Notes on the Apollonian Problem and the allied theory.

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1. This paper contains a number of investigations, more or less connected, on the theory of systems of circles. In such a well-worn field one does not expect to have hit upon much that is absolutely new, but it may be hoped that there is sufficient freshness of treatment to give the paper some interest even where it deals with results already known.

The possibility of some of the elements of a figure being imaginary is contemplated throughout, not only in the analytical proofs but also in the few which are purely geometrical in form. It need not be said that, if we are building on the foundation of the ordinary real geometry, say, as contained in Euclid, much is required in the way of definition and deduction before proofs of the latter kind can be considered complete, and unfortunately it is still the practice in our elementary text-books to leave this to be supplied by the reader. Partly to fill some of the blanks, but chiefly to put in relief the point of view from which the subject is considered, one or two paragraphs dealing with the most elementary matters have been inserted at the commencement of the paper.

The ambiguity of certain elements associated with a circle or a system of circles, as, *e.g.*, the radius of a circle, the common tangent of two circles, the axis of similitude of three circles, is the source of inconvenience in the statement of many general theorems, and an attempt has been made to remove this ambiguity by laying down suitable definitions. As one application the inversion invariant of two circles is investigated and a current error pointed out in the statement of Casey's important extension of Ptolemy's Theorem.

Several methods are given for determining the centres and radii of the circles touching three given circles. Two generalisations of the problem are also discussed, and the circles are found which have given common tangents with three given circles, or which intersect them at given angles. The solution of the latter problem is based upon a relation, which appears not to have been noticed before, between the angles of intersection of any five circles. This relation leads directly to

(i) the equation of the two circles satisfying the conditions,

(ii) the equation for the radii of these circles,

(iii) the equation of either circle in a form involving its radius.

The single equation of the two circles is a homogeneous quadratic in what may be called *tricircular coordinates*, any such coordinate being the square of the tangent from the variable point to one of the given circles, divided by the radius of that circle, and the equation involves, besides these coordinates, only the angles of intersection of the given circles with each other and with the required circles.

Several particular cases are worked out, as, for instance, the equations of the four pairs of circles which go through the points of intersection of the given circles, and the equations of the four pairs of Hart circles, each pair of which touches every Apollonian pair.

It is proved that the four pairs of circumscribing circles are also touched by other four pairs of circles. The corresponding proposition in spherical geometry is easy to prove, and perhaps known, but what suggested the theorem was a certain result due, I think, to Cayley and given by Salmon,* in the theory of conics having double contact with a given conic. In fact, the equation in tricircular coordinates of a pair of circles inverse to each other with respect to a given circle is identical in form with the equation in trilinears of a conic touching a given conic at two points. The relation between the theory of such conics and that of circles on a sphere has been noticed and used to advantage by Casey.* Some of the aspects of the connection between the three theories I hope to consider in a supplement to the present paper.

I have to thank my friend, Dr Muirhead, for his kindness in looking up some references, and in placing at my disposal a collection of abstracts of papers on the subject which he drew up for his own use some years ago.

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* Salmon's *Conic Sections*, Sixth Edition, Arts. 386, 387.

Section I.

Definitions and Theorems.

2. (*a*) *A right line* is the assemblage of pairs of values (x, y), or, as we say, of *points* (x, y), satisfying an equation of the first degree.

(*b*) Two lines $Ax + By + C = 0, \quad A'x + B'y + C' = 0$

are *parallel* if $AB' - A'B = 0$; they are *at right angles*, or *perpendicular*, to each other if $AA' + BB' = 0$.

(*c*) The *square of the distance* between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is
$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

The distance itself is two-valued, but we shall distinguish beween PQ and QP so as to have $PQ + QP = 0$, and, if P, Q, R are in a line, $PQ + QR + RP = 0$, as follows.

In the first place, if PQ is parallel to one of the axes, say, to Ox, we take $PQ = x_2 - x_1$.

The line through P, Q being $Ax + By + C = 0$, then, if $A^2 + B^2 = 0$, the formula gives $PQ^2 = 0$ and therefore $PQ = 0$.

In any other case, since $A(x_2 - x_1) + B(y_2 - y_1) = 0$,

we have $(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_2 - x_1)^2(1 + A^2/B^2)$.

Fix upon one of the square roots of $1 + A^2/B^2$ and take
$$PQ = (x_2 - x_1)\sqrt{(1 + A^2/B^2)}.$$

If $R(x_3, y_3)$ and $S(x_4, y_4)$ lie on the line PQ or a parallel to it, take also
$$RS = (x_4 - x_3)\sqrt{(1 + A^2/B^2)},$$
and the product $PQ \,.\, RS$ and ratio $PQ : RS$ are definite, whichever square root of $1 + A^2/B^2$ we may have fixed upon.

(*d*) From Def. (*b*) the line through $P(x_1, y_1)$ perpendicular to $Ax + By + C = 0$, is
$$B(x - x_1) - A(y - y_1) = 0.$$

The *foot of the* $\perp^r$, N, satisfies both equations.

Writing $Ax + By + C = 0$ in the form
$$A(x - x_1) + B(y - y_1) = -(Ax_1 + By_1 + C)$$

we get by squaring and adding

$$(A^2+B^2)\{(x-x_1)^2+(y-y_1)^2\}=(Ax_1+By_1+C)^2$$

that is $$PN=\frac{Ax_1+By_1+C}{\surd(A^2+B^2)}.$$

As in (*c*) one of the square roots is to be selected, and adhered to.

(*e*) If P is (x_1, y_1), $Q(x_2, y_2)$ and $N(x_4, y_4)$ the equation of PN is

$$(x-x_1)(y_1-y_4)-(y-y_1)(x_1-x_4)=0$$

and of QN $$(x-x_2)(y_2-y_4)-(y-y_2)(x_2-x_4)=0.$$

PN, QN will therefore be at right angles, by (*b*), if

$$(x_1-x_4)(x_2-x_4)+(y_1-y_4)(y_2-y_4)=0,$$

which is equivalent to

$$(x_1-x_2)^2+(y_1-y_2)^2=(x_1-x_4)^2+(y_1-y_4)^2+(x_2-x_4)^2+(y_2-y_4)^2$$

or $PQ^2=PN^2+QN^2$. (Euc. I. 47).

Now let N be the foot of the $\perp^r$ from P to QR, where R is (x_3, y_3), then, attending to (*c*)

$$\begin{aligned}2QN\,.\,QR &= QR^2+\ QN^2-(QR-QN)^2\\ &= QR^2+\ QN^2-NR^2\\ &= QR^2+(QN^2+NP^2)-(NR^2+NP^2)\\ &= QR^2+\ QP^2-PR^2. \quad \text{(Euc. II. 12, 13).}\end{aligned}$$

(*f*) A *circle* is the locus of points satisfying an equation of the form $$x^2+y^2+2gx+2fy+c=0.$$

When the equation is brought to the form

$$(x-\alpha)^2+(y-\beta)^2-r^2=0$$

(α, β) is the centre, and r^2 is the square of the radius.

The radius itself, which is $\sqrt{g^2+f^2-c}$ is two-valued. We fix on one of these values arbitrarily, and call that the radius r.

(*g*) The *centre of similitude* of two circles, radii a, b and centres A, B is that point S in the line AB for which

$$SA:SB=a:b.$$

Hence if A is (x_1, y_1) and $B(x_2, y_2)$, S is

$$\left(\frac{ax_2-bx_1}{a-b},\quad \frac{ay_2-by_1}{a-b}\right).$$

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For two real circles with positive radii, S is thus what is usually called the external centre of similitude. In the same case the internal centre of similitude is *the* centre of similitude of the circles a, $-b$, or of $-a$, b. Note that the centre of similitude of a circle, radius a and the same circle, radius $-a$, is the centre of the circle. The centre of similitude of a, a is indeterminate, that is, any point is a centre of similitude.

The *axis of similitude* of three circles a, b, c is that line the $\perp^{rs}$ on which from the centres are proportional to the radii. It therefore passes through the three centres of similitude of the circles taken in pairs.

(*h*) Two circles of radii a, b, or, as we shall usually say for brevity, two circles a, b will be said to *touch* if the square of the distance between the centres, D^2, is equal to $(a-b)^2$.

The circles have two coincident points in common not only when $D^2 = (a-b)^2$ but also when $D^2 = (a+b)^2$, but in the latter case we say that it is the circles a, $-b$ or $-a$, b that *touch*. It may sometimes be convenient to state, in the ordinary sense, that two circles touch, but in such a case we shall take care that the radii are not specifically mentioned.

It follows from the definition that when two circles, with radii assigned, touch, the point of contact is the centre of similitude. Also, if two real circles touch internally (concavely), their radii have the same sign; if they touch externally (convexly), their radii have opposite signs.

(*i*) There are four lines which cut each circle in two coincident points.

The *common tangents* of a, b are the two which pass through the centre of similitude.

The square of the length of a common tangent is defined to be $D^2 - (a-b)^2$.

It vanishes when, and only when, the circles touch.

For the length itself, either root of its square may be taken.

(j) The *cosine of the angle between two circles* a, b is

$$\frac{a^2 + b^2 - D^2}{2ab}.$$

Here we follow Salmon, but for some purposes it would be a good deal more convenient to take the cosine with the opposite sign. For instance, the analogy between certain formulæ for circles, and the corresponding formulæ for right lines would thus be more apparent.

For two real intersecting circles, the cosine as defined belongs to the angle subtended at a common point by the line joining the centres.

For the angle itself, we fix on any one of the infinite number of angles whose cosines have the value specified.

The angle between two circles which touch may thus be taken as zero, the angle between a circle and itself as zero, and the angle between a and $-a$ as π.

(k) *The angle θ between a line* L *and a circle a* is given by

$$p = a\cos\theta$$

where p is the $\perp^r$ (of definite sign) from the centre to the line.

A line L *touches* a circle a if it meets it at angle zero, that is, if $p = a$.

If the signs of all the $\perp^{rs}$ to the line L be changed, we may speak of the line in the altered circumstances as the line $-$L. If L touches a, then $-$L touches $-a$, but not a.

(l) Two points P, Q are inverse to each other with respect to a circle of centre O and radius k, if O, P, Q are in a line and OP . OQ $= k^2$.

If O is the origin and P is (x, y), Q is $\left(\dfrac{k^2x}{x^2+y^2}, \dfrac{k^2y}{x^2+y^2}\right)$.

The inverse of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

is thus the circle

$$x^2 + y^2 + 2gxk^2/c + 2fyk^2/c + k^4/c = 0. \qquad (2)$$

Consider another pair of inverse circles

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0 \qquad (3)$$

and $$x^2 + y^2 + 2g'xk^2/c' + 2f'yk^2/c' + k^4/c' = 0. \qquad (4)$$

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Let the radii of the four circles be r_1, r_2, r_3, r_4; the angle between (1) and (3) θ_1 and that between (2) and (4) θ_2.

Then by (j) $2r_1r_3\cos\theta_1 = 2gg' + 2ff' - c - c'$,

$$2r_2r_4\cos\theta_2 = (2gg' + 2ff' - c - c')k^4/cc'.$$

Now $r_2 = \pm k^2r_1/c$ and $r_4 = \pm k^2r_3/c'$.

If we take $r_2 = +k^2r_1/c$ and $r_4 = +k^2r_3/c'$

we shall have $\cos\theta_1 = \cos\theta_2$; also according to ($g$) O will be the centre of similitude of each inverse pair.

Two circles will not be called inverse to each other, when their radii are definite, unless the centre of inversion is their centre of similitude.

Then we have the theorem that two circles cut at the *same* angle as their inverses.

By a remark made under (g) the circle of inversion k is not inverse to itself but to the circle $-k$; also a circle will be inverse to itself if the square of the tangent from O is equal to k^2. Hence the circle of inversion k cuts two inverse circles at *supplementary* angles; a self-inverse circle cuts them at equal angles.

(m) If, in (1) of (l), $c = 0$, instead of (2) we have the line

$$2gx + 2fy + k^2 = 0.$$

We prove in the same way that $\cos\theta_1 = \cos\theta_2$ provided $k^2 = -2r_1p$, where p is the $\perp^r$ from O on the line. The line will be said to be inverse to the circle (1) when the signs of its $\perp^{rs}$ are so determined that that relation is fulfilled.

(n) Let D_1 be the distance between the centres of (1), (3) of (l); D_2 that between the centres of (2), (4).

Since $\cos\theta_1 = \cos\theta_2$ we have

$$\frac{D_1^2 - r_1^2 - r_3^2}{2r_1r_3} = \frac{D_2^2 - r_2^2 - r_4^2}{2r_2r_4}$$

and therefore $$\frac{D_1^2 - (r_1 - r_3)^2}{2r_1r_3} = \frac{D_2^2 - (r_2 - r_4)^2}{2r_2r_4}$$

or the quotient of the square of the common tangent of two circles by the product of their radii is not altered by inversion.

The statement of the theorem in the usual unconventional language is much less simple.

Thus let a, b, c, d be the real positive radii of the real circles (1), (2), (3), (4).

To fix the ideas, we may suppose k^2 positive, but the final statement will not be affected if k^2 be negative. Then if O is outside (1) it will be the *external* centre of similitude of (1), (2), but if O is inside (1) it will be their *internal* centre of similitude. Similarly with (3), (4).

If O is outside both of (1), (3) then for the purposes of the theorem the radii of (2), (4) are $+b$, $+d$; if O is inside both of (1), (3) the radii of (2), (4) are $-b$, $-d$; it being supposed that the radii a, c are kept fixed. In these cases, then, we have

$$\frac{D_1^2 - (a-c)^2}{ac} = \frac{D_2^2 - (b-d)^2}{bd}.$$

If we had taken the radius of (3) as $-c$, that of (4) would have been $-d$, and therefore

$$\frac{D_1^2 - (a+c)^2}{ac} = \frac{D_2^2 - (b+d)^2}{bd}.$$

These two relations, which obviously are not independent, show that if the centre of inversion is *inside all the circles, or outside them all*, then

(sq. of either com. tang.)/prod. of radii inverts into
(sq. of similar com. tang.)/prod. of radii.

But if O is outside one of the two original circles, and inside the other, the radii of the inverse circles will have opposite signs, and therefore

$$\frac{D_1^2 - (a-c)^2}{ac} = -\frac{D_2^2 - (b+d)^2}{bd}$$

and
$$\frac{D_1^2 - (a+c)^2}{ac} = -\frac{D_2^2 - (b-d)^2}{bd},$$

that is to say, if the centre of inversion is inside one inverse pair, but outside the other inverse pair, then

(sq. of either com. tang.)/prod. of radii inverts into
– (sq. of the dissimilar com. tang.)/prod. of radii.

In the case of two intersecting circles, it is easy to verify the result from a figure, by comparing the angles between the original and inverted circles.

In the first case it will be readily seen that $\cos\theta_1 = \cos\theta_2$, but in the second that $\cos\theta_1 = -\cos\theta_2$.

Casey, to whom the theorem is due, states simply that "if two circles be inverted into two others, the square of the common tangent of the first pair, divided by the rectangle contained by their diameters, is equal to the square of the common tangent of the second pair, divided by the rectangle contained by their diameters." This is vague, but from the way in which he applies the theorem it seems clear that he understands the common tangent to be direct in both cases, or else transverse in both cases.

He is thus led into an inaccurate statement of his theorem

$$12\cdot34 \pm 13\cdot24 \pm 14\cdot23 = 0$$

respecting the common tangents of four circles touched by a fifth. He states that the direct tangent is to be taken between two circles which are *on the same side* of the fifth circle, the transverse between two which are on opposite sides of it. This is wrong and ought to be that the direct tangent is taken when two circles are both touched concavely, or both convexly, by the fifth circle; the transverse when one is touched concavely and the other convexly.* Two circles touched convexly are necessarily on the same side of the fifth circle, namely, the outside; but a circle touched concavely may be either outside or inside of it. Salmon, in giving an account of the matter, uses different methods from Casey, but repeats the defective enunciation of the theorem. (Salmon's *Conic Sections*, last 3 Arts. of Chap. VIII., Sixth Edition.)

SECTION II.

First solution of the Apollonian and allied problems.

3. *To find the centres and radii of the circles touching three given circles a, b, c.*

We do not assume that a, b, c are positive, even if they are real.

First Method.

Let $D(x_1, y_1)$, $E(x_2, y_2)$, $F(x_3, y_3)$ be the centres of a, b, c.

We take the origin at the radical centre O so that

$$x_1^2 + y_1^2 - a^2 = x_2^2 + y_2^2 - b^2 = x_3^2 + y_3^2 - c^2 = k^2.$$

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* See Art. 16 below.

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The circle, centre O and radius k, will be referred to throughout as the *orthogonal circle* simply.

Let $P(\xi, \eta)$ be the centre of a circle ρ touching a, b, c.

Then from Art. 2 (h) we have three equations, of which the first is

$$(\xi - x_1)^2 + (\eta - y_1)^2 = (\rho - a)^2$$

$$\text{or} \quad 2\xi x_1 + 2\eta y_1 + (\rho^2 - \xi^2 - \eta^2 - k^2) = 2a\rho. \qquad (1)$$

This and the two similar equations may be regarded as asserting that the $\perp^{rs}$ from (x_1, y_1), (x_2, y_2), (x_3, y_3) on the line

$$2\xi x + 2\eta y + (\rho^2 - \xi^2 - \eta^2 - k^2) = 0 \qquad (2)$$

are as $a : b : c$.

The line (2) is therefore the axis of similitude.

Let the actual $\perp^{rs}$ on it from D, E, F, O be a/λ, b/λ, c/λ, p.

Then by 2 (d) $\quad 2a\rho : a/\lambda = \rho^2 - \xi^2 - \eta^2 - k^2 : p$

that is
$$2\lambda\rho p = \rho^2 - \xi^2 - \eta^2 - k^2 \qquad (3)$$

and
$$\lambda^2\rho^2 = \xi^2 + \eta^2. \qquad (4)$$

Eliminating $\xi^2 + \eta^2$ from (3), (4) we get the equation for ρ, viz.,

$$(\lambda^2 - 1)\rho^2 + 2\lambda p\rho + k^2 = 0. \qquad (5)$$

The line OP which is $\xi y - \eta x = 0$, is $\perp^r$ to (2), the axis of similitude.

The $\perp^r$ p_1 from P on (2) is

$$(\rho^2 + \xi^2 + \eta^2 - k^2)/2\lambda\rho$$

and if $OP = \sigma$, taken with definite sign as a segment on the same line as p and p_1,

$$\begin{aligned} \sigma &= p - p_1 \\ &= -(\xi^2 + \eta^2)/\lambda\rho \\ &= -\lambda\rho. \qquad (6) \end{aligned}$$

We have thus defined precisely the centres and radii of two circles ρ_1, ρ_2 touching a, b, c.

Since σ/ρ is the same for both, O is their centre of similitude.

Also from (3) which is $\quad 2\sigma p = \sigma^2 + k^2 - \rho^2$

it follows from Euclid II., 12, 13 proved in 2 (e) that the points of intersection of ρ, k are on the axis of similitude.

The circles ρ_1, ρ_2 are therefore inverse to each other with respect to k, and the axis of similitude is their radical axis. The first part of this statement is proved more simply by observing that the inverse of ρ_1 with respect to k will, by 2 (l), cut a, b, c which are self-inverses at the same angle as ρ_1, that is, will also touch them.

Again if X is the point of contact of ρ, a; and if the $\perp^r$ from D to the axis of similitude meets it at L, and OX at G; we have from the similar triangles OPX, GDX, attending to signs

$$\mathrm{OP}:\mathrm{GD}=\mathrm{XP}:\mathrm{XD}$$
$$=\rho \quad : a \quad \text{, by 2 } (g), (h).$$
$$\therefore \quad \mathrm{GD}=a\sigma/\rho = -\lambda a, \text{ by (6).}$$
$$\text{Also} \quad \mathrm{DL}=a/\lambda;$$

$\therefore$ $\mathrm{DL}\,.\,\mathrm{DG}=a^2$, and G is the pole of the axis of similitude with respect to the circle a. Hence Gergonne's construction, viz., we find X_1, X_2 as the intersections of a with OG; then P_1, P_2 are the intersections of the $\perp^r$ to the axis of similitude through O with X_1D, X_2D.

4. *Second Method.*

If X, Y be the centres of two circles x, y;

T_1^2, T_2^2 the squares of the tangents from a point U, that is

$$T_1^2=\mathrm{UX}^2-x^2,\quad T_2^2=\mathrm{UY}^2-y^2,$$

and if UN be the $\perp^r$ from U to the radical axis, we have the fundamental theorem

$$T_1^2-T_2^2=2\mathrm{UN}\,.\,\mathrm{YX}.$$

Apply the theorem to the circles k, ρ taking U, first at D, then at O, and putting p_a, p for the $\perp^{rs}$ from D, O on the radical axis of k, ρ.

$$\left.\begin{aligned} &\text{Thus} && a^2-\{(a-\rho)^2-\rho^2\}=2p_a(-\sigma)\\ &\text{and} && -k^2-(\sigma^2-\rho^2)=2p\,(-\sigma)\end{aligned}\right\}$$

that is

$$\left.\begin{aligned} a\rho &= -\sigma p_a\\ \sigma^2+k^2-\rho^2 &= 2\sigma p\end{aligned}\right\}.$$

The first of these, and the two similar equations

$$b\rho=-\sigma p_b,\quad c\rho=-\sigma p_c$$

show that the radical axis of k, ρ is the axis of similitude of a, b, c; and if we put $p_a=a/\lambda$ we have the two equations

$$\left.\begin{aligned} \sigma &= -\lambda\rho\\ \sigma^2+k^2-\rho^2 &= 2\sigma p\end{aligned}\right\} \text{ and the whole theory as before.}$$

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5. *Third Method.*

Take P an undefined point on the $\perp^r$ from O to the axis of similitude; $OP = \sigma$; $\perp^{rs}$ from O, D to axis of similitude $= p$, a/λ.

By Euclid II. 12, 13

$$PD^2 = OD^2 + OP^2 - 2OP(p - a/\lambda)$$
$$= a^2 + k^2 + \sigma^2 - 2\sigma p + 2\sigma pa/\lambda.$$

We shall have

$$PD^2 = (a - \rho)^2$$

if $$0 = (k^2 + \sigma^2 - 2\sigma p - \rho^2) + 2a(\rho + \sigma/\lambda).$$

We shall therefore also have

$$PE^2 = (b - \rho)^2 \quad \text{and} \quad PF^2 = (c - \rho)^2$$

if we take

$$\left.\begin{aligned} k^2 + \sigma^2 - 2\sigma p - \rho^2 &= 0 \\ \rho + \sigma/\lambda \quad &= 0 \end{aligned}\right\}.$$

6. The last method may be used to solve a more general problem, viz., if the radii of the three circles with centres D, E, F are changed to $a - x$, $b - x$, $c - x$, to find the new axis of similitude and the new orthogonal circle.

We assert in the first place that as x varies the axis of similitude moves parallel to itself, and the radical centre in a line $\perp^r$ to it. This is obvious if we assume the results just proved, for the centres of the circles touching $a - x$, $b - x$, $c - x$ will remain fixed, their radii being $\rho_1 - x$, $\rho_2 - x$. For an independent proof, note first that the parallel to the axis of similitude of a, b, c at distance $-x/\lambda$ from it will be at distances $(a - x)/\lambda$, $(b - x)/\lambda$, $(c - x)/\lambda$ from D, E, F, and will therefore be the axis of similitude of $a - x$, $b - x$, $c - x$.

Next take O′ a point on the $\perp^r$ to the axis of similitude from O; $OO' = y$.

We have $$O'D^2 = OD^2 + OO'^2 - 2OO'(p - a/\lambda)$$
$$= a^2 + k^2 + y^2 - 2yp + 2ya/\lambda$$

and $$O'D^2 - (a - x)^2 = (k^2 + y^2 - 2yp - x^2) + 2a(x + y/\lambda).$$

Hence if we define y by the equation $x + y/\lambda = 0$

we shall have

$$O'D^2 - (a - x)^2 = O'E^2 - (b - x)^2 = O'F^2 - (c - x)^2$$
$$= (\lambda^2 - 1)x^2 + 2\lambda px + k^2;$$

$\therefore$ O′ is the radical centre of $a-x$, $b-x$, $c-x$ and the square of the radius of their orthogonal circle is given by

$$k'^2 = (\lambda^2-1)x^2 + 2\lambda px + k^2.$$

k' will be zero when the circles have a common point which must be either P_1 or P_2.

The radius of the touching circle is then zero, *i.e.*, x is equal to ρ_1 or ρ_2; hence the equations for σ, ρ as before.

The new radical centre O′ will be on the new axis of similitude when

$$OO' = p - x/\lambda$$

$$\text{or} \quad (\lambda^2-1)x = -\lambda p.$$

Since with this value of x, the centre of similitude of the touching circles is on their radical axis, it is geometrically obvious that their radii are equal and of opposite signs, that is,

$$(\rho_1 - x) + (\rho_2 - x) = 0.$$

Of course this also follows immediately from the quadratic for ρ. The result will be used in Art. 13.

7. The equation $\quad (\lambda^2-1)\rho^2 + 2\lambda p\rho + k^2 = 0$

may be partially verified as follows.

(i) When $k^2 = 0$; the three circles have a common point, which is a touching circle of radius zero; the equation gives one value of $\rho = 0$.

(ii) When $\lambda = 1$; the $\perp^{rs}$ from the centres to the axis of similitude are a, b, c and the axis of similitude is a common tangent.

When $\lambda = -1$; the $\perp^{rs}$ are $-a$, $-b$, $-c$ and the axis of similitude with the signs of all its $\perp^{rs}$ changed is a common tangent.

In either case one of the touching circles is a line; the equation gives one value of ρ infinite.

(iii) When $p = 0$; the radical centre lies on the axis of similitude, and therefore as at the end of Art. 6, the radii of the touching circles are equal but of opposite signs; the equation gives $\rho_1 + \rho_2 = 0$.

8. Suppose that the three circles are real, and that the absolute values of their radii are r_1, r_2, r_3.

If in the above analysis we take $a=r_1$, $b=r_2$, $c=r_3$ the axis of similitude is that which has D, E, F all on one side of it. To a positive real root of the equation for ρ corresponds a circle touching all the circles concavely; to a negative real root a circle touching them all convexly. (2 (h), last remark.)

If we take $a=-r_1$, $b=r_2$, $c=r_3$ the axis of similitude is that which has D on one side of it, and E, F on the other. A positive real value of ρ gives a circle touching the first circle convexly, the other two concavely; a negative real value of ρ gives a circle touching the first circle concavely, the other two convexly.

Similarly for the cases

$$a=r_1,\ b=-r_2,\ c=r_3 \quad \text{and} \quad a=r_1,\ b=r_2,\ c=-r_3.$$

Since a, b, c have the same axis of similitude and orthogonal circle as $-a$, $-b$, $-c$, the other four permutations of signs of the radii do not yield further solutions.

A very persistent form of erroneous statement of these results should be noticed. For the first case, as an instance, it is frequently said that when the values of ρ are real, a pair of tangent circles exists each of which has the three given circles *all on the same side of it*. The error is analogous to that which was noticed at the end of Art. 2.

9. *To find the centres and radii of the circles cutting a, b, c at given angles α, β, γ.*

We can proceed precisely as in Art. 3 till we come to equation (1) which will now be

$$2\xi x_1 + 2\eta y_1 + (\rho^2 - \xi^2 - \eta^2 - k^2) = 2a\rho\cos\alpha.$$

The $\perp^{rs}$ on the line (2) will therefore be as $a\cos\alpha : b\cos\beta : c\cos\gamma$.

This property defines that line; it cuts the circle at angles with cosines proportional to $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and we may call it the α, β, γ axis.

If the actual $\perp^{rs}$ on it from D, E, F, O are

$$a\cos\alpha/\lambda',\ b\cos\beta/\lambda',\ c\cos\gamma/\lambda',\ p',$$

we find equations of the same form as before

$$\left.\begin{array}{c} \sigma = -\lambda'\rho \\ (\lambda'^2-1)\rho^2 + 2\lambda' p'\rho + k^2 = 0 \end{array}\right\}.$$

The circles ρ_1, ρ_2 are inverse with respect to k, and the α, β, γ axis is their radical axis.

10. *To find the centres and radii of the circles the squares of whose common tangents with a, b, c have given values u^2, v^2, w^2.*

Following again the lines of Art. 3, we choose the origin O′ so that

$$x_1^2+y_1^2-a^2-u^2=x_2^2+y_2^2-b^2-v^2=x_3^2+y_3^2-c^2-w^2 \quad (=k'^2 \text{ say}).$$

O′ is therefore the radical centre of the circles with centres D, E, F and squares of radii a^2+u^2, b^2+v^2, c^2+w^2, and k'^2 is the square of the radius of their orthogonal circle.

The line (2), with k'^2 for k^2, is still the axis of similitude of a, b, c. If the $\perp^r$ from O′ to it is p'', we have

$$\left.\begin{array}{l}\sigma=-\lambda\rho \\ (\lambda^2-1)\rho^2+2\lambda p''\rho+k'^2=0\end{array}\right\}.$$

The circles ρ_1, ρ_2 are inverses with respect to k' and the axis of similitude of a, b, c is their radical axis.

Either of the two problems just discussed may be reduced at once to the other, for

$$\rho^2-2a\rho\cos\alpha+a^2=(\rho-a\cos\alpha)^2+a^2\sin^2\alpha$$

and therefore a circle which cuts a, b, c at angles α, β, γ will have $a^2\sin^2\alpha$, $b^2\sin^2\beta$, $c^2\sin^2\gamma$ for the squares of its common tangents with the circles whose centres are D, E, F and radii $a\cos\alpha$, $b\cos\beta$, $c\cos\gamma$.

The other methods given for the case of contact may easily be adapted to the more general problems.

11. The methods of this section fail in certain cases; notably when the centres of the circles are in a line, a case requiring exceptional treatment in most general methods for the contact problem, including Gergonne's. The other cases of failure arise when the axis of similitude or the α, β, γ axis is at an infinite distance, but these cases are easily met. Thus in 3, when $a=b=c$, equations (1) give at once

$$\xi=0,\ \eta=0\ ;\quad \rho^2-k^2=2a\rho.$$

In 9 when $a\cos\alpha=b\cos\beta=c\cos\gamma$, we have similarly

$$\xi=0,\ \eta=0\ ;\quad \rho^2-k^2=2a\rho\cos\alpha.$$

The equations obtained in this section involve coefficients whose geometrical meaning, as we have seen, is very simple; but results may be required in terms of more fundamental constants of the system of given circles, as for instance, the radii and distances between the centres. It is not difficult, but certainly tedious, to deduce such results directly from those found here. One way is to use trilinear coordinates with the triangle of centres as triangle of reference. In the investigation I found the following formula useful; it gives the trilinear coordinates α, β, γ of a point P in terms of the squares of its distances from the vertices A, B, C of the triangle of reference, whose sides are a, b, c.

$$4\Delta\alpha = -a\,.\,AP^2 + b\cos C\,.\,BP^2 + c\cos B\,.\,CP^2 + abc\cos A$$

with similar expressions for β and γ.

But much more powerful methods are available.

Section III.

Miscellaneous methods for the Apollonian problem.

12. *To find the radii of the circles touching a, b, c.*

It has been remarked by numerous writers that the relation between the six mutual distances of four points in a plane furnishes a natural and immediate solution of this problem.

If $PD^2 = x^2$, $PE^2 = y^2$, $PF^2 = z^2$ and $EF^2 = d^2$, $FD^2 = e^2$, $DE^2 = f^2$ then

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & f^2 & e^2 & x^2 \\ 1 & f^2 & 0 & d^2 & y^2 \\ 1 & e^2 & d^2 & 0 & z^2 \\ 1 & x^2 & y^2 & z^2 & 0 \end{vmatrix} = 0. \quad (1).$$

To find the radii we have only to put $x^2 = (a-\rho)^2$, $y^2 = (b-\rho)^2$, $z^2 = (c-\rho)^2$. But the equation will be much simpler if we first transform this determinant D as follows.

From the 2nd, 3rd, and 4th columns subtract the last; then deal in the same way with the rows. Thus

$$D = \begin{vmatrix} 2x^2, & x^2+y^2-f^2, & x^2+z^2-e^2 \\ x^2+y^2-f^2, & 2y^2, & y^2+z^2-d^2 \\ x^2+z^2-e^2, & y^2+z^2-d^2, & 2z^2 \end{vmatrix}.$$

Divide the rows by $2x$, $2y$, $2z$ and the columns by x, y, z and we may write

$$D = 8x^2y^2z^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & \dfrac{x^2+y^2-f^2}{2xy} & \dfrac{x^2+z^2-e^2}{2xz} \\ 1 & \dfrac{x^2+y^2-f^2}{2xy} & 1 & \dfrac{y^2+z^2-d^2}{2yz} \\ 1 & \dfrac{x^2+z^2-e^2}{2xz} & \dfrac{y^2+z^2-d^2}{2yz} & 1 \end{vmatrix}$$

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and on subtracting the first row from each of the others

$$D = -8x^2y^2z^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & \frac{n^2}{2xy} & \frac{m^2}{2xz} \\ 1 & \frac{n^2}{2xy} & 0 & \frac{l^2}{2yz} \\ 1 & \frac{m^2}{2xz} & \frac{l^2}{2yz} & 0 \end{vmatrix} = -2 \begin{vmatrix} \frac{1}{2} & x & y & z \\ x & 0 & n^2 & m^2 \\ y & n^2 & 0 & l^2 \\ z & m^2 & l^2 & 0 \end{vmatrix}, \quad (2)$$

where $\quad l^2 = d^2 - (y-z)^2, \;\; m^2 = e^2 - (z-x)^2, \;\; n^2 = f^2 - (x-y)^2.$

If now we put $a-\rho$ for x, $b-\rho$ for y, $c-\rho$ for z, so that l^2, m^2, n^2 are now the squares of the common tangents of a, b, c, we find

$$\begin{vmatrix} \frac{1}{2} & a-\rho & b-\rho & c-\rho \\ a-\rho & 0 & n^2 & m^2 \\ b-\rho & n^2 & 0 & l^2 \\ c-\rho & m^2 & l^2 & 0 \end{vmatrix} = 0. \quad (3).$$

This determinant is of a familiar type and its expansion gives

$$\begin{aligned} & l^2m^2n^2 \\ & + l^4(a-\rho)^2 + m^4(b-\rho)^2 + n^4(c-\rho)^2 \\ & - 2m^2n^2(b-\rho)(c-\rho) - 2n^2l^2(c-\rho)(a-\rho) - 2l^2m^2(a-\rho)(b-\rho) = 0 \end{aligned}$$

or

$$\begin{aligned} & \rho^2(l^4 + m^4 + n^4 - 2m^2n^2 - 2n^2l^2 - 2l^2m^2) \\ & + 2\rho\{(b+c)m^2n^2 + (c+a)n^2l^2 + (a+b)l^2m^2 - al^4 - bm^4 - cn^4\} \\ & + (l^2m^2n^2 + a^2l^4 + b^2m^4 + c^2n^4 - 2bcm^2n^2 - 2can^2l^2 - 2abl^2m^2) = 0. \quad (4) \end{aligned}$$

This is the equation found geometrically by Mr Alex. Holm in last year's *Proceedings*, except that, in accordance with our conventions, we have $-\rho$ instead of his x.

13. *Deduction of the equation found in Section II.*

If we write the left sides of equations (3), (4) just found as

$$L\rho^2 + M\rho + N,$$

L, M, N must from Art. 3 be proportional to $\lambda^2 - 1$, $2\lambda p$, k^2.

This we shall now prove independently by a development of the remarks in 7.

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(i) The circles with centres D, E, F and squares of radii a^2+k^2, b^2+k^2, c^2+k^2 have a common point, viz., the radical centre of a, b, c.

In 12 (1) put $x^2=a^2+k^2$, $y^2=b^2+k^2$, $z^2=c^2+k^2$. Then from last column and last row subtract the first $\times k^2$. Thus

$$-2k^2\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & f^2 & e^2 \\ 1 & f^2 & 0 & d^2 \\ 1 & e^2 & d^2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & f^2 & e^2 & a^2 \\ 1 & f^2 & 0 & d^2 & b^2 \\ 1 & e^2 & d^2 & 0 & c^2 \\ 1 & a^2 & b^2 & c^2 & 0 \end{vmatrix} = 0.$$

The first determinant is $-16\Delta^2$ where $\Delta =$ area of Δ DEF; the second, by the transformation by which (2) was derived from (1) in 12, is $-2\mathrm{N}$.

$$\therefore \quad \mathrm{N}=k^2 \, . \, 16\Delta^2.$$

(ii) The circles with centres D, E, F and radii a/λ, b/λ, c/λ have a common tangent. Hence if we put a/λ for a, and so on, in (3) or (4) above the coefficient of the highest power of ρ must vanish, that is

$$l'^4+m'^4+n'^4-2m'^2n'^2-2n'^2l'^2-2l'^2m'^2=0$$

where $\qquad l'^2=d^2-(b-c)^2/\lambda^2$, etc.

Write the left side of this equation $\mathrm{A}+\mathrm{B}/\lambda^2+\mathrm{C}/\lambda^4$.

Then $\mathrm{C}=0$, for it is $-16\times$ square of area of Δ with sides $b-c$, $c-a$, $a-b$;

$\mathrm{A}=-16\Delta^2$, by putting $1/\lambda=0$.

To get B, note that when $\lambda=1$, $l'^2=l^2$ and therefore $\mathrm{A}+\mathrm{B}=\mathrm{L}$; but $\mathrm{A}+\mathrm{B}/\lambda^2=0$; hence $\lambda^2-1=-(\mathrm{A}+\mathrm{B})/\mathrm{A}$,

$$i.e., \quad \mathrm{L}=(\lambda^2-1) \, . \, 16\Delta^2.$$

(iii) By Art. 6 if we take $a-x$, $b-x$, $c-x$ for radii the coefficient of ρ in the resulting quadratic will vanish when

$$x(\lambda^2-1)=-\lambda p.$$

But when a, b, c are replaced by $a-x$, $b-x$, $c-x$ in 12 (4) the coefficient of ρ becomes $\mathrm{L}x+\frac{1}{2}\mathrm{M}$. $\therefore$ $\mathrm{L} \, . \, 2\lambda p=\mathrm{M}(\lambda^2-1)$, and, using (ii),

$$\mathrm{M}=2\lambda p \, . \, 16\Delta^2.$$

14. *The equation for ρ deduced from considerations of Solid Geometry.*

Considered analytically, the problem of finding a circle, centre $P(\xi, \eta)$, radius ρ, to touch the circles,

centres $D(x_1, y_1)$, $E(x_2, y_2)$, $F(x_3, y_3)$ and radii a, b, c
is the same as that of solving the three equations of the form

$$(\xi - x_1)^2 + (\eta - y_1)^2 = (\rho - a)^2.$$

These equations express that the point in space $(\xi, \eta, i\rho)$ is at distance zero from each of the points (x_1, y_1, ia), etc. The problem is therefore equivalent to that of finding the centre of a sphere of given radius passing through three given points, and this can be solved geometrically.

In order to obtain a figure with as many real elements as possible, we shall suppose that the radii are all pure imaginaries, but the formula obtained for ρ will still be true even if a, b, c are all real.

Let L, M, N be the points (x_1, y_1, ia), etc.

Then $$MN^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 - (b - c)^2 = l^2;$$
$$NL^2 = m^2; \; LM^2 = n^2.$$

If H (X, Y, Z) is the centre and R the radius of the circum-circle of LMN, then a point Q $(\xi, \eta, i\rho)$ at zero distance from L, M, N is on the normal through H to the plane LMN at distance $\pm i$R from H.

A well-known formula gives the coordinates of H in terms of those of L, M, N.

In particular,

$$Z = \frac{l\cos L \,.\, ia + m\cos M \,.\, ib + n\cos N \,.\, ic}{l\cos L + m\cos M + n\cos N}$$
$$= \{l^2(m^2 + n^2 - l^2)a + m^2(n^2 + l^2 - m^2)b + n^2(l^2 + m^2 - n^2)c\}i/16\Delta_{lmn}^2.$$

Then, for the z coordinate of Q,

$$i\rho = Z \pm i R\cos\phi$$

where ϕ is the angle between the planes LMN, DEF, and therefore

$$\cos\phi = \Delta_{def}/\Delta_{lmn}.$$
$$\text{Also} \quad R = lmn/4\Delta_{lmn}.$$

Hence $\rho = \{l^2(m^2 + n^2 - l^2)a + \ldots + \ldots \pm 4lmn\Delta_{def}\}/16\Delta_{lmn}^2.$

This is the formula obtained by Mr Holm in his paper of last year.

We can also derive the results of Art. 3.

(i) If p_1 is the $\perp^r$ DT from D to the line of intersection ST of the planes LMN, DEF then $ia = p_1 \tan\phi$; similarly $ib = p_2\tan\phi$ and $ic = p_3\tan\phi$.

Hence ST is the axis of similitude, and $\tan\phi = i\lambda$.

$$\therefore \quad \lambda^2 - 1 = -\sec^2\phi$$

$$\text{and} \quad (\lambda^2 - 1)\Delta^2_{def} = -\Delta^2_{lmn}.$$

(ii) Let the normal HQ to the plane LMN meet the plane DEF in K, and let KUS be $\perp^r$ to ST.
Then UH $\perp^r$ to KUS is Z.

Also $\quad \mathrm{KD}^2 - a^2 = \mathrm{KD}^2 + \mathrm{DL}^2 = \mathrm{KH}^2 + \mathrm{R}^2.$

$$\therefore \quad \mathrm{KD}^2 - a^2 = \mathrm{KE}^2 - b^2 \quad = \mathrm{KF}^2 - \mathrm{C}^2.$$

$\therefore$ K is the radical centre and $k^2 = \mathrm{KH}^2 + \mathrm{R}^2$

$$= \mathrm{Z}^2\sec^2\phi + \mathrm{R}^2.$$

(iii) KS is p, and $\mathrm{Z} = p\sin\phi\cos\phi = p\tan\phi\cos^2\phi$
so that $\quad (\lambda^2 - 1)\mathrm{Z} = -i\lambda p.$

Now $\quad \rho = -i\mathrm{Z} \pm \mathrm{R}\cos\phi$

and the quadratic for ρ is

$$(\rho + i\mathrm{Z})^2 = \mathrm{R}^2\cos^2\phi,$$

that is $\quad \rho^2 + 2i\mathrm{Z}\rho - (\mathrm{Z}^2 + \mathrm{R}^2\cos^2\phi) = 0,$

or, multiplying by $\lambda^2 - 1$, which is equal to $-\sec^2\phi$,

$$(\lambda^2 - 1)\rho^2 + 2i(\lambda^2 - 1)\mathrm{Z}\rho + \mathrm{Z}^2\sec^2\phi + \mathrm{R}^2 = 0$$

$$i.e., \quad (\lambda^2 - 1)\rho^2 + 2\lambda p\rho + k^2 = 0.$$

It is also obvious from the figure that the centre P of ρ lies on KS and that $\sigma = -\lambda\rho$.

Section IV.

Application of general theorems relating to given circles.

Equations of the circles required by the general problems, in terms of tricircular coordinates.

15. We write S_1 for $x^2 + y^2 + 2g_1x + 2f_1y + c_1$,

S_2 for $x^2 + y^2 + 2g_2x + 2f_2y + c_2$, and so on.

S_1 is the square of the tangent from (x, y) to the circle $\mathrm{S}_1 = 0$.

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Also we shall habitually use S_1, S_2, S_3 with reference to the three circles a, b, c round which the problems hinge.

When we speak of inverse points, or inverse circles, we shall understand the circle of inversion to be the orthogonal circle k.

The values of S_1, S_2, S_3 at the point (x, y) are proportional to their values at the inverse point (x', y'); for, taking the origin at the radical centre, we find at once

$$S_1(x', y') = S_1(x, y) \,.\, k^2/(x^2+y^2), \text{ etc., since } c_1 = c_2 = c_3 = k^2.$$

Hence a *homogeneous* equation in S_1, S_2, S_3 represents a locus which, if it contains any point, contains the inverse point also.

As an immediate consequence, a homogeneous equation of the first degree in S_1, S_2, S_3 represents a self-inverse circle. Such a circle is cut orthogonally by k; it is co-orthogonal with a, b, c, and for brevity we shall call it an *orthogonal* simply.

When the ratios $S_1 : S_2 : S_3$ are given, a pair of inverse points are determined, namely, the intersections of two orthogonals as

$$S_2 = C_1 S_1, \quad S_3 = C_2 S_1.$$

When a homogeneous equation of the second degree in S_1, S_2, S_3 is known to represent a circle as part of the locus, the complete locus must be this circle and its inverse.

It will often be convenient to write X, Y, Z instead of S_1/a, S_2/b, S_3/c.

The ratios of the *tricircular coordinates* X, Y, Z determine an *inverse pair* of points.

16. *The circles having given common tangents with a, b, c.*

By an application of a remarkably powerful method, due to Cayley, Salmon proves the relation between the common tangents 12, etc., of any five circles

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 12^2 & 13^2 & 14^2 & 15^2 \\ 1 & 12^2 & 0 & 23^2 & 24^2 & 25^2 \\ 1 & 13^2 & 23^2 & 0 & 34^2 & 35^2 \\ 1 & 14^2 & 24^2 & 34^2 & 0 & 45^2 \\ 1 & 15^2 & 25^2 & 35^2 & 45^2 & 0 \end{vmatrix} = 0. \quad - \quad - \quad (1)$$

For 1, 2, 3 take the circles a, b, c; for 4 a circle ρ the squares of whose common tangents with them are u^2, v^2, w^2.

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The centre square of the determinant is then

$$\begin{matrix} 0 & n^2 & m^2 & u^2 \\ n^2 & 0 & l^2 & v^2 \\ m^2 & l^2 & 0 & w^2 \\ u^2 & v^2 & w^2 & 0 \end{matrix} \qquad - \quad - \quad - \quad - \qquad (2)$$

This will remain unchanged in many of the manipulations now to be made, and we need then only write the border constituents.

Note in passing that if the circles 1, 2, 3, 4 are touched by a circle, say 5, then $15^2 = 25^2 = 35^2 = 45^2 = 0$ and (1) then reduces to, its centre square $= 0$, which is equivalent to

$$12{\cdot}34 \pm 13{\cdot}24 \pm 14{\cdot}23 = 0.^* \qquad - \quad - \quad - \qquad (3)$$

Now two real circles with radii of the same sign must, by 2 (*h*), be touched similarly, *i.e.*, both concavely, or both convexly, by a real circle touching them both; but dissimilarly when the radii are of opposite signs. Hence the proof of the statement at the end of Art. 2 that in (3) direct tangents are to be taken between circles touched similarly, transverse between circles touched dissimilarly, by the fifth circle.

Returning to (1), (2), for the fifth circle take the circle with centre (x, y) and radius r.

Then $$15^2 = (x - x_1)^2 + (y - y_1)^2 - (r - a)^2$$
$$= S_1 + 2ar - r^2, \text{ etc.}$$

Substitute these values of 15^2, 25^2, ..., in (1); to the last row add r^2. the first, and similarly with the last column. We have then

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & S_1 + 2ar \\ 1 & & & & & S_2 + 2br \\ 1 & & & & & S_3 + 2cr \\ 1 & & & & & S_4 + 2\rho r \\ 1, & S_1 + 2ar, & S_2 + 2br, & S_3 + 2cr, & S_4 + 2\rho r, & 2r^2 \end{vmatrix} = 0. \qquad (3)$$

* Conversely, if (3) is satisfied, the circles 1, 2, 3, 4 have a common tangent circle. To prove this, take the circle 5 so that $15^2 = 25^2 = 35^2 = 0$. The equation (1) is a quadratic for 45^2, one of the roots of which is zero, in virtue of (3). Hence one of the two circles touching 1, 2, 3 touches 4 also.

This is an identical quadratic in r, and we obtain three useful equations by equating the absolute term, and the coefficients of r and r^2 to zero.

(i) From the constant term

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & S_1 \\ 1 & & & & & S_2 \\ 1 & & & & & S_3 \\ 1 & & & & & S_4 \\ 1 & S_1 & S_2 & S_3 & S_4 & 0 \end{vmatrix} = 0 ; \quad (4)$$

but if (x, y) is a point on the circle (4) then $S_4 = 0$; hence the equation of the two circles whose common tangents with a, b, c are u, v, w, viz.,

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & S_1 \\ 1 & & & & & S_2 \\ 1 & & & & & S_3 \\ 1 & & & & & 0 \\ 1 & S_1 & S_2 & S_3 & 0 & 0 \end{vmatrix} = 0. \quad (5)$$

This equation is not homogeneous in the S's, but if from the last row and column we subtract the last but one, the equation is homogeneous in

$$S_1 - u^2, \ S_2 - v^2, \ S_3 - w^2 ;$$

showing that the two circles are inverse to each other with respect to the orthogonal circle of

$$S_1 - u^2 = 0, \quad S_2 - v^2 = 0, \quad S_3 - w^2 = 0.$$

If $u^2 = v^2 = w^2 = 0$, we have Casey's equation for the two touching circles

$$\begin{vmatrix} 0 & n^2 & m^2 & S_1 \\ n^2 & 0 & l^2 & S_2 \\ m^2 & l^2 & 0 & S_3 \\ S_1 & S_2 & S_3 & 0 \end{vmatrix} = 0. \quad - \quad - \quad - \quad (6)$$

(ii) From the coefficient of r^2 in (3)

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & a \\ 1 & & & & & b \\ 1 & & & & & c \\ 1 & & & & & \rho \\ 0 & a & b & c & \rho & \frac{1}{2} \end{vmatrix} = 0. \quad (7)$$

For the touching circles, put $u^2 = v^2 = w^2 = 0$; then from 2nd, 3rd and 4th rows subtract the 5th, and similarly with columns, and we obtain the equation already found in Art. 12.

(iii) From the coefficient of r in (3)

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & a \\ 1 & & & & & b \\ 1 & & & & & c \\ 1 & & & & & \rho \\ 1 & S_1 & S_2 & S_3 & S_4 & 0 \end{vmatrix} = 0, \qquad (8)$$

and if (x, y) is a point on the circle 4, the equation obtained by writing 0 instead of S_4 in (8) gives us individually the two circles required, ρ being one of the roots of (7).

For the touching circle of radius ρ put $u^2 = v^2 = w^2 = 0$, and from the 2nd, 3rd, 4th, and 6th rows subtract the 5th, and obtain

$$\begin{vmatrix} 0 & n^2 & m^2 & a-\rho \\ n^2 & 0 & l^2 & b-\rho \\ m^2 & l^2 & 0 & c-\rho \\ S_1 & S_2 & S_3 & -\rho \end{vmatrix} = 0, \qquad (9)$$

or
$$l^2S_1\{m^2(b-\rho) + n^2(c-\rho) - l^2(a-\rho)\}$$
$$+ m^2S_2\{n^2(c-\rho) + l^2(a-\rho) - m^2(b-\rho)\}$$
$$+ n^2S_3\{l^2(a-\rho) + m^2(b-\rho) - n^2(c-\rho)\} + 2l^2m^2n^2\rho = 0. \qquad (10)$$

The equation (6) of the two touching circles together is

$$l^4S_1^2 + \ldots + \ldots - 2m^2n^2S_2S_3 - \ldots - \ldots = 0.$$

Where this meets $S_1 = 0$, we have $m^2S_2 - n^2S_3 = 0$,

and therefore from (10) $S_2 = n^2\rho/(\rho - a)$; $S_3 = m^2\rho/(\rho - a)$.

Hence the points of contact of the two touching circles are separated, for obviously when the actual values of S_1, S_2, S_3 are given and not merely their ratios, a single point is determined.

As an interesting special case, suppose that two of the given circles, say b and c, touch.

Then $l^2 = 0$ and the equation of the two touching circles is

$$(m^2S_2 - n^2S_3)^2 = 0.$$

The two touching circles coincide, and touch b, c at their point of contact.

17. *The circles cutting a, b, c at given angles.*

The solution just given was deduced directly from the relation between the common tangents of five circles. The solution of the problem now proposed comes equally naturally from a similar relation connecting the cosines of their angles of intersection. To obtain this, let (x_1, y_1) be the centre and r_1 the radius of the first circle, and so on.

Multiply together the two matrices

$$\begin{vmatrix} x_1^2+y_1^2-r_1^2, & -2x_1, & -2y_1, & 1 \\ x_2^2+y_2^2-r_2^2, & -2x_2, & -2y_2, & 1 \\ \cdots\cdots & & & \end{vmatrix} \quad \begin{vmatrix} 1, & x_1, & y_1, & x_1^2+y_1^2-r_1^2 \\ 1, & x_2, & y_2, & x_2^2+y_2^2-r_2^2 \\ \cdots\cdots & & & \end{vmatrix}.$$

There being five rows and only four columns, the resulting determinant is zero, that is

$$\begin{vmatrix} -2r_1^2, & -2r_1r_2\cos 12, & -2r_1r_3\cos 13, & -2r_1r_4\cos 14, & -2r_1r_5\cos 15 \\ & & \cdots\cdots & & \end{vmatrix} = 0$$

or dividing the rows and columns in order by r_1, r_2, r_3, r_4, r_5,

$$\begin{vmatrix} 1 & \cos 12 & \cos 13 & \cos 14 & \cos 15 \\ \cos 12 & 1 & \cos 23 & \cos 24 & \cos 25 \\ \cos 13 & \cos 23 & 1 & \cos 34 & \cos 35 \\ \cos 14 & \cos 24 & \cos 34 & 1 & \cos 45 \\ \cos 15 & \cos 25 & \cos 35 & \cos 45 & 1 \end{vmatrix} = 0. \qquad (1)$$

This relation is of even greater interest than that between the common tangents. For one thing, it can be transferred at once by inversion to circles on a sphere, subject to a certain convention, or definition. Indeed in the whole of the following theory we might, with very slight changes, be dealing with circles on a sphere instead of on a plane.

For 1, 2, 3 take the circles a, b, c cutting at angles A, B, C and for 4 take a circle ρ cutting these at angles α, β, γ. For 5 take the circle with centre (x, y) and radius r.

Then $$S_1 = (x-x_1)^2 + (y-y_1)^2 - a^2$$
$$= r^2 - 2ar\cos 15 \qquad (2)$$

or $$\cos 15 = (r^2 - S_1)/2ar, \text{ and so on.}$$

Substitute in (1) and multiply the last row and column by r.

Thus $$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & (r^2-S_1)/2a \\ \cos C & 1 & \cos A & \cos\beta & (r^2-S_2)/2b \\ \cos B & \cos A & 1 & \cos\gamma & (r^2-S_3)/2c \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & (r^2-S_4)/2\rho \\ \dfrac{r^2-S_1}{2a} & \dfrac{r^2-S_2}{2b} & \dfrac{r^2-S_3}{2c} & \dfrac{r^2-S_4}{2\rho} & r^2 \end{vmatrix} = 0. \qquad (3)$$

The coefficients of r^4 and r^2, and the constant term are separately zero.

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(i) Putting $r=0$ and taking (x, y) on the circle $S_4=0$, we have *the equation of the two circles cutting a, b, c at angles* α, β, γ in the form

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & X \\ \cos C & 1 & \cos A & \cos\beta & Y \\ \cos B & \cos A & 1 & \cos\gamma & Z \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & 0 \\ X & Y & Z & 0 & 0 \end{vmatrix} = 0, \quad (4) \qquad \text{where } X=S_1/a,\ Y=S_2/b,\ Z=S_3/c.$$

From the 1st, 2nd, and 3rd rows subtract the 4th multiplied by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively, and we get another form which is sometimes convenient

$$\begin{vmatrix} \sin^2\alpha & \cos C-\cos\alpha\cos\beta & \cos B-\cos\gamma\cos\alpha & X \\ \cos C-\cos\alpha\cos\beta & \sin^2\beta & \cos A-\cos\beta\cos\gamma & Y \\ \cos B-\cos\gamma\cos\alpha, & \cos A-\cos\beta\cos\gamma & \sin^2\gamma & Z \\ X & Y & Z & 0 \end{vmatrix} = 0. \quad (5)$$

By putting $\cos\alpha=\cos\beta=\cos\gamma=1$ we get the equation of the touching circles in a form which is the same as 16 (6), seeing that $l^2=4bc\sin^2\dfrac{A}{2}$, etc.

(ii) From the coefficient of r^4 in (3)

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & 1/a \\ \cos C & 1 & \cos A & \cos\beta & 1/b \\ \cos B & \cos A & 1 & \cos\gamma & 1/c \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & 1/\rho \\ 1/a & 1/b & 1/c & 1/\rho & 0 \end{vmatrix} = 0. \quad (6)$$

This equation for the radii has been given by Salmon (*Conics*, Chap. IX., last example).

(iii) From the coefficient of r^2 in (3)

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & 1/a \\ \cos C & 1 & \cos A & \cos\beta & 1/b \\ \cos B & \cos A & 1 & \cos\gamma & 1/c \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & 1/\rho \\ S_1/a & S_2/b & S_3/c & S_4/\rho & -2 \end{vmatrix} = 0. \quad (7)$$

This is the linear (non-homogeneous) relation connecting the equations of any four circles. If we take (x, y) on $S_4=0$, we get the equations of the two circles cutting a, b, c at angles α, β, γ in separate form. Also we might obtain (6) by putting the coefficient of $x^2+y^2=0$ in the identity (7).

18. *Some deductions from the equations of Art. 17.*

(i) From (1) if the circles 1, 2, 3, 4 are co-orthogonal, we may take 5 as their orthogonal circle and we get the determinant (D say), obtained by cutting out the last row and column of (1), equal to zero.

The equation (7) is then homogeneous in S_1, S_2, S_3, S_4; by a well-known theorem in determinants the left of (4) is a perfect square, and the equation (6) for $1/\rho$ has equal roots. The two circles ρ coincide in this case.

(ii) By putting $\cos\alpha = \cos\beta = \cos\gamma = 0$ in (6) so that the circle ρ is the orthogonal circle, we get

$$k^2 \begin{vmatrix} 0 & 1/a & 1/b & 1/c \\ 1/a & 1 & \cos C & \cos B \\ 1/b & \cos C & 1 & \cos A \\ 1/c & \cos B & \cos A & 1 \end{vmatrix} = \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$$

The determinant which multiplies k^2 is easily shown to be $-4\Delta^2_{def}/a^2b^2c^2$.

If the radii and Δ_{def} are finite the circles will have a common point if

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0 \quad \text{or} \quad \sin\tfrac{1}{2}(A+B+C)\sin\tfrac{1}{2}(B+C-A)\sin\tfrac{1}{2}(C+A-B)\sin\tfrac{1}{2}(A+B-C) = 0.$$

This also follows from putting

$$r = 0 \text{ and } S_1 = S_2 = S_3 = 0 \text{ in (3).}$$

If in addition $\Delta = 0$, k will be indeterminate and the circles a, b, c will be coaxal.

If a, b, c are all infinite, then from (6), for example,

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0 \quad \text{again.}$$

This is the relation connecting the supplements of the angles of a rectilineal triangle when their signs are undetermined (see Art. 21).

(iii) Putting $r=0$ in (3) and taking (x, y) at the centre of ρ we find, after slight manipulation

$$\begin{vmatrix} 1 & \cos C & \cos B & 1/a & X \\ \cos C & 1 & \cos A & 1/b & Y \\ \cos B & \cos A & 1 & 1/c & Z \\ 1/a & 1/b & 1/c & 0 & -2 \\ X & Y & Z & -2 & 0 \end{vmatrix} = 0.$$

This is the identical relation connecting the absolute tri-circular coordinates of a point pair.

(iv) Denote by Σ what the determinant of (4) becomes when we put $\cos\alpha$, $\cos\beta$, $\cos\gamma$ all equal to zero. $\Sigma=0$ is then the equation of the orthogonal circle.

Also let S denote the determinant of (4) as it stands.

$$\text{Then} \quad S+\kappa\Sigma = \begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & X \\ \cos C & 1 & \cos A & \cos\beta & Y \\ \cos B & \cos A & 1 & \cos\gamma & Z \\ \cos\alpha & \cos\beta & \cos\gamma, & 1+\kappa, & 0 \\ X & Y & Z & 0 & 0 \end{vmatrix}$$

If now we determine κ so that the determinant obtained from this by leaving out the last row and column is zero, which is possible unless

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0,$$

then the determinant itself is, by the theorem already cited, the square of a linear function of X, Y, Z, say of $pX+qY+rZ$.

$$\text{Thus} \quad S \equiv -\kappa\Sigma + (pX+qY+rZ)^2.$$

The equation of two inverse circles in tricirculars is thus of the same form as the equation of a conic having double contact with a given conic.

Similarly it may be shown that by adding a certain multiple of the determinant of (iii) to Σ, we obtain a square of the form $$(p'X+q'Y+r'Z+s)^2.$$

This obviously ought to be the case since $\Sigma=0$ represents a single circle.

(v) Referring to (1), suppose that the angles of intersection of 4 and 5 with 1, 2, 3 are given. Then (1) gives a quadratic for cos45. The inverse pairs 4 and 5 will *touch*, that is, one of 4 will touch one of 5, and the other of 4 the other of 5, provided the determinant obtained by putting cos45 equal to 1 in (1) is zero. Modify the determinant so obtained by subtracting from the last column the last but one, and similarly with rows; and we see from (4) that the circles 4 and 5 will touch provided the point pair

$$X : Y : Z = \cos 15 - \cos 14 : \cos 25 - \cos 24 : \cos 35 - \cos 34$$

lies on 4, or, similarly, on 5.

This result will be of use later, and its converse will be proved in next Article.

19. *The point of contact of two touching circles given by their angles of section with a, b, c.*

Theorem. If two circles ρ, r which touch each other cut a circle a at angles α, θ_1, and if Σ_1 = square of tangent to a from the point of contact of ρ, r,

then $$(\rho - r)\Sigma_1 = 2ar\rho(\cos\alpha - \cos\theta_1). \quad \text{(1)}$$

Let P, Q, D be the centres of ρ, r, a; R the point of contact of ρ, r.

The theorem comes from the well-known relation between the mutual distances of D and the three collinear points P, Q, R, viz.,

$$QR \, . \, PD^2 + RP \, . \, QD^2 + PQ \, . \, RD^2 + QR \, . \, RP \, . \, PQ = 0.$$

It follows that

$$QR(PD^2 - a^2) + RP(QD^2 - a^2) + PQ(RD^2 - a^2) + QR . RP . PQ = 0. \quad (2)$$

But here $$QR = r, \ RP = -\rho, \ PQ = \rho - r;$$

$$PD^2 - a^2 = \rho^2 - 2a\rho\cos\alpha,$$

$$QD^2 - a^2 = r^2 - 2ar\cos\theta_1,$$

$$RD^2 - a^2 = \Sigma_1.$$

Substitute these values in (2) and the theorem follows at once.

20. *An inverse pair of circles as an envelope of orthogonals.*

Another solution of the problem of section at given angles.

If $$S_4 \equiv (\lambda S_1/a + \mu S_2/b + \nu S_3/c)/(\lambda/a + \mu/b + \nu/c)$$

the circle $S_4 = 0$ is an orthogonal; let its radius be r, and its angles with a, b, c be θ_1, θ_2, θ_3.

Let any fifth circle R cut S_1, S_2, S_3, S_4 at angles α', β', γ', δ.

Then for the values of S_1, S_2, S_3, S_4 at the centre of R we have

$$S_1 = R^2 - 2aR\cos\alpha',$$
$$S_2 = R^2 - 2bR\cos\beta',$$
$$S_3 = R^2 - 2cR\cos\gamma',$$
$$S_4 = R^2 - 2rR\cos\delta.$$

Multiply these equations by λ/a, μ/b, ν/c, $-(\lambda/a+\mu/b+\nu/c)$ and add.

$$\therefore\quad (\lambda/a+\mu/b+\nu/c)r\cos\delta = \lambda\cos\alpha' + \mu\cos\beta' + \nu\cos\gamma'. \qquad (1)$$

For the circle R take in turn the circles a, b, c, r.

$$\therefore\quad \left.\begin{aligned}(\lambda/a+\mu/b+\nu/c)r\cos\theta_1 &= \lambda + \mu\cos C + \nu\cos B\\ (\quad\ldots\quad)r\cos\theta_2 &= \lambda\cos C + \mu + \nu\cos A\\ (\quad\ldots\quad)r\cos\theta_3 &= \lambda\cos B + \mu\cos A + \nu\end{aligned}\right\} \qquad (2)$$

$$\text{and}\quad (\quad\ldots\quad)r = \lambda\cos\theta_1 + \mu\cos\theta_2 + \nu\cos\theta_3. \qquad (3)$$

By elimination of θ_1, θ_2, θ_3 from (2), (3)

$$(\lambda/a+\mu/b+\nu/c)^2r^2 = \lambda^2+\mu^2+\nu^2+2\mu\nu\cos A+2\nu\lambda\cos B+2\lambda\mu\cos C. \qquad (4)$$

Finally in (1) for R take a circle ρ which cuts a, b, c at angles α, β, γ and suppose that S_4 touches ρ.

$$\therefore\quad (\lambda/a+\mu/b+\nu/c)r = \lambda\cos\alpha+\mu\cos\beta+\nu\cos\gamma. \qquad (5)$$

From (4) and (5) it follows that the circle ρ will touch S_4 (of radius r determined by (5)) provided

$$\lambda^2+\mu^2+\nu^2+2\mu\nu\cos A+2\nu\lambda\cos B+2\lambda\mu\cos C = (\lambda\cos\alpha+\mu\cos\beta+\nu\cos\gamma)^2. \qquad (6)$$

Hence the inverse pair of circles cutting a, b, c at α, β, γ may be considered as the envelope of the orthogonal $\lambda X+\mu Y+\nu Z=0$, subject to (6).

In the system of coordinates X, Y, Z, (6) is the *tangential equation* of the inverse pair of circles.

It would be easy to show that the condition (6) is equivalent to this, that the radius of the orthogonal is proportional to the $\perp^r$ from its centre on the radical axis of the inverse pair.

As to the *form* of (6), compare 18 (iv).

The equation of the two circles ρ might now be found by following precisely the lines of the method by which, in Conics, the trilinear equation of a conic is deduced from its tangential equation. But we may use the theorem of 19, which gives, at the point of contact of (6) with its envelope

$$\frac{\rho - r}{2\rho}\,\frac{S_1}{a} = r\cos\alpha - r\cos\theta_1$$

or multiplying by $\lambda/a + \mu/b + \nu/c$, writing X for S_1/a, and using the first of (2),

$$\left.\begin{array}{l} (\lambda/a + \mu/b + \nu/c)(\rho - r)X/2\rho = (\lambda/a + \mu/b + \nu/c)r\cos\alpha - \lambda - \mu\cos C - \nu\cos B \\ \text{Similarly} \\ (\quad)(\quad)Y/2\rho = (\quad)r\cos\beta - \lambda\cos C - \mu - \nu\cos A \\ (\quad)(\quad)Z/2\rho = (\quad)r\cos\gamma - \lambda\cos B - \mu\cos A - \nu \end{array}\right\}. \quad (7)$$

$$\left.\begin{array}{ll} \text{Also from (5)} & 0 = (\quad)r - \lambda\cos\alpha - \mu\cos\beta - \nu\cos\gamma \\ \text{and we have of course} & 0 = \lambda X + \mu Y + \nu Z \end{array}\right\}. \quad (8)$$

By linear elimination of

$$(\lambda/a + \mu/b + \nu/c)(\rho - r)/2\rho,\ (\lambda/a + \mu/b + \nu/c)r,\ \lambda,\ \mu,\ \nu$$

we find 17 (4), the equation of the two circles ρ.

To get the equation of the tangent orthogonal at (X′, Y′, Z′) write X′, Y′, Z′ for X, Y, Z in (7) and then eliminate between (7) and (8). The equation is 17 (4) but with X, Y, Z accented either in last column or last row.

If instead of the last of (8) we take the obvious identity

$$(\lambda/a + \mu/b + \nu/c)(\rho - r)(-2)/2\rho = (\lambda/a + \mu/b + \nu/c)r/\rho - \lambda/a - \mu/b - \nu/c$$

and eliminate as before, we obtain 17 (7) with $S_4 = 0$.

Lastly, to get the equation for the radii 17 (6). Put $S_4 = 0$ in 17 (7) and let the minors of the constituents of the last row be L, M, N, P, Q.

Then the determinant

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & 1/a \\ \cos C & 1 & \cos A & \cos\beta & 1/b \\ \cos B & \cos A & 1 & \cos\gamma & 1/c \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & 1/\rho \\ S_1/a & S_2/b & S_3/c & 0 & -2 \end{vmatrix} \quad (9) \quad \text{is equal to} \quad LS_1/a + MS_2/b + NS_3/c - 2Q.$$

Now the value of any S at the centre of $S = 0$ is $-$square of radius.

In particular the value of the determinant (9) at the centre of ρ is $-(L/a + M/b + N/c)\rho^2$.

But since, at the centre of ρ, $S_1 = \rho^2 - 2a\rho\cos\alpha$ and so on, it follows that the value of the determinant (9) at the centre of ρ is

$$(\rho^2 - 2a\rho\cos\alpha)L/a + (\rho^2 - 2b\rho\cos\beta)M/b + (\rho^2 - 2c\rho\cos\gamma)N/c - 2Q.$$

Equating the two values, we have

$$L\left(\frac{1}{a} - \frac{1}{\rho}\cos\alpha\right) + M\left(\frac{1}{b} - \frac{1}{\rho}\cos\beta\right) + N\left(\frac{1}{c} - \frac{1}{\rho}\cos\gamma\right) - Q \cdot \frac{1}{\rho^2} = 0 \quad (10)$$

which is a determinant obtained from (9) by replacing the last row by the coefficients of L, M, N, P, Q in (10). 17 (6) follows by adding to the last row $(1/\rho)$. last but one.

21. *Method of finding corresponding results for a rectilineal triangle. The circumscribing circles.*

It might be interesting to discuss in detail the limiting forms of the equations of Art. 17 when one or more of the given circles become right lines, but we shall merely notice here the form which equation (4) takes for the case of a rectilineal triangle.

Suppose we have three real intersecting circles with positive radii a, b, c. Keep three of the points of intersection A, B, C fixed and let the centre of a pass to infinity on the side of BC remote from A, and similarly with the other two. It is geometrically obvious that in the limit $S_1/2a$ or $\frac{1}{2}X$ becomes the $\perp^r$ a, or x let us say, from the variable point to the line BC. Hence we have only to write x, y, z for X, Y, Z in 17 (4) to get the equation in trilinears of the circle cutting BC, CA, AB at angles α, β, γ.

It is essential to notice, however, that the angles A, B, C of the circles become, not the angles A, B, C of the triangle, but their *supplements.*

Returning to the circles a, b, c, 17 (5) is

$$X^2\{(\cos\beta\cos\gamma - \cos A)^2 - \sin^2\beta\sin^2\gamma\} + \dots + \dots$$
$$- 2YZ\{\sin^2\alpha(\cos\beta\cos\gamma - \cos A)$$
$$+ (\cos\alpha\cos\beta - \cos C)(\cos\alpha\cos\gamma - \cos B)\} - \dots - \dots = 0. \quad (1)$$

If we choose α, β, γ so that the coefficients of X^2, Y^2 and Z^2 in this equation are all zero, the equation will be satisfied if any two of X, Y, Z are zero, and will therefore represent a pair of circles one or other of which passes through each of the six points of intersection of a, b, c. If b, c intersect at A, A′; c, a at B, B′; and

a, b at C, C′, then A, A′ are inverse points as also B, B′ and C, C′. There will clearly be four inverse pairs of *circumscribing circles*, viz.,

ABC, A′B′C′; ABC′, A′B′C; AB′C, A′BC′; AB′C′, A′BC.

To determine α, β, γ we have

$$\left.\begin{array}{l}\cos(\beta \pm \gamma) = \cos A \\ \cos(\gamma \pm \alpha) = \cos B \\ \cos(\alpha \pm \beta) = \cos C\end{array}\right\}.$$

As explained more fully in connection with equation (6) of next Article, we reject certain of these solutions as involving that the three circles have a common point.

Further, since it is the *cosines* only of α, β, γ that it is material to know, it is easy to see that all the solutions are in effect included in the four

$$\left.\begin{array}{l}\beta+\gamma=A \\ \gamma+\alpha=B \\ \alpha+\beta=C\end{array}\right\}\text{(i)};\ \left.\begin{array}{l}\beta+\gamma=-A \\ \gamma+\alpha=\ \ B \\ \alpha+\beta=\ \ C\end{array}\right\}\text{(ii)};\ \left.\begin{array}{l}\beta+\gamma=\ \ A \\ \gamma+\alpha=-B \\ \alpha+\beta=\ \ C\end{array}\right\}\text{(iii)};\ \left.\begin{array}{l}\beta+\gamma=\ \ A \\ \gamma+\alpha=\ \ B \\ \alpha+\beta=-C\end{array}\right\}\text{(iv)}.$$

In (1) the coefficient of $-2YZ$ is

$$\sin^2\alpha(\cos\beta\cos\gamma - \cos A) + (\cos\alpha\cos\beta - \cos C)(\cos\alpha\cos\gamma - \cos B)$$

which, when $\cos(\beta+\gamma)$ is put for $\cos A$, and so on, becomes

$$2\sin^2\alpha\sin\beta\sin\gamma.$$

Hence the equation of any one of the four circumscribing pairs has the form

$$\frac{\sin\alpha}{X} + \frac{\sin\beta}{Y} + \frac{\sin\gamma}{Z} = 0.$$

For (i) $\qquad \alpha = M - A,\ \beta = M - B,\ \gamma = M - C,$

where $\qquad M = \frac{1}{2}(A + B + C),$

and the equation is

$$\frac{\sin(M-A)}{X} + \frac{\sin(M-B)}{Y} + \frac{\sin(M-C)}{Z} = 0.$$

For the others we have to change the signs of A, B, C respectively both in the expressions for the angles of intersection and in the equation of the pair of circles. In the case of 3 real intersecting circles, the values of the angles of intersection can easily be verified from a figure.

For a rectilineal triangle write $\pi - A$, $\pi - B$, $\pi - C$ for A, B, C.

The equation of the pair (i) becomes

$$\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}=0,$$

the well-known equation of the circumscribing circle.

The equation of the second circumscribing pair is

$$\sin\tfrac{1}{2}(A+B+C)/X+\sin\tfrac{1}{2}(C-A-B)/Y+\sin\tfrac{1}{2}(B-A-C)/Z=0.$$

This becomes for a triangle

$$x(y\sin B+z\sin C)=0$$

which represents the base BC and the parallel to it through A.

We find in the same way the equation of the inverse pair of circles with respect to which a, b, c are *self-polar* (in a sense which we do not explain at present), namely,

$$\frac{X^2\cos A}{\cos A-\cos B\cos C}+\frac{Y^2\cos B}{\cos B-\cos C\cos A}+\frac{Z^2\cos C}{\cos C-\cos A\cos B}=0.$$

This becomes for a triangle, $x^2\sin 2A+y^2\sin 2B+z^2\sin 2C=0$, the known equation.

22. *The Hart Circles.*

Besides the Apollonian circles and the circumscribing circles, there are some other sets of four inverse pairs related in an interesting way to the given tri-circle, notably the circles discovered by Hart, each pair of which touches each Apollonian pair. We shall investigate these by determining their angles of intersection with a, b, c.

The condition, 16 (3), that four circles should be co-tangible by a circle is, in terms of the common tangents,

$$12\cdot34\pm13\cdot24\pm14\cdot23=0, \qquad (1)$$

or, in terms of the angles between the circles,

$$\sin\tfrac{1}{2}12\cdot\sin\tfrac{1}{2}34\pm\sin\tfrac{1}{2}13\cdot\sin\tfrac{1}{2}24\pm\sin\tfrac{1}{2}14\cdot\sin\tfrac{1}{2}23=0. \qquad (2)$$

We propose to enquire whether a circle ρ, cutting a, b, c at α, β, γ can be found such that each of the four sets of circles

$$\rho,\ a,\ b,\ c\ ;\ \ -\rho,\ -a,\ b,\ c\ ;\ \ -\rho,\ a,\ -b,\ c\ ;\ \ -\rho,\ a,\ b,\ -c \qquad (3)$$

is a cotangible set.

ρ, a, b, c will be cotangible if

$$\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}A\pm\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}B\pm\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}C=0.$$

The angles between $-a$ and b, c, $-\rho$ are $\pi-C$, $\pi-B$, a respectively, and therefore $-\rho$, $-a$, b, c will be cotangible if

$$\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}A \pm \cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}B \pm \cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}C = 0.$$

Each of the sets (3) will therefore be cotangible, provided we can find α, β, γ so that

$$\left.\begin{aligned} \sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}A + \sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}B + \sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}C = 0 \\ \sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}A + \cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}B - \cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}C = 0 \\ -\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}A + \sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}B + \cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}C = 0 \\ \cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}A - \cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}B + \sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}C = 0 \end{aligned}\right\} \quad . \quad - \quad (4)$$

There are here only three independent equations, as the first is found on adding the other three. This would happen with various other distributions of the signs of the terms, but it will be found that the altered equations are really the same as (4), with different initial determinations of α, β, γ from their cosines. (2) (j)).

The equations (4) are equivalent to

$$\left.\begin{aligned} \cos\tfrac{1}{2}(\beta+B) - \cos\tfrac{1}{2}(\gamma-C) = 0 \\ \cos\tfrac{1}{2}(\gamma+C) - \cos\tfrac{1}{2}(\alpha-A) = 0 \\ \cos\tfrac{1}{2}(\alpha+A) - \cos\tfrac{1}{2}(\beta-B) = 0 \end{aligned}\right\} \quad - \quad - \quad - \quad (5)$$

or to

$$\left.\begin{aligned} \beta+B = 4n_1\pi \pm (\gamma - C) \\ \gamma + C = 4n_2\pi \pm (\alpha - A) \\ \alpha + A = 4n_3\pi \pm (\beta - B) \end{aligned}\right\} \quad (6) \quad \text{where the } n\text{'s are integers.}$$

If we take all the ambiguous signs in (6) + we get by addition $A+B+C=2N\pi$; if we take the first ambiguity + and the other two $-$ we get $B+C-A=2N'\pi$.

But if $A \pm B \pm C = 2M\pi$ the circles a, b, c have a common point, (18 (ii)).

Neglecting consideration of this special case, we have to deal with four sets of equations, viz.,

$$\left.\begin{aligned} \beta+B = 4n_1\pi - \gamma + C \\ \gamma + C = 4n_2\pi - \alpha + A \\ \alpha + A = 4n_3\pi - \beta + B \end{aligned}\right\} \quad \text{and three sets of the type} \quad \left.\begin{aligned} \beta+B = 4n_1\pi - \gamma + C \\ \gamma + C = 4n_2\pi + \alpha - A \\ \alpha + A = 4n_3\pi + \beta - B \end{aligned}\right\}.$$

Now it makes no difference to the determination of the circles α, β, γ if the signs of any or all of α, β, γ are changed, or if any multiple of 2π is added to any of them.

Hence we may take as the four solutions

	H_0	H_1	H_2	H_3
α	$B-C$	$B-C$	$B+C$	$B+C$
β	$C-A$	$C+A$	$C-A$	$C+A$
γ	$A-B$	$A+B$	$A+B$	$A-B$

- - (7)

and we have proved that each of the eight circles thus specified, with the sign of its radius determined as in the scheme (3), touches four Apollonian circles, one from each inverse pair. Obviously when a Hart circle touches four Apollonian circles, its inverse touches the other four. From this it follows easily that each Apollonian circle touches four Hart circles, one from each pair.

For, denote the Appollonian pairs by A_0, A_1, A_2, A_3; A_0 touching a, b, c; A_1 touching $-a$, b, c, etc., and let their radii be a_0, a_0'; a_1, a_1', etc. Also let h_0, h_0'; h_1, h_1', etc., be the radii of the Hart circles. Each *pair* of radii is perfectly definite, being the roots of 17 (6) with the proper values of α, β, γ.

Now since h_0 is cotangible with a, b, c it touches either a_0 or a_0'; if it is a_0', then h_0', a_0, being the inverses of h_0, a_0', touch. Hence either h_0 or h_0' touches a_0. Similarly every Apollonian circle is touched by one from each Hart pair.

We can indicate the nature of the different contacts more definitely.

Thus, attaching the symbol a_0 arbitrarily to one of the pair of radii of A_0, denote by h_0, h_1, h_2, h_3 the radii of the circles which a_0 touches; then by a_1, a_2, a_3 the radii of the circles which $-h_0$ touches.

Then a_0 touches h_0, h_1, h_2, h_3

and ∴ a_0' ...... h_0', h_1', h_2', h_3'.

Also $-h_0'$ ...... a_1', a_2', a_3'.

Next a_1 touches either $-h_1$ or $-h_1'$; *the question is, which?* The following method not only answers this question, but it gives an interesting expression for the angle between h_0 and h_1. (The *two* possible values could be found from 17 (1)).

We know that a, b, c, h_0, h_1 are cotangible; hence from (2), taking

$$\left.\begin{aligned}
&b, c, h_0, h_1;\ \sin\tfrac{1}{2}A\sin\tfrac{1}{2}h_0h_1 \pm \sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A+B) \pm \sin\tfrac{1}{2}(C+A)\sin\tfrac{1}{2}(A-B)=0,\\
&c, a, h_0, h_1;\ \sin\tfrac{1}{2}B\sin\tfrac{1}{2}h_0h_1 \pm \sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(B-C) \pm \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(B-C)=0,\\
&a, b, h_0, h_1;\ \sin\tfrac{1}{2}C\sin\tfrac{1}{2}h_0h_1 \pm \sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C+A) \pm \sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)=0,
\end{aligned}\right\}\ (8$$

The common solution of these equations is

$$\sin\tfrac{1}{2}h_0h_1 = 2\sin\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}A, \quad - \quad - \quad - \quad (9)$$

(strictly, $\pm$ this, but the sign is immaterial.)

In the first of the identities (8)

change B into $\pi + B$, and C into $\pi + C$.

This gives

$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}h_0h_1 \pm \cos\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(A+B) \mp \cos\tfrac{1}{2}(C+A)\cos\tfrac{1}{2}(A-B) = 0,$$

which is just the condition that b, c, $-h_0$, $-h_1$ should be cotangible.

Similarly we show that c, $-a$, $-h_0$, $-h_1$ and $-a$, b, $-h_0$, $-h_1$ are cotangible.

Hence $-a$, b, c, $-h_0$, $-h_1$ are cotangible.

We can prove in the same way that

$$\left.\begin{aligned} \sin\tfrac{1}{2}h_0h_1' &= 2\cos\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}A \\ \sin\tfrac{1}{2}h_2h_3 &= 2\sin\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}A \\ \sin\tfrac{1}{2}h_2h_3' &= 2\cos\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}A \end{aligned}\right\} \quad - \quad - \quad - \quad (10)$$

and the rest can be written down from symmetry.

We have then the following scheme of contacts

$$\begin{array}{lll} \begin{cases} a_0 & \text{touches} \\ a_0' & \end{cases} & \begin{array}{l} a,\ b,\ c,\ h_0,\ h_1,\ h_2,\ h_3 \\ a,\ b,\ c,\ h_0',\ h_1',\ h_2',\ h_3' \end{array} & \\ \begin{cases} a_1 \\ a_1' \end{cases} & \begin{array}{l} -a,\ b,\ c,\ -h_0,\ -h_1,\ -h_2',\ -h_3' \\ -a,\ b,\ c,\ -h_0',\ -h_1',\ -h_2,\ -h_3 \end{array} & (11) \\ \begin{cases} a_2 \\ a_2' \end{cases} & \begin{array}{l} a,\ -b,\ c,\ -h_0,\ -h_1',\ -h_2,\ -h_3' \\ a,\ -b,\ c,\ -h_0',\ -h_1,\ -h_2',\ -h_3 \end{array} & \\ \begin{cases} a_3 \\ a_3' \end{cases} & \begin{array}{l} a,\ b,\ -c,\ -h_0,\ -h_1',\ -h_2',\ -h_3 \\ a,\ b,\ -c,\ -h_0',\ -h_1,\ -h_2,\ -h_3'. \end{array} & \end{array}$$

From the scheme we see that the eight Apollonian circles can be arranged in sets of four, one from each pair, in eight ways, so that the four circles of each set are touched by another circle besides

$\pm a, \pm b, \pm c$, viz., by h_0 or h_0' or h_1, etc.

$$\text{For instance}\quad \left.\begin{array}{llll} -a_0, & a_1, & a_2, & a_3 \text{ are touched by } -h_0 \\ a_0, & -a_1, & a_2, & a_3 \quad \ldots\ldots\ldots \quad a \\ a_0, & a_1, & -a_2, & a_3 \quad \ldots\ldots\ldots \quad b \\ a_0, & a_1, & a_2, & -a_3 \quad \ldots\ldots\ldots \quad c \end{array}\right\}. \ (12)$$

Any three of the Apollonian circles fall into one of these sets of four, but the fourth is determined when the three are chosen.

Further, when three are chosen, the four circles which touch them belong, one to each of the Apollonian pairs of the three.

We can thus specify all eight Apollonian circles of any three of the Apollonian circles of $\pm a$, $\pm b$, $\pm c$. Thus, taking $\pm a_1$, $\pm a_2$, $\pm a_3$ we see from (12) that $-h_0$, a, b, c are four of their Apollonian circles, one from each pair, and the other four are therefore the inverses of these with respect to the orthogonal circle of a_1, a_2, a_3.

23. The equation of any Hart pair is found from 21 (1), viz.,

$$X^2\{(\cos\beta\cos\gamma - \cos A)^2 - \sin^2\beta\sin^2\gamma\} + \ldots + \ldots$$
$$-2YZ\{\sin^2\alpha(\cos\beta\cos\gamma - \cos A)$$
$$+(\cos\alpha\cos\beta - \cos C)(\cos\alpha\cos\gamma - \cos B)\} - \ldots - \ldots = 0$$

by substituting the values of α, β, γ from 22 (7).

For H_0, the cofficient of X^2 is

$$(\cos\overline{\beta - \gamma} - \cos A)(\cos\overline{\beta + \gamma} - \cos A)$$
$$= 4\sin\frac{3A - B - C}{2}\sin\frac{B + C - A}{2}\sin\frac{C + A - B}{2}\sin\frac{A + B - C}{2};$$

the coefficient of $-2YZ$ is

$$\cos\beta\cos\gamma - \cos A + \cos\alpha(\cos A\cos\alpha - \cos B\cos\beta - \cos C\cos\gamma) + \cos B\cos C$$
$$= \cos\beta\cos\gamma - \cos A - \cos\alpha\cos(B + C - A) + \cos B\cos C$$
$$= (\cos\overline{B - C} - \cos A)(1 - \cos\overline{B + C - A})$$
$$= 4\sin^2\frac{B + C - A}{2}\sin\frac{C + A - B}{2}\sin\frac{A + B - C}{2}.$$

Hence the equation of H_0 is

$$X^2\sin\frac{B+C-3A}{2} + Y^2\sin\frac{C+A-3B}{2} + Z^2\sin\frac{A+B-3C}{2}$$
$$+2YZ\sin\frac{B+C-A}{2} + 2ZX\sin\frac{C+A-B}{2} + 2XY\sin\frac{A+B-C}{2} = 0. \quad (1)$$

To get the equations of H_1, H_2, H_3 it is clear from 22 (7) that we have only to change the signs of A, B, C respectively in this.

When the circles become right lines, (1) becomes, as in Art. 21,

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C - 2yz\sin A - 2zx\sin B - 2xy\sin C = 0$$

which is the known equation of the nine-points circle.

Again, the equation of H_1 is

$$X^2\sin\frac{B+C+3A}{2} + Y^2\sin\frac{C-A-3B}{2} + Z^2\sin\frac{B-A-3C}{2}$$

$$+2YZ\sin\frac{A+B+C}{2} + 2ZX\sin\frac{C-A-B}{2} + 2XY\sin\frac{B-A-C}{2} = 0 \quad (2)$$

which degenerates into

$$x^2\sin A + y^2\sin(B-C) - z^2\sin(B-C) + 2zx\sin C + 2xy\sin B = 0$$

or $$(x\sin A + y\sin B + z\sin C)^2 - (y\sin C + z\sin B)^2 = 0,$$

which represents two parallel lines, which are easily seen to be, one the fourth common tangent to the inscribed and the first escribed circle, the other the fourth common tangent to the other two escribed circles.

24. The absolute tricircular coordinates of the point of contact of any Hart circle with an Apollonian circle which touches it are given by the theorem of Art. 19. This enables us to distinguish the whole 32 points of contact. For example, where a_0, h_0 touch we have

$$\frac{X}{\sin^2\frac{1}{2}(B-C)} = \frac{Y}{\sin^2\frac{1}{2}(C-A)} = \frac{Z}{\sin^2\frac{1}{2}(A-B)} = \frac{4a_0h_0}{a_0-h_0}. \quad (1)$$

This gives for an ordinary triangle at the point of contact of the nine-point and the inscribed circle (since X becomes $2x$, etc.)

$$\frac{x}{\sin^2\frac{1}{2}(B-C)} = \frac{y}{\sin^2\frac{1}{2}(C-A)} = \frac{z}{\sin^2\frac{1}{2}(A-B)} = \frac{2(-r)(-\frac{1}{2}R)}{-r+\frac{1}{2}R} \quad (2)$$

the radii being both negative, when the circles proceed to their limiting forms in such a way as to make X reduce to $+2x$, etc. (Art. 21).

The orthogonal which touches H_0, A_0 at their pair of points of contact might now be found by the very same method as that by which we find the line which touches a conic at a given point, in trilinear coordinates. (The result, in fact, is proved in Art. 20.)

Or we may proceed thus :

let $\lambda X + \mu Y + \nu Z = 0$ be an orthogonal touching H_0, A_0.

Then by Art. 20 (1), (4), (6),

$$\left.\begin{aligned}(\lambda/a + \mu/b + \nu/c)r &= \lambda\cos(B - C) + \mu\cos(C - A) + \nu\cos(A - B)\\ &= \lambda \qquad\qquad + \mu \qquad\qquad + \nu\\ \text{and}\quad (\lambda + \mu + \nu)^2 &= \lambda^2 + \mu^2 + \nu^2 + 2\mu\nu\cos A + 2\nu\lambda\cos B + 2\lambda\mu\cos C\end{aligned}\right\} \quad (3)$$

Hence, to determine $\lambda : \mu : \nu$ we have

$$\left.\begin{aligned}\lambda\sin^2\tfrac{1}{2}(B - C) + \mu\sin^2\tfrac{1}{2}(C - A) + \nu\sin^2\tfrac{1}{2}(A - B) &= 0\\ \text{and}\quad \mu\nu\sin^2\tfrac{1}{2}A + \nu\lambda\sin^2\tfrac{1}{2}B + \lambda\mu\sin^2\tfrac{1}{2}C &= 0\end{aligned}\right\} \quad (4)$$

In general, this method of determining the common tangent orthogonals of two given inverse pairs would give us *two* common tangents. When the inverse pairs touch, these coincide.

The obvious identity

$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(B - C) + \sin\tfrac{1}{2}B\sin\tfrac{1}{2}(C - A) + \sin\tfrac{1}{2}C\sin\tfrac{1}{2}(A - B) = 0$$

will be found useful in eliminating, say, ν from (4) and finding

$$\left(\lambda\sin\frac{B}{2}\sin\frac{B - C}{2} - \mu\sin\frac{A}{2}\sin\frac{C - A}{2}\right)^2 = 0.$$

Hence λ, μ, ν are as $\sin\frac{1}{2}A/\sin\frac{1}{2}(B - C)$, etc., and the equation of the tangent orthogonal is

$$\frac{X\sin\frac{1}{2}A}{\sin\frac{1}{2}(B - C)} + \frac{Y\sin\frac{1}{2}B}{\sin\frac{1}{2}(C - A)} + \frac{Z\sin\frac{1}{2}C}{\sin\frac{1}{2}(A - B)} = 0. \quad (5)$$

25. *The four circumscribing pairs are touched by other four pairs.*

The analogy between the theory of inverse pairs of circles, and the theory of conics having double contact with a given conic (*Salmon, Conic Sections, Chapter on Invariants*) suggests the theorem that, like the four Apollonian pairs, the four circumscribing pairs are touched by other four pairs of circles.

This is from more than one point of view the *reciprocal*, or *polar*, counterpart of the theorem which we have just discussed at some length, as we hope to explain in detail, along with some other matters, in a supplement to this paper. Meanwhile we merely verify that the theorem is true, by the methods which have been given here.

The equations of the circumscribing pairs (Art. 21) are

$$\left.\begin{array}{l}\sin\overline{M-A}/X+\sin\overline{M-B}/Y+\sin\overline{M-C}/Z=0\\ -\sin M\;/X+\sin\overline{M-C}/Y+\sin\overline{M-B}/Z=0\\ \sin\overline{M-C}/X-\sin M\;/Y+\sin\overline{M-A}/Z=0\\ \sin\overline{M-B}/X+\sin\overline{M-A}/Y-\sin M\;/Z=0\end{array}\right\} \qquad (1)$$

where $M=\frac{1}{2}(A+B+C)$. The angles of intersection of these with a, b, c are for the first pair $M-A$, $M-B$, $M-C$; and for the others, these with the signs of A, B, C respectively changed.

Now we have seen in Art. 18 (v) that the pair of circles cutting a, b, c at angles θ_1, θ_2, θ_3 will touch the pair

$$\sin\alpha/X+\sin\beta/Y+\sin\gamma/Z=0$$

which cut a, b, c at angles α, β, γ if

$$\frac{\sin\alpha}{\cos\alpha-\cos\theta_1}+\frac{\sin\beta}{\cos\beta-\cos\theta_2}+\frac{\sin\gamma}{\cos\gamma-\cos\theta_3}=0. \qquad (2)$$

If we have

$$\frac{\sin\alpha}{\cos\alpha+\cos\theta_1}+\frac{\sin\beta}{\cos\beta+\cos\theta_2}+\frac{\sin\gamma}{\cos\gamma+\cos\theta_3}=0 \qquad (3)$$

the one pair will still touch the other, provided we take the radii of one of the pairs with their signs changed.

Now we have the identity

$$\frac{\sin(M-A)}{\cos(M-A)-\dfrac{\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A}}+\frac{\sin(M-B)}{\cos(M-B)-\dfrac{\cos\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}B}}$$
$$+\frac{\sin(M-C)}{\cos(M-C)-\dfrac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C}}=0. \qquad (4)$$

For the first term on the left is

$$\frac{2\cos\frac{1}{2}A\sin(M-A)}{\cos(\frac{1}{2}B+\frac{1}{2}C-A)-\cos\frac{1}{2}(B-C)}=\frac{\cos\frac{1}{2}A\sin(M-A)}{\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(C-A)},$$

so that the identity proposed is equivalent to

$$\sin\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}A\sin\tfrac{1}{2}(B+C-A)+\text{two similar terms}=0$$

or $\{\sin\frac{1}{2}(A+B-C)-\sin\frac{1}{2}(A-B+C)\}\sin\frac{1}{2}(B+C-A)$
$+\text{two similar terms}=0,$

which is obviously true.

In (4) change the signs of A, B, C respectively and we get 3 other identities, say (5), (6), (7).

In (4), (5), (6) and (7) change B into $\pi + B$ and C into $\pi + C$; in the same four equations change C into $\pi + C$ and A into $\pi + A$; and again in the same four A into $\pi + A$ and B into $\pi + B$.

As will be seen at once on writing them down, the 16 identities thus obtained are in virtue of (2) and (3) just the conditions that each of the four pairs (1) with radii properly assigned should touch each of the four pairs of circles whose angles of intersection θ_1, θ_2, θ_3 with a, b, c are given by the table

$$\begin{array}{l|c|c|c|c}
\cos\theta_1 = & \dfrac{\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A} & -\dfrac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A} & \dfrac{\cos\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A} & \dfrac{\sin\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A} \\
\cos\theta_2 = & \dfrac{\cos\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}B} & \dfrac{\sin\frac{1}{2}C\cos\frac{1}{2}A}{\sin\frac{1}{2}B} & -\dfrac{\sin\frac{1}{2}C\sin\frac{1}{2}A}{\cos\frac{1}{2}B} & \dfrac{\cos\frac{1}{2}C\sin\frac{1}{2}A}{\sin\frac{1}{2}B} \\
\cos\theta_3 = & \dfrac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C} & \dfrac{\cos\frac{1}{2}A\sin\frac{1}{2}B}{\sin\frac{1}{2}C} & \dfrac{\sin\frac{1}{2}A\cos\frac{1}{2}B}{\sin\frac{1}{2}C} & -\dfrac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C}
\end{array}.$$