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Analytic extension of riemannian manifolds

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The object of this thesis is to construct a unique simply connected inextendable analytic manifold from a knowledge of its geometric properties in the neighbourhood of a single point. There is an obvious obstacle to the construction, namely that if one removes an arbitrary point from a simply connected inextendable manifold any covering manifold of the resulting manifold will again be simply connected and inextendable. In order to avoid such trivialities and at the same time obtain the largest possible structure, the definition of completeness has to be reformulated. The following example suggests how this should be done. Consider the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ with a, b, c distinct. Remove its six axis points to form a manifold M and construct the universal covering manifold N of M. Then N has the following properties:

(1) it is inextendable;

(2) a small neighbourhood of an arbitrary point of N possesses at most one reflection symmetry;

(3) neither N nor any of its quotient manifolds has a proper extension satisfying condition (2).

Our definition of completeness in the case of a manifold which does not admit continuous motions is essentially that such a manifold is complete if it satisfies the above three conditions. In the case where continuous motions are admitted, completeness has to be defined by considering symmetries of the orbit space.

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The following is an outline of the method used in constructing extensions.

Consider a simply connected open set U in euclidean n-space E^n. Suppose U is provided with an analytic riemannian metric and that there are no local continuous motions on U. Define a chain $C_0f_0 \ldots C_kf_k$ to be a sequence $C_0, \ldots, C_k$ of curves in E^n with the initial point of C_0 in U, together with a sequence $f_0, \ldots, f_k$ of analytic, invertible E^n-valued functions such that for $i = 0, \ldots, k$ the domain of f_i includes the terminal point of C_i and for $i = 0, \ldots, k-1$, f_i maps the terminal point of C_i onto the initial point of C_{i+1}.

A chain is called admissible if it is possible to continue the metric on U analytically along the chain in a precisely defined manner. An equivalence relation is defined on the set of admissible chains and it is shown that the resulting set of equivalence classes can be given the structure of a complete simply connected extension of U. Equivalent admissible chains are represented by homotopic curves on the extension.

If U does admit local continuous motions then to exclude pathological cases one has to assume that all orbits have the same dimension. The set of equivalence classes of admissible chains then yields the orbit space of the desired extension which can be lifted to the extension itself provided that the Killing vector fields on U satisfy a certain condition. This condition amounts to assuming that the Lie subalgebra of Killing vector fields which leaves a point of U fixed generates a closed subgroup of the simply connected group of motions.

The construction can be readily extended to spaces endowed only with a symmetric connection provided that U does not admit local affine motions. The case where U does admit such motions is more difficult and no general solution has been obtained. However, we show that if the chain construction breaks down then no extension having the desired properties exists.