 n$. Thus, the first part of Lemma 2.3 includes the cases $d/\alpha \geq 1$ and $-\log n/\log_2 n \leq t \leq 0$.

Proof of Lemma 2.3. For the case where $d/\alpha \geq 1$ and $\lambda r_n^\alpha - a_n \leq t \leq 0$, we have, by (2.16),

$$g_n(t) \leq \left(\frac{0+a_n}{\log n}\right)^{d/\alpha-1}\mathrm{e}^{-t} \leq 2^{d/\alpha-1}\mathrm{e}^{-t}.$$

Again using (2.16), if $d/\alpha < 1$ and $-\log n/\log_2 n \leq t \leq 0$, then

$$\begin{aligned} g_n(t) &\leq \left(\frac{-\log n/\log_2 n + \log n/2}{\log n}\right)^{d/\alpha-1}\mathrm{e}^{-t} \\ &\leq \left(\frac{-\log n/4 + \log n/2}{\log n}\right)^{d/\alpha-1}\mathrm{e}^{-t} \\ &\leq 4^{1-d/\alpha}\mathrm{e}^{-t}. \end{aligned}$$

From (2.12), it follows that, for large n, $\lambda r_n^\alpha \geq ((d-1)\log_2 n/2\alpha(\log n)^{1-1/\alpha})^\alpha$. Thus, if $d/\alpha < 1$ and $\lambda r_n^\alpha - a_n \leq t \leq -\log n/\log_2 n$,

$$g_n(t) \leq \left(\frac{\lambda r_n^\alpha - a_n + a_n}{\log n}\right)^{d/\alpha-1}\mathrm{e}^{-t} \leq \left(\frac{(d-1)\log_2 n}{2\alpha\log n}\right)^{d-\alpha}\mathrm{e}^{-t}.$$

Proposition 2.1. *Let the sequence $\{r_n\}_{n\geq 1}$ satisfy (1.4), and let C_d be as defined in (1.6). Then*

$$\lim_{n\to\infty} \mathrm{E}[W_n'] = \frac{\mathrm{e}^{-\beta}}{C_d}.$$

Proof. From Lemma 2.2 and (2.9), for each $t \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \exp(-nI(\rho_n(t), r_n))g_n(t) = \exp(-C_d \mathrm{e}^{\beta-t})\mathrm{e}^{-t}. \tag{2.17}$$

Suppose that we can find integrable bounds for $\exp(-nI(\rho_n(t), r_n))g_n(t)$ that hold for all large n. Then, from (2.8), (2.17), and the dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \mathrm{E}[W_n'(r_n)] &= \lim_{n\to\infty} \int_{-a_n}^{\infty} \exp(-nI(\rho_n(t), r_n))g_n(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \exp(-C_d \mathrm{e}^{\beta-t})\mathrm{e}^{-t}\,\mathrm{d}t \\ &= \frac{\mathrm{e}^{-\beta}}{C_d}.\end{aligned}$$

We find integrable bounds for $\exp(-nI(\rho_n(t), r_n))g_n(t)$ by dividing the range of t into four parts.

Part 1. First consider $t \geq 0$. For large n, since $0.5\log n < a_n < 2\log n$, we have

$$g_n(t) \leq \begin{cases} \left(\dfrac{t+2\log n}{\log n}\right)^{d/\alpha-1} \mathrm{e}^{-t} \leq \mathrm{e}^{-t}2^{d/\alpha}\max(t,1)^{d/\alpha-1}, & d/\alpha > 1, \\ 2^{1-d/\alpha}\mathrm{e}^{-t}, & d/\alpha \leq 1. \end{cases} \tag{2.18}$$

By the above bound on $g_n(t)$, it follows that

$$\exp(-nI(\rho_n(t), r_n))g_n(t) \leq g_n(t) \tag{2.19}$$

is integrable over $[0, \infty)$.

Part 2. Now consider the range $-\log n/\log_2 n \leq t \leq 0$. As $\lambda\rho_n(t)^\alpha = t + a_n$, from (2.13) we obtain

$$\begin{aligned}\lambda\alpha r_n\rho_n(t)^{\alpha-1} &= \left((d-1)\log_2 n - \frac{d-1}{2}\log_3 n + \beta + o(1)\right) \\ &\quad \times \left(1 + \frac{\alpha-1}{\alpha}\left(\frac{t+(d/\alpha-1)\log_2 n - \log(\Gamma(d/\alpha))}{\log n}\right)(1+\zeta_n(t))^{-1/\alpha}\right),\end{aligned}$$

where $|\zeta_n(t)| \leq |t + (d/\alpha-1)\log_2 n - \log(\Gamma(d/\alpha))|(\log n)^{-1}$. Hence,

$$\zeta_n(t)\,\mathbf{1}_{[-\log n/\log_2 n, 0]}(t) \to 0$$

uniformly as $n \to \infty$. Since $-1 \leq t\log_2 n/\log n \leq 0$ in the above range of t, we can find constants c_1 and c_2 such that, for large n,

$$(d-1)\log_2 n - \frac{d-1}{2}\log_3 n - c_1 \leq \lambda\alpha r_n\rho_n(t)^{\alpha-1} \leq (d-1)\log_2 n - \frac{d-1}{2}\log_3 n + c_2. \tag{2.20}$$

Hence, for all large n, we have

$$\exp(\lambda\alpha r_n\rho_n^{\alpha-1}) \geq \frac{(\log n)^{d-1}}{(\log_2 n)^{(d-1)/2}}\mathrm{e}^{-c_1}. \tag{2.21}$$

From Lemma 2.1,

$$\begin{aligned} nI(\rho_n, r_n) &\geq C_1 n\bigg(\Gamma\bigg(\frac{d+1}{2}\bigg) + E_n\bigg)\mathrm{e}^{-\lambda w_1(n)} g(\rho_n, r_n) \\ &= C_1 n\bigg(\Gamma\bigg(\frac{d+1}{2}\bigg) + E_n\bigg) r_n^d \frac{\Gamma(d/\alpha)\mathrm{e}^{-t}}{n(\log n)^{d/\alpha-1}}\mathrm{e}^{-\lambda w_1(n)} \\ &\quad \times \exp(\lambda\alpha r_n\rho_n^{\alpha-1})(\lambda\alpha r_n\rho_n^{\alpha-1})^{-(d+1)/2}, \end{aligned} \tag{2.22}$$

where $C_1 = A_d\theta_{d-1}2^{(d-1)/2}$. Substituting (2.12), (2.20), and (2.21) into the above expression we obtain, for some constant C and large n,

$$\begin{aligned} nI(\rho_n, r_n) &\geq A_d\theta_{d-1}2^{(d-1)/2}\bigg(\Gamma\bigg(\frac{d+1}{2}\bigg) + E_n\bigg)\mathrm{e}^{-\lambda w_1(n)} \\ &\quad \times \bigg(\frac{(d-1)\log_2 n}{\lambda\alpha(\lambda^{-1}\log n)^{1-1/\alpha}}(1+o(1))\bigg)^d \frac{\Gamma(d/\alpha)\mathrm{e}^{-t}}{(\log n)^{d/\alpha-1}} \frac{(\log n)^{d-1}}{(\log_2 n)^{(d-1)/2}} \\ &\quad \times \mathrm{e}^{-c_1}\bigg((d-1)\log_2 n - \frac{d-1}{2}\log_3 n + c_2\bigg)^{-(d+1)/2} \\ &\geq C\bigg(\Gamma\bigg(\frac{d+1}{2}\bigg) + E_n\bigg)\mathrm{e}^{-\lambda w_1(n)}\mathrm{e}^{-t}. \end{aligned} \tag{2.23}$$

Suppose that we show, as $n \to \infty$, $r_n/\rho_n(t)$ and $r_n^2\rho_n(t)^{\alpha-2}$ converge uniformly to 0, and that $r_n\rho_n(t)^{\alpha-1} \to \infty$ uniformly for $-\log n/\log_2 n \leq t \leq 0$. It then follows from (2.2) and (2.3) that $w_1(n)$ and E_n converge uniformly to 0. Hence, we can find a constant $c' > 0$ such that

$$nI(\rho_n, r_n) \geq c'\mathrm{e}^{-t}. \tag{2.24}$$

From the above inequality and part 1 of Lemma 2.3 (see also Remark 2.1), there exists a constant c such that, for all large n, we have

$$\exp(-nI(\rho_n(t), r_n))g_n(t) \leq c\exp(-c'\mathrm{e}^{-t})\mathrm{e}^{-t}, \qquad -\frac{\log n}{\log_2 n} \leq t \leq 0. \tag{2.25}$$

This upper bound is integrable over $t \in (-\infty, 0)$. We now verify the three conditions assumed above. That $r_n\rho_n(t)^{\alpha-1} \to \infty$ uniformly follows from (2.20). From (2.12), for some constant c_1 and large n, we have $r_n \leq c_1\log_2 n/(\log n)^{1-1/\alpha}$. Since $-\log n/\log_2 n \leq t \leq 0$, by (2.6), for all sufficiently large n,

$$\lambda\rho_n(t)^\alpha \geq \log n - \frac{\log n}{\log_2 n} + \bigg(\frac{d}{\alpha} - 1\bigg)\log_2 n - \log\bigg(\Gamma\bigg(\frac{d}{\alpha}\bigg)\bigg) \geq \frac{1}{2}\log n.$$

So, for large n, we can find a constant c_2 such that $r_n/\rho_n(t) \leq c_2\log_2 n/\log n \to 0$ as $n \to \infty$. Next we show that $r_n^2\rho_n(t)^{\alpha-2} \to 0$ as $n \to \infty$ uniformly over $-\log n/\log_2 n \leq t \leq 0$. By

(1.4) and (2.6), we obtain

$$
\begin{aligned}
(\lambda\alpha)^2 r_n^2 \rho_n(t)^{\alpha-2} &= \left(\frac{(d-1)\log_2 n}{(\lambda^{-1}\log n)^{1-1/\alpha}} - \frac{d-1}{2}\frac{\log_3 n}{(\lambda^{-1}\log n)^{1-1/\alpha}} + \frac{\beta+o(1)}{(\lambda^{-1}\log n)^{1-1/\alpha}} \right)^2 \\
&\quad \times \left(\frac{1}{\lambda}\left(t + \log n + \left(\frac{d}{\alpha}-1\right)\log_2 n - \log\left(\Gamma\left(\frac{d}{\alpha}\right)\right)\right)\right)^{(\alpha-2)/\alpha} \\
&= \lambda \left(\frac{((d-1)\log_2 n - \log_3 n(d-1)/2 + \beta + o(1))^2}{\log n} \right) \\
&\quad \times \left(1 + \frac{t + (d/\alpha - 1)\log_2 n - \log(\Gamma(d/\alpha))}{\log n} \right)^{(\alpha-2)/\alpha}.
\end{aligned}
$$

Since $-\log n/\log_2 n \le t \le 0$, the right-hand side of the above equation is bounded by

$$
\begin{aligned}
&\lambda \frac{((d-1)\log_2 n - \log_3 n(d-1)/2 + \beta + o(1))^2}{\log n} \\
&\quad \times \left(1 + \frac{(d/\alpha - 1)\log_2 n - \log(\Gamma(d/\alpha))}{\log n} \right)^{(\alpha-2)/\alpha}
\end{aligned}
$$

for $\alpha \ge 2$, and by

$$
\begin{aligned}
&\lambda \frac{((d-1)\log_2 n - \log_3 n(d-1)/2 + \beta + o(1))^2}{\log n} \\
&\quad \times \left(1 - \frac{\log n/\log_2 n - (d/\alpha - 1)\log_2 n + \log(\Gamma(d/\alpha))}{\log n} \right)^{(\alpha-2)/\alpha}
\end{aligned}
$$

for $0 < \alpha < 2$. Both these bounds are independent of t and converge to 0 as $n \to \infty$.

Part 3. Next, consider the range $\lambda r_n^\alpha - a_n \le t \le -\log n/\log_2 n$ (see Remark 2.1). From the first inequality we have $r_n \le \rho_n(t)$, and, hence,

$$
\begin{aligned}
I(\rho_n(t), r_n) &= \int_{B(\rho_n(t)e, r_n)} A_d \mathrm{e}^{-\lambda\|x\|^\alpha}\,\mathrm{d}x \\
&> \int_{B(\rho_n(t)e, r_n),\ \|x\|\le\rho_n(t)} A_d \mathrm{e}^{-\lambda\|x\|^\alpha}\,\mathrm{d}x \\
&\ge A_d \mathrm{e}^{-\lambda\rho_n(t)^\alpha} |B(\rho_n(t)e, r_n) \cap B(0, \rho_n(t))|, \qquad (2.26)
\end{aligned}
$$

where $|\cdot|$ denotes the volume and $e = (1, 0, \ldots, 0) \in \mathbb{R}^d$. Inscribe a sphere of diameter r_n inside $B(\rho_n(t)e, r_n) \cap B(0, \rho_n(t))$ (see Figure 1). Hence,

$$
|B(\rho_n(t)e, r_n) \cap B(0, \rho_n(t))| \ge \frac{\theta_d r_n^d}{2^d}. \qquad (2.27)
$$

Substituting (2.6), (2.12), and (2.27) into (2.26), we have, for large n,

$$
\begin{aligned}
I(\rho_n(t), r_n) &\ge c'' \mathrm{e}^{-\lambda\rho_n(t)^\alpha} r_n^d \\
&= \frac{c''' \mathrm{e}^{-t}}{n(\log n)^{d/\alpha-1}} \frac{(\log_2 n)^d}{(\log n)^{d-d/\alpha}} \left(1 - \frac{\log_3 n}{2\log_2 n} + \frac{\beta+o(1)}{(d-1)\log_2 n} \right)^d \\
&\ge c^* n^{-1}(\log n)^{1-d}(\log_2 n)^d \mathrm{e}^{-t} \\
&= q_n \mathrm{e}^{-t}, \qquad (2.28)
\end{aligned}
$$

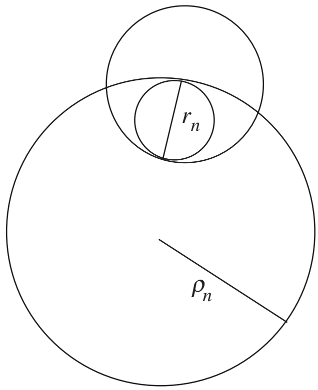

FIGURE 1.

where

$$q_n = c^*(\log n)^{1-d}(\log_2 n)^d n^{-1}. \tag{2.29}$$

From Lemma 2.3 and (2.28), we obtain,

$$\begin{aligned}
&\int_{\lambda r_n^\alpha - a_n}^{-\log n/\log_2 n} \exp(-nI(\rho_n(t), r_n))g_n(t)\,\mathrm{d}t \\
&\quad\le \begin{cases} M\displaystyle\int_{\lambda r_n^\alpha - a_n}^{-\log n/\log_2 n} \exp(-nq_n\mathrm{e}^{-t})\mathrm{e}^{-t}\,\mathrm{d}t, & d/\alpha \ge 1, \\ M\left(\dfrac{\log_2 n}{\log n}\right)^{d-\alpha} \displaystyle\int_{\lambda r_n^\alpha - a_n}^{-\log n/\log_2 n} \exp(-nq_n\mathrm{e}^{-t})\mathrm{e}^{-t}\,\mathrm{d}t, & d/\alpha < 1, \end{cases} \\
&\quad\le \begin{cases} M\displaystyle\int_{\exp(\log n/\log_2 n)}^{\exp(a_n - \lambda r_n^\alpha)} \mathrm{e}^{-nq_n y}\,\mathrm{d}y, & d/\alpha \ge 1, \\ M\left(\dfrac{\log_2 n}{\log n}\right)^{d-\alpha} \displaystyle\int_{\exp(\log n/\log_2 n)}^{\exp(a_n - \lambda r_n^\alpha)} \mathrm{e}^{-nq_n y}\,\mathrm{d}y, & d/\alpha < 1, \end{cases} \\
&\quad\le \begin{cases} \dfrac{M}{nq_n}\exp(-nq_n\mathrm{e}^{\log n/\log_2 n}), & d/\alpha \ge 1, \\ \dfrac{M}{nq_n}\left(\dfrac{\log_2 n}{\log n}\right)^{d-\alpha} \exp(-nq_n\mathrm{e}^{\log n/\log_2 n}), & d/\alpha < 1. \end{cases}
\end{aligned} \tag{2.30}$$

We have

$$\begin{aligned}
\frac{M}{nq_n}\exp(-nq_n\mathrm{e}^{\log n/\log_2 n}) &= \frac{M}{nq_n}\exp(-n^{1+1/\log_2 n}q_n) \\
&= C\frac{(\log n)^{d-1}}{(\log_2 n)^d}\exp(-c^* n^{1/\log_2 n}(\log n)^{1-d}(\log_2 n)^d).
\end{aligned} \tag{2.31}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

Consider the exponent $c^* n^{1/\log_2 n} (\log n)^{1-d} (\log_2 n)^d$. Taking logarithms we obtain, for large n,

$$\log(c^*) + \frac{\log n}{\log_2 n} + (1-d)\log_2 n + d\log_3 n \geq \frac{\log n}{2\log_2 n}.$$

Hence,

$$c^* n^{1/\log_2 n} (\log n)^{1-d} (\log_2 n)^d \geq \mathrm{e}^{\log n/2\log_2 n}. \tag{2.32}$$

Using (2.32) in (2.31), we obtain

$$\frac{M}{nq_n}\exp(-nq_n \mathrm{e}^{\log n/\log_2 n}) \leq C\left(\frac{\log n}{\log_2 n}\right)^{d-1} \frac{1}{\log_2 n}\exp(-\mathrm{e}^{\log n/2\log_2 n}) \to 0, \tag{2.33}$$

since the exponent is decaying exponentially fast in $\log n/\log_2 n$. Using the inequality from (2.33) in (2.30) for the case $d/\alpha < 1$, we obtain

$$\frac{M}{nq_n}\left(\frac{\log_2 n}{\log n}\right)^{d-\alpha} \exp(-nq_n \mathrm{e}^{\log n/\log_2 n}) \leq \frac{C(\log n)^{\alpha-1}}{(\log_2 n)^{\alpha}}\exp(-\mathrm{e}^{\log n/2\log_2 n}), \tag{2.34}$$

which converges to 0 as $n \to \infty$, by the same argument as above. From (2.30), (2.33), and (2.34), we have

$$\int_{\lambda r_n^{\alpha} - a_n}^{-\log n/\log_2 n} \exp(-nI(\rho_n(t), r_n)) g_n(t)\,\mathrm{d}t \to 0. \tag{2.35}$$

Part 4. Finally, consider the case $-a_n \leq t \leq \lambda r_n^{\alpha} - a_n$. The second inequality implies that $r_n \geq \rho_n(t)$. Hence, for large n, we have

$$nI(\rho_n(t), r_n) = n\int_{B(\rho_n(t)e, r_n)} A_d \mathrm{e}^{-\lambda\|x\|^{\alpha}}\,\mathrm{d}x \geq n\int_{B(r_n e, r_n)} A_d \mathrm{e}^{-\lambda\|x\|^{\alpha}}\,\mathrm{d}x \geq c_1 n \mathrm{e}^{-\lambda(2r_n)^{\alpha}} r_n^d. \tag{2.36}$$

For large n from (2.12), we have

$$\frac{(d-1)\log_2 n}{2\lambda^{1/\alpha}\alpha(\log n)^{1-1/\alpha}} \leq r_n \leq \frac{2(d-1)\log_2 n}{\lambda^{1/\alpha}\alpha(\log n)^{1-1/\alpha}}. \tag{2.37}$$

Fix $0 < \varepsilon_1, \varepsilon_2 < 1$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2 < 1$. Substituting (2.37) into (2.36) we obtain, for large n and some positive constants c_2 and c_3,

$$\begin{aligned}
nI(\rho_n(t), r_n) &\geq c_2 n \exp\left(-c_3\frac{(\log_2 n)^{\alpha}}{(\log n)^{\alpha-1}}\right)\frac{(\log_2 n)^d}{(\log n)^{d-d/\alpha}} \\
&\geq c_2 n^{1-\varepsilon_1}\exp\left(-c_3\left(\frac{\log_2 n}{\log n}\right)^{\alpha}\log n\right) \\
&= c_2 n^{1-\varepsilon_1 - c_3(\log_2 n/\log n)^{\alpha}} \\
&\geq c_2 n^{1-\varepsilon_1-\varepsilon_2} \\
&= c_2 n^{1-\varepsilon}.
\end{aligned} \tag{2.38}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

From (2.9), (2.38), and the fact that, for large n, $a_n < 2\log n$, we obtain

$$\begin{aligned}\int_{-a_n}^{\lambda r_n^\alpha - a_n} \mathrm{e}^{-nI(\rho_n(t),r_n)} g_n(t)\,\mathrm{d}t &\le \frac{\mathrm{e}^{-c_2 n^{1-\varepsilon}}}{(\log n)^{d/\alpha-1}} \int_{-a_n}^{\lambda r_n^\alpha - a_n} (t+a_n)^{d/\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &\le \frac{\mathrm{e}^{a_n}\mathrm{e}^{-c_2 n^{1-\varepsilon}}}{(\log n)^{d/\alpha-1}} \int_0^\infty u^{d/\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u \\ &\le \frac{cn^2\mathrm{e}^{-c_2 n^{1-\varepsilon}}}{(\log n)^{d/\alpha-1}} \\ &\to 0. \end{aligned} \tag{2.39}$$

This completes the proof of Proposition 2.1.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, and let r_n be as defined in (1.4). Then*

$$W_n'(r_n) \xrightarrow{\mathcal{D}} \mathrm{Po}(\mathrm{e}^{-\beta}/C_d),$$

where C_d is as defined in (1.6) and $\mathrm{Po}(\mathrm{e}^{-\beta}/C_d)$ is the Poisson random variable with mean $\mathrm{e}^{-\beta}/C_d$.

Proof. From Theorem 6.7 of Penrose (2003) and Proposition 2.1, the total variation distance, $\mathrm{d_{TV}}(W_n'(r_n), \mathrm{Po}(\mathrm{E}[W_n'(r_n)]))$ is bounded by a constant times $J_1(n) + J_2(n)$, where $J_1(n)$ and $J_2(n)$ are defined as

$$\begin{aligned} J_1(n) &= n^2 \int_{\mathbb{R}^d} \exp(-nI(x,r_n)) f(x)\,\mathrm{d}x \int_{B(x,3r_n)} \exp(-nI(y,r_n)) f(y)\,\mathrm{d}y, \\ J_2(n) &= n^2 \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \int_{B(x,3r_n)\setminus B(x,r_n)} \exp(-nI^{(2)}(x,y,r_n)) f(y)\,\mathrm{d}y, \end{aligned} \tag{2.40}$$

where $I^{(2)}(x,y,r) = \int_{B(x,r)\cup B(y,r)} f(z)\,\mathrm{d}z$. Theorem 2.1 follows from Proposition 2.1 if we show that $J_i(n) \to 0$ as $n \to \infty$, $i = 1, 2$. We first analyze J_1. Let $\rho_n(t)$ and $g_n(t)$ be as defined in Lemma 2.2 and (2.9), respectively. We have

$$J_1(n) = n^2 \int_{-a_n}^\infty \exp(-nI(\rho_n(t),r_n)) g_n(t)\,\mathrm{d}t \int_{B(\rho_n(t)e,3r_n)} \exp(-nI(y,r_n)) f(y)\,\mathrm{d}y.$$

Write $J_1(n) = J_{11}(n) + J_{12}(n)$, where

$$\begin{aligned} J_{11}(n) &= \int_{-a_n}^{-\log n/\log_2 n} \exp(-nI(\rho_n(t),r_n)) g_n(t)\,\mathrm{d}t \\ &\quad \times \int_{B(\rho_n(t)e,3r_n)} \exp(-nI(y,r_n)) nf(y)\,\mathrm{d}y, \\ J_{12}(n) &= \int_{-\log n/\log_2 n}^\infty \exp(-nI(\rho_n(t),r_n)) g_n(t)\,\mathrm{d}t \\ &\quad \times \int_{B(\rho_n(t)e,3r_n)} \exp(-nI(y,r_n)) nf(y)\,\mathrm{d}y. \end{aligned}$$

From (2.8) and Proposition 2.1, the inner integral in J_{11},

$$\begin{aligned}\int_{B(\rho_n(t)e,3r_n)} \exp(-nI(y,r_n))nf(y)\,\mathrm{d}y &\le \int_{-a_n}^{\infty} \exp(-nI(\rho_n(t'),r_n))g_n(t')\,\mathrm{d}t' \\ &= \mathrm{E}[W_n'(r_n)] \\ &\to \frac{\mathrm{e}^{-\beta}}{C_d}\end{aligned}$$

as $n\to\infty$. Thus, for any $\varepsilon>0$ and all large n, we have

$$J_{11}(n) \le (1+\varepsilon)\frac{\mathrm{e}^{-\beta}}{C_d}\int_{-a_n}^{-\log n/\log_2 n} \exp(-nI(\rho_n(t),r_n))g_n(t)\,\mathrm{d}t.$$

It follows from (2.35) and (2.39) that $J_{11}(n)\to 0$. Next we will show that $J_{12}(n)\to 0$ as $n\to\infty$. Define $B_n(t)=\{t'\colon \rho_n(t)-3r_n\le\rho_n(t')\le\rho_n(t)+3r_n\}$. Note that, for $t\ge -\log n/\log_2 n$, $\rho_n(t)-3r_n\ge 0$. The inner integral in $J_{12}(n)$ reduces to

$$\begin{aligned}&\int_{B(\rho_n(t)e,3r_n)} \exp(-nI(y,r_n))nf(y)\,\mathrm{d}y \\ &\quad\le \left(2\sin^{-1}\left(\frac{3r_n}{\rho_n(t)}\right)\right)^{d-1}\int_{B_n(t)}\exp(-nI(\rho_n(t'),r_n))g_n(t')\,\mathrm{d}t' \\ &\quad\le \left(2\sin^{-1}\left(\frac{3r_n}{\rho_n(t)}\right)\right)^{d-1}\int_{-a_n}^{\infty}\exp(-nI(\rho_n(t'),r_n))g_n(t')\,\mathrm{d}t' \\ &\quad\le 2^{d-1}(1+\varepsilon)\frac{\mathrm{e}^{-\beta}}{C_d}\left(\sin^{-1}\left(\frac{3r_n}{\rho_n(t)}\right)\right)^{d-1} \\ &\quad\le C\left(\frac{\log_2 n}{\log n}\right)^{d-1},\end{aligned} \tag{2.41}$$

since, for all large n and $t\in(-\log n/\log_2 n,\infty)$, we can find constants c, c', and $\varepsilon>0$ such that $0\le 3r_n/\rho_n(t)\le c\log_2 n/\log n\to 0$, and $\sin^{-1}(x)\le c'x$ for all $x\in[0,\varepsilon]$. Thus, the inner integral in J_{12} converges uniformly to 0 as $n\to\infty$. Hence, J_{12} converges to 0 from the last statement and the fact that the bounds in (2.19) and (2.25) are integrable over $[0,\infty)$ and $(-\infty,0]$, respectively.

We now show that J_2 as defined in (2.40) converges to 0. Write $J_2(n)=\sum_{k=1}^{3}J_{2k}(n)$, where

$$J_{2k}(n)=n^2\int_{\mathbb{R}^d}f(x)\,\mathrm{d}x\int_{A_k(n)}\exp(-nI^{(2)}(x,y,r_n))f(y)\,\mathrm{d}y,\qquad k=1,2,3,$$

with $A_1(n)=\{2r_n\le\|x-y\|\le 3r_n\}$, $A_2(n)=\{r_n\le\|x-y\|\le 2r_n,\ \|x\|\le\|y\|\}$, and $A_3(n)=\{r_n\le\|x-y\|\le 2r_n,\ \|y\|\le\|x\|\}$. Since on $A_1(n)$, $I^{(2)}(x,y,r_n)=I(x,r_n)+I(y,r_n)$, we obtain

$$\begin{aligned}J_{21}(n)&=n^2\int_{\mathbb{R}^d}\exp(-nI(x,r_n))f(x)\,\mathrm{d}x\int_{\{y\colon 2r_n\le\|x-y\|\le 3r_n\}}\exp(-nI(y,r_n))f(y)\,\mathrm{d}y \\ &\le n^2\int_{\mathbb{R}^d}\exp(-nI(x,r_n))f(x)\,\mathrm{d}x\int_{B(x,3r_n)}\exp(-nI(y,r_n))f(y)\,\mathrm{d}y \\ &= J_1(n),\end{aligned}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

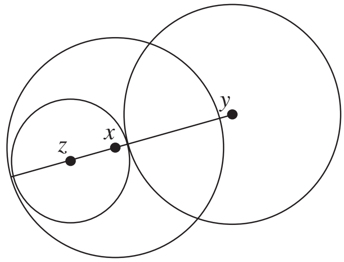

FIGURE 2.

which has already been shown to converge to 0. Next we analyze $J_{22}(n)$ as $n \to \infty$. The proof for $J_{23}(n)$ is the same and so we omit it.

Let $B(z(x, y), \rho_1)$ be the ball with center $z = z(x, y)$ (see Figure 2) and radius $\rho_1 = \rho_1(x, y) \geq r_n/2$ inscribed inside $B(x, r_n) \setminus B(y, r_n)$. Then

$$\begin{aligned} I^{(2)}(x, y, r_n) &\geq I(z(x, y), \rho_1) + I(y, r_n) \\ &\geq I(z(x, y), r_n/2) + I(y, r_n) \\ &\geq I(x, r_n/2) + I(y, r_n), \end{aligned}$$

where the last inequality follows since $\|z\| < \|x\|$. Thus,

$$\begin{aligned} J_{22}(n) &\leq n^2 \int_{\mathbb{R}^d} \exp\left(-nI\left(x, \frac{r_n}{2}\right)\right) f(x)\,\mathrm{d}x \int_{A_2(n)} \exp(-nI(y, r_n)) f(y)\,\mathrm{d}y \\ &\leq n^2 \int_{\mathbb{R}^d} \exp\left(-nI\left(x, \frac{r_n}{2}\right)\right) f(x)\,\mathrm{d}x \int_{B(x,3r_n)} \exp(-nI(y, r_n)) f(y)\,\mathrm{d}y. \end{aligned}$$

Write $J_{22}(n) = J_1^*(n) + J_2^*(n) + J_3^*(n)$, where

$$J_i^*(n) = \int_{D_i} \exp\left(-nI\left(\rho_n(t), \frac{r_n}{2}\right)\right) g_n(t)\,\mathrm{d}t \int_{B(\rho_n(t)e,3r_n)} \exp(-nI(y, r_n)) nf(y)\,\mathrm{d}y$$

for $i = 1, 2, 3$, where $D_1 = [-a_n, -\log n/\log_2 n)$, $D_2 = [-\log n/\log_2 n, 0)$, and $D_3 = [0, \infty)$. The proof of $J_i^* \to 0$ as $n \to \infty$ for $i = 1, 3$ proceeds in exactly the same manner as in the cases of J_{11} and J_{12} by replacing r_n by $r_n/2$ while estimating the outer integrals. In the case of J_2^*, we proceed exactly as in the case of J_{12} (see (2.41)) to obtain

$$J_2^*(n) \leq C\left(\frac{\log_2 n}{\log n}\right)^{d-1} \int_{D_2} \exp\left(-nI\left(\rho_n(t), \frac{r_n}{2}\right)\right) g_n(t)\,\mathrm{d}t. \tag{2.42}$$

We will show that there exists a constant C_1 such that, for large n,

$$\int_{D_2} \exp\left(-nI\left(\rho_n(t), \frac{r_n}{2}\right)\right) g_n(t)\,\mathrm{d}t \leq C_1 \frac{(\log n)^{(d-1)/2}}{(\log_2 n)^{(d-1)/4}}. \tag{2.43}$$

Substituting (2.43) into (2.42) we obtain

$$J_2^*(n) \leq C'\left(\frac{\log_2 n}{\log n}\right)^{d-1} \frac{(\log n)^{(d-1)/2}}{(\log_2 n)^{(d-1)/4}} \to 0 \quad \text{as } n \to \infty.$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

This completes the proof of Theorem 2.1 once we show (2.43). We first obtain a bound for the integrand in (2.43) as in (2.25) by replacing r_n by $r_n/2$. Using Lemma 2.1 (see also (2.22)), we can find a constant $c_0 > 0$ such that

$$nI\left(\rho_n(t), \frac{r_n}{2}\right) \geq c_0 n\left(\Gamma\left(\frac{d+1}{2}\right) + E_n\right) r_n^d \mathrm{e}^{-\lambda w_1(n)} (r_n \rho_n^{\alpha-1})^{-(d+1)/2} \times \exp(-\lambda(\rho_n^\alpha - \alpha r_n \rho_n^{\alpha-1})) \exp\left(-\frac{\lambda\alpha}{2} r_n \rho_n^{\alpha-1}\right), \tag{2.44}$$

where the functions E_n and $w_1(n)$ are as in Lemma 2.1, but with r_n replaced by $r_n/2$. It is easy to see that the conditions for the uniform convergence to 0 of E_n and $w_1(n)$ that were verified below (2.25) for r_n hold for $r_n/2$ as well. Hence, the computations leading to (2.24) can be used to estimate the right-hand side of (2.44) leaving out the last factor. This gives, for large n and some constant $c'' > 0$,

$$nI\left(\rho_n(t), \frac{r_n}{2}\right) \geq c'' \mathrm{e}^{-t} \exp\left(-\frac{\lambda\alpha}{2} r_n \rho_n(t)^{\alpha-1}\right).$$

From (2.20) we have $\exp(-\lambda\alpha r_n \rho_n(t)^{\alpha-1}/2) \geq (\log_2 n)^{(d-1)/4} (\log n)^{-(d-1)/2} \mathrm{e}^{-c_2/2}$. Therefore,

$$nI\left(\rho_n(t), \frac{r_n}{2}\right) \geq c_3 \frac{(\log_2 n)^{(d-1)/4}}{(\log n)^{(d-1)/2}} \mathrm{e}^{-t}. \tag{2.45}$$

From (2.45) and part 1 of Lemma 2.3, we obtain

$$\exp\left(-nI\left(\rho_n(t), \frac{r_n}{2}\right)\right) g_n(t) \leq M \exp\left(-c_3 \frac{(\log_2 n)^{(d-1)/4}}{(\log n)^{(d-1)/2}} \mathrm{e}^{-t}\right) \mathrm{e}^{-t}$$

for all large n. Hence, for all large n, we have

$$\int_{D_2} \exp\left(-nI\left(\rho_n(t), \frac{r_n}{2}\right)\right) g_n(t)\,\mathrm{d}t \leq M \int_{-\infty}^{0} \exp\left(-c_3 \frac{(\log_2 n)^{(d-1)/4}}{(\log n)^{(d-1)/2}} \mathrm{e}^{-t}\right) \mathrm{e}^{-t}\,\mathrm{d}t \leq \frac{M}{c_3} \frac{(\log n)^{(d-1)/2}}{(\log_2 n)^{(d-1)/4}}.$$

This proves (2.43).

Proof of Theorem 1.1. For each positive integer n, set $m_1(n) = n - n^{3/4}$ and $m_2(n) = n + n^{3/4}$. Recall that the Poisson sequence N_n is assumed to be nondecreasing. Let r_n be as in the statement of the theorem. Since $m_i(n) \sim n$, it is easy to see that Proposition 2.1 and Theorem 2.1 hold with n replaced by $m_i(n)$, that is,

$$\mathrm{E}[W'_{m_i(n)}(r_n)] \to \frac{\mathrm{e}^{-\beta}}{C_d}, \qquad W'_{m_i(n)}(r_n) \xrightarrow{\mathcal{D}} \mathrm{Po}\left(\frac{\mathrm{e}^{-\beta}}{C_d}\right), \qquad i = 1, 2.$$

Let $\mathcal{P}_n^- = \mathcal{P}_{m_1(n)}$ and $\mathcal{P}_n^+ = \mathcal{P}_{m_2(n)}$. Let A^{c} denote the complement of set A. Let $H_n = \{\mathcal{P}_n^- \subseteq \mathcal{X}_n \subseteq \mathcal{P}_n^+\}$. Let A_n be the event that there exists a point $Y \in \mathcal{P}_n^+ \setminus \mathcal{P}_n^-$ such that Y is isolated in $G(\mathcal{P}_n^- \cup \{Y\}, r_n)$. Let B_n be the event that one or more points of $\mathcal{P}_n^+ \setminus \mathcal{P}_n^-$ lie within distance r_n of a point X of $\mathcal{P}_n^-$ with degree 0 in $G(\mathcal{P}_n^-, r_n)$. Then $\{W_n(r_n) \neq W'_n(r_n)\} \subseteq A_n \cup B_n \cup H_n^{\mathrm{c}}$. Thus, the proof is complete if we show that $\mathrm{P}[A_n]$, $\mathrm{P}[B_n]$, $\mathrm{P}[H_n^{\mathrm{c}}]$ all converge to 0. We have

$$\mathrm{P}[H_n^{\mathrm{c}}] \leq \mathrm{P}[|N_{m_1(n)} - m_1(n)| \geq n^{3/4}] + \mathrm{P}[|N_{m_2(n)} - m_2(n)| \geq n^{3/4}] \to 0$$

as $n \to \infty$ by Chebyshev's inequality.

Let $Y \in \mathbb{R}^d$ be a point distributed according to the density f independent of $\mathcal{P}_n^-$. By the Palm theory we have

$$\mathrm{P}[A_n] \le 2n^{3/4}\,\mathrm{P}[Y \text{ is isolated in } G(\mathcal{P}_n^- \cup \{Y\}, r_n)] = 2n^{3/4} m_1(n)^{-1}\,\mathrm{E}[W'_{m_1(n)}(r_n)],$$

which converges to 0 as $n \to \infty$. By Boole's inequality and the Palm theory,

$$\begin{aligned}\mathrm{P}[B_n] &\le 2n^{3/4}\,\mathrm{P}[\text{there is a isolated point of } G(\mathcal{P}_n^-, r_n) \text{ in } B(Y, r_n)]\\ &\le 2n^{7/4}\int_{\mathbb{R}^d} f(y)\,\mathrm{d}y \int_{B(y,r_n)} \exp(-m_1(n)I(x, r_n)) f(x)\,\mathrm{d}x.\end{aligned}$$

By interchanging the order of integration, we obtain

$$\begin{aligned}\mathrm{P}[B_n] &\le 2n^{7/4}\int_{\mathbb{R}^d} I(x, r_n)\exp(-m_1(n)I(x, r_n)) f(x)\,\mathrm{d}x\\ &= 2n^{3/4}\int_{-a_n}^{\infty} I(\rho_n(t), r_n)\exp(-m_1(n)I(\rho_n(t), r_n)) g_n(t)\,\mathrm{d}t. \qquad (2.46)\end{aligned}$$

From (2.9) and (2.10), we obtain

$$2n^{3/4} I(\rho_n(t), r_n)\exp(-m_1(n)I(\rho_n(t), r_n)) g_n(t) \to 0.$$

Thus, the integrand in (2.46) converges pointwise to 0 as $n \to \infty$. To complete the proof, we need to find integrable bounds for the left-hand side of the above equation. Let $0 < \varepsilon < 1$ be fixed. For large n, we have $m_1(n) \ge (1-\varepsilon)n$. Hence, for large n,

$$\begin{aligned}&n^{3/4} I(\rho_n(t), r_n)\exp(-m_1(n)I(\rho_n(t), r_n)) g_n(t)\\ &\qquad \le n^{3/4} I(\rho_n(t), r_n)\exp(-(1-\varepsilon)nI(\rho_n(t), r_n)) g_n(t). \qquad (2.47)\end{aligned}$$

Thus, it suffices to find integrable bounds for the right-hand side expression in (2.47). The procedure for doing this is the same as in the proof of Proposition 2.1, where we obtained integrable bounds for $\exp(-nI(\rho_n(t), r_n)) g_n(t)$ by dividing the range of the integral $[-a_n, \infty)$ into four parts, and in each part finding a lower bound for $nI(\rho_n(t), r_n)$ of the form $c_i h_i(n, t)$ and upper bounds for $g_n(t)$, $i = 1, \dots, 4$. We can use the same bounds for the factor $\exp(-(1-\varepsilon)nI(\rho_n(t), r_n)) g_n(t)$ by replacing the constants c_i by $(1-\varepsilon)c_i$, $i = 1, \dots, 4$ (see below). Thus, for each of the four domains analyzed in Proposition 2.1, we need to find an upper bound for $n^{3/4} I(\rho_n(t), r_n)$, and then verify that the product of this bound and the one obtained for $\exp(-(1-\varepsilon)nI(\rho_n(t), r_n)) g_n(t)$ is integrable.

Part 1. Let $t \ge 0$. From (2.15) we obtain, for large n,

$$n^{3/4} I(\rho_n(t), r_n) \le n^{3/4} I(\rho_n(0), r_n) \le (1+\varepsilon) C_d \mathrm{e}^{\beta} n^{-1/4}.$$

Hence, for large n,

$$n^{3/4} I(\rho_n(t), r_n)\exp(-(1-\varepsilon)nI(\rho_n(t), r_n)) g_n(t) \le (1+\varepsilon) C_d \mathrm{e}^{\beta} n^{-1/4} g_n(t),$$

which is integrable over $[0, \infty)$ by (2.18) for each n, and converges to 0 as $n \to \infty$.

Part 2. Next suppose that $-\log n/\log_2 n \le t \le 0$. Using Lemma 2.1 and proceeding as in (2.22) and (2.23), with $w_1(n)$ replaced by $w_2(n)$ and E_n replaced by 0, we obtain, for large n and some constant c'',

$$n^{3/4} I(\rho_n(t), r_n) \le n^{-1/4} c'' \mathrm{e}^{-t}.$$

This together with (2.24) and part 1 of Lemma 2.3 yields

$$n^{3/4} I(\rho_n(t), r_n) \exp(-(1-\varepsilon) n I(\rho_n(t), r_n)) g_n(t) \le c_1 n^{-1/4} \exp(-(1-\varepsilon) c' \mathrm{e}^{-t}) \mathrm{e}^{-2t}$$

for large n and some constant c_1. This bound is integrable in t over the interval $(-\infty, 0)$, and converges to 0 as $n \to \infty$.

Part 3. Next, consider the range $\lambda r_n^{\alpha} - a_n \le t \le -\log n / \log_2 n$. For large n, we have

$$I(\rho_n(t), r_n) \le \int_{B(0, r_n)} A_d \mathrm{e}^{-\lambda \|x\|^{\alpha}} \,\mathrm{d}x \le A_d r_n^d \le c_2 \frac{(\log_2 n)^d}{(\log n)^{d - d/\alpha}}, \tag{2.48}$$

where the last inequality in (2.48) follows from (2.37). Note that the above bound is independent of t. Hence, for large n,

$$\int_{\lambda r_n^{\alpha} - a_n}^{-\log n / \log_2 n} n^{3/4} I(\rho_n(t), r_n) \exp(-(1-\varepsilon) n I(\rho_n(t), r_n)) g_n(t) \,\mathrm{d}t$$

will be bounded by $n^{3/4}$ times the last expression in (2.48) times the bound obtained in (2.30), with the constant c^* (see (2.29)) replaced by $(1-\varepsilon)c^*$. The bounds obtained in (2.30) are in turn bounded above in (2.33) or (2.34), depending on whether $d/\alpha \ge 1$ or $d/\alpha \le 1$. If $d/\alpha \ge 1$ then, for large n, the product of the bounds in (2.33) and (2.48) will be less than a constant times

$$n^{3/4} (\log n)^{d/\alpha - 1} \exp(-\mathrm{e}^{\log n / 2 \log_2 n}) \le n^{3/4} (\log n)^{d/\alpha - 1} \exp(-\mathrm{e}^{\log_2 n}) = n^{-1/4} (\log n)^{d/\alpha - 1},$$

which converges to 0 as $n \to \infty$. The same reasoning applies to the case $d/\alpha < 1$ by using (2.34) instead of (2.33).

Part 4. Finally, consider the case $-a_n \le t \le \lambda r_n^{\alpha} - a_n$. Using (2.39) and (2.48) with c_2 replaced by $(1-\varepsilon)c_2$, we obtain, for large n and some constant c,

$$\begin{aligned} &\int_{-a_n}^{\lambda r_n^{\alpha} - a_n} n^{3/4} I(\rho_n(t), r_n) \exp(-(1-\varepsilon) n I(\rho_n(t), r_n)) g_n(t) \,\mathrm{d}t \\ &\qquad \le \frac{c n^3 (\log_2 n)^d \exp(-(1-\varepsilon) c_2 n^{1-\varepsilon})}{(\log n)^{d-1}}, \end{aligned}$$

which converges to 0 as $n \to \infty$.

Proof of Theorem 1.2. Define a sequence $\{r_n\}_{n \ge 1}$ by

$$\lambda\alpha(\lambda^{-1} \log n)^{1 - 1/\alpha} r_n - (d-1) \log_2 n + \frac{d-1}{2} \log_3 n = \beta.$$

Then by Theorem 1.1 we have

$$\lim_{n \to \infty} \mathrm{P}[d_n \le r_n] = \lim_{n \to \infty} \mathrm{P}[W_n(r_n) = 0] = \exp\biggl(-\frac{\mathrm{e}^{-\beta}}{C_d}\biggr).$$

The result now follows by taking $\beta = \gamma - \log(C_d)$.

3. Proof of Theorem 1.3

In order to prove strong laws for the LNND for graphs with densities having compact support, we cover the support of the density using an appropriate collection of concentric balls and then show summability of certain events involving the distribution of the points of $\mathcal{X}_n$ on these balls. The results then follow by an application of the Borel–Cantelli lemma. In the case of densities having unbounded support, the region to be covered changes with n and must be determined first. The following lemma gives us the regions of interest when the points in $\mathcal{X}_n = \{X_1, X_2, \dots, X_n\}$, $n \geq 1$, are distributed according to the probability density function f given by (1.1).

For any set A, let A^{c} denote its complement. For any two real sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, $a_n \gtrsim b_n$ means that $a_n \geq c_n$, $n \geq 1$, for some sequence $\{c_n\}_{n\geq 1}$ with $c_n \sim b_n$.

For any $c \in \mathbb{R}$ and large enough n, define

$$R_n(c) = \left(\frac{1}{\lambda}\left(\log n + \frac{c+d-\alpha}{\alpha}\log_2 n\right)\right)^{1/\alpha}. \tag{3.1}$$

Define the events $U_n(c) = \{\mathcal{X}_n \subset B(0, R_n(c))\}$, and, for any $c < 0$, let $V_n(c) = \{\mathcal{X}_n \cap (B(0, R_n(0)) \setminus B(0, R_n(c))) \neq \varnothing\}$ for all large enough n for which $R_n(c) > 0$, and arbitrarily otherwise.

Lemma 3.1. *Let the events $U_n(c)$ and $V_n(c)$, $n \geq 1$, be as defined above. Then*

1. $\mathrm{P}[U_n^{\mathrm{c}}(c)$ *infinitely often* $] = 0$ *for any $c > \alpha$, and*
2. $\mathrm{P}[V_n^{\mathrm{c}}(c)$ *infinitely often* $] = 0$ *for any $c < 0$.*

The above results are also true with $\mathcal{X}_n$ replaced by $\mathcal{P}_{\lambda_n}$ provided that $\lambda_n \sim n$.

Thus, for almost all realizations of the sequence $\{\mathcal{X}_n\}_{n\geq 1}$, all points of $\mathcal{X}_n$ will lie within the ball $B(0, R_n(c))$ for any $c > \alpha$ eventually, and, for any $c < 0$, there will be at least one point of $\mathcal{X}_n$ in $B(0, R_n(0)) \setminus B(0, R_n(c))$ eventually.

Proof of Lemma 3.1. From (1.3) we obtain

$$\int_{\tilde{R}}^{\infty} f_R(r)\,\mathrm{d}r \sim A'_d(\lambda\alpha)^{-1}\tilde{R}^{d-\alpha}\mathrm{e}^{-\lambda\tilde{R}^{\alpha}} \quad \text{as } \tilde{R} \to \infty. \tag{3.2}$$

Fix an integer $a > 1$, and define the subsequence $\{n_k\}_{k\geq 1}$ by $n_k = a^k$. For large k, we have

$$\begin{aligned}
\mathrm{P}\left[\bigcup_{n=n_k}^{n_{k+1}} U_n^{\mathrm{c}}(c)\right] &\leq \mathrm{P}[\text{at least one vertex of } \mathcal{X}_{n_{k+1}} \text{ is in } B^{\mathrm{c}}(0, R_{n_k}(c))] \\
&\leq n_{k+1}\int_{R_{n_k}(c)}^{\infty} f_R(r)\,\mathrm{d}r \\
&\sim A'_d(\lambda\alpha)^{-1} n_{k+1} R_{n_k}^{d-\alpha}(c)\mathrm{e}^{-\lambda R_{n_k}^{\alpha}(c)} \\
&\leq \frac{C}{k^{c/\alpha}}.
\end{aligned}$$

Thus, the above probability is summable for $c > \alpha$, and the first part of Lemma 3.1 follows from the Borel–Cantelli lemma.

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

Next, suppose that $c < 0$ and let $\{n_k\}_{k\geq 1}$ be as above. Note that, for all large k, $R_{n_{k+1}}(c) < R_{n_k}(0)$. From (3.1) we obtain, for all large n,

$$\frac{1}{2\lambda^{(d-\alpha)/\alpha}}\frac{1}{n(\log n)^{b/\alpha}} \leq R_n^{d-\alpha}(b)\mathrm{e}^{-\lambda R_n^\alpha(b)} \leq \frac{2}{\lambda^{(d-\alpha)/\alpha}}\frac{1}{n(\log n)^{b/\alpha}}, \qquad b = c, 0.$$

Using the inequality $1 - x \leq \mathrm{e}^{-x}$, (3.2), and the above inequalities, we obtain, for large k,

$$\begin{aligned}
\mathrm{P}\Bigg[\bigcup_{n=n_k}^{n_{k+1}} V_n^{\mathrm{c}}(c)\Bigg] &\leq \mathrm{P}[\mathcal{X}_{n_k} \cap (B(0, R_{n_k}(0)) \setminus B(0, R_{n_{k+1}}(c))) = \varnothing] \\
&\leq \exp\bigg(-n_k \int_{R_{n_{k+1}}(c)}^{R_{n_k}(0)} A'_d \mathrm{e}^{-\lambda r^\alpha} r^{d-1}\,\mathrm{d}r\bigg) \\
&\leq \exp\big(-n_k A'_d(\lambda\alpha)^{-1}\big(\tfrac{1}{2}R_{n_{k+1}}^{d-\alpha}(c)\mathrm{e}^{-\lambda R_{n_{k+1}}^\alpha(c)} - 2R_{n_k}^{d-\alpha}(0)\mathrm{e}^{-\lambda R_{n_k}^\alpha(0)}\big)\big) \\
&= \exp\bigg(-\frac{A'_d(\lambda\alpha)^{-1}}{\lambda^{(d-\alpha)/\alpha}}\bigg(\frac{1}{4a(\log a)^{c/\alpha}}(k+1)^{-c/\alpha} - 4\bigg)\bigg),
\end{aligned}$$

which is summable for all $c < 0$. The second part of Lemma 3.1 now follows from the Borel–Cantelli lemma. If $\mathcal{X}_n$ is replaced by $\mathcal{P}_{\lambda_n}$, where $\lambda_n \sim n$, then

$$\begin{aligned}
\mathrm{P}[U_n^{\mathrm{c}}(c)] &= 1 - \exp(-\lambda_n(1 - I(0, R_n(c)))) \\
&\lesssim \lambda_n A'_d(\lambda\alpha)^{-1} R_n^{d-\alpha}(c)\exp(-\lambda R_n^\alpha(c)) \\
&\sim n A'_d(\lambda\alpha)^{-1} R_n^{d-\alpha}(c)\exp(-\lambda R_n^\alpha(c)),
\end{aligned}$$

which is the same as the asymptotic behavior of $\mathrm{P}[U_n^{\mathrm{c}}(c)]$ in case of $\mathcal{X}_n$. Similarly, we can show that $\mathrm{P}[V_n^{\mathrm{c}}(c)]$ has the same asymptotic behavior as in the case of $\mathcal{X}_n$. Thus, the results stated for $\mathcal{X}_n$ also hold for $\mathcal{P}_{\lambda_n}$.

Proposition 3.1. *For any $t > d/\alpha\lambda$, let $r_n(t) = t(\lambda^{-1}\log n)^{1/\alpha-1}\log_2 n$. Then, with probability 1, $d_n \leq r_n(t)$ for all large enough n.*

Proof. Fix $t > d/\alpha\lambda$, and choose $u \in (d/\alpha\lambda, t)$. Pick $c > \alpha$ and ε satisfying $0 < \varepsilon < u/(2+u)$, $\varepsilon + u < t$, and

$$\frac{c + \alpha(d-1)}{\alpha^2\lambda} = \frac{d}{\alpha\lambda} + \frac{c-\alpha}{\alpha^2\lambda} < (1-\varepsilon)u < t. \tag{3.3}$$

From Lemma 3.1, a.s., $\mathcal{X}_n \subset B(0, R_n(c))$ for all large enough n, where $R_n(c)$ is as defined in (3.1). For $m = 1, 2, \ldots$, let $\nu(m) = a^m$ for some integer $a > 1$. Define the sequence of functions $\{\tilde{r}_m(v)\}_{m\geq 1}$ by

$$\tilde{r}_m(v) = \begin{cases} r_{\nu(m+1)}(v) & \text{if } \alpha > 1, \\ r_{\nu(m)}(v) & \text{if } \alpha \leq 1. \end{cases}$$

This is required since $r_n(u)$ is decreasing in n if $\alpha > 1$, and increasing if $\alpha \leq 1$. Let κ_m (the covering number) be the minimum number of balls of radii $\tilde{r}_m(\varepsilon)$ required to cover the ball $B(0, R_{\nu(m+1)}(c))$. Since

$$\frac{r_{\nu(m+1)}(\varepsilon)}{r_{\nu(m)}(\varepsilon)} \to 1 \quad \text{as } m \to \infty, \tag{3.4}$$

we can find constants C, C_1, and C_2 such that, for all sufficiently large m,

$$\begin{aligned}\kappa_m &\le C\frac{R_{\nu(m+1)}(c)^d}{\tilde{r}_m(\varepsilon)^d}\\ &\le C_1\frac{R_{\nu(m+1)}(c)^d}{r_{\nu(m+1)}(\varepsilon)^d}\\ &= \frac{C_1}{\lambda^d\varepsilon^d}\frac{(\log(\nu(m+1))+\log_2(\nu(m+1))(c+d-\alpha)/\alpha)^{d/\alpha}}{(\log(\nu(m+1)))^{d/\alpha-d}(\log_2(\nu(m+1)))^d}\\ &\le C_2\bigg(\frac{m+1}{\log(m+1)}\bigg)^d. \end{aligned}\tag{3.5}$$

Consider a deterministic set $\{x_1^m,\dots,x_{\kappa_m}^m\}\subset B(0,R_{\nu(m+1)}(c))$, such that

$$B(0,R_{\nu(m+1)}(c))\subset\bigcup_{i=1}^{\kappa_m}B(x_i^m,\tilde{r}_m(\varepsilon)).$$

Given $x\in\mathbb{R}^d$, define $A_m(x)$ to be the annulus $B(x,\tilde{r}_m(u))\setminus B(x,\tilde{r}_m(\varepsilon))$, and let $F_m(x)$ be the event such that no vertex of $\mathcal{X}_{\nu(m)}$ lies in $A_m(x)$, i.e.

$$F_m(x)=\{\mathcal{X}_{\nu(m)}[A_m(x)]=0\},\tag{3.6}$$

where $\mathcal{X}[B]$ denotes the number of points of the finite set $\mathcal{X}$ that lie in B. For any $x\in B(0,R_{\nu(m+1)}(c))$, we have, by the radial symmetry of f,

$$\mathrm{P}[X_i\in A_m(x)]=\int_{A_m(x)}f(y)\,\mathrm{d}y=\int_{A_m(|x|e)}f(y)\,\mathrm{d}y,\tag{3.7}$$

where $e=(1,0,\dots,0)\in\mathbb{R}^d$. The aim is to find a lower bound for the above probability. Note that

$$\frac{r_n(u)}{R_n(c)}=\frac{u(\lambda^{-1}\log n)^{1/\alpha-1}\log_2 n}{(\lambda^{-1}(\log n+\log_2 n(c+d-\alpha)/\alpha))^{1/\alpha}}\to 0\tag{3.8}$$

as $n\to\infty$. Thus, in (3.7) we integrate the density f over a relatively small annulus $A_m(x)$ centered at x which lies in a ball $B(0,R_{\nu(m+1)}(c))$ of relatively larger radius. Since f is radially symmetric and decreasing, it should be possible to obtain a lower bound for $x=R_{\nu(m+1)}(c)e$. To show this, first consider the case in which $|x|>\tilde{r}_m(u)$. Then, $y=(y_1,\dots,y_d)\in A_m(|x|e)$ implies that $y_1>0$. Combining this with the fact that $|x|\le R_{\nu(m+1)}(c)$, we have

$$|y+(R_{\nu(m+1)}(c)-|x|)e|=|(y_1+(R_{\nu(m+1)}(c)-|x|),y_2,\dots,y_d)|\ge|y|.\tag{3.9}$$

On the other hand, if $|x|\le\tilde{r}_m(u)$ then, for any $y\in A_m(x)$, $|y|\le 2\tilde{r}_m(u)$. Hence, for all large m, we obtain

$$|y+(R_{\nu(m+1)}(c)-|x|)e|\ge R_{\nu(m+1)}(c)-\tilde{r}_m(u)-|y|\ge|y|+(R_{\nu(m+1)}(c)-5\tilde{r}_m(u))\ge|y|,\tag{3.10}$$

where the last inequality follows using (3.8). From (3.7), (3.9), (3.10), and the fact that the density f is radially symmetric and decreasing, we obtain, for all large m,

$$\begin{aligned}\mathrm{P}[X_i \in A_m(x)] &\geq \int_{A_m(|x|e)} f((R_{\nu(m+1)}(c) - |x|)e + y)\,\mathrm{d}y \\ &= \int_{A_m(R_{\nu(m+1)}(c)e)} f(y)\,\mathrm{d}y \\ &= I(R_{\nu(m+1)}(c), \tilde{r}_m(u)) - I(R_{\nu(m+1)}(c), \tilde{r}_m(\varepsilon)). \end{aligned} \tag{3.11}$$

To apply Lemma 2.1 with $\rho_n = R_n(c)$ and $r_n = r_n(v)$, $v = \varepsilon, u$, we first check the four conditions of the lemma. As $n \to \infty$, by definition, $R_n^\alpha(c) \to \infty$, and

$$\frac{r_n(v)}{R_n(c)} = \lambda^{1-1/\alpha} v \log_2 n \Big/ \log n \bigg(\frac{1}{\lambda}\bigg(1 + \frac{c+d-\alpha}{\alpha}\frac{\log_2 n}{\log n}\bigg)\bigg)^{1/\alpha} \to 0,$$

$$r_n^2(v) R_n^{\alpha-2}(c) = \frac{v^2 \lambda^{2-2/\alpha}(\log_2 n)^2}{\log n}\bigg(\frac{1}{\lambda}\bigg(1 + \frac{c+d-\alpha}{\alpha}\frac{\log_2 n}{\log n}\bigg)\bigg)^{1-2/\alpha} \to 0,$$

$$r_n(v) R_n^{\alpha-1}(c) = v\lambda^{1-1/\alpha}\log_2 n\bigg(\frac{1}{\lambda}\bigg(1 + \frac{c+d-\alpha}{\alpha}\frac{\log_2 n}{\log n}\bigg)\bigg)^{1-1/\alpha} \to \infty.$$

The above limits can be easily seen to hold if we take R_n and r_n to be $R_{\nu(m+1)}$ and $r_{\nu(m)}$, respectively, by using (3.4). Hence, by Lemma 2.1 (noting from the last line of the lemma that $w_i(n)$, $i = 1, 2$, and E_n converge to 0), we can find positive constants c_1 and c_2 (depending on u) such that, for large m,

$$c_1 g(R_{\nu(m+1)}(c), \tilde{r}_m(u)) \leq (\lambda\alpha)^{-(d+1)/2} I(R_{\nu(m+1)}(c), \tilde{r}_m(u)) \leq c_2 g(R_{\nu(m+1)}(c), \tilde{r}_m(u)). \tag{3.12}$$

Substituting (3.12) and (2.4) into (3.11), we have, for large m,

$$\begin{aligned}&\mathrm{P}[X_i \in A_m(x)] \\ &\quad\geq \exp(-\lambda R_{\nu(m+1)}^\alpha(c))(R_{\nu(m+1)}^{\alpha-1}(c))^{-(d+1)/2} \\ &\qquad\times (c_1(u)\exp(\lambda\alpha\tilde{r}_m(u)R_{\nu(m+1)}^{\alpha-1}(c))(\tilde{r}_m(u))^{(d-1)/2} \\ &\qquad\quad - c_2(\varepsilon)\exp(\lambda\alpha\tilde{r}_m(\varepsilon)R_{\nu(m+1)}^{\alpha-1}(c))(\tilde{r}_m(\varepsilon))^{(d-1)/2}) \\ &\quad= q_m. \end{aligned} \tag{3.13}$$

We now compute a lower bound for q_m. For large n, we can find a constant C_3 such that

$$\begin{aligned}&\exp(-\lambda R_n^\alpha(c))(R_n^{\alpha-1}(c))^{-(d+1)/2} \\ &\quad= \frac{1}{n(\log(n))^{(c+d-\alpha)/\alpha}}\bigg(\frac{1}{\lambda}\bigg(\log n + \frac{c+d-\alpha}{\alpha}\log_2 n\bigg)\bigg)^{-(1-1/\alpha)(d+1)/2} \\ &\quad\geq C_3 \frac{(\log n)^{-(1-1/\alpha)(d+1)/2}}{n(\log n)^{(c+d-\alpha)/\alpha}}. \end{aligned} \tag{3.14}$$

Furthermore,

$$\lambda\alpha r_n(u) R_n^{\alpha-1}(c) = \lambda\alpha u \log_2 n\bigg(1 + \frac{\log_2 n(c+d-\alpha)/\alpha}{\log n}\bigg)^{1-1/\alpha}. \tag{3.15}$$

From the above equation (and using (3.4) for the $\alpha \le 1$ case), we obtain, for large m,

$$(1-\varepsilon)\lambda\alpha u \log_2 \nu(m+1) \le \lambda\alpha\tilde{r}_m(u) R^{\alpha-1}_{\nu(m+1)}(c) \le 2\lambda\alpha u \log_2 \nu(m+1). \tag{3.16}$$

By the above inequality and (3.4), we can find a constant $C_4 = C_4(u)$ such that, for large m,

$$\exp(\lambda\alpha\tilde{r}_m(u) R^{\alpha-1}_{\nu(m+1)}(c))(\tilde{r}_m(u))^{(d-1)/2} \tag{3.17}$$

$$\ge C_4 \frac{\exp((1-\varepsilon)\lambda\alpha u \log_2(\nu(m+1)))(\log_2 \nu(m+1))^{(d-1)/2}}{(\log \nu(m+1))^{(1-1/\alpha)(d-1)/2}}. \tag{3.18}$$

From (3.16), with u replaced by ε, for some constant $C_5 = C_5(\varepsilon)$ and large m, we obtain

$$\exp(\lambda\alpha\tilde{r}_m(\varepsilon) R^{\alpha-1}_{\nu(m+1)}(c))(\tilde{r}_m(\varepsilon))^{(d-1)/2}$$

$$\le C_5 \frac{\exp(2\lambda\alpha\varepsilon \log_2(\nu(m+1)))(\log_2(\nu(m+1)))^{(d-1)/2}}{(\log(\nu(m+1)))^{(1-1/\alpha)(d-1)/2}}. \tag{3.19}$$

Substituting (3.14), (3.18), and (3.19) into (3.13), we obtain, for large m,

$$\begin{aligned} q_m \ge{} & C_3 \frac{(\log(\nu(m+1)))^{-(1-1/\alpha)(d+1)/2}}{\nu(m+1)(\log(\nu(m+1)))^{(c+d-\alpha)/\alpha}} \\ & \times \frac{(\log_2(\nu(m+1)))^{(d-1)/2}\exp((1-\varepsilon)\lambda\alpha u \log_2(\nu(m+1)))}{(\log(\nu(m+1)))^{(1-1/\alpha)(d-1)/2}} \\ & \times (c_1C_4 - c_2C_5\exp(\lambda\alpha(2\varepsilon-(1-\varepsilon)u)\log_2(\nu(m+1)))). \end{aligned}$$

Since $\varepsilon < u/(2+u)$, $2\varepsilon - (1-\varepsilon)u < 0$, and, hence, for large m, the term on the last line above is bounded below by $c_1C_4/2$. Hence, for large m, we have

$$\begin{aligned} q_m \ge{} & \frac{C_3c_1C_4}{2} \frac{(\log(\nu(m+1)))^{-(1-1/\alpha)(d+1)/2}}{\nu(m+1)(\log(\nu(m+1)))^{(c+d-\alpha)/\alpha}} \\ & \times \frac{(\log_2(\nu(m+1)))^{(d-1)/2}\exp((1-\varepsilon)\lambda\alpha u \log_2(\nu(m+1)))}{(\log(\nu(m+1)))^{(1-1/\alpha)(d-1)/2}} \\ \ge{} & C \frac{(\log(m+1))^{(d-1)/2}}{a^m(m+1)^{c/\alpha+d-(1-\varepsilon)\lambda\alpha u-1}} \end{aligned} \tag{3.20}$$

for some constant C. Hence, for large m, from (3.6), (3.13), (3.20), and the inequality $1-x \le \mathrm{e}^{-x}$, we obtain, for any $x \in B(0, R_{\nu(m+1)}(c))$,

$$\begin{aligned} \mathrm{P}[F_m(x)] &= (1-\mathrm{P}[X_1 \in A_m(x)])^{\nu(m)} \\ &\le (1-q_m)^{\nu(m)} \\ &\le \exp(-\nu(m)q_m) \\ &\le \exp\biggl(-C\frac{(\log(m+1))^{(d-1)/2}}{(m+1)^{c/\alpha+d-(1-\varepsilon)\lambda\alpha u-1}}\biggr). \end{aligned} \tag{3.21}$$

Set $G_m = \bigcup_{i=1}^{\kappa_m} F_m(x_i^m)$. From (3.5) and (3.21), we have, for large m,

$$\begin{aligned} \mathrm{P}[G_m] &\le \sum_{i=1}^{\kappa_m} \mathrm{P}[F_m(x_i^m)] \\ &\le C_2\biggl(\frac{m+1}{\log(m+1)}\biggr)^d \exp\biggl(-C\frac{(\log(m+1))^{(d-1)/2}}{(m+1)^{\lambda\alpha((c+\alpha(d-1))/\lambda\alpha^2-(1-\varepsilon)u)}}\biggr), \end{aligned}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

which is summable in m by (3.3). By the Borel–Cantelli lemma, a.s., G_m occurs only for finitely many m. Choose $n > a$, and take m such that $a^m \leq n \leq a^{m+1}$. If $d_n > r_n(t)$ then there exists an $X \in \mathcal{X}_n$ such that $\mathcal{X}_n[B(X, r_n(t)) \setminus \{X\}] = 0$. By Lemma 3.1, X will a.s. be in $B(0, R_{\nu(m+1)}(c))$ for all large enough m. Hence, a.s., if n is large enough then there is some $i \leq \kappa_m$ such that $X \in B(x_i^m, \tilde{r}_m(\varepsilon))$. Since

$$\tilde{r}_m(\varepsilon) + \tilde{r}_m(u) \leq \tilde{r}_m(t) \leq r_n(t),$$

it follows that $F_m(x_i)$ and, hence, G_m occur. Since, a.s., G_m occurs only finitely often, it follows that $d_n \leq r_n(t)$ a.s. for all large n. This completes the proof of Proposition 3.1.

Proposition 3.2. *For any $t \in (0, (d-1)/\alpha\lambda)$, let $r_n(t) = t \log_2 n(\lambda^{-1} \log n)^{1/\alpha - 1}$. Then, with probability* 1*, $d_n \geq r_n(t)$ eventually.*

We prove Proposition 3.2 via the Poissonization technique, which uses the following lemma (see Lemma 1.4 of Penrose (2003)).

Lemma 3.2. *Let N be a Poisson random variable with mean λ. Then there exist constants c and λ_1 such that, for all $\lambda > \lambda_1$,*

$$\max\big\{\mathrm{P}\big[N > \lambda + \tfrac{1}{2}\lambda^{3/4}\big], \mathrm{P}\big[N < \lambda - \tfrac{1}{2}\lambda^{3/4}\big]\big\} \leq c \exp\big(-\tfrac{1}{9}\sqrt{\lambda}\big),$$

Proof of Proposition 3.2. Enlarging the probability space, assume that there exists nondecreasing sequences of Poisson variables $\{N(n)\}_{n\geq 1}$ and $\{M(n)\}_{n \geq 1}$ with $\mathrm{E}[N(n)] = n - n^{3/4}$ and $\mathrm{E}[M(n)] = 2n^{3/4}$, independent of each other and of the sequence $\{X_1, X_2, \dots\}$. Define the point processes

$$\mathcal{P}_n^- = \{X_1, X_2, \dots, X_{N(n)}\}, \qquad \mathcal{P}_n^+ = \{X_1, X_2, \dots, X_{N(n)+M(n)}\}.$$

Then, $\mathcal{P}_n^-$ and $\mathcal{P}_n^+$ are Poisson point processes on $\mathbb{R}^d$ with intensity functions $(n - n^{3/4}) f(\cdot)$ and $(n + n^{3/4}) f(\cdot)$, respectively. The point processes $\mathcal{P}_n^-$ and $\mathcal{P}_n^+$ are coupled in such a way that $\mathcal{P}_n^- \subseteq \mathcal{P}_n^+$. Furthermore, if $H_n = \{\mathcal{P}_n^- \subseteq \mathcal{X}_n \subseteq \mathcal{P}_n^+\}$ then, by the Borel–Cantelli lemma and Lemma 3.2, $\mathrm{P}[H_n^{\mathrm{c}}$ infinitely often$] = 0$. Hence, a.s., the event H_n happens eventually.

Fix $t \in (0, (d-1)/\alpha\lambda)$. Choose u and c such that $c < 0$, $t < u < (d-1)/\alpha\lambda$, and $u < (c + \alpha(d-1))/\alpha^2\lambda$. Pick $\varepsilon > 0$ small enough such that $(1+\varepsilon)u < (c + \alpha(d-1))/\alpha^2\lambda$ and $\varepsilon + t < u$. Fix an integer $a > 1$, and let $\nu(m) = a^m$, $m = 1, 2, \dots$. Define the annulus

$$A_m(c) = B(0, R_{\nu(m)}(0)) \setminus B(0, R_{\nu(m)}(c)),$$

where $R_n(c)$ is as defined in (3.1) (note that $R_n(c) < R_n(0)$ since $c < 0$). Define the sequence of functions $\{\hat{r}_m(v)\}_{m\geq 1}$ by

$$\hat{r}_m(v) = \begin{cases} r_{\nu(m)}(v) & \text{if } \alpha > 1, \\ r_{\nu(m+1)}(v) & \text{if } \alpha \leq 1. \end{cases} \tag{3.22}$$

For each m, choose a nonrandom set $\{x_1^m, x_2^m, \dots, x_{\sigma_m}^m\} \subset A_m(c)$, such that the balls

$$B(x_i^n, \hat{r}_m(u)), \qquad 1 \leq i \leq \sigma_m,$$

are disjoint. The packing number σ_m is the maximum number of disjoint balls $B(x, \hat{r}_m(u))$ with $x \in A_m(c)$. Using (3.4), we can find constants c_0 and c_1 such that, for all large m, we

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

have

$$\begin{aligned}\sigma_m &\geq c_0 \frac{R^d_{\nu(m)}(0) - R_{\nu(m)}{}^d(c)}{\hat{r}^d_m(u)} \\ &\geq c_1 \frac{R^d_{\nu(m)}(0) - R_{\nu(m)}{}^d(c)}{r^d_{\nu(m)}(u)} \\ &= \frac{c_1}{(\lambda u)^d}\bigg(\frac{\log \nu(m)}{\log_2 \nu(m)}\bigg)^d \bigg(\bigg(1 + \frac{d-\alpha}{\alpha}\frac{\log_2 \nu(m)}{\log \nu(m)}\bigg)^{d/\alpha} \\ &\qquad - \bigg(1 + \frac{c+d-\alpha}{\alpha}\frac{\log_2 \nu(m)}{\log \nu(m)}\bigg)^{d/\alpha}\bigg).\end{aligned}$$

The function $g(x) = (1 + x(d-\alpha)/\alpha)^{d/\alpha} - (1 + x(c+d-\alpha)/\alpha)^{d/\alpha}$, $x \geq 0$, can satisfy $g(0) = 0$ and $g'(0) = -cd/\alpha^2 > 0$, since $c < 0$. Hence, for all sufficiently small $x > 0$, we have $g(x) \geq \delta x$ for some constant $\delta > 0$. Using this inequality in the above lower bound for σ_m, we obtain

$$\sigma_m \geq c_2 \bigg(\frac{\log \nu(m)}{\log_2 \nu(m)}\bigg)^{d-1} \tag{3.23}$$

for large m and some constant c_2. By part 2 of Lemma 3.1, there will a.s. be points of $\mathcal{P}^-_{\nu(m)}$ in A_m for all large enough m. Consider the sequence of sets $(\bigcup_{i=1}^{\sigma_m} E_{m,i})^{\mathrm{c}}$, where

$$E_{m,i} = \{\mathcal{P}^-_{\nu(m)}[B(x_i^{\nu(m)}, \hat{r}_m(\varepsilon))] = 1\} \cap \{\mathcal{P}^+_{\nu(m+1)}[B(x_i^{\nu(m)}, \hat{r}_m(u))] = 1\}$$

for $i = 1, 2, \dots, \sigma_m$, $m = 1, 2, \dots$. From an earlier argument $\mathrm{P}[H_n^{\mathrm{c}}]$ is summable and, hence, with probability 1, H_n happens eventually. For any $n > a$, let m be such that $a^m \leq n \leq a^{m+1}$. Recall that $\{N(n)\}_{n\geq 1}$ and $\{M(n)\}_{n\geq 1}$ are nondecreasing. Hence, if H_n and $E_{m,i}$ happen, then there is a point $X \in \mathcal{P}^-_{\nu(m)} \subset \mathcal{P}^-_n \subset \mathcal{X}_n$ such that $X \in B(x_i^{\nu(m)}, \hat{r}_m(\varepsilon))$ with no other point of $\mathcal{P}^+_{\nu(m+1)} \supset \mathcal{P}^+_n$ (and, hence, of $\mathcal{X}_n$) in $B(x_i^{\nu(m)}, \hat{r}_m(u))$. This would imply that $d_n \geq \hat{r}_m(u) - \hat{r}_m(\varepsilon) \geq \hat{r}_m(t) \geq r_n(t)$. Thus, by the Borel–Cantelli lemma, the proof of Proposition 3.2 is complete if we show that

$$\sum_{m=1}^{\infty} \mathrm{P}\bigg[\bigg(\bigcup_{i=1}^{\sigma_m} E_{m,i}\bigg)^{\mathrm{c}}\bigg] < \infty. \tag{3.24}$$

To this end, we first estimate $\mathrm{P}[E_{m,i}]$. For $i = 1, 2, \dots, \sigma_m$, $m = 1, 2, \dots$, define the sets

$$\mathcal{I}_m = \mathcal{P}^+_{\nu(m+1)} \setminus \mathcal{P}^-_{\nu(m)}, \qquad U_{m,i} = B(x_i^{\nu(m)}, \hat{r}_m(\varepsilon)), \qquad V_{m,i} = B(x_i^{\nu(m)}, \hat{r}_m(u)) \setminus U_{m,i}.$$

Then, $E_{m,i} = \{\mathcal{P}^-_{\nu(m)}[U_{m,i}] = 1\} \cap \{\mathcal{P}^-_{\nu(m)}[V_{m,i}] = 0\} \cap \{\mathcal{I}_m[U_{m,i}] = 0\} \cap \{\mathcal{I}_m[V_{m,i}] = 0\}$. Let $\alpha(m) = \nu(m) - \nu(m)^{3/4}$ and $\beta(m) = \nu(m+1) + \nu(m+1)^{3/4}$. Note that each of the four events appearing in the above equation are independent. Hence,

$$\begin{aligned}\mathrm{P}[E_{m,i}] &= \bigg(\alpha(m)\int_{U_{m,i}} f(y)\,\mathrm{d}y\bigg)\exp\bigg(-\beta(m)\int_{U_{m,i}\cup V_{m,i}} f(y)\,\mathrm{d}y\bigg) \\ &= \alpha(m) I(x_i^{\nu(m)}, \hat{r}_m(\varepsilon)) \exp(-\beta(m) I(x_i^{\nu(m)}, \hat{r}_m(u))) \\ &\geq \alpha(m) I(R_{\nu(m)}(0), \hat{r}_m(\varepsilon)) \exp(-\beta(m) I(R_{\nu(m)}(c), \hat{r}_m(u))),\end{aligned} \tag{3.25}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

where the last inequality follows since $R_{\nu(m)}(c) \leq |x_i^{\nu(m)}| \leq R_{\nu(m)}(0)$ and the density is decreasing radially. Using Lemma 2.1 (that this lemma is applicable is shown in the proof of Proposition 3.1; see the arguments below (3.11) leading to (3.12)) and noting that $\alpha(m) \sim \nu(m)$ and $\beta(m) \sim a\nu(m)$, we can find constants C_1' and C_2', such that, for large m,

$$\begin{aligned}\mathrm{P}[E_{m,i}] \geq C_1'\nu(m)\hat{r}_m^{(d-1)/2}(\varepsilon)\exp(-\lambda(R^{\alpha}_{\nu(m)}(0) - \alpha\hat{r}_m(\varepsilon)R^{\alpha-1}_{\nu(m)}(0)))(R^{\alpha-1}_{\nu(m)}(0))^{-(d+1)/2} \\ \times \exp(-C_2'\nu(m)\hat{r}_m^{(d-1)/2}(u)\exp(-\lambda(R^{\alpha}_{\nu(m)}(c) - \alpha\hat{r}_m(u)R^{\alpha-1}_{\nu(m)}(c))) \\ \times (R^{\alpha-1}_{\nu(m)}(c))^{-(d+1)/2}).\end{aligned} \tag{3.26}$$

We now estimate the right-hand side of the above inequality. In what follows, we will use (3.4) to obtain the desired inequalities for the $\alpha \leq 1$ case, as in the proof of Proposition 3.1, without mentioning it explicitly. From (3.15), with $c = 0$ and u replaced by ε, we obtain $\lambda\alpha\hat{r}_m(\varepsilon)R^{\alpha-1}_{\nu(m)}(0) \geq \log_2(\nu(m))\lambda\alpha\varepsilon/2$ for large m. Hence, we can find a constant c_3' such that, for large m,

$$\exp(\lambda\alpha\hat{r}_m(\varepsilon)R^{\alpha-1}_{\nu(m)}(0))(\hat{r}_m(\varepsilon))^{(d-1)/2} \geq c_3'\frac{\mathrm{e}^{\log_2\nu(m)\lambda\alpha\varepsilon/2}(\log_2\nu(m))^{(d-1)/2}}{(\log\nu(m))^{(1-1/\alpha)(d-1)/2}}.$$

Using (3.14) with $c = 0$ and the above inequality, we obtain, for large m and some constant C_3',

$$\begin{aligned}&C_1'\nu(m)\hat{r}_m^{(d-1)/2}(\varepsilon)\exp(-\lambda(R^{\alpha}_{\nu(m)}(0) - \alpha\hat{r}_m(\varepsilon)R^{(\alpha-1)}_{\nu(m)}(0)))(R^{(\alpha-1)}_{\nu(m)}(0))^{-(d+1)/2} \\ &\quad\geq C_3'\nu(m)\frac{(\log\nu(m))^{-(1-1/\alpha)(d+1)/2}}{\nu(m)(\log\nu(m))^{(d-\alpha)/\alpha}}\frac{\mathrm{e}^{\log_2\nu(m)\lambda\alpha\varepsilon/2}(\log_2\nu(m))^{(d-1)/2}}{(\log\nu(m))^{(1-1/\alpha)(d-1)/2}} \\ &\quad= C_3'\frac{(\log_2\nu(m))^{(d-1)/2}}{(\log\nu(m))^{d-1-\alpha\varepsilon\lambda/2}}.\end{aligned} \tag{3.27}$$

From (3.15), we obtain, for large m, $\lambda\alpha\hat{r}_m(u)R^{\alpha-1}_{\nu(m)}(c) \leq (1+\varepsilon)\lambda\alpha u\log_2\nu(m)$, from which we obtain, for some constant C_4',

$$\begin{aligned}\exp(\lambda\alpha\hat{r}_m(u)R^{\alpha-1}_{\nu(m)}(c))(\hat{r}_m(u))^{(d-1)/2} &\leq C_4'\frac{\mathrm{e}^{(1+\varepsilon)\lambda\alpha u\log_2\nu(m)}(\log_2\nu(m))^{(d-1)/2}}{(\log\nu(m))^{(1-1/\alpha)(d-1)/2}} \\ &= C_4'\frac{(\log\nu(m))^{(1+\varepsilon)\lambda\alpha u}(\log_2\nu(m))^{(d-1)/2}}{(\log\nu(m))^{(1-1/\alpha)(d-1)/2}}.\end{aligned}$$

As in (3.14), we can find a constant C_5' such that, for large m,

$$\exp(-\lambda R^{\alpha}_{\nu(m)}(c))(R^{\alpha-1}_{\nu(m)}(c))^{-(d+1)/2} \leq C_5'\frac{(\log\nu(m))^{-(1-1/\alpha)(d+1)/2}}{\nu(m)(\log\nu(m))^{(c+d-\alpha)/\alpha}}.$$

Using the two bounds obtained above, we obtain, for large m and some constant C_6',

$$\begin{aligned}&C_2'\nu(m)\hat{r}_m^{(d-1)/2}(u)\exp(-\lambda(R^{\alpha}_{\nu(m)}(c) - \alpha\hat{r}_m(u)R^{\alpha-1}_{\nu(m)}(c)))(R^{\alpha-1}_{\nu(m)}(c))^{-(d+1)/2} \\ &\quad\leq C_6'\frac{(\log_2(\nu(m)))^{(d-1)/2}}{(\log(\nu(m)))^{d+c/\alpha-(1+\varepsilon)\alpha u\lambda-1}}.\end{aligned}$$

Using (3.27) and the above inequality in (3.26), we obtain, for large m,

$$\begin{aligned}\mathrm{P}[E_{m,i}] &\geq C_3' \frac{(\log_2(\nu(m)))^{(d-1)/2}}{(\log(\nu(m)))^{d-1-\alpha\varepsilon\lambda/2}} \exp\biggl(-C_6' \frac{(\log_2(\nu(m)))^{(d-1)/2}}{(\log(\nu(m)))^{d+c/\alpha-(1+\varepsilon)\lambda\alpha u-1}}\biggr) \\ &\sim C_3' \frac{(\log_2(\nu(m)))^{(d-1)/2}}{(\log(\nu(m)))^{d-1-\alpha\varepsilon\lambda/2}}, \end{aligned} \tag{3.28}$$

where the last relation follows since $(1+\varepsilon)u < (\alpha d+c-\alpha)/\alpha^2\lambda$. The events $E_{m,i}$, $1 \leq i \leq \sigma_m$, are independent, since the balls $B(x_i^{\nu(m)}, \hat{r}_m(u))$ are disjoint. So, by (3.23), (3.28), and the inequality $1 - x \leq \mathrm{e}^{-x}$, we can find constants C' and C'' such that, for all large enough m,

$$\begin{aligned}\mathrm{P}\biggl[\biggl(\bigcup_{i=1}^{\sigma_m} E_{m,i}\biggr)^{\mathrm{c}}\biggr] &\leq \prod_{i=1}^{\sigma_m} \exp(-\mathrm{P}[E_{m,i}]) \\ &\leq \exp\biggl(-C'\sigma_m \frac{(\log_2(\nu(m)))^{(d-1)/2}}{(\log(\nu(m)))^{d-1-\alpha\varepsilon\lambda/2}}\biggr) \\ &\leq \exp\biggl(-C''\biggl(\frac{m}{\log m + \log_2 a}\biggr)^{d-1} \frac{(\log m + \log_2 a)^{(d-1)/2}}{m^{d-1-\alpha\varepsilon\lambda/2}}\biggr) \\ &= \exp\biggl(-C'' \frac{m^{\alpha\varepsilon\lambda/2}}{(\log m + \log_2 a)^{(d-1)/2}}\biggr),\end{aligned}$$

which is summable in m. This proves (3.24).

Proof of Theorem 1.3. The proof is immediate from Propositions 3.1 and 3.2.

Appendix A. Proof of Lemma 2.1

In the definition of $I(\rho_n, r_n) = I(\rho_n e, r_n)$, set $y = (\rho_n + r_n t, r_n s)$, $t \in (-1, 1)$, $s \in \mathbb{R}^{d-1}$. This gives

$$I(\rho_n, r_n) = A_d \int_{-1}^{1} \int_{\|s\|^2 \leq (1-t^2),\, s \in \mathbb{R}^{d-1}} \exp(-\lambda((\rho_n + r_n t)^2 + (\|s\| r_n)^2)^{\alpha/2}) r_n^d \,\mathrm{d}s \,\mathrm{d}t. \tag{A.1}$$

First consider the case in which $0 < \alpha \leq 2$. Using the Taylor expansion, we obtain

$$\begin{aligned}((\rho_n + r_n t)^2 + (\|s\| r_n)^2)^{\alpha/2} &= ((\rho_n^2 + 2r_n t\rho_n) + (t^2 + \|s\|^2) r_n^2)^{\alpha/2} \\ &= (\rho_n^2 + 2r_n\rho_n t)^{\alpha/2} + h_1(n, s, t),\end{aligned} \tag{A.2}$$

where $h_1(n, s, t) = (\alpha/2) r_n^2 (t^2 + ||s||^2)(\rho_n^2 + 2r_n\rho_n t + \xi)^{\alpha/2-1}$ and $\xi \in (0, r_n^2(t^2 + ||s||^2))$. Since $0 < \alpha \leq 2$, $(t, s) \in B(0, 1)$, and $0 \leq \xi \leq r_n^2$, we have

$$0 \leq h_1(n, s, t) \leq \frac{\alpha}{2} r_n^2 (t^2 + ||s||^2)(\rho_n^2 + 2r_n\rho_n t)^{\alpha/2-1} \leq w_1(n),$$

where

$$0 \leq w_1(n) = \frac{\alpha}{2} r_n^2 (\rho_n^2 - 2r_n\rho_n)^{\alpha/2-1} = \frac{\alpha}{2} r_n^2 \rho_n^{\alpha-2} \biggl(1 - \frac{2r_n}{\rho_n}\biggr)^{\alpha/2-1} \to 0, \tag{A.3}$$

https://doi.org/10.1239/aap/1282924057 Published online by Cambridge University Press

since $r_n^2\rho_n^{\alpha-2} \to 0$, and $r_n/\rho_n \to 0$ as $n \to \infty$. Again, from the Taylor expansion applied to $(\rho_n^2 + 2r_n\rho_n t)^{\alpha/2}$ in (A.2), we obtain

$$(\rho_n^2 + 2r_n\rho_n t)^{\alpha/2} = \rho_n^\alpha + \alpha r_n t\rho_n^{\alpha-1} + h_2(n, t), \tag{A.4}$$

where $h_2(n, t) = \frac{1}{2}\alpha(\alpha - 2)(r_n t\rho_n)^2(\rho_n^2 + \zeta)^{\alpha/2-2}$ and $\zeta \in (\min(0, 2\rho_n r_n t), \max(0, 2\rho_n r_n t))$. Since $0 < \alpha \le 2$ and $-1 \le t \le 1$, we obtain

$$w_2(n) = \frac{\alpha(\alpha - 2)}{2} r_n^2\rho_n^{\alpha-2}\left(1 - 2\frac{r_n}{\rho_n}\right)^{\alpha/2-2} \le h_2(n, t) \le 0. \tag{A.5}$$

Since $r_n^2\rho_n^{\alpha-2} \to 0$ and $r_n/\rho_n \to 0$, it follows that $w_2(n) \to 0$ as $n \to \infty$. From (A.2)–(A.5), we obtain

$$\rho_n^\alpha + \alpha r_n t\rho_n^{\alpha-1} + w_2 \le ((\rho_n + r_n t)^2 + (\|s\| r_n)^2)^{\alpha/2} \le \rho_n^\alpha + \alpha r_n t\rho_n^{\alpha-1} + w_1.$$

Using the above in (A.1), we obtain

$$A_d r_n^d \mathrm{e}^{-\lambda w_1} G_n \le I(\rho_n, r_n) \le A_d r_n^d \mathrm{e}^{-\lambda w_2} G_n, \tag{A.6}$$

where

$$G_n = \int_{-1}^{1} \int_{\|s\|^2 \le (1-t^2),\, s\in\mathbb{R}^{d-1}} \exp(-\lambda(\rho_n^\alpha + \alpha r_n t\rho_n^{\alpha-1}))\, \mathrm{d}s\, \mathrm{d}t, \tag{A.7}$$

and w_1 and w_2, as defined in (A.3) and (A.5), respectively, converge to 0 as $n \to \infty$.

If $\alpha > 2$ then $h_2(n, t) \ge 0$, and we take w_1 and w_2 to be the upper and lower bounds of $h_1(n, s, t) + h_2(n, t)$, respectively. We then obtain (A.6) with $w_2(n) = 0$, and

$$\begin{aligned} w_1(n) &= \frac{\alpha}{2} r_n^2(\rho_n^2 + 2r_n\rho_n)^{\alpha/2-1}[1 + (\alpha - 2)\rho_n^2(\rho_n^2 - 2r_n\rho_n)^{-1}] \\ &= \frac{\alpha}{2} r_n^2\rho_n^{\alpha-2}\left(1 + 2\frac{r_n}{\rho_n}\right)^{\alpha/2-1}\left[1 + (\alpha - 2)\left(1 - 2\frac{r_n}{\rho_n}\right)^{-1}\right], \end{aligned}$$

which converges to 0 by the conditions of the lemma. Now consider the integral in (A.7). First make the change of variable $u = t + 1$ and then set $v = \lambda\alpha r_n\rho_n^{\alpha-1}u$ to obtain

$$\begin{aligned} G_n &= \theta_{d-1}\mathrm{e}^{-\lambda\rho_n^\alpha} \int_{-1}^{1} \exp(-\lambda\alpha r_n\rho_n^{\alpha-1}t)(1 - t^2)^{(d-1)/2}\, \mathrm{d}t \\ &= \theta_{d-1}\exp(-\lambda(\rho_n^\alpha - \alpha r_n\rho_n^{\alpha-1})) \int_0^2 \exp(-\lambda\alpha r_n\rho_n^{\alpha-1}u)u^{(d-1)/2}(2 - u)^{(d-1)/2}\, \mathrm{d}u \\ &= \theta_{d-1}\exp(-\lambda(\rho_n^\alpha - \alpha r_n\rho_n^{\alpha-1}))(\lambda\alpha r_n\rho_n^{\alpha-1})^{-(d+1)/2}2^{(d-1)/2}M_n, \end{aligned} \tag{A.8}$$

where

$$M_n = \int_0^{2\lambda\alpha r_n\rho_n^{\alpha-1}} \mathrm{e}^{-v}v^{(d-1)/2}\left(1 - \frac{v}{2\lambda\alpha r_n\rho_n^{\alpha-1}}\right)^{(d-1)/2} \mathrm{d}v \le \Gamma\left(\frac{d+1}{2}\right). \tag{A.9}$$

We will show that the integral in (A.9) converges to $\Gamma((d + 1)/2)$ as $n \to \infty$, and also estimate the error in this approximation. Note that $r_n\rho_n^{\alpha-1} \to \infty$ as $n \to \infty$. Write $E_n = M_n - \Gamma((d+1)/2) = A_n - B_n$, where

$$A_n = \int_0^{2\lambda\alpha r_n\rho_n^{\alpha-1}} \mathrm{e}^{-v} v^{(d-1)/2} \left[\left(1 - \frac{v}{2\lambda\alpha r_n\rho_n^{\alpha-1}}\right)^{(d-1)/2} - 1\right] \mathrm{d}v,$$

$$B_n = \int_{2\lambda\alpha r_n\rho_n^{\alpha-1}}^{\infty} \mathrm{e}^{-v} v^{(d-1)/2} \,\mathrm{d}v,$$

and

$$|A_n| \le \sup_{0\le v\le 2\lambda\alpha r_n\rho_n^{\alpha-1}} \left\{\mathrm{e}^{-v/2}\left|1 - \left(1 - \frac{v}{2\lambda\alpha r_n\rho_n^{\alpha-1}}\right)^{(d-1)/2}\right|\right\} \int_0^\infty \mathrm{e}^{-v/2} v^{(d-1)/2} \,\mathrm{d}v.$$

Since $(1-x)^a \ge 1 - Cx,\ 0 \le x \le 1$, with $C = \mathbf{1}_{\{0<a\le 1\}} + a\,\mathbf{1}_{\{a>1\}}$, we obtain

$$0 \le 1 - \left(1 - \frac{v}{2\lambda\alpha r_n\rho_n^{\alpha-1}}\right)^{(d-1)/2} \le \frac{Cv}{2\lambda\alpha r_n\rho_n^{\alpha-1}}, \qquad 0 \le v \le 2\lambda\alpha r_n\rho_n^{\alpha-1}.$$

Therefore,

$$|A_n| \le \frac{C}{2\lambda\alpha r_n\rho_n^{\alpha-1}} \sup_{0\le v<\infty} \{v\mathrm{e}^{-v/2}\} \int_0^\infty \mathrm{e}^{-v/2} v^{(d-1)/2} \,\mathrm{d}v = \frac{C'}{r_n\rho_n^{\alpha-1}},$$

where C' is some constant. Furthermore, $|B_n| \le \exp(-\lambda\alpha r_n\rho_n^{\alpha-1}/2) \int_0^\infty \mathrm{e}^{-v/2} v^{(d-1)/2} \,\mathrm{d}v$, and, hence, decays exponentially fast in $r_n\rho_n^{\alpha-1}$. Putting the above two estimates together, we obtain

$$|E_n| \le \frac{C_1}{r_n\rho_n^{\alpha-1}} \to 0 \quad \text{as } n \to \infty. \tag{A.10}$$

The result now follows from (A.6), (A.8), and (A.10).

Acknowledgement

The authors would like to thank an anonymous referee for numerous suggestions that significantly improved this paper.

References

Appel, M. J. B. and Russo, R. P. (1997). The minimum vertex degree of a graph on the uniform points in $[0, 1]^d$. *Adv. Appl. Prob.* **29,** 582–594.

Dette, H. and Henze, N. (1989). The limit distribution of the largest nearest-neighbour link in the unit d-cube. *J. Appl. Prob.* **26,** 67–80.

Gupta, B., Iyer, S. K. and Manjunath, D. (2005). On the topological properties of one dimensional exponential random geometric graphs. *Random Structures and Algorithms* **32,** 181–204.

Hsing, T. and Rootzén, H. (2005). Extremes on trees. *Ann. Prob.* **33,** 413–444.

Penrose, M. D. (1997). The longest edge of the minimal spanning tree. *Ann. Appl. Prob.* **7,** 340–361.

Penrose, M. D. (1998). Extremes for the minimal spanning tree on normally distributed points. *Adv. Appl. Prob.* **30,** 628–639.

Penrose, M. D. (1999). A strong law for the largest nearest-neighbour link between random points. *J. London Math. Soc.* **60,** 951–960.

Penrose, M. (2003). *Random Geometric Graphs*. Oxford University Press.

Steele, J. M. and Tierney, L. (1986). Boundary domination and the distribution of the largest nearest-neighbor link in higher dimensions. *J. Appl. Prob.* **23,** 524–528.