On some generalisations of Laguerre polynomials

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1. Let $L_n^{(a)}(x)$, $(n = 0, 1, \ldots)$ be the Laguerre, $H_n(x)$ the Hermite polynomial. Let $\mathfrak{L}_p(a, b)$, $(1 \leqq p < \infty)$, be the space of all functions $f(x)$ the pth powers of which are integrable over (a, b), with the norm

$$\|f(x)\| = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}.$$

Let $\mathfrak{L}_p(a, b)$, $(p = \infty)$, be the space of all functions $f(x)$ which are measurable and essentially bounded in (a, b) where the norm $\|f(x)\|$ is the essential upper bound of $|f(x)|$, $(a \leqq x \leqq b)$. Let

$$\psi_n^{(a)}(x) = x^{\frac{1}{2}a} e^{-\frac{1}{2}x} L_n^{(a)}(x).$$

Then

$$\{\psi_n^{(a)}(x)\,[n!/\Gamma(n + a + 1)]^{1/2}\}, \quad (n = 0, 1, \ldots),$$

is a complete orthogonal and normal system of $\mathfrak{L}_2(0, \infty)$, when a is real and $a > -1$. Of course[1] every system $\{\phi_n\}$, $(n = 0, 1, \ldots)$, which is complete with respect to $\mathfrak{L}_q(0, \infty)$, $(1 < q \leqq \infty)$, is also closed in $\mathfrak{L}_p(0, \infty)$, $(p^{-1} = 1 - q^{-1})$. This means that, if $\phi_n \in \mathfrak{L}_p(0, \infty)$ and if

$$\int_0^\infty f(x)\,\bar{\phi}_n(x)\,dx = 0, \; [f(x) \in \mathfrak{L}_q(0, \infty);\; n = 0, 1, \ldots],$$

implies that $f(x) \equiv 0$, then every function $g(x) \in \mathfrak{L}_p(0, \infty)$ can be approximated in the mean with index p by integral linear finite aggregates of the ϕ_n, and conversely.

When a is not real, $\mathfrak{R}(a) > -1$, then the sequence $\{\psi_n^{(a)}\}$ is only a closed, and hence complete, system in $\mathfrak{L}_2(0, \infty)$. It has not yet been proved hitherto[2] that the system $\{\psi_n^{(a)}\}$ is a complete system with respect to $\mathfrak{L}_p(0, \infty)$ for $\mathfrak{R}(a) > 2/p - 2$ in the general case $1 \leqq p \leqq \infty$, or is a closed system in $\mathfrak{L}_p(0, \infty)$ for $\mathfrak{R}(a) > -2/p$, $(1 \leqq p < \infty)$. The proof of these theorems is given in the present paper.

The case $\Re(a) < -2/p$ has not yet been discussed at all for any value of p, not even for $p = 1, 2, \infty$, since then

$$\| \psi_n^{(a)} \| = \left(\int_0^\infty | \psi_n^{(a)}(x) |^p \, dx \right)^{1/p}$$

is infinite. The problem of this paper is to fill this gap. For this purpose the functions $\psi_n^{(a)}$ need some modification. Let m be any fixed integer and let

$$E_m(x) = e^{-x} - \sum_{h=0}^{m-1} (-x)^h / h!, \tag{1.1}$$

the sum being interpreted as zero when $m \leqq 0$, so that $E_m(x) = e^{-x}$ for $m \leqq 0$; for $n = 0, 1, 2, \ldots,$ let

$$L_{n,m}^{(a)}(x) = e^{\frac{1}{2}x} \sum_{h=0}^{n} \binom{n+a}{n-h} \frac{(-x)^h}{h!} E_{m-h}(\tfrac{1}{2}x); \tag{1.2}$$

$$\psi_{n,m}^{(a)}(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}a} L_{n,m}^{(a)}(x).$$

Hence

$$\psi_{n,m}^{(a)}(x) = \psi_n^{(a)}(x), \quad (m \leqq 0),$$

$$\psi_{n,1}^{(a)}(x) = \psi_n^{(a)}(x) - \binom{n+a}{n} x^{\frac{1}{2}a},$$

and so on. The reason for modifying $L^{(a)}$ and $\psi^{(a)}$ in this way will be given in the second section of the paper.

In the third and following sections we shall give some equations connecting the $\psi_{n,m}^{(a)}$, and, subject to certain restrictions on a and m, we will calculate

$$P_{r,s}^{(a)} = \int_0^\infty \psi_{r,m}^{(a)}(x) \, \psi_{s,m}^{(a)}(x) \, dx,$$

and prove some other properties of $\psi_{n,m}^{(a)}$. The main property of the sequence $\{\psi_{n,m}^{(a)}\}$, $(n = 0, 1, \ldots)$, is that it is a closed system in $\mathfrak{L}_p(0, \infty)$, $(1 \leqq p < \infty)$, under such restrictions. In the sixth section we shall discuss the eigen-functions of the cut Hankel transform and obtain some integral equations for $L_{n,m}^{(a)}$ and for another, more general, modification $\Lambda_{n,m}^{(a)}$ of $L_n^{(a)}$.

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2. When modifying the functions $\psi_n^{(a)}$ into $\psi_{n,m}^{(a)}$ for $\Re(a) < 0$ we restrict the modification by requiring that the Mellin transform

of $\psi^{(a)}$ shall remain unchanged formally. Now the functions $x^{1/2}\psi_n^{(a)}(x^2)$, $(n = 0, 1, \ldots)$, are eigen-functions of the ordinary Hankel transform, and the Mellin transform of the Bessel function $J_a(x)$ remains unchanged formally under certain restrictions when we cut the first terms of the power series[3] for $J_a(x)$; therefore from the theory[4] of "General Transforms" it follows immediately that $x^{1/2}\psi_{n,m}^{(a)}(x^2)$, $(n = 0, 1, \ldots)$, are eigen-functions of the cut Hankel transform. When $\Re(s) > -\frac{1}{2}\Re(a)$ we have

$$(2.1)\quad M_n^{(a)}(s) = \int_0^\infty x^{s-1}\psi_n^{(a)}(x)\,dx = \sum_{h=0}^{n}\binom{n+a}{n-h}\frac{(-1)^h}{h!}\int_0^\infty x^{\frac{1}{2}a+h+s-1}e^{-\frac{1}{2}x}\,dx$$

$$= \sum_{h=0}^{n}\binom{n+a}{n-h}\frac{(-1)^h}{h!}2^{\frac{1}{2}a+h+s}\Gamma(\tfrac{1}{2}a+h+s).$$

Now let

$$(2.2)\qquad -m+1-\Re(s) > \tfrac{1}{2}\Re(a) > -m-\Re(s),$$

where m is a positive integer. Then $\Re(a) < 0$ for $\Re(s) \geqq -m+1$, and some of the integrals diverge. But the Gamma integral $\Gamma(z) = \int e^{-x}x^{z-1}\,dx$, $[\Re(z) > 0]$, is to be replaced by

$$\Gamma(z) = \int_0^\infty \left\{e^{-x} - \sum_{h=0}^{k-1}\frac{(-x)^h}{h!}\right\}x^{z-1}\,dx = \int_0^\infty E_k(x)\,x^{z-1}\,dx,$$

where k is an arbitrary integer and where $-k+1 > \Re(z) > -k$. Hence here we have to replace

$$\int_0^\infty x^{\frac{1}{2}a+h+s-1}e^{-\frac{1}{2}x}\,dx \quad\text{by}\quad \int_0^\infty x^{\frac{1}{2}a+h+s-1}E_{m-h}(\tfrac{1}{2}x)\,dx,$$

since $-(m-h)+1 > \frac{1}{2}\Re(a)+\Re(s)+h > -(m-h)$. Thus (1.2) is justified.

We shall now compute the Mellin transform and shall give a formula which has not yet been proved for any value of m. Let

$$(2.3)\quad \psi_{n,m}^{(a)}(x; a) = x^{\frac{1}{2}a}\sum_{h=0}^{n}\binom{n+a}{n-h}\frac{(-ax)^h}{h!}E_{m-h}(\tfrac{1}{2}x),\ (n = 0, 1, \ldots);$$

let a be an arbitrary complex number, and let

$$(2.21)\quad \Re(s) > -\tfrac{1}{2}\Re(a),\ (m=0),\ -\tfrac{1}{2}\Re(a)-m < \Re(s) < -\tfrac{1}{2}\Re(a)-m+1,\ (m>0).$$

Then

$$x^{s-1}\psi_{n,m}^{(a)}(x; a) = O(x^\delta),\ [x\to 0,\ \delta = m+\Re(s+\tfrac{1}{2}a)-1 > -1]$$

$$x^{s-1}\psi_{n,m}^{(a)}(x; a) = O(x^\eta),\ [x\to\infty,\ \eta = m+\Re(s+\tfrac{1}{2}a)-2 < -1].$$

Hence the integral $\int_0^\infty x^{s-1}\psi_{n,m}^{(a)}(x; a)\,dx = M_n^{(a)}(s; a)$ converges, and by (2.1)

(2.4) $$M_n^{(a)}(s;\,a) = 2^{\frac{1}{2}a+s}\,\Gamma(\tfrac{1}{2}a+s)\sum_{h=0}^{n}(2a)^h\binom{n+a}{n-h}\binom{-\frac{1}{2}a-s}{h}.$$

Since

$$\sum_{h=0}^{n} u^h\binom{v}{n-h}\binom{w}{h} = (1-u)^n\sum_{h=0}^{n}\left(\frac{u}{u-1}\right)^h\binom{v}{n-h}\binom{n-v-w-1}{h},$$

when $u \neq 1$, and is equal to $\binom{v+w}{n}$, when $u = 1$, we get

(2.5) $$M_n^{(a)}(s;\,a) = \begin{cases}(1-2a)^n\,2^{\frac{1}{2}a+s}\,\Gamma(\tfrac{1}{2}a+s)\sum_{h=0}^{n}\left(\frac{2a}{2a-1}\right)^h\binom{n+a}{n-h}\binom{-\frac{1}{2}a+s-1}{h}\\ (-1)^n\,2^{\frac{1}{2}a+s}\,\Gamma(\tfrac{1}{2}a+s)\binom{-\frac{1}{2}a+s-1}{n}\end{cases}$$

for $a \neq \frac{1}{2}$ or $a = \frac{1}{2}$ respectively. Hence, when both s and $s' = 1 - s$ are subjected to the restrictions (2.21),

(2.6) $$\frac{M_n^{(a)}(s;\,a)}{M_n^{(a)}(1-s;\,a/[2a-1])} = (1-2a)^n\,2^{2s-1}\,\frac{\Gamma(\frac{1}{2}a+s)}{\Gamma(\frac{1}{2}a+1-s)},\ [a \neq \tfrac{1}{2}].$$

When $a = 1$ using (3.4.1) we obtain also

(2.4.1) $$M_n^{(a)}(s;\,1) = 2^{\frac{1}{2}a+s}\,\Gamma(\tfrac{1}{2}a+s)\sum_{h=0}^{n}(-1)^h\binom{-\frac{1}{2}a-s}{n-h}\binom{-\frac{1}{2}a+s-1}{h}.$$

3. Let α, β be any complex numbers, m an integer, and let $\psi_{n,m}^{(\alpha)}$, $(n = 0, 1, \ldots)$, be defined by (1.2), and E_m by (1.1); then

(3.1) $$\psi_{n,m}^{(\alpha)}(x) - \psi_{n-1,m}^{(\alpha)}(x) = x^{\frac{1}{2}}\,\psi_{n,m}^{(\alpha-1)}(x),$$

(3.2) $$\frac{d}{dx}\{x^{-\frac{1}{2}\alpha}\,\psi_{n,m}^{(\alpha)}(x)\} = -\,x^{-\frac{1}{2}(\alpha+1)}\,\psi_{n-1,m-1}^{(\alpha+1)}(x) - \tfrac{1}{2}x^{-\frac{1}{2}\alpha}\,\psi_{n,m-1}^{(\alpha)}(x),$$

(3.3) $$\psi_{n,m}^{(\alpha+\beta)}(x) = x^{\frac{1}{2}\beta}\sum_{h=0}^{n}\binom{n+\beta-1-h}{n-h}\psi_{h,m}^{(\alpha)}(x) = (-1)^n\,x^{\frac{1}{2}\beta}\sum_{h=0}^{n}(-1)^h\binom{-\beta}{n-h}\psi_{h,m}^{(\alpha)}(x),$$

(3.4) $$\psi_{n,m}^{(\alpha)}(x) - \psi_{n,m-1}^{(\alpha)}(x) = -(-\tfrac{1}{2}x)^{m-1}\,\frac{x^{\frac{1}{2}\alpha}}{\Gamma(m)}\,g(n;\,m;\,\alpha),$$

where

(3.4.1) $$g(n;\,m;\,\alpha) = \sum_{h=0}^{n}2^h\binom{m-1}{h}\binom{n+\alpha}{n-h} = \sum_{h=0}^{n}\binom{m-1}{h}\binom{\alpha+m+n-h-1}{n-h} = \sum_{h=0}^{n}(-1)^h\binom{m-1}{n-h}\binom{-\alpha-m}{h},$$

(3.5) $$\sum_{n=0}^{\infty}t^n\,\psi_{n,m}^{(\alpha)}(x) = (1-t)^{-\alpha-1}\,x^{\frac{1}{2}\alpha}\,E_m\!\left(\frac{x}{2}\cdot\frac{1+t}{1-t}\right),\ (|t|<1),$$

(3.6) $$x^{n+\frac{1}{2}\alpha}\,E_{m-n}(\tfrac{1}{2}x) = \Gamma(n+1)\sum_{h=0}^{n}(-1)^h\binom{n+\alpha}{n-h}\psi_{h,m}^{(\alpha)}(x).$$

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We omit the proofs of these formulae. When $m \leqq 0$, they are well known. The formula (3.5) gives the generating function of the $\psi^{(a)}_{n,m}$.

4. *The problem of orthogonality.* Let $x \geqq 0$, $1 \leqq p \leqq \infty$, let $\Re(a) + 2/p$ be neither a negative even integer nor zero, and let $j = j(a, p)$ be an integer defined by

$$(4.1) \qquad -2j - 2/p < \Re(a) < -2j - 2/p + 2$$

when $\Re(a) < 2/p$. From (1.2) we easily see that

$$(4.2) \quad \psi^{(a)}_{n,j}(x) = O(x^{\frac{1}{2}\Re(a)+j}),\ (x \to 0),\ \psi^{(a)}_{n,j} = O(x^{\frac{1}{2}\Re(a)+j-1}),\ (x \to \infty).$$

When $\Re(a) > -2/p$ we take $j = 0$; then

$$(4.21) \quad \psi^{(a)}_{n,j} = \psi^{(a)}_{n} = O(x^{\frac{1}{2}\Re(a)}),\ (x \to 0),\ \psi^{(a)}_{n} = O(e^{-\frac{1}{2}x} x^{\frac{1}{2}\Re(a)}),\ (x \to \infty).$$

Plainly $\psi^{(a)}_{n,j}(x)$ belongs to $\mathfrak{L}_p(0, \infty)$; when $p = \infty$, a is not restricted, and we then permit equality on the right hand side of (4.1). We now put $p = 2$ in order to calculate $\int \{\psi^{(a)}_{n,j}(x)\}^2 dx$. Let $j = j(a, 2) > 0$, $r \geqq 0$, $s \geqq 0$,

$$I^{(a)}_{r,s} = \int_0^\infty E_{j-r}(x)\, E_{j-s}(x)\, x^{a+r+s}\, dx.$$

We first take $\Re(a) > -2j$, $r + s < 2j - 1$, $j > 0$. Since, for $x \to 0$ and $x \to \infty$,

$$x^{a+r+s} E_{j-r}(x)\, E_{j-s}(x) = O(x^{\Re(a)+2j}),\ O(x^{\Re(a)+2j-2})$$

respectively, the integral $I^{(a)}_{r,s}$ converges. By $2j - 1 - r - s$ partial integrations we get

$$I^{(a)}_{r,s} = \frac{\Gamma(a+r+s+1)}{\Gamma(a+2j)} \sum_{h=0}^{2j-1-r-s} \binom{2j-1-r-s}{h} \int_0^\infty x^{a+2j-1} E_{j-r-h}(x)\, E_{r+1-j+h}(x)\, dx$$

$$= \frac{\Gamma(a+r+s+1)}{\Gamma(a+2j)} \left\{ \sum_{h=0}^{j-r-1} + \sum_{h=j-r}^{2j-r-s-1} \right\} = \frac{\Gamma(a+r+s+1)}{\Gamma(a+2j)} \{\Sigma_1 + \Sigma_2\}.$$

In all the integrals in Σ_1 plainly $E_{r+1-j+h} = e^{-x}$, and in Σ_2 plainly $E_{j-r-h} = e^{-x}$, when Σ_2 exists at all. Since

$$\Re(a) + 2j - 1 > -1$$

every term can easily be calculated, and so we have

$$(4.3)\ I^{(a)}_{r,s} = \Gamma(a+r+s+1) \left\{ 2^{-a-r-s-1} - \left(\sum_{h=0}^{j-r-1} + \sum_{h=0}^{j-s-1} \right) \binom{-a-r-s-1}{h} \right\}.$$

When $r + s \geqq 2j - 1$, then one of r, s (say r) must exceed $j - 1$, and since $E_{j-r} = e^{-x}$ we can calculate $I^{(a)}_{r,s}$ directly. We obtain the same result. By analytic continuation we see that (4.3) also holds for

$$-2j-1<\Re(a)<-2j+1,$$

though we have to take the limit of the right hand side for $a \to -2j$, when $a=-2j$ and $r+s \leqq 2j-1$. Hence (4.3) is valid for

$$-2j-1<\Re(a)<-2j+1,\ r \geqq 0,\ s \geqq 0,\ j>0,$$

while it is trivial for $j \leqq 0$. Now we have

$$\int_0^\infty \psi_{r,j}^{(a)} \psi_{s,j}^{(a)} dx = \sum_{h=0}^{r} \sum_{k=0}^{s} \binom{r+a}{r-h}\binom{s+a}{s-k} \frac{(-1)^{h+k}}{h!\,k!} \int_0^\infty x^{a+h+k} E_{j-h}\left(\frac{x}{2}\right) E_{j-k}\left(\frac{x}{2}\right) dx$$

$$= \sum_{h=0}^{r} \sum_{k=0}^{s} \binom{r+a}{r-h}\binom{s+a}{s-k} \frac{(-1)^{h+k}}{h!\,k!}\, 2^{a+h+k+1} I_{h,k}^{(a)}$$

$$= Z_1(r,s)-Z_2(r,s)-Z_3(r,s),$$

where Z_1, Z_2, Z_3 correspond to the right-hand terms of (4.3). Thus $Z_1(r,s)$ is zero when $r \neq s$ and $\Gamma(a+r+1)/\Gamma(r+1)$ when $r=s$.

$$Z_2(r,s)=\sum_{h,k} \binom{r+a}{r-h}\binom{s+a}{s-k} \frac{(-1)^{h+k}}{h!\,k!}\, 2^{a+h+k+1}\,\Gamma(a+h+k+1) \sum_{l=0}^{j-k-1} \binom{-a-h-k-1}{l}.$$

$Z_3(r,s)=Z_2(s,r)$.

By (3.41) we have finally:

$$Z_2(r,s)=(-1)^s\, 2^{a+1}\,\Gamma(a+1) \sum_{q=0}^{j-1} (-1)^q \binom{a+q}{q} g(r;q+1;a)\, g(s;q+1;a),$$

$$(4.4) \quad \int_0^\infty \psi_{r,j}^{(a)} \psi_{s,j}^{(a)} dx = \{(-1)^{r+1}+(-1)^{s+1}\}\, 2^{a+1}\,\Gamma(a+1) \times \sum_{q=0}^{j-1} \binom{-a-1}{q} g(r;q+1,a)\, g(s;q+1,a)+R,$$

where R is zero when $r \neq s$ and $\Gamma(a+r+1)/\Gamma(r+1)$ when $r=s$. Hence the integral vanishes when $r+s$ is odd, and then $\psi_{r,j}^{(a)}$ and $\bar{\psi}_{s,j}^{(a)}$ are mutually orthogonal. This also can be deduced from (2.41). Taking $s=\frac{1}{2}+it$, as a consequence of a well known property of the Mellin transform, we have

$$\int_0^\infty \psi_{r,j}^{(a)}(x)\, \psi_{s,j}^{(a)}(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_r^{(a)}(\tfrac{1}{2}+it)\, M_s^{(a)}(\tfrac{1}{2}-it)\, dt$$

$$= \frac{2^{a+1}}{2\pi} \int_{-\infty}^{\infty} \Gamma\left(\frac{a+1}{2}+it\right) \Gamma\left(\frac{a+1}{2}-it\right) \sum_{h=0}^{r} (-1)^h \binom{-\frac{a+1}{2}-it}{r-h}$$

$$\times \binom{-\frac{a+1}{2}+it}{h} \sum_{k=0}^{s} (-1)^k \binom{-\frac{a+1}{2}+it}{s-k} \binom{-\frac{a+1}{2}-it}{k} dt,$$

and the integrand is an odd function of t, when $r+s$ is odd.

When $j>0$ and $r+s$ is not odd, then generally $\psi_{r,j}^{(a)}$ and $\bar{\psi}_{s,j}^{(a)}$ are not mutually orthogonal. Take for instance $j=1$, $r=2$, $s=0$; then

$$\int_0^\infty \psi_{r,1}^{(a)}\,\psi_{s,1}^{(a)}\,dx=(-1)^{r+1}\,2^{a+2}\,\Gamma(a+1)\binom{r+a}{r}\binom{s+a}{s}$$
$$=-2^{a+1}\,\Gamma(a+3)\neq 0.$$

When we put $j=0$ in (4.4), we get a well known formula.

Of course the system $\{\psi_{n,j}^{(a)}\}$, $(n=0,1,\ldots.)$, can also be orthogonalised when $j>0$ by E. Schmidt's method, but the results are complicated.

5. *Completeness and closedness.*

THEOREM I. *Let* $1\leqq p\leqq\infty$, $q=p/(p-1)$, *and let* $\psi_{n,j}^{(a)}$ *be defined by* (1.2), j *by* (4.1), $(j>0)$. *Then the system* $\{\psi_{n,j}^{(a)}(x)\}$, $(n=0,1,\ldots.)$, *belongs to* $\mathfrak{L}_p(0,\infty)$ *and is complete with respect to the space* $\mathfrak{L}_q(0,\infty)$.

Proof. Let f belong to $\mathfrak{L}_q$ and satisfy the conditions

$$\int_0^\infty f(x)\,\bar{\psi}_{n,j}^{(a)}(x)\,dx=0,\quad(n=0,1,\ldots.). \tag{5.1}$$

We have to show that $f(x)\equiv 0$. From (5.1) and (3.6) it follows that

$$\int_0^\infty x^{n+\beta-i\gamma}\,E_{j-n}(\tfrac{1}{2}x)\,f(x)\,dx=0,\quad(n=0,1,\ldots.), \tag{5.2}$$

when we put $\tfrac{1}{2}a=\beta+i\gamma$, so that $-j-p^{-1}<\beta<-j-p^{-1}+1$, $1\leqq p\leqq\infty$. Let

$$-\int_x^\infty t^{\beta-i\gamma}f(t)\,dt=f_1(x),\ -\int_x^\infty f_k(t)\,dt=f_{k+1}(x),\ (k=1,2,\ldots.j-1);$$

then evidently $-1<\beta+j+1/p-1<0$,

$$|f_1(x)|\leqq Ax^{\beta+1/p},\ |f_k(x)|\leqq Ax^{\beta+k+1/p-1},\ (k=1,2,\ldots.,j). \tag{5.3}$$

Now for $n=1,2,\ldots.$

$$Z_0^{(n)}=\int_0^\infty x^{n+\beta-i\gamma}\,E_{j-n}(\tfrac{1}{2}x)\,f(x)\,dx$$
$$=[x^n f_1(x)\,E_{j-n}(\tfrac{1}{2}x)]_0^\infty+\int_0^\infty f_1(x)\{\tfrac{1}{2}x^n E_{j-n-1}(\tfrac{1}{2}x)-nx^{n-1}E_{j-n}(\tfrac{1}{2}x)\}\,dx,$$

$$Z_s^{(n)}=\tfrac{1}{2}Z_{s+1}^{(n)}-n\,Z_{s+1}^{(n-1)}(n>0),\ Z_s^{(0)}=\tfrac{1}{2}Z_{s+1}^{(0)}(n=0),\ (s=0,1,\ldots,j-1), \tag{5.4}$$

when we put

$$Z_s^{(n)}=\int_0^\infty x^n f_s(x)\,E_{j-n-s}(\tfrac{1}{2}x)\,dx,\ (n=0,1,\ldots.,s=1,2,\ldots.,j).$$

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It follows from (5.4) that, for $n = 0, 1, \ldots ; s = 1, 2, \ldots j$,

(5.41) $Z_s^{(n)} = 2Z_{s-1}^{(n)} + 4nZ_{s-1}^{(n-1)} + 8n(n-1)Z_{s-1}^{(n-2)} + \ldots + 2^{n+1}n!\, Z_{s-1}^{(0)}$.

Now all $Z_0^{(n)}$, $(n = 0, 1, \ldots)$, vanish in consequence of (5.2), and hence all $Z_1^{(n)}, Z_2^{(n)}, \ldots, Z_j^{(n)}$ also vanish in consequence of (5.41). Therefore, since $E_{j-n-s}(x) = e^{-x}$ for $s = j$,

$$(5.42) \qquad \int_0^\infty f_j(x)\, e^{-\frac{1}{2}x} x^n\, dx = 0, \quad (n = 0, 1, \ldots).$$

We now need the following lemma.

LEMMA 1. *Let* (i) $g(x)$ *be* $O(e^{-cx})$, $(x \to \infty)$, *for some* $c > 0$, *and* (ii) $\int_0^\infty g(t)\, t^n\, dt = 0$, $(n = 0, 1, \ldots)$; *then* $g(t) \equiv 0$ *in* $(0, \infty)$.

The proof is sketched in Footnote 5.

Here take $g(x) = e^{-\frac{1}{2}x} f_j(x)$, $c = \frac{1}{2}$, and it follows from the lemma, that $f_j(x) \equiv 0$, whence $f_{j-1}(x) \equiv 0, \ldots, f(x) \equiv 0$.

When $p = \infty$ and $q = 1$, α is not restricted since the value $\mathfrak{R}(\alpha) = -2j + 2$ is also admissible. In this case $\beta = -j + 1$, $f(x) \in \mathfrak{L}(0, \infty)$, $f_k(x)\, x^{j-k-1} \in \mathfrak{L}(0, \infty)$, $f_k(x) = o(x^{k-j})$, $k = 1, 2, \ldots, j-1$; when $x \to 0$ and when $x \to \infty$, and

$$|f_j(x)| < A \ (0 < x < \infty), \ f_j(x) \to 0 \ (x \to \infty),$$

in consequence of

LEMMA 2. *Let* $g(x)$ *belong to* $\mathfrak{L}(0, \infty)$, *let* $\mathfrak{R}(u) > 0$, $x > 0$, *and let* $h(x) = \int_x^\infty g(t)\, t^{-u}\, dt$. *Then* $h(x)\, x^{u-1}$ *belongs to* $\mathfrak{L}(0, \infty)$, $h(x) = o(x^{-\mathfrak{R}(u)})$, $(x \to 0$ *and* $x \to \infty)$.

Hence (5.4), (5.41), (5.42) hold in this case also, and therefore $f(x) = 0$ almost everywhere.

When $\mathfrak{R}(\alpha) > -2/p$, $(1 \leqq p < \infty)$, or when $\mathfrak{R}(\alpha) \geqq -2/p = 0$, $(p = \infty)$ and $j = 0$, Theorem I is also valid. We then take $g(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}\alpha} f(x) = e^{-\frac{1}{2}x} \phi(x)$, so that $\phi(x) \in \mathfrak{L}(0, \infty)$. Now Lemma 1 holds also when we replace the condition (i) by

$$(\text{i}') \ g(x) = e^{-cx} \phi(x), \ [c > 0, \ \phi(x) \in \mathfrak{L}(0, \infty)],$$

and from (5.2) it follows immediately that $f(x) \equiv 0$.

From the theorem cited above (section 1) we obtain

THEOREM II. *Let* $1 \leqq p < \infty$, *let* $\psi_{n,j}^{(\alpha)}$ *be defined by* (1.2) *and* j *by* (4.1); *then the sequence* $\{\psi_{n,j}^{(\alpha)}\}$, $(n = 0, 1, \ldots)$, *is a closed system in the space* $\mathfrak{L}_p(0, \infty)$.

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When $H_n(x)$, $(n = 0, 1, \ldots)$, are the Hermite polynomials, $\Phi_n(x) = e^{-\frac{1}{2}x^2} H_n(x)$, then $\{\Phi_n\}$, $(n = 0, 1, \ldots)$, is complete[6] with respect to the space $\mathfrak{L}_p (1 \leqq p \leqq \infty)$, and closed in the space $\mathfrak{L}_p (1 \leqq p < \infty)$. This will be proved in a very similar way putting

$$f(x) = G_1(x) + G_2(x),\ G_1(x) = \tfrac{1}{2}\{f(x) - f(-x)\} = -G_1(-x),$$
$$G_2(x) = \tfrac{1}{2}\{f(x) + f(-x)\} = G_2(-x).$$

6. *The Hankel transform.* Throughout this section we take $p = 2$.

THEOREM III. *Let $p = 2$, let $j = j(a, p)$ be defined by (4.1), $j > 0$, and let*

$$(6.1)\quad J_{a,j}(x) = J_a(x) - \sum_{k=0}^{j-1} \frac{(-1)^k (\frac{1}{2}x)^{a+2k}}{\Gamma(k+1)\Gamma(a+k+1)} = \sum_{k=j}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{a+2k}}{\Gamma(k+1)\Gamma(a+k+1)};$$

then, for $n = 0, 1, \ldots,$

$$(6.2)\qquad T^{(a)}\{y^{\frac{1}{2}} \psi^{(a)}_{n,j}(y^2)\} = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_0^N J_{a,j}(xy)(xy)^{\frac{1}{2}}\{y^{\frac{1}{2}}\psi^{(a)}_{n,j}(y^2)\}\,dy$$
$$= (-1)^n x^{\frac{1}{2}} \psi^{(a)}_{n,j}(x^2),$$

and a function $f(x) \in \mathfrak{L}_2(0, \infty)$ is self or skew reciprocal with respect to the cut Hankel transform $T^{(a)}$ if, and only if, it can be approximated by finite linear combinations of the $x^{\frac{1}{2}}\psi^{(a)}_{2n,j}(x^2)$ or $x^{\frac{1}{2}}\psi^{(a)}_{2n+1,j}(x^2)$, $(n = 0, 1, \ldots)$, respectively.

The transform $g = T^{(a)} f\ (f \in \mathfrak{L}_2)$ exists, is bounded and involutory and is equivalent to the equation[7]

$$(6.3)\quad G(t) = \omega(t) F(-t),\ [F(t) = \mathcal{M}f,\ G(t) = \mathcal{M}g,\ \omega(t) = \int_0^\infty J_{a,j}(x)\, x^{it}\, dx],$$

where we define the integral operator $\mathcal{M}$ by

$$\mathcal{M}h = \underset{N\to\infty}{\mathrm{l.\,i.\,m.}} \int_{1/N}^{N} h(x)\, x^{-\frac{1}{2}+it}\, dx,\ (h \in \mathfrak{L}_2;\ -\infty < t < \infty).$$

Now

$$(6.4)\qquad \omega(t) = 2^{it}\,\Gamma\{\tfrac{1}{2}(a+1+it)\}/\Gamma\{\tfrac{1}{2}(a+1-it)\},$$

and since, for $s = \frac{1}{2}(1 \pm it)$, it is clear that

$$-\tfrac{1}{2}\Re(a) - j < \Re(s) < -\tfrac{1}{2}\Re(a) - j + 1$$

holds, we have

$$\mathcal{M}\{T^{(a)}[y^{\frac{1}{2}}\psi^{(a)}_{n,j}(y^2;\, a)]\}$$
$$= (1-2a)^n\, 2^{\frac{1}{2}(a+it-1)}\,\Gamma\{\tfrac{1}{2}(a+it-1)\} \sum_{h=0}^{n} \left(\frac{2a}{2a-1}\right)^h \binom{n+a}{n-h} \binom{-\frac{1}{2}[a+it+1]}{h}$$
$$= (1-2a)^n\, \mathcal{M}\left\{x^{\frac{1}{2}}\psi^{(a)}_{n,j}\left(x^2;\ \frac{a}{2a-1}\right)\right\},\ [a \neq \tfrac{1}{2}],$$

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in consequence of (2.5), (2.4), (6.3) and (6.4) when $\psi^{(\alpha)}_{n,m}(x; a)$ is defined by (2.3). Hence for $a \neq \frac{1}{2}$

$$(6.5) \quad \int_0^\infty J_{\alpha,j}(xy)\,(xy)^{\frac{1}{2}}\,y^{\frac{1}{2}}\,\psi^{(\alpha)}_{n,j}(y^2; a)\,dy = (1-2a)^n x^{\frac{1}{2}}\,\psi^{(\alpha)}_{n,j}\left(x^2; \frac{a}{2a-1}\right).$$

When $j \leqq 0$ this is a well known formula due to Erdélyi[8]. When $a = \frac{1}{2}$ we easily obtain[9] from (2.5)

$$(6.51) \int_0^\infty J_{\alpha,j}(xy)\,(xy)^{\frac{1}{2}}\,y^{\frac{1}{2}}\,\psi^{(\alpha)}_{n,j}(y^2; \tfrac{1}{2})\,dy$$
$$= \mathcal{M}^{-1}\left\{(-1)^n\,2^{\frac{1}{2}(\alpha+it-1)}\,\Gamma\left(\frac{\alpha+it+1}{2}\right).\binom{-\frac{1}{2}[\alpha+it+1]}{n}\right\}$$
$$= \frac{2^{-n}}{n!}\,x^{2n+\alpha+\frac{1}{2}}\,E_{j-n}(\tfrac{1}{2}x^2).$$

Now (6.2) follows immediately from (6.5) when we take $a = 1$, and the sequences $\{\psi^{(\alpha)}_{2n;j}\}$, $\{\psi^{(\alpha)}_{2n+1,j}\}$, $(n = 0, 1 \ldots .)$, belong to the spaces $\mathfrak{R}$ or $\mathfrak{R}'$ consisting of all functions which are self or skew reciprocal in the Hankel transform. We now make use of the following theorem[10]:

Let T be a linear involutory and bounded transform in $\mathfrak{L}_2(a, b)$, and let $\{X_n\}$ be a closed system in $\mathfrak{L}_2$. Further let X_{2n} be self-reciprocal and let X_{2n+1} be skew-reciprocal; then $\{X_{2n}\}$, $\{X_{2n+1}\}$, $(n = 0, 1, \ldots .)$, are closed systems in $\mathfrak{R}$ and $\mathfrak{R}'$ respectively.

Hence from Theorem II Theorem III now follows.

Let r be any number $(0 < r < 1)$ and substitute

$$(6.6) \quad x^2 = x_1^2(1 + i\cot \pi r),\ y^2 = y_1^2(1 - i\cot\pi r),\ a = a_1(1 - i\cot\pi r)^{-1}$$

in (6.5) and (6.51). Then, by some calculation and omitting the indices, for $2a \neq 1 - i\cot\pi r$ and $2a = 1 - i\cot\pi r$ we see that formally

$$(6.7) \qquad T_r^{(\alpha)}\left\{y^{\frac{1}{2}+\alpha}\,e^{-\frac{1}{2}y^2}\,\Lambda^{(\alpha)}_{n,j}\left(ay^2, \frac{1-i\cot\pi r}{a}\right)\right\}$$

reduces to

$$\{1 - 2b/(1 + i\cot\pi r)\}^{-n}\,x^{\alpha+\frac{1}{2}}\,e^{-\frac{1}{2}x^2}\,\Lambda^{(\alpha)}_{n,j}\{bx^2; (1 + i\cot\pi r)/b\}$$

and

$$(1 - e^{2i\pi r})^{-n}\,(n!)^{-1}\,x^{2n+\alpha+\frac{1}{2}}\,e^{\frac{1}{2}ix^2\cot\pi r}\,E_{j-n}\{\tfrac{1}{2}x^2(1 + i\cot\pi r)\}$$

respectively, where we denote by $T_r^{(\alpha)}f$ the operator[11]

$$T_r^{(\alpha)}f = e^{\frac{1}{2}i\pi(r-\frac{1}{2})(\alpha+1)}\ \operatorname{cosec}\pi r\ \underset{N\to\infty}{\mathrm{l.\,i.\,m.}}\int_0^N J_{\alpha,j}(xy\operatorname{cosec}\pi r)\,(xy)^{\frac{1}{2}}\,e^{\frac{1}{2}i(x^2+y^2)\cot\pi r}\,f(y)$$

and where we put $b = (1 + i\cot\pi r)/\{2 - a^{-1}(1 - i\cot\pi r)\}$ and

$$\Lambda^{(\alpha)}_{n,m}(x; v) = e^{\frac{1}{2}xv}\sum_{h=0}^{n}\binom{n+\alpha}{n-h}\frac{(-x)^h}{h!}\,E_{m-h}\left(\frac{vx}{2}\right).$$

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These formulae are generalizations of (6.5) and (6.5.1), and plainly $\Lambda_{n,m}$ is a modification of L_n which is more general than $L_{n,m}$, since

$$\Lambda^{(a)}_{n,m}(x;\, v) = L^{(a)}_n(x),\ [m \leqq 0],\ \Lambda^{(a)}_{n,m}(x;\, 1) = L^{(a)}_{n,m}(x).$$

The formal deduction of (6.7) from (6.5) and (6.5.1) can be justified. The substitution (6.6) is due to Erdélyi who communicated it to me to enable me to deduce (6.7) in the case $j \leqq 0$ from his formula which is mentioned above.

Taking $j \leqq 0$, $a = \pm \frac{1}{2}$, from (6.8) we have by some calculation

$$F_s\{e^{-\frac{1}{2}x^2} H_n(ax)\} = e^{-\frac{1}{2}x^2}\{(1 - a^2 + a^2 e^{-4i\pi s})^{\frac{1}{2}}\}^n H_n\{axe^{-2i\pi s}(1 - a^2 + a^2 e^{-4i\pi s})^{-\frac{1}{2}}\}$$

when $a^2 \neq 1 - e^{-4i\pi s}$, and

$$F_s\{e^{-\frac{1}{2}x^2} H_n(ax)\} = \{2e^{-2i\pi s}(1 - e^{-4i\pi s})^{-\frac{1}{2}}\}^n e^{-\frac{1}{2}x^2} x^n$$

when $a^2 = 1 - e^{-4i\pi s}$, where we denote by $F_s f$ the operator[11]

$$F_s f = e^{\frac{1}{2}i\pi(2s-\frac{1}{2})}(2\pi \sin 2\pi s)^{-\frac{1}{2}} \operatorname*{l.i.m.}_{N\to\infty} \int_{-N}^{N} \exp\{-ixy \operatorname{cosec} 2\pi s + \tfrac{1}{2}i(x^2+y^2)\cot 2\pi s\} f(y)\,dy,$$

for $0 < s < \frac{1}{2}$ and $f \in \mathfrak{L}_2(-\infty, \infty)$.

FOOTNOTES.

1. S. Banach, *Théorie des opérations linéaires* (Warszawa, 1932), 58, théorème 7, S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen* (Warszawa-Lwòw, 1935), [625] and [624]. There the theorem is proved for a finite interval, but it is also true for an infinite one, since the well known theorem [1.71] on moments is also valid for an infinite interval.

   The theorem is also valid for complex-valued functions.

2. *Cf.* Kaczmarz-Steinhaus, *l. c.* This book contains a proof, page 280 *et seq.*, but Kaczmarz's definition of a complete system is different from that used here. In order to apply their general theorem [874] on Laguerre polynomials, the condition $n^{-1} v(nq)^{1/(nq)} \to 0$, $(n \to \infty,\ q = 1/p')$, has to be replaced by

   $$\overline{\lim}\ \{n^{-1} v(nq)^{1/(nq)}\} < \infty\ (n \to \infty).$$

   Then the theorem covers the completeness of $\{\psi^{(a)}_n\}$, $(n=0, 1, \ldots\ldots)$, in the cases $1 < p < \infty$.

3. H. Kober, *Quart. J. of Math.* (*Oxford*) 8 (1937), 186-99. Full details are given in the sixth section of the present paper.

4. G. N. Watson, *Proc. London Math. Soc.* (2), 35 (1933), 156-99.
   I. W. Busbridge, *Journal London Math. Soc.* 9 (1934), 179-87.

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5. We can obtain the proof of this lemma by following the lines of the proof [483] given by Kaczmarz-Steinhaus. Let $0 < b < c$, $h(t) = e^{bt} g(t)$; then (i) and (ii) imply $g(t) \in \mathfrak{L}(0, \infty)$, $h(t) \in \mathfrak{L}(0, \infty)$. The function

$$F(s) = \int_0^\infty e^{-st} g(t)\, dt = \int_0^\infty e^{-(b+s)t} h(t)\, dt$$

is regular for $\mathfrak{R}(s) > -b$; and for, $|s| < b$,

$$\sum_{n=0}^\infty \frac{|s|^n}{n!} \int_0^\infty e^{-bt} \,|h(t)|\, t^n dt \leq A \sum_{n=0}^\infty \frac{|s|^n}{n!} \max_{0<t<\infty} (e^{-bt} t^n)$$

$$= A \sum_{n=0}^\infty \frac{|s|^n}{n!} \left(\frac{n}{be}\right)^n$$

$$\leq A \sum_{n=1}^\infty \left|\frac{s}{b}\right|^n (2\pi n)^{-\frac{1}{2}} + A < \infty .$$

Therefore for, $|s| < b$,

$$F(s) = \sum_{n=0}^\infty \frac{s^n}{n!} \int_0^\infty e^{-bt} h(t)\, t^n\, dt = \sum_{n=0}^\infty \frac{s^n}{n!} \int_0^\infty g(t)\, t^n\, dt = 0.$$

Hence $F(s)$ vanishes for $\mathfrak{R}(s) > -b$, and by Lerch's theorem so also does $g(x)$.

6. *Cf.* Kaczmarz-Steinhaus, *l. c.* [874]; the proof covers the cases $1 < p < \infty$. Here I outline another proof.

7. H. Kober, *l. c.*

8. *Quart. J. of Math.* (*Oxford*), **9** (1938), 196-8.

9. When $j \leq 0$ *cf.* G. Szegö, *Math. Zeitschrift*, **25** (1926), 87-115.

10. H. Kober, *Annals of Math.*, **40** (1939), 549-59.

11. Originally we deduced (6.7) for $j \leq 0$ from the properties of the operator $T_r^{(a)}$, with which we have already dealt in a former paper, *Quart. J. of Math.* (*Oxford*), **10** (1939), 45-59. The assertion stated on page 52, line 19-21, is untrue in general; when $r = \frac{1}{2}$, it can be replaced by some results given in this paper. The operator F_s has also been dealt with in that paper.

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