Bull. Aust. Math. Soc. **108** (2023), 107–113
doi:10.1017/S0004972722001241

DISTRIBUTION OF *r*-FREE INTEGERS OVER A FLOOR FUNCTION SET

PINTHIRA TANGSUPPHATHAWAT, TEERAPAT SRICHAN and VICHIAN LAOHAKOSOL

(Received 15 August 2022; accepted 19 September 2022; first published online 23 November 2022)

Abstract

For a positive integer $r \geq 2$, a natural number n is r-free if there is no prime p such that $p^r \mid n$. Asymptotic formulae for the distribution of r-free integers in the floor function set $S(x) := \{\lfloor x/n \rfloor : 1 \leq n \leq x\}$ are derived. The first formula uses an estimate for elements of $S(x)$ belonging to arithmetic progressions. The other, more refined, formula makes use of an exponent pair and the Riemann hypothesis.

2020 *Mathematics subject classification*: primary 11N64; secondary 11N69.

Keywords and phrases: floor function, r-free integer.

1. Introduction and results

Let $\lfloor t \rfloor$ be the integral part of $t \in \mathbb{R}$ and let r be a fixed integer ≥ 2. A positive integer n is called r-free if in its canonical prime representation, each exponent is $< r$; a 2-free integer is also called square-free. Let μ_r be the characteristic function of the r-free integers. There is considerable research on the distribution of r-free integers over certain special sets, such as the set of integer parts, the Beatty sequence $\lfloor \alpha n + \beta \rfloor$, [1, 8, 9, 19] and the Piatetski-Shapiro sequence $\lfloor n^c \rfloor$, [5–7, 13–15, 17, 18, 21]. In 2019, Bordellés *et al.* [4] established an asymptotic formula for a sum of the form

$$\sum_{n \leq x} f\Big(\Big\lfloor \frac{x}{n} \Big\rfloor\Big),$$

where f is an arithmetic function subject to some growth condition, and applied it in particular to Euler's totient function. It is thus natural to consider such a sum for various other functions. For example, in [3], Bordellés proved that

$$S_{\mu_2}(x) = x \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n(n+1)} + O(x^{1919/4268+\epsilon}), \tag{1.1}$$

This work was financially supported by the Office of the Permanent Secretary, Ministry of Higher Education, Science, Research and Innovation, Grant No. RGNS 63-40.

where $\mu_2(n)$ is the characteristic function of the set of square-free numbers. Later, Liu *et al.* [12] improved the O-term in (1.1) to $O(x^{2/5+\epsilon})$. In 2022, Stucky [16] generalised the sum in (1.1) to the case of r-free integers and showed that for $r \geq 3$,

$$S_{\mu_r}(x) = \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n(n+1)} x + O(x^{\theta_r}), \quad \theta_r := \frac{r+1}{3r+1},$$

where $\mu_r(n)$ is the characteristic function of the set of r-free numbers. Very recently, Heyman [10] considered the floor function set

$$S(x) := \left\{ \left\lfloor \frac{x}{n} \right\rfloor : 1 \leq n \leq x \right\} \tag{1.2}$$

and studied the number of primes in $S(x)$. Our objective here is to investigate how the r-free integers are distributed over the set $S(x)$, that is, to ask for an asymptotic estimate for the function

$$T_{\mu_r}(x) := \sum_{m \in S(x)} \mu_r(m). \tag{1.3}$$

There first arises the question whether the sum (1.3) is identical or related to the sum

$$S_{\mu_r}(x) := \sum_{n \leq x} \mu_r\left(\left\lfloor \frac{x}{n} \right\rfloor \right). \tag{1.4}$$

To answer this question, we rewrite the sum (1.3) as

$$T_{\mu_r}(x) = \sum_{\substack{m \leq x \\ \exists n \in \mathbb{N} \text{ such that } \lfloor x/n \rfloor = m}} \mu_r(m).$$

Observe that each argument appears once in the sum (1.3), but usually appears several times in the sum (1.4), as seen in the following example. Taking $x = 20$ and $r = 2$, for the sum (1.4), we get

$$\begin{aligned} S_{\mu_2}(20) &= \sum_{n \leq 20} \mu_2\left(\left\lfloor \frac{20}{n} \right\rfloor \right) \\ &= \mu_2(20) + \mu_2(10) + \mu_2(6) + \mu_2(5) + \mu_2(4) + \mu_2(3) + \mu_2(2) + \mu_2(2) + \mu_2(2) + \mu_2(2) \\ &\quad + \mu_2(1) + \mu_2(1) + \mu_2(1) + \mu_2(1) + \mu_2(1) + \mu_2(1) + \mu_2(1) + \mu_2(1) + \mu_2(1) + \mu_2(1) \\ &= \mu_2(20) + \mu_2(10) + \mu_2(6) + \mu_2(5) + \mu_2(4) + \mu_2(3) + 4\mu_2(2) + 10\mu_2(1) = 18, \end{aligned}$$

while for the sum (1.3), we get $S(20) = \{\lfloor 20/n \rfloor : 1 \leq n \leq 20\} = \{1, 2, 3, 4, 5, 6, 10, 20\}$ and

$$\begin{aligned} T_{\mu_2}(20) &= \sum_{m \in S(20)} \mu_2(m) \\ &= \mu_2(20) + \mu_2(10) + \mu_2(6) + \mu_2(5) + \mu_2(4) + \mu_2(3) + \mu_2(2) + \mu_2(1) = 6. \end{aligned}$$

Our first main result is proved using the following estimate of Yu and Wu [20].

https://doi.org/10.1017/S0004972722001241 Published online by Cambridge University Press

LEMMA 1.1 [20]. *Let $x \in \mathbb{R}$, $x > 0$, let $q, a \in \mathbb{Z}$ such that $0 \le a < q \le x^{1/4} \log^{-3/2} x$ and let $S(x)$ be as defined in (1.2). Then,*

$$\sum_{\substack{n \in S(x) \\ n \equiv a \pmod q}} 1 = \frac{2x^{1/2}}{q} + O\Big(\frac{x^{1/3}}{q^{1/3}} \log x\Big).$$

REMARK 1.2. For real large x, let $S(x)$ be as defined in (1.2). We shall need the estimate $|S(x)| = O(x^{1/2})$ in our proofs. To verify this, note that for $n \in \{1, 2, \dots \lfloor x \rfloor\}$:

- if $\lfloor x/n \rfloor = 1$, then $n \le x$;
- if $\lfloor x/n \rfloor = 2$, then $n \le x/2$;

  $\vdots$
- if $\lfloor x/n \rfloor = \lfloor x^{1/2} \rfloor$, then $n < x^{1/2}$.

It follows that for $n < x^{1/2}$, the function $\lfloor x/n \rfloor$ takes at most $\lfloor x^{1/2} \rfloor$ distinct integral values. However, if $n \ge x^{1/2}$, the function $\lfloor x/n \rfloor \le x^{1/2}$ can then take at most $\lfloor x^{1/2} \rfloor$ integral values. Thus, $|S(x)| \le 2\lfloor x^{1/2} \rfloor = O(x^{1/2})$.

Our first theorem reads as follows.

THEOREM 1.3. *Let $S(x)$ and T_{μ_r} be as defined in (1.2) and (1.3). Then,*

$$T_{\mu_r}(x) = \frac{2x^{1/2}}{\zeta(r)} + \begin{cases} O(x^{3/8} \log^{3/4} x) & \text{for } r = 2, \\ O(x^{1/3} \log x) & \text{for } r \ge 3. \end{cases}$$

Regarding our second main result, very recently, Zhang [22] improved the results of Bordellés (1.1), Stuctky [16] and Liu *et al.* [12] by proving using the exponent pair method, that

$$S_{\mu_r}(x) = x \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n(n+1)} + \begin{cases} O(x^{11/29} \log^2 x) & \text{for } r = 2, \\ O(x^{1/3} \log^2 x) & \text{for } r = 3, \\ O(x^{1/3} \log x), & \text{for } r \ge 3. \end{cases}$$

We use the same idea as in [22] to give another formula, which, in the case $r = 2$, is slightly weaker than that in Theorem 1.3.

THEOREM 1.4. *For an exponent pair (κ, λ) such that $1/2 < \lambda/(1 + \kappa)$, we have*

$$T_{\mu_r}(x) = \frac{2x^{1/2}}{\zeta(r)} + \begin{cases} O(x^{1/4+\lambda/4(1+\kappa)} (\log x)^{3/2 - 3\lambda/2(1+\kappa)}) & \text{for } r = 2, \\ O(x^{1/3} \log x) & \text{for } r \ge 3. \end{cases}$$

We note in passing that a result better than that in Theorem 1.4 for the case $r = 2$ can be derived, assuming the Riemann hypothesis, by taking the exponent pair $(\kappa, \lambda) = (2/7, 4/7)$ to get

$$T_{\mu_2}(x) = \frac{2x^{1/2}}{\zeta(2)} + O(x^{13/36} (\log x)^{1/6}).$$

https://doi.org/10.1017/S0004972722001241 Published online by Cambridge University Press

Under the Riemann hypothesis, we can omit the restriction on the exponent pair (κ, λ) such that $1/2 < \lambda/(1+\kappa)$ and obtain the following result.

THEOREM 1.5. *Assume the Riemann hypothesis. For an exponent pair* (κ, λ) *such that* $1/4 < \lambda/(1+\kappa)$, *we have*

$$T_{\mu_2}(x) = \frac{2x^{1/2}}{\zeta(2)} + O(x^{1/4+\lambda/4(1+\kappa)}(\log x)^{3/2-3\lambda/2(1+\kappa)}).$$

2. Proofs

PROOF OF THEOREM 1.3. Since $\mu_r(n)$ is the characteristic function of the set of r-free numbers, by the well-known identity $\mu_r(n) = \sum_{d^r|n} \mu(d)$, we have

$$\begin{aligned} T_{\mu_r}(x) &= \sum_{n\in S(x)} \mu_r(n) = \sum_{n\in S(x)} \sum_{d^r|n} \mu(d) = \sum_{d\le x^{1/r}} \mu(d) \sum_{\substack{n\in S(x) \\ n\equiv 0\,(\mathrm{mod}\ d^r)}} 1 \\ &= \sum_{d\le x^{1/4r}\log^{-3/2r} x} \mu(d) \sum_{\substack{n\in S(x) \\ n\equiv 0\,(\mathrm{mod}\ d^r)}} 1 + \sum_{x^{1/4r}\log^{-3/2r} x<d\le x^{1/r}} \mu(d) \sum_{\substack{n\in S(x) \\ n\equiv 0\,(\mathrm{mod}\ d^r)}} 1. \end{aligned}$$

Using Lemma 1.1 to compute the first sum, we have

$$\begin{aligned} \sum_{d\le x^{1/4r}\log^{-3/2r} x} \mu(d) \sum_{\substack{n\in S(x) \\ n\equiv 0\,(\mathrm{mod}\ d^r)}} 1 &= \sum_{d\le x^{1/4r}\log^{-3/2r} x} \mu(d)\Big(\frac{2x^{1/2}}{d^r} + O\Big(\frac{x^{1/3}}{d^{r/3}}\log x\Big)\Big) \\ &= \frac{2x^{1/2}}{\zeta(r)} + \begin{cases} O(x^{3/8}\log^{3/4} x) & \text{for } r = 2, \\ O(x^{1/3}\log x) & \text{for } r \ge 3. \end{cases} \end{aligned}$$

By the remark preceding the statement of Theorem 1.3, we have $|S(x)| = O(x^{1/2})$ and so the second sum is bounded by

$$\begin{aligned} \sum_{x^{1/4r}\log^{-3/2r} x<d\le x^{1/r}} \mu(d) \sum_{\substack{n\in S(x) \\ n\equiv 0\,(\mathrm{mod}\ d^2)}} 1 &= O\Big(x^{1/2} \sum_{x^{1/4r}\log^{-3/2r} x<d\le x^{1/r}} \frac{1}{d^2}\Big) \\ &= \begin{cases} O(x^{3/8}\log^{3/4} x) & \text{for } r = 2, \\ O(x^{1/4}\log^{1/2} x) & \text{for } r \ge 3, \end{cases} \end{aligned}$$

which completes the proof of Theorem 1.3. □

PROOF OF THEOREM 1.4. In view of [11, (14.23)],

$$\mu_r(n) = \sum_{d|n} g(d), \quad \text{where } g(d) = \begin{cases} \mu(\ell) & \text{if } d = \ell^r, \text{ for some } \ell \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/S0004972722001241 Published online by Cambridge University Press

From the definition of $g(d)$,

$$\sum_{d\le x}|g(d)| = \begin{cases} O(x^{1/2}) & \text{if } r=2, \\ O(x^{1/3}) & \text{if } r\ge 3. \end{cases} \tag{2.1}$$

Let (κ,λ) be an exponent pair such that $1/2 < \lambda/(1+\kappa)$ (<1). Trivially, from (2.1),

$$\sum_{d\le x}|g(d)| = O(x^{\lambda/(1+\kappa)}). \tag{2.2}$$

For $x>1$,

$$\begin{aligned}\sum_{n\in S(x)}\mu_r(n) &= \sum_{n\in S(x)}\sum_{d|n} g(d) = \sum_{d\le x} g(d) \sum_{\substack{n\in S(x)\\ n\equiv 0\,(\text{mod } d)}} 1 \\ &= \sum_{d\le x^{1/4}\log^{-3/2}x} g(d) \sum_{\substack{n\in S(x)\\ n\equiv 0\,(\text{mod } d)}} 1 + \sum_{x^{1/4}\log^{-3/2}x<d\le x} g(d) \sum_{\substack{n\in S(x)\\ n\equiv 0\,(\text{mod } d)}} 1.\end{aligned}$$

We use Lemma 1.1 to compute the first sum. We have

$$\begin{aligned}\sum_{d\le x^{1/4}\log^{-3/2}x} g(d) \sum_{\substack{n\in S(x)\\ n\equiv 0\,(\text{mod } d)}} 1 &= \sum_{d\le x^{1/4}\log^{-3/2}x} g(d)\Big(\frac{2x^{1/2}}{d} + O\Big(\frac{x^{1/3}}{d^{1/3}}\log x\Big)\Big) \\ &= 2x^{1/2}\sum_{d\le x^{1/4}\log^{-3/2}x}\frac{g(d)}{d} + O\Big(x^{1/3}\log x \sum_{d\le x^{1/4}\log^{-3/2}x}\frac{|g(d)|}{d^{1/3}}\Big).\end{aligned}$$

Denote the first and the second sums on the right-hand side by Σ_1 and Σ_2, respectively. Using partial summation (or Abel's identity [2, Theorem 4.2]) and (2.2),

$$\begin{aligned}\sum_{d\le x^{1/4}\log^{-3/2}x}\frac{g(d)}{d} &= \sum_{d=1}^{\infty}\frac{g(d)}{d} - \sum_{d>x^{1/4}\log^{-3/2}x}\frac{g(d)}{d} = \sum_{d=1}^{\infty}\frac{g(d)}{d} + O\Big(\sum_{d>x^{1/4}\log^{-3/2}x}\frac{|g(d)|}{d}\Big) \\ &= \sum_{d=1}^{\infty}\frac{g(d)}{d} + O(x^{-1/4+\lambda/4(1+\kappa)}(\log x)^{3/2-3\lambda/2(1+\kappa)})\end{aligned}$$

and so

$$\Sigma_1 = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d} + O(x^{1/4+\lambda/4(1+\kappa)}(\log x)^{3/2-3\lambda/2(1+\kappa)}).$$

Again from (2.2), by Abel's identity,

$$\sum_{d\le x^{1/4}\log^{-3/2}x}\frac{|g(d)|}{d^{1/3}} \ll x^{-1/12+\lambda/4(1+\kappa)}(\log x)^{1/2-3\lambda/2(1+\kappa)}$$

and so

$$\Sigma_2 = O(x^{1/4+\lambda/4(1+\kappa)}(\log x)^{3/2-3\lambda/2(1+\kappa)}).$$

https://doi.org/10.1017/S0004972722001241 Published online by Cambridge University Press

Then,

$$\sum_{d\le x^{1/4}\log^{-3/2}x} g(d) \sum_{\substack{n\in S(x)\\ n\equiv 0\,(\mathrm{mod}\ d)}} 1 = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d} + O(x^{1/4+\lambda/4(1+\kappa)}(\log x)^{3/2-3\lambda/2(1+\kappa)}).$$

Next, we bound the second sum. Using $|S(x)| = O(x^{1/2})$,

$$\sum_{x^{1/4}\log^{-3/2}x<d\le x} g(d) \sum_{\substack{n\in S(x)\\ n\equiv 0\,(\mathrm{mod}\ d)}} 1 = O\Big(x^{1/2}\sum_{x^{1/4}\log^{-3/2}x<d\le x}\frac{|g(d)|}{d}\Big).$$

By Abel's identity and (2.2), we arrive at

$$x^{1/2}\sum_{x^{1/4}\log^{-3/2}x<d\le x}\frac{|g(d)|}{d} \ll x^{1/4+\lambda/4(1+\kappa)}(\log x)^{3/2-3\lambda/2(1+\kappa)},$$

and Theorem 1.4 follows. □

Proof of Theorem 1.5. The proof is the same as the proof of Theorem 1.4. We assume the Riemann hypothesis. Thus, we can replace (2.1) by

$$\sum_{d\le x}|g(d)| = O(x^{1/4+\epsilon}). \tag{2.3}$$

Using (2.3), we choose the exponent pairs (κ,λ) such that $1/4 < \lambda/(1+\kappa)$. The result follows. □

References

[1] A. G. Abercrombie, W. D. Banks and I. E. Shparlinski, 'Arithmetic functions on Beatty sequences', *Acta Arith.* **136** (2009), 81–89.

[2] T. M. Apostol, *Introduction to Analytic Number Theory* (Springer, New York, 1976).

[3] O. Bordellés, 'On certain sums of number theory', *Int. J. Number Theory* **18**(9) (2022), 2053–2074.

[4] O. Bordellés, L. Dai, R. Heyman, H. Pan and I. E. Shparlinski, 'On a sum involving the Euler function', *J. Number Theory* **202** (2019), 278–297.

[5] X. D. Cao and W. G. Zhai, 'The distribution of square-free numbers of the form $\lfloor n^c\rfloor$', *J. Théor. Nombres Bordeaux* **10**(2) (1998), 287–299.

[6] X. D. Cao and W. G. Zhai, 'On the distribution of square-free numbers of the form $\lfloor n^c\rfloor$(II)', *Acta Math. Sinica (Chin. Ser.)* **51** (2008), 1187–1194.

[7] J. M. Deshouillers, 'A remark on cube-free numbers in Segal–Piatetski-Shapiro sequences', *Hardy-Ramanujan J.* **41** (2019), 127–132.

[8] D. V. Goryashin, 'Squarefree numbers in the sequence $\lfloor \alpha n\rfloor$', *Chebyshevskiĭ Sb.* **14**(3) (2013), 42–48.

[9] A. M. Gűloğlu and C. W. Nevans, 'Sums of multiplicative functions over a Beatty sequence', *Bull. Aust. Math. Soc.* **78** (2008), 327–334.

[10] R. Heyman, 'Cardinality of a floor function set', *Integers* **19** (2019), Article no. A67.

[11] A. Ivić, *The Theory of the Riemann Zeta Function* (Wiley, New York, 1985).

[12] K. Liu, J. Wu and Z.-S. Yang, 'On some sums involving the integral part function', Preprint, 2021, arXiv:2109.01382v1.

[13] G. J. Rieger, 'Remark on a paper of Stux concerning squarefree numbers in non-linear sequences', *Pacific J. Math.* **78**(1) (1978), 241–242.

https://doi.org/10.1017/S0004972722001241 Published online by Cambridge University Press

[14] T. Srichan, 'On the distribution of (k, r)-integers in Piatetski-Shapiro sequences', *Czechoslovak Math. J.* **71**(4) (2021), 1063–1070.

[15] T. Srichan, 'Multiplicative functions of special type on Piatetski-Shapiro sequences', *Math. Slovaca* **72**(5) (2022), 1145–1150.

[16] J. Stucky, 'The fractional sum of small arithmetic functions', *J. Number Theory* **238** (2022), 731–739.

[17] I. E. Stux, 'Distribution of squarefree integers in non-linear sequences', *Pacific J. Math.* **59**(2) (1975), 577–584.

[18] P. Tangsupphathawat, T. Srichan and V. Laohakosol, 'Consecutive square-free numbers in Piatetski-Shapiro sequences', *Bull. Aust. Math. Soc.* **105**(2) (2022), 217–222.

[19] K. Veasna, T. Srichan and S. Mavecha, 'On r-free integers in Beatty sequences', *Bol. Soc. Mat. Mex. (3)* **28**(2) (2022), 1–10.

[20] Y. Yu and J. Wu, 'Distribution of elements of a floor function set in arithmetical progressions', *Bull. Aust. Math. Soc.*, to appear. Published online (1 March 2022).

[21] M. Zhang and L. Jinjiang, 'Distribution of cube-free numbers with form $[n^c]$', *Front. Math. China* **12**(6) (2017), 1515–1525.

[22] W. Zhang, 'On general sums involving the floor function with applications to k-free numbers', Preprint, 2021, arXiv:2112.06156.

PINTHIRA TANGSUPPHATHAWAT, Department of Mathematics,
Faculty of Science and Technology, Phranakhon Rajabhat University,
Bangkok 10220, Thailand
e-mail: t.pinthira@hotmail.com

TEERAPAT SRICHAN, Department of Mathematics,
Faculty of Science, Kasetsart University, Bangkok 10900, Thailand
e-mail: fscitrp@ku.ac.th

VICHIAN LAOHAKOSOL, Department of Mathematics,
Faculty of Science, Kasetsart University, Bangkok 10900, Thailand
e-mail: fscivil@ku.ac.th