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CONTRIBUTIONS TO COMPUTATIONAL ANALYSIS

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In this paper some new results concerning computational analysis are established. The fundamental concepts of interval analysis and fixed point theorems suitable for computational purposes are developed and applied to concrete examples illustrating this new method of proving mathematical statements.

1. INTRODUCTION

Computational analysis is still in its initial stage. The first attempts were made, by, among others, Lanford (see [4]), who attacked the Feigenbaum conjectures with computational methods and Rump (see [5]), who succeeded in proving the nonsingularity of matrices in a computational framework. In this article we treat problems in functional analysis, considering the integral operators

$$Kx(s) = \int_{\alpha}^{\beta} k(s,t)x(t)dt, \text{ where } \alpha \leqslant s \leqslant \beta, \tag{1}$$

with L^2-kernels $k(s,t)$. In general it is hard to determine with standard methods whether or not the spectral radius of this compact operator K is less than one. Therefore methods of computational analysis are used to investigate these problems.

2. FUNDAMENTAL CONCEPTS

The set of all real intervals

$$[a] = [\underline{a}, \overline{a}] = \{x \in \mathsf{R} \mid \underline{a} \leqslant x \leqslant \overline{a}\}, \tag{2}$$

is denoted by $I\mathsf{R}$. In $I\mathsf{R}$ the operations $+, -, \cdot, /$ are defined by

$$[a] \star [b] = \{x \mid x = a \star b, \quad a \in [a], \quad b \in [b]\}, \text{ where } \star \in \{+, -, \cdot, /\}, \tag{3}$$

with $0 \notin [b]$ in the case of division.

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An interval screen of $L^2((\alpha,\beta)\times(\alpha,\beta))$ consists of all interval polynomials

$$[f] = [f](s,t) = \sum_{i=0}^{n}\sum_{j=0}^{m}[a_{ij}]s^i t^j, \text{ where } [a_{ij}] \in I\mathbb{R}, \tag{4}$$

with fixed degree $n \times m$. We see that $[f]$ is closed, bounded and convex. The screen together with the approximated operations $[+], [-], [\cdot], [/], [\int]$ is called a *functiod* and denoted by F, where these operations are defined in such a way that

$$[f][\star][g] \in F, \text{ for } [f],[g] \in F, \qquad \star \in \{+,-,\cdot,/\}, \tag{5a}$$

$$[\textstyle\int][f] \in F \text{ for } [f] \in F, \tag{5b}$$

holds. An interval screen (respectively, a functiod) for $L^2(\alpha,\beta)$ is defined analogously. Functiods can be implemented very easily on a computer (see [**2**, **3**]).

REMARK 1. $[f]$ contains all L^2-functions g for which

$$g \subseteq [f], \tag{6}$$

in the graph sense, is true. If $g_1 \in [f_1]$ and $g_2 \in [f_2]$ then $g_1 \star g_2 \in [f_1][\star][f_2]$, for $\star \in \{+,-,\cdot,/,\int\}$ is also valid in the graph sense.

DEFINITION 1: $[f]$ is called an *interval extension* of the kernel $k(s,t)$ if and only if $k \subset [f]$ holds; that is

$$k(s,t) \in [f](s,t), \text{ for } \alpha \leqslant s,t \leqslant \beta. \tag{7}$$

It is well-known that $k(s,t)$ can be approximated by polynomials with any required accuracy. For estimating at the same time the approximation error, interval polynomials are chosen, where extensive use is made of truncated Taylor or Fourier expansions with an automatic estimation of the truncation error. The norm of a bounded set $A \subseteq L^2(\alpha,\beta)$ is defined by $\|A\| = \sup_{a\in A}\|a\|$. As a trivial consequence we can formulate

LEMMA 1. *For A, $B \subseteq L^2(\alpha,\beta)$ with $A \subseteq B$ we have $\|A\| \leqslant \|B\|$.*

3. THEOREMS FOR COMPUTIONAL ANALYSIS

We shall give now theorems formulated in such a manner that they are applicable for computational purposes and treat the following problems:

PROBLEM 1. Are all eigenvalues of the operator K less than one?

PROBLEM 2. Has the equation

$$(I-K)x = g, \qquad \text{where } g \in L^2(\alpha,\beta), \tag{8}$$

a solution x and is it possible to determine quickly close and reliable bounds for this solution?

The answers will be given by theorems suitable for computation, below.

THEOREM 1. *Let $[g]$, $[K]$ be interval extensions of g and K in (8). If we could achieve the inclusion condition*

$$[x_{i+1}] \subseteq [x_i], \tag{9}$$

for an element $[x_i]$ of the iteration process

$$[x_{i+1}] = [g][+][K][x_i], \qquad i = 0,1,2,\dots, \tag{10}$$

then we can deduce for the spectral radius $\rho(K)$ of K

$$\rho(K) < 1. \tag{11}$$

PROOF: Using induction, it is clear that

$$[x_{i+n}] \subseteq [x_{i+n-1}] \subseteq \dots \subseteq [x_{i+1}] \subseteq [x_i], \tag{12}$$

holds for all $n = 1,2,3,\dots$. By Lemma 1 we can conclude that

$$\|[x_{i+n}]\| \leqslant \|[x_{i+n-1}]\| \leqslant \dots \leqslant \|[x_{i+1}]\| \leqslant \|[x_i]\|, \qquad n = 1,2,3,\dots \tag{13}$$

Furthermore we have

$$[x_{i+n}] = [\sum_{\nu=0}^{i+n}][K^\nu][g] \subseteq \dots \subseteq [\sum_{\nu=0}^{i}][K^\nu][g] = [x_i], \qquad n = 1,2,3,\dots \tag{14}$$

From (13) and (14) we see that the sequence $\|[\sum_{\nu=0}^{i+n}][k^\nu][g]\|$ is monotone decreasing and bounded, hence the series $[\sum_{\nu=o}^{\infty}][K^\nu][g]$ is convergent. This implies $\lim_{\nu\to\infty} \|[K^\nu]\|^{1/\nu} < 1$ and $\rho([K]) < 1$, because $[g]$ is absorbing, $[K]$ was provided as an interval extension of K, and therefore we have in addition $\rho(K) < 1$. □

THEOREM 2. *If for an iterate of the computer performed process (10) the strong inclusion condition*

$$[x_{i+1}] \overset{\circ}{\subset} [x_i], \tag{15}$$

(that is, the closure of $[x_{i+1}]$ lies in the interior of $[x_i]$) holds, then the following statements are true.

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EXISTENCE. *There exists a solution* $\hat{x}$ *of (8).*

UNIQUENESS. $\hat{x}$ *is uniquely determined.*

INCLUSION. $\hat{x} \in [x_{i+1}]$.

PROOF: Since $K \in [K]$ and K is a compact operator, mapping a closed, bounded and convex set into itself, the existence of a fixed point $\hat{x}$, that is, a solution of (8) follows at once by applying Schauder's fixed point theorem.

Suppose y and z are two different solutions of (8). Then we get $y - z = K(y - z)$ and 1 turns out to be an eigenvalue of K, in contradiction to Theorem 1. The inclusion of $\hat{x}$ is now a trivial consequence. ▯

REMARK 2. The inclusion conditions (9) and (15) are proved for $[x_i] = \sum_{\nu=0}^{n} [a_\nu^{(i)}] s^\nu$, $[x_{i+1}] = \sum_{\nu=0}^{n} [a_\nu^{(i+1)}] s^\nu$ by means of the inequalities

$$\underline{a}_\nu^{(i)} \leqslant \underline{a}_\nu^{(i+1)}, \qquad \bar{a}_\nu^{(i+1)} \leqslant \bar{a}_\nu^{(i)},$$

for $\nu = 0, 1, 2, \dots, n$ in the case of (9) and

$$\underline{a}_\nu^{(i)} < \underline{a}_\nu^{(i+1)}, \qquad \bar{a}_\nu^{(i+1)} < \bar{a}_\nu^{(i)},$$

for $\nu = 0, 1, \dots, n$ in the case of (15).

4. EXAMPLES

Finally, we give some examples for computer assisted proofs. This is done by implementing the iteration (10) together with the functiod F and controlling the inclusion conditions (9) (respectively, (15)) by computational means.

So it has been proved on a computer, that the spectral radius of the operator

$$K(s,t) = \left(s^3 + st\right) \cos \sqrt{st}, \qquad \text{where } 0 \leqslant s,t \leqslant 1,$$

is less than one and that the integral equation

$$x(s) = s\left(1 - 2.325/\left(4 - s^2\right)\right) + \int_{-1}^{1} \left(s/\left(4 - s^2\right)\right)\left(t/\left(4 - t^2\right)\right) x(t)\,dt,$$

is uniquely solvable, where the solution has been computed with an accuracy of 12 guaranteed correct digits.

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