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**RESEARCH ARTICLE**

# Categorical and K-theoretic Donaldson–Thomas theory of $\mathbb{C}^3$ (part II)

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**Abstract**

Quasi-BPS categories appear as summands in semiorthogonal decompositions of DT categories for Hilbert schemes of points in the three-dimensional affine space and in the categorical Hall algebra of the two-dimensional affine space. In this paper, we prove several properties of quasi-BPS categories analogous to BPS sheaves in cohomological DT theory.

We first prove a categorical analogue of Davison's support lemma, namely that complexes in the quasi-BPS categories for coprime length and weight are supported over the small diagonal in the symmetric product of the three-dimensional affine space. The categorical support lemma is used to determine the torsion-free generator of the torus equivariant K-theory of the quasi-BPS category of coprime length and weight.

We next construct a bialgebra structure on the torsion free equivariant K-theory of quasi-BPS categories for a fixed ratio of length and weight. We define the K-theoretic BPS space as the space of primitive elements with respect to the coproduct. We show that all localized equivariant K-theoretic BPS spaces are one-dimensional, which is a K-theoretic analogue of the computation of (numerical) BPS invariants of the three-dimensional affine space.

## 1. Introduction

### *1.1. Quasi-BPS categories*

In [PTa, Păda, Păd23], we studied quasi-BPS (named after Bogomol'nyi–Prasad–Sommerfield states) categories $\mathbb{S}(d)_w$ for $d \in \mathbb{N}$ (length) and $w \in \mathbb{Z}$ (weight) in relation to categorical Donaldson–Thomas (DT) theory and to categorical Hall algebras (of surfaces and of quivers with potentials). They are defined to be full subcategories of the category of matrix factorizations

$$\mathbb{S}(d)_w := \mathrm{MF}(\mathbb{M}(d)_w, \mathrm{Tr}\, W_d) \subset \mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W_d), \tag{1.1}$$

where $\mathbb{M}(d)_w$ is a twisted noncommutative resolution first considered by Špenko–Van den Bergh [ŠdB17]; see Subsection 2.6 for more details. Here, the stack $\mathcal{X}(d)$ and the regular function $\mathrm{Tr}\, W_d$ are given by

$$\mathcal{X}(d) := \mathrm{Hom}(V,V)^{\oplus 3}/GL(V),\ \ \mathrm{Tr}\, W_d(A,B,C) = \mathrm{Tr}\, A[B,C], \tag{1.2}$$

where $V$ is a $d$-dimensional vector space.

There is also a graded quasi-BPS category $\mathbb{S}^{\mathrm{gr}}(d)_v$, which is equivalent (via Koszul duality) to a subcategory

$$\mathbb{T}(d)_v \subset D^b(\mathscr{C}(d)),$$

where $\mathscr{C}(d)$ is the derived moduli stack of zero-dimensional sheaves on $\mathbb{C}^2$ with length $d$. We will also consider $T$-equivariant versions of these categories, where $T = (\mathbb{C}^*)^2$ is the Calabi–Yau torus of $\mathbb{C}^3$. We denote by $\mathbb{K} = K(BT) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$ and by $\mathbb{F}$ the fractional field of $\mathbb{K}$. The ($T$-equivariant or not) Grothendieck groups of $\mathbb{S}(d)_w$ and $\mathbb{S}^{\mathrm{gr}}(d)_w \cong \mathbb{T}(d)_w$ are isomorphic.

The purpose of this paper is to prove several properties of quasi-BPS categories analogous to BPS sheaves in cohomological DT theory and use these properties to compute the $T$-equivariant K-theory of quasi-BPS categories.

### 1.2. Semiorthogonal decompositions

We briefly review semiorthogonal decompositions with summands given by quasi-BPS categories proved in [PTa, Păda].

In our previous paper [PTa], we constructed a semiorthogonal decomposition of the categorification $\mathcal{DT}(d)$ of the DT invariant $\mathrm{DT}_d$, which is a virtual count of zero-dimensional closed subschemes in $\mathbb{C}^3$. The category $\mathcal{DT}(d)$ is defined by

$$\mathcal{DT}(d) := \mathrm{MF}(\mathrm{NHilb}(d), \mathrm{Tr}\, W_d);$$

see [PTa, Subsection 1.5]. Here, $\mathrm{NHilb}(d)$ is the noncommutative Hilbert scheme of points

$$\mathrm{NHilb}(d) := \left(V \oplus \mathrm{Hom}(V,V)^{\oplus 3}\right)^{\mathrm{ss}} / GL(V).$$

More precisely, in [PTa, Theorem 1.1] we showed that there is a semiorthogonal decomposition

$$\mathcal{DT}(d) = \left\langle \boxtimes_{i=1}^{k} \mathbb{S}(d_i)_{v_i + d_i(\sum_{i>j} d_j - \sum_{j>i} d_j)} \;\middle|\; \begin{array}{c} 0 \leqslant v_1/d_1 < \cdots < v_k/d_k < 1 \\ d_1 + \cdots + d_k = d \end{array} \right\rangle. \tag{1.3}$$

There is also a semiorthogonal decomposition of $D^b(\mathscr{C}(d))$ in graded quasi-BPS categories; see [Păda, Theorem 1.1]:

$$D^b(\mathscr{C}(d)) = \left\langle \boxtimes_{i=1}^{k} \mathbb{T}(d_i)_{v_i} \;\middle|\; \begin{array}{c} v_1/d_1 < \cdots < v_k/d_k \\ d_1 + \cdots + d_k = d \end{array} \right\rangle. \tag{1.4}$$

### 1.3. Numerical and cohomological BPS invariants

It is expected that, for any smooth Calabi–Yau threefold $X$, there are certain deformation invariant integers called *BPS invariants* which determine the DT and Gromov–Witten invariants of $X$; see [PT14, Section 2 and a half]. Denote by $\Omega_d$ the BPS invariants of $\mathbb{C}^3$ for $d \geqslant 1$. Then

$$\Omega_d = -1 \text{ for all } d \geqslant 1. \tag{1.5}$$

The wall-crossing formula for the DT invariants of $\mathbb{C}^3$, see [JS12, Section 6.3], [Tod10, Remark 5.14], says that

$$\sum_{d \geqslant 0} \mathrm{DT}_d q^d = \prod_{d \geqslant 1} (1 - (-q)^d)^{d\Omega_d} = \prod_{d \geqslant 1} \frac{1}{(1 - (-q)^d)^d}; \tag{1.6}$$

see [PTa, Subsection 1.6] for more details.

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Davison–Meinhardt [DM20] constructed a perverse sheaf $\mathcal{BPS}_d$ on $\mathrm{Sym}^d(\mathbb{C}^3)$, called *BPS sheaf*, whose Euler characteristic recovers the BPS invariant $\Omega_d$. Davison [Davb, Theorem 5.1] showed that

$$\mathcal{BPS}_d = \Delta_* \mathrm{IC}_{\mathbb{C}^3}, \tag{1.7}$$

where $\Delta\colon \mathbb{C}^3 \hookrightarrow \mathrm{Sym}^d(\mathbb{C}^3)$ is the small diagonal. Davison [Davb, Lemma 4.1] also proved restrictions on the support of BPS sheaves for tripled quivers with potentials and used [Davb, Lemma 4.1] to prove purity results about stacks of representations of preprojective algebras [Davb, Theorem A]. We refer to Equation (1.7) as Davison's support lemma. The (cohomological) BPS spaces are the cohomology of the BPS sheaf.

Properties (and computations in special cases) of BPS sheaves have applications in the study of Hodge theory of various cohomological DT spaces (in particular, in proving purity of the Borel-Moore homology of moduli of objects in K3 categories [Davc]) and in the study of Cohomological Hall algebras [Dava, KV].

Our point of view is that the semiorthogonal decomposition (1.3) may be regarded as a categorification of the wall-crossing formula (1.6) and the category $\mathbb{S}(d)_w$ may be regarded as a categorical analogue of the BPS invariant (1.5) or BPS sheaf (1.7). In this paper, we make the above heuristic more rigorous. Let $(d, w) \in \mathbb{N} \times \mathbb{Z}$ be integers with $\gcd(d, w) = 1$. We first prove a categorical analogue of Equation (1.7), namely any object of $\mathbb{S}(d)_w$ is supported over the small diagonal in $\mathrm{Sym}^d(\mathbb{C}^3)$. Further, for $n \in \mathbb{N}$, we define a K-theoretic analogue of the cohomological BPS space:

$$\mathrm{P}(nd)_{nw} \subset K_T(\mathbb{S}(nd)_{nw}) \cong K_T(\mathbb{T}(nd)_{nw})$$

and show that $\mathrm{P}(nd)_{nw,\mathbb{F}} := \mathrm{P}(nd)_{nw} \otimes_{\mathbb{K}} \mathbb{F}$ is a one-dimensional $\mathbb{F}$-vector space, compare with Equation (1.5).

### *1.4. Support of matrix factorizations in quasi-BPS categories*

We prove a version of the support lemma (1.7) for categories $\mathbb{S}(d)_w$ with $\gcd(d, w) = 1$. We consider the quotient stack $\mathcal{X}(d)$ defined in Equation (1.2) together with its good moduli space

$$\pi\colon \mathcal{X}(d) \to X(d) := \mathrm{Hom}(V, V)^{\oplus 3} /\!\!/ GL(V).$$

Consider the diagram

$$\begin{tikzcd}
\mathcal{C}oh(\mathbb{C}^3, d) \arrow[r, equal] \arrow[d, "\pi"'] & \mathrm{Crit}(\mathrm{Tr}\, W_d) \arrow[r, hook] & \mathcal{X}(d) \arrow[d, "\pi"'] \arrow[dr, "\mathrm{Tr}\, W_d"] & \\
\mathrm{Sym}^d(\mathbb{C}^3) \arrow[rr, hook] & & X(d) \arrow[r] & \mathbb{C},
\end{tikzcd}$$

where $\mathcal{C}oh(\mathbb{C}^3, d)$ is the stack of sheaves with zero-dimensional support and length $d$ on $\mathbb{C}^3$. In Section 3, we prove the following (see Theorem 3.1 for a more precise statement):

**Theorem 1.1** (Theorem 3.1). *For a pair $(d, w) \in \mathbb{N} \times \mathbb{Z}$ with $\gcd(d, w) = 1$, any object in $\mathbb{S}(d)_w$ is supported on $\pi^{-1}(\Delta)$.*

The categorical support restriction in Theorem 1.1 implies a strong constraint on $T$-equivariant K-theory classes of objects of quasi-BPS categories. Let $\Theta$ be the forget-the-potential map

$$\Theta\colon K_T(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W_d)) \to K_T(\mathrm{MF}(\mathcal{X}(d), 0)) = \mathbb{K}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]^{\mathfrak{S}_d}; \tag{1.8}$$

see [Păd22, Proposition 3.6]. In Lemma 4.1, we show that if a complex $\mathcal{F}$ is supported on $\pi^{-1}(\Delta)$, then $\Theta([\mathcal{F}])$ is divisible by the element

$$(q_1-1)^{d-1}(q_2-1)^{d-1}(q_1q_2-1)^{d-1}\in\mathbb{K}. \tag{1.9}$$

In [PTa], we introduced certain complexes $\mathcal{E}_{d,w}\in\mathbb{T}(d)_w$ using a derived stack of pairs of commuting matrices, both of which have spectrum of cardinality one and have an explicit shuffle description (2.38). In particular, Lemma 4.1 applies to $\mathcal{E}_{d,w}$ for $\gcd(d,w)=1$. We remark that the divisibility by Equation (1.9) is not obvious from the shuffle description of $\Theta([\mathcal{E}_{d,w}])$.

### 1.5. Integral generator of K-theory of quasi-BPS categories

In [PTa, Theorem 1.2], we showed that the localized (i.e., taking $\otimes_{\mathbb{K}}\mathbb{F}$) $T$-equivariant K-theory of $\mathbb{T}(d)_w$ (and thus of $\mathbb{S}(d)_w$) is generated by monomials in $[\mathcal{E}_{d',w'}]$ for $d'\leqslant d$ and $\frac{w}{d}=\frac{w'}{d'}$. The divisibility by Equation (1.9) will be useful to compute the $T$-equivariant K-theory of quasi-BPS categories *without* localization. For a $\mathbb{K}$-module $M$, we will use the notation $M':=M/(\mathbb{K}\text{-torsion})$. Using Lemma 4.1 and Theorem 1.1, we show that:

**Theorem 1.2** (Theorem 4.4). *Let $(d,w)\in\mathbb{N}\times\mathbb{Z}$ with $\gcd(d,w)=1$. The $\mathbb{K}$-module $K_T(\mathbb{S}(d)_w)'$ is free of rank one with generator $\left[\mathcal{E}_{d,w}\right]$.*

For $(d,w)\in\mathbb{N}\times\mathbb{Z}$ with $\gcd(d,w)=1$, we believe the category $\mathbb{T}(d)_w$ is generated by $\mathcal{E}_{d,w}$, which in particular implies Theorem 1.2. Further, for $*\in\{\emptyset,T\}$, we suspect there are equivalences $\mathbb{T}_*(1)_0\xrightarrow{\sim}\mathbb{T}_*(d)_w$, but we do not have much evidence supporting this belief.

### 1.6. The coproduct

For the reminder of the introduction, we consider a pair $(d,v)\in\mathbb{N}\times\mathbb{Z}$ with $\gcd(d,v)=1$. The Grothendieck group of the category $\mathbb{T}(nd)_{nv}$ for $n>1$ contains a contribution from partitions $a+b=n$ because the Hall product restricts to a functor

$$m_{a,b}\colon\mathbb{T}(ad)_{av}\otimes\mathbb{T}(bd)_{bd}\to\mathbb{T}(nd)_{nv}$$

for $a,b\geqslant 1$ with $a+b=n$; see [PTa, Lemma 4.8]. The category

$$\mathcal{D}_{d,v}:=\bigoplus_{n\geqslant 0}\mathbb{T}(nd)_{nv} \tag{1.10}$$

is thus monoidal. For $n=a+b$, there is a coproduct

$$\Delta_{a,b}\colon\mathbb{T}(nd)_{nv}\to\mathbb{T}(ad)_{av}\otimes\mathbb{T}(bd)_{bv}; \tag{1.11}$$

see also [Păda, Section 5]. The construction also provides a $T$-equivariant version. In Section 5, we prove that the Hall product is compatible with the above coproduct:

**Theorem 1.3** (Corollary 5.6). *Let $(d,v)\in\mathbb{N}\times\mathbb{Z}$ be coprime, let $a,b,c,e,n\in\mathbb{N}$ such that $a+b=c+e=n$, and let $S$ be the set of tuples $(f_1,f_2,f_3,f_4)$ such that $a=f_1+f_2$, $b=f_3+f_4$, $c=f_1+f_3$, $e=f_2+f_4$. The following diagram commutes:*

$$\begin{tikzcd}
K_T(\mathbb{T}(ad)_{av})'\otimes K_T(\mathbb{T}(bd)_{bv})' \arrow[r, "m_{a,b}"] \arrow[d, "\Delta"] & K_T(\mathbb{T}(nd)_{nv})' \arrow[d, "\Delta_{c,e}"] \\
\bigoplus_S\bigotimes_{i=1}^4 K_T(\mathbb{T}(f_id)_{f_iv})' \arrow[r, "m'"] & K_T(\mathbb{T}(cd)_{cv})\otimes K_T(\mathbb{T}(ed)_{ev})',
\end{tikzcd}$$

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*where we have used the shorthand notations* $\Delta := \bigoplus_S \Delta_{f_1,f_2} \otimes \Delta_{f_3,f_4}$ *and* $m' := \bigoplus_S (m \otimes m)(1 \otimes sw_{f_2,f_3} \otimes 1)$.

In the proof, we first construct a version of $\Delta_{a,b}$ for the subcategories $\mathbb{M}(nd)_{nv} \subset D^b(\mathcal{X}(nd))_{nv}$; see Equation (1.1). We then construct the functor (1.11) by applying matrix factorizations and using the Koszul equivalence.

In Propositions 5.7 and 5.8, we show that the product and the coproduct on $K_T(\mathcal{D}_{d,v})'$ are commutative and cocommutative. We then obtain an isomorphism

$$K_T(\mathcal{D}_{d,v})_{\mathbb{F}} \cong \Lambda_{\mathbb{F}}, \tag{1.12}$$

where $\Lambda_{\mathbb{F}}$ is the $\mathbb{F}$-algebra of symmetric functions; see Subsection 2.12 for its definition. The elementary functions $e_n \in \Lambda_{\mathbb{F}}$ are sent, up to a factor in $\mathbb{F}$, to $\mathcal{E}_{nd,nv}$.

### *1.7. K-theoretic BPS spaces*

We define the K-theoretic BPS space to be the space of primitive elements of $K_T(\mathbb{T}(nd)_{nv})$ with respect to the above coproduct:

$$\mathrm{P}(nd)_{nv} := \ker\left(\bigoplus_{\substack{a+b=n\\ a,b\geqslant 1}} \Delta_{a,b} : K_T(\mathbb{T}(nd)_{nv}) \to \bigoplus_{\substack{a+b=n\\ a,b\geqslant 1}} K_T(\mathbb{T}(ad)_{av} \otimes \mathbb{T}(bd)_{bv})\right). \tag{1.13}$$

We show that the dimension over $\mathbb{F}$ of localized K-theoretic BPS spaces for all pairs $(nd, nv) \in \mathbb{N} \times \mathbb{Z}$ is the same as the (numerical) BPS invariants (1.5) up to a sign:

**Proposition 1.4** (Corollary 5.11). *The* $\mathbb{F}$*-vector space* $\mathrm{P}(nd)_{nv,\mathbb{F}}$ *is one-dimensional.*

Using the above proposition and [PTa, Theorem 1.1], we obtain a K-theoretic analogue of the wall-crossing formula (1.6); see Subsection 5.5 and [PTa, Subsection 1.6] for more details.

**Corollary 1.5** (Corollary 5.13). *There is an isomorphism of* $\mathbb{N}$*-graded* $\mathbb{F}$*-vector spaces:*

$$\bigoplus_{d\geqslant 0} K_T(\mathcal{DT}(d))_{\mathbb{F}} \cong \bigotimes_{\substack{0\leqslant v<d\\ \gcd(d,v)=1}} \left(\bigotimes_{n\geqslant 1} \mathrm{Sym}\big(\mathrm{P}(nd)_{nv,\mathbb{F}}\big)\right). \tag{1.14}$$

We conjecture an integral version of Proposition 1.4, which is a version of Theorem 1.2 for pairs $(nd, nv)$ for all $n \in \mathbb{N}$:

**Conjecture 1.6.** *The* $\mathbb{K}$*-module* $\mathrm{P}(nd)_{nv}$ *is free of rank one.*

The torsion-free version of the above conjecture for $(d, v) = (1, 0)$ and $n = 2$ follows from the discussion in Subsection 4.3. We finally conjecture an analogue of Theorem 1.1 for all $n \in \mathbb{N}$.

**Conjecture 1.7.** *The subspace* $\mathrm{P}(nd)_{nv} \subset K_T(\mathbb{T}(nd)_{nv}) \cong K_T(\mathbb{S}(nd)_{nv})$ *is supported over* $\pi^{-1}(\Delta)$*, alternatively, the following composition is zero:*

$$\mathrm{P}(nd)_{nv} \to K_T(\mathbb{S}(nd)_{nv}) \to K_T(\mathcal{X}(nd) \setminus \pi^{-1}(\Delta)).$$

## 2. Preliminaries

### *2.1. Notations*

The spaces considered in this paper are defined over the complex field $\mathbb{C}$, and they are quotient stacks $\mathcal{X} = A/G$, where $A$ is a dg scheme, the derived zero locus of a section $s$ of a finite rank bundle vector

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bundle $\mathcal{E}$ on a finite type separated scheme $X$ over $\mathbb{C}$, and $G$ is a reductive group. For such a dg scheme $A$, let $\dim A := \dim X - \operatorname{rank}(\mathcal{E})$, let $A^{\mathrm{cl}} := Z(s) \subset X$ be the (classical) zero locus, and let $\mathcal{X}^{\mathrm{cl}} := A^{\mathrm{cl}}/G$. We denote by $\mathcal{O}_{\mathcal{X}}$ or $\mathcal{O}_A$ the structure sheaf of $\mathcal{X}$.

For $G$ a reductive group and $A$ a dg scheme as above, denote by $A/G$ the corresponding quotient stack and by $A/\!\!/G$ the quotient dg scheme with dg-ring of regular functions $\mathcal{O}_A^G$.

For a classical stack $\mathcal{X}$ with a morphism $\mathcal{X} \to X$ to a scheme $X$ and for a closed point $p \in X$, we denote by $\mathcal{X}_p$ its formal fiber $\mathcal{X}_p := \mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,p}$.

For a (derived) stack $\mathcal{X}$ with an action of a torus $T$, we denote by $D^b_T(\mathcal{X})$ the bounded derived category of $T$-equivariant coherent sheaves on $\mathcal{X}$ and by $G_T(\mathcal{X})$ its Grothendieck group. We denote by $\operatorname{Perf}_T(\mathcal{X})$ the subcategory of $T$-equivariant perfect complexes and by $K_T(\mathcal{X})$ its Grothendieck group. When $T$ is trivial, we drop $T$ from the notation of the above Grothendieck groups. We introduce more notations for categories and K-theory in Subsection 2.6.2.

Consider the two-dimensional torus

$$(\mathbb{C}^*)^2 \xrightarrow{\cong} T := \{(t_1, t_2, t_3) \in (\mathbb{C}^*)^{\times 3} \mid t_1 t_2 t_3 = 1\} \subset (\mathbb{C}^*)^3. \tag{2.1}$$

The isomorphism above is given by $(t_1, t_2) \mapsto (t_1, t_2, t_1^{-1} t_2^{-1})$. We denote by $\mathbb{K} := K_T(\mathrm{pt}) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$ and let $\mathbb{F}$ be the fraction field of $\mathbb{K}$. For $V$ a $\mathbb{K}$-module, we use the notations $V' := V/(\mathbb{K}\text{-torsion})$ and $V_{\mathbb{F}} := V \otimes_{\mathbb{K}} \mathbb{F}$.

For a dg-category $\mathcal{D}$, a full dg-subcategory $\mathcal{C} \subset \mathcal{D}$ is called dense if any object in $\mathcal{D}$ is a direct summand of an object in $\mathcal{C}$.

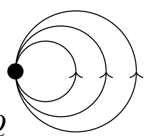
### 2.2. *Weights and partitions*

**2.2.1.**

For $d \in \mathbb{N}$, let $V$ be a $\mathbb{C}$-vector space of dimension $d$, and let $\mathfrak{g} = \mathfrak{gl}(V) := \operatorname{End}(V)$. When the dimension is clear from the context, we drop $d$ from its notation. Let

$$\pi\colon \mathcal{X}(d) := R(d)/G(d) := \mathfrak{gl}(V)^{\oplus 3}/GL(V) \to X(d) := \mathfrak{gl}(V)^{\oplus 3}/\!\!/GL(V).$$

Alternatively, $\mathcal{X}(d)$ is the stack of representations of dimension $d$ of the quiver $Q$ with one vertex and three loops $\{x, y, z\}$:



$Q$

Consider the superpotential $W = x[y, z]$ of $Q$ and the regular function

$$\operatorname{Tr} W = \operatorname{Tr} W_d := \operatorname{Tr} A[B, C]\colon \mathcal{X}(d) \to \mathbb{C}, \tag{2.2}$$

where $(A, B, C) \in \mathfrak{gl}(V)^{\oplus 3}$.

Fix the maximal torus $T(d) \subset GL(d)$ to be consisting of diagonal matrices. Denote by $M = \oplus_{i=1}^d \mathbb{Z}\beta_i$ the weight space of $T(d)$, and let $M(d)_{\mathbb{R}} := M(d) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\beta_1, \ldots, \beta_d$ is the set of simple roots. A weight $\chi = \sum_{i=1}^d c_i \beta_i$ is dominant if

$$c_1 \leqslant \ldots \leqslant c_d,$$

We denote by $M^+ \subset M$ and $M^+_{\mathbb{R}} \subset M_{\mathbb{R}}$ the dominant chambers. When we want to emphasize the dimension vector, we write $M(d)$ and so on. Denote by $N$ the coweight lattice of $T(d)$ and by $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\langle\, ,\, \rangle$ be the natural pairing between $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$.

Let $W = \mathfrak{S}_d$ be the Weyl group of $GL(d)$. For $\chi \in M(d)^+$, let $\Gamma_{GL(d)}(\chi)$ be the irreducible representation of $GL(d)$ of highest weight $\chi$. We drop $GL(d)$ from the notation if the dimension vector $d$ is clear from the context. Let $w * \chi := w(\chi+\rho)-\rho$ be the Weyl-shifted action of $w \in W$ on $\chi \in M(d)_{\mathbb{R}}$. We denote by $\ell(w)$ the length of $w \in W$.

Denote by $\mathcal{W}$ the multiset of $T(d)$-weights of $R(d)$. If there is a natural action of a torus $T$ on $R(d)$, we abuse notation and write $\mathcal{W}$ for the multiset of $(T \times T(d))$-weights of $R(d)$. For $\lambda$ a cocharacter of $T(d)$, we denote by $N^{\lambda>0}$ the sum of weights $\beta$ in $\mathcal{W}$ such that $\langle \lambda, \beta \rangle > 0$.

**2.2.2.**
We denote by $\rho$ half the sum of positive roots of $GL(d)$. In our convention of the dominant chamber, it is given by

$$\rho = \frac{1}{2}\sum_{j<i}(\beta_i - \beta_j).$$

We denote by $1_d := z \cdot \mathrm{Id}$ the diagonal cocharacter of $T(d)$. Define the weights

$$\sigma_d := \sum_{j=1}^{d} \beta_j \in M,\ \tau_d := \frac{\sigma_d}{d} \in M_{\mathbb{R}}.$$

**2.2.3.**
Let $G$ be a reductive group (in this paper, $G$ will be a Levi subgroup of $GL(d)$ for some positive integer $d$), let $T_G$ be a maximal torus of $G$, let $X$ be a $G$-representation and let

$$\mathcal{X} = X/G$$

be the corresponding quotient stack. Let $\mathcal{W}$ be the multiset of $T_G$-weights of $X$. For $\lambda$ a cocharacter of $T_G$, let $X^\lambda \subset X$ be the subspace generated by weights $\beta \in \mathcal{W}$ such that $\langle \lambda, \beta \rangle = 0$, let $X^{\lambda \geqslant 0} \subset X$ be the subspace generated by weights $\beta \in \mathcal{W}$ such that $\langle \lambda, \beta \rangle \geqslant 0$ and let $G^\lambda$ and $G^{\lambda \geqslant 0}$ be the Levi and parabolic groups associated to $\lambda$. Consider the fixed and attracting stacks

$$\mathcal{X}^\lambda := X^\lambda/G^\lambda,\ \mathcal{X}^{\lambda\geqslant 0} := X^{\lambda \geqslant 0}/G^{\lambda \geqslant 0}$$

with maps

$$\mathcal{X}^\lambda \xleftarrow{q_\lambda} \mathcal{X}^{\lambda \geqslant 0} \xrightarrow{p_\lambda} \mathcal{X}.$$

Define the integer

$$n_\lambda := \langle \lambda, [X^{\lambda \geqslant 0}] - [\mathfrak{g}^{\lambda \geqslant 0}] \rangle. \tag{2.3}$$

**2.2.4.**
Let $d \in \mathbb{N}$, and recall the definition of $\mathcal{X}(d)$ from Subsection 2.2.1. For a cocharacter $\lambda : \mathbb{C}^* \to T(d)$, consider the maps of fixed and attracting loci

$$\mathcal{X}(d)^\lambda \xleftarrow{q_\lambda} \mathcal{X}(d)^{\lambda \geqslant 0} \xrightarrow{p_\lambda} \mathcal{X}(d). \tag{2.4}$$

We say that two cocharacters $\lambda$ and $\lambda'$ are equivalent and write $\lambda \sim \lambda'$ if $\lambda$ and $\lambda'$ have the same fixed and attracting stacks as above.

We call $\underline{d} := (d_i)_{i=1}^k$ a partition of $d$ if $d_i \in \mathbb{N}$ are all nonzero and $\sum_{i=1}^k d_i = d$. In Section 5, we allow partitions $\underline{d} = (d_i)_{i=1}^k$ to have terms $d_i$ equal to zero. We similarly define partitions of $(d, w) \in \mathbb{N} \times \mathbb{Z}$. For a cocharacter $\lambda$ of $T(d)$, there is an associated partition $(d_i)_{i=1}^k$ such that

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$$\mathcal{X}(d)^{\lambda\geqslant 0} \xrightarrow{q_\lambda} \mathcal{X}(d)^{\lambda} \cong \times_{i=1}^{k}\mathcal{X}(d_i).$$

Define the length $\ell(\lambda) := k$.

Equivalence classes of antidominant cocharacters are in bijection with ordered partitions $(d_i)_{i=1}^k$ of $d$. For an ordered partition $\underline{d} = (d_i)_{i=1}^k$ of $d$, fix a corresponding antidominant cocharacter $\lambda = \lambda_{\underline{d}}$ of $T(d)$ which induces the maps

$$\mathcal{X}(d)^{\lambda} \cong \times_{i=1}^{k}\mathcal{X}(d_i) \xleftarrow{q_\lambda} \mathcal{X}(d)^{\lambda\geqslant 0} \xrightarrow{p_\lambda} \mathcal{X}(d).$$

We also use the notations $p_\lambda = p_{\underline{d}}$, $q_\lambda = q_{\underline{d}}$. The categorical Hall algebra is given by the functor $p_{\lambda *}q_\lambda^* = p_{\underline{d}*}q_{\underline{d}}^*$:

$$m = m_{\underline{d}} \colon D^b(\mathcal{X}(d_1)) \boxtimes \cdots \boxtimes D^b(\mathcal{X}(d_k)) \to D^b(\mathcal{X}(d)). \tag{2.5}$$

We may drop the subscript $\lambda$ or $\underline{d}$ in the functors $p_*$ and $q^*$ when the cocharacter $\lambda$ or the partition $\underline{d}$ is clear. We also use the notation $*$ for the Hall product.

**2.2.5.**

Let $(d_i)_{i=1}^k$ be a partition of $d$. There is an identification

$$\bigoplus_{i=1}^{k} M(d_i) \cong M(d),$$

where the simple roots $\beta_j$ in $M(d_1)$ correspond to the first $d_1$ simple roots $\beta_j$ of $d$ etc.

**2.2.6.**

Let $\underline{e} = (e_i)_{i=1}^l$ and $\underline{d} = (d_i)_{i=1}^k$ be two partitions of $d \in \mathbb{N}$. We write $\underline{e} \geqslant \underline{d}$ if there exist integers

$$a_0 = 0 < a_1 < \cdots < a_{k-1} \leqslant a_k = l$$

such that for any $0 \leqslant j \leqslant k-1$, we have

$$\sum_{i=a_j+1}^{a_{j+1}} e_i = d_{j+1}.$$

We say $\underline{e}$ is a refinement of $\underline{d}$. There is a similarly defined order on pairs $(d, w) \in \mathbb{N}\times\mathbb{Z}$.

**2.2.7.**

Let $A$ be a partition $(d_i, w_i)_{i=1}^k$ of $(d, w)$, and consider its corresponding antidominant cocharacter $\lambda$. Define the weights

$$\chi_A := \sum_{i=1}^{k} w_i\tau_{d_i},\ \chi'_A := \chi_A + \mathfrak{g}^{\lambda>0}.$$

Consider weights $\chi'_i \in M(d_i)_{\mathbb{R}}$ such that

$$\chi'_A = \sum_{i=1}^{k}\chi'_i.$$

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Let $v_i$ be the sum of coefficients of $\chi'_i$ for $1 \leqslant i \leqslant k$; alternatively, $v_i := \langle 1_{d_i}, \chi'_i \rangle$. We denote the above transformation by

$$A \mapsto A',\ (d_i, w_i)_{i=1}^k \mapsto (d_i, v_i)_{i=1}^k. \tag{2.6}$$

Explicitly, the weights $v_i$ for $1 \leqslant i \leqslant k$ are given by

$$v_i = w_i + d_i\left(\sum_{j>i} d_j - \sum_{j<i} d_j\right). \tag{2.7}$$

### *2.3. Polytopes*

The polytope $\mathbf{W}(d)$ is defined as

$$\mathbf{W}(d) := \frac{3}{2}\mathrm{sum}[0, \beta_i - \beta_j] + \mathbb{R}\tau_d \subset M(d)_{\mathbb{R}}, \tag{2.8}$$

where the Minkowski sum is after all $1 \leqslant i, j \leqslant d$. For $w \in \mathbb{Z}$, consider the hyperplane

$$\mathbf{W}(d)_w := \frac{3}{2}\mathrm{sum}[0, \beta_i - \beta_j] + w\tau_d \subset \mathbf{W}(d). \tag{2.9}$$

For $r > 0$ and $\lambda$ a cocharacter of $T(d)$, let $F_r(\lambda)$ be the face of the polytope $2r\mathbf{W}(d)$ corresponding to the cocharacter $\lambda$, so the set of weights $\chi$ in $M(d)_{\mathbb{R}}$ such that

$$\chi \in 2r\mathbf{W}(d),\ \langle \lambda, \chi \rangle = r\langle \lambda, R(d)^{\lambda>0} \rangle.$$

When $r = \frac{1}{2}$, we use the notations $F(\lambda)$. For $\chi \in M(d)^+_{\mathbb{R}}$, its $r$-invariant $r(\chi)$ is the smallest real number $r$ such that

$$\chi \in 2r\mathbf{W}(d).$$

For a cocharacter $\lambda$ of $T(d)$, denote by

$$\mathbf{W}(\lambda)_0 := \frac{3}{2}\mathrm{sum}[0, \beta_i - \beta_j] \subset M(d)_{\mathbb{R}},$$

where the sum is after all weights $1 \leqslant i, j \leqslant d$ such that $\langle \lambda, \beta_i - \beta_j \rangle = 0$.

### *2.4. A corollary of the Borel–Weyl–Bott theorem*

For future reference, we state a result from [HLS20, Section 3.2]. We continue with the notations from Subsection 2.2.3. Let $M$ be the weight lattice of $T_G$. We assume that $X$ is a symmetric $G$-representation, meaning that for any weight $\beta$ of $X$, the weights $\beta$ and $-\beta$ appear with the same multiplicity in $X$. Let $\chi$ be a weight in $M$. Let $\chi^+$ be the dominant Weyl-shifted conjugate of $\chi$ if it exists and zero otherwise. For a multiset $J \subset \mathcal{W}$, let

$$\sigma_J := \sum_{\beta \in J} \beta.$$

Let $w$ be the element of the Weyl group of minimal length such that $w * (\chi - \sigma_J)$ is dominant or zero. We let $\ell(J) := \ell(w)$.

**Proposition 2.1.** *Let $X$ be a symmetric $G$-representation, and let $\lambda$ be an antidominant cocharacter of $T_G$. Recall the fixed and attracting stacks and the corresponding maps*

$$X^{\lambda}/G^{\lambda} \xleftarrow{q_\lambda} X^{\lambda\geqslant 0}/G^{\lambda\geqslant 0} \xrightarrow{p_\lambda} X/G.$$

*For a weight $\chi$ in $M$, there is a quasi-isomorphism*

$$\left(\bigoplus_J \mathcal{O}_X \otimes \Gamma_G\big((\chi-\sigma_J)^+\big)[|J|-\ell(J)], d\right) \xrightarrow{\sim} p_{\lambda *}q_\lambda^*(\mathcal{O}_{X^\lambda}\otimes \Gamma_{G^\lambda}(\chi)),$$

*where the complex on the left-hand side has terms (shifted) vector bundles for all multisets $J \subset \{\beta \in \mathcal{W} \mid \langle \lambda, \beta\rangle < 0\}$.*

### *2.5. Matrix factorizations*

In the notations from Subsection 2.2.1, we denote by

$$\mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W_d)$$

the dg-category of matrix factorizations of the regular function $\operatorname{Tr} W_d$ on the smooth stack $\mathcal{X}(d)$. Its objects consist of tuples

$$(\alpha\colon F \rightleftarrows G\colon \beta) \text{ such that } \alpha\circ\beta = \beta\circ\alpha = \cdot \operatorname{Tr} W_d,$$

where $F, G \in \mathrm{Coh}(\mathcal{X}(d))$; see [PTa, Subsection 2.6] for details. For an object $\mathcal{F} \in \mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W_d)$, its internal homomorphism $R\mathcal{H}om(\mathcal{F},\mathcal{F})$ is an object of the $\mathbb{Z}/2$-graded derived category of $\mathrm{Coh}(\mathcal{X}(d))$. The support of $\mathcal{F}$

$$\mathrm{Supp}(\mathcal{F}) \subset \mathcal{X}(d)$$

is defined to be the support of $R\mathcal{H}om(\mathcal{F},\mathcal{F})$, which is a closed substack of $\mathcal{X}(d)$. Alternatively, it is the smallest closed substack $\mathcal{Z} \subset \mathcal{X}(d)$ such that $\mathcal{F}|_{\mathcal{X}(d)\setminus\mathcal{Z}} \cong 0$ in $\mathrm{MF}(\mathcal{X}(d)\setminus\mathcal{Z}, \operatorname{Tr} W_d)$.

Similarly to Equation (2.5), for $d = d_1 + d_2$ we have the categorical Hall product

$$m = m_{d_1,d_2}\colon \mathrm{MF}(\mathcal{X}(d_1), \operatorname{Tr} W_{d_1}) \boxtimes \mathrm{MF}(\mathcal{X}(d_2), \operatorname{Tr} W_{d_2}) \longrightarrow \mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W_d);$$

see [Păd22] for details. We sometimes write $a * b$ instead of $m(a,b)$.

We also consider equivariant and graded matrix factorizations for the regular function (2.2); see [PTa, Subsection 2.6.2] for details. The group $(\mathbb{C}^*)^3$ acts on the linear maps corresponding to the edges $(x,y,z)$ of the quiver $Q$ by scalar multiplication. Consider the two-dimensional subtorus

$$T \cong (\mathbb{C}^*)^2 \subset (\mathbb{C}^*)^3$$

which preserves the superpotential $W = x[y,z]$; see Equation (2.1). Then $T$ acts on $\mathcal{X}(d)$ and preserves $\operatorname{Tr} W$. We will also consider graded matrix factorizations, where the grading is given by scaling with weight 2 the space $\mathfrak{gl}(V)$ for $V$ a vector space. For example, we can choose an edge $e \in \{x,y,z\}$ of $Q$ and let $\mathbb{C}^*$ scale with weight 2 the linear map corresponding to $e$. Contrary to the $T$-action, the regular function $\operatorname{Tr} W$ has weight 2 with respect to such a grading. The corresponding categories of matrix factorizations are denoted by

$$\mathrm{MF}^{\bullet}_{*}(\mathcal{X}(d), \operatorname{Tr} W_d) \text{ for } * \in \{\emptyset, T\},\ \bullet \in \{\emptyset, \mathrm{gr}\}.$$

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### 2.6. Quasi-BPS categories

**2.6.1.**
For $w \in \mathbb{Z}$, we denote by $D^b(\mathcal{X}(d))_w$ the subcategory of $D^b(\mathcal{X}(d))$ consisting of objects of weight $w$ with respect to the diagonal cocharacter $1_d$ of $T(d)$. We have the direct sum decomposition

$$D^b(\mathcal{X}(d)) = \bigoplus_{w \in \mathbb{Z}} D^b(\mathcal{X}(d))_w.$$

We define the dg subcategories

$$\mathbb{M}(d) \subset D^b(\mathcal{X}(d)), \text{ (resp. } \mathbb{M}(d)_w \subset D^b(\mathcal{X}(d))_w)$$

to be generated by the vector bundles $\mathcal{O}_{\mathcal{X}(d)} \otimes \Gamma_{GL(d)}(\chi)$, where $\chi$ is a dominant weight of $T(d)$ such that

$$\chi + \rho \in \mathbf{W}(d), \text{ (resp. } \chi + \rho \in \mathbf{W}(d)_w). \tag{2.10}$$

Note that $\mathbb{M}(d)$ decomposes into the direct sum of $\mathbb{M}(d)_w$ for $w \in \mathbb{Z}$. The following is an alternative description of the category $\mathbb{M}(d)_w$:

**Lemma 2.2** [HLS20, Lemma 2.9]. *The category $\mathbb{M}(d)_w$ is generated by the vector bundles $\mathcal{O}_{\mathcal{X}(d)} \otimes \Gamma$ for $\Gamma$ a $GL(d)$-representation such that the $T(d)$-weights of $\Gamma$ are contained in the set $\nabla_w$ defined by*

$$\nabla_w := \left\{ \chi \in M_{\mathbb{R}} \,\middle|\, -\frac{1}{2} n_\lambda \leqslant \langle \lambda, \chi \rangle \leqslant \frac{1}{2} n_\lambda \text{ for all } \lambda \colon \mathbb{C}^* \to T(d) \right\} + w\tau_d.$$

For a partition $A = (d_i, w_i)_{i=1}^k$ of $(d, w)$, define

$$\mathbb{M}_A := \boxtimes_{i=1}^k \mathbb{M}(d_i)_{w_i}. \tag{2.11}$$

**2.6.2.**
Recall the regular function (2.2). We define the subcategory

$$\mathbb{S}(d) := \mathrm{MF}(\mathbb{M}(d), \operatorname{Tr} W_d) \subset \mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W_d)$$

to be the subcategory of matrix factorizations $(\alpha \colon F \rightleftarrows G \colon \beta)$ with $F$ and $G$ in $\mathbb{M}(d)$. It decomposes into the direct sum of $\mathbb{S}(d)_w$ for $w \in \mathbb{Z}$, where $\mathbb{S}(d)_w$ is defined similarly to $\mathbb{S}(d)$ using $\mathbb{M}(d)_w$.

We also consider subcategories for $* \in \{\emptyset, T\}$, $\bullet \in \{\emptyset, \mathrm{gr}\}$ defined in a similar way

$$\mathbb{S}^\bullet_*(d) := \mathrm{MF}^\bullet_*(\mathbb{M}(d), \operatorname{Tr} W_d) \subset \mathrm{MF}^\bullet_*(\mathcal{X}(d), \operatorname{Tr} W_d).$$

The subcategory $\mathbb{S}^\bullet_*(d)_w$ is also defined in a similar way. For a partition $A = (d_i, w_i)_{i=1}^k$ of $(d, w)$, the category $\mathbb{S}^\bullet_{*,A}$ is also defined similarly to Equation (2.11). We denote the Grothendieck group of $\mathbb{S}^\bullet_*(d)_w$ by

$$K_*(\mathbb{S}^\bullet(d)_w), \; * \in \{\emptyset, T\}, \; \bullet \in \{\emptyset, \mathrm{gr}\}.$$

By [Tod23, Corollary 3.13], there are natural isomorphisms (which hold for all graded matrix factorizations as in Subsection 2.5):

$$K(\mathbb{S}^{\mathrm{gr}}(d)_w) \xrightarrow{\cong} K(\mathbb{S}(d)_w), \; K_T(\mathbb{S}^{\mathrm{gr}}(d)_w) \xrightarrow{\cong} K_T(\mathbb{S}(d)_w). \tag{2.12}$$

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### 2.7. Complexes in quasi-BPS categories

Let $V$ be a $d$-dimensional complex vector space, and recall that we denote by $\mathfrak{g} = \mathrm{Hom}(V,V)$ the Lie algebra of $GL(V)$. We set

$$\mathcal{Y}(d) := \mathfrak{g}^{\oplus 2}/GL(V),$$

where $GL(V)$ acts on $\mathfrak{g}$ by conjugation. The stack $\mathcal{Y}(d)$ is the moduli stack of representations of dimension $d$ of the quiver with one vertex and two loops. Let $s$ be the morphism

$$s\colon \mathcal{Y}(d) \to \mathfrak{g},\ (X,Y) \mapsto [X,Y]. \tag{2.13}$$

The morphism $s$ induces a map of vector bundles $\partial : \mathfrak{g}^\vee \otimes \mathcal{O}_{\mathfrak{g}^{\oplus 2}} \to \mathcal{O}_{\mathfrak{g}^{\oplus 2}}$. Let $s^{-1}(0)$ be the derived scheme with the dg-ring of regular functions

$$\mathcal{O}_{s^{-1}(0)} := \mathcal{O}_{\mathfrak{g}^{\oplus 2}}\left[\mathfrak{g}^\vee \otimes \mathcal{O}_{\mathfrak{g}^{\oplus 2}}[1]; d_s\right], \tag{2.14}$$

where the differential $d_s$ is induced by the map $\partial$. Consider the (derived) stack

$$\mathscr{C}(d) := s^{-1}(0)/GL(V) \hookrightarrow \mathcal{Y}(d). \tag{2.15}$$

For a smooth variety $X$, we denote by $\mathscr{C}oh(X,d)$ the derived moduli stack of zero-dimensional sheaves on $X$ with length $d$ and by $\mathcal{C}oh(X,d)$ the classical truncation of $\mathscr{C}oh(X,d)$. Then $\mathscr{C}(d)$ is equivalent to $\mathscr{C}oh(\mathbb{C}^2,d)$.

For a decomposition $d = d_1 + \cdots + d_k$, let $\mathscr{C}(d_1,\ldots,d_k)$ be the derived moduli stack of filtrations of coherent sheaves on $\mathbb{C}^2$:

$$0 = Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_k \tag{2.16}$$

such that each subquotient $Q_i/Q_{i-1}$ is a zero-dimensional sheaf on $\mathbb{C}^2$ with length $d_i$. There exist evaluation morphisms

$$\mathscr{C}(d_1)\times\cdots\times\mathscr{C}(d_k) \xleftarrow{q} \mathscr{C}(d_1,\ldots,d_k) \xrightarrow{p} \mathscr{C}(d),$$

where $p$ is proper and $q$ is quasi-smooth. The above diagram for $k=2$ defines the categorical Hall product

$$m = m_{d_1,d_2} = p_*q^*\colon D^b(\mathscr{C}(d_1)) \boxtimes D^b(\mathscr{C}(d_2)) \to D^b(\mathscr{C}(d)), \tag{2.17}$$

which is a special case of the product of categorical Hall algebras for surfaces defined by Porta–Sala [PS23].

Let $T$ be the two-dimensional torus in Equation (2.1) which acts on $\mathbb{C}^2$ by $(t_1,t_2)\cdot(x,y) = (t_1x, t_2y)$. It naturally induces an action on $\mathscr{C}(d)$. There is also a $T$-equivariant Hall product

$$m = m_{d_1,d_2} = p_*q^*\colon D^b_T(\mathscr{C}(d_1)) \boxtimes D^b_T(\mathscr{C}(d_2)) \to D^b_T(\mathscr{C}(d)). \tag{2.18}$$

Here, the box product is taken over $BT$. In what follows, whenever we take a box-product in the $T$-equivariant setting, we take it over $BT$. We also use the notation $*$ for the Hall product.

### 2.8. Subcategories $\mathbb{T}(d)_v$

Let

$$i\colon \mathscr{C}(d) \hookrightarrow \mathcal{Y}(d) \tag{2.19}$$

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be the natural closed immersion. Define the full triangulated subcategory

$$\widetilde{\mathbb{T}}(d)_v \subset D^b(\mathcal{Y}(d))$$

generated by the vector bundles $\mathcal{O}_{\mathcal{Y}(d)} \otimes \Gamma_{GL(d)}(\chi)$ for a dominant weight $\chi$ satisfying

$$\chi + \rho \in \mathbf{W}(d)_v.$$

Define the full triangulated subcategory

$$\mathbb{T}(d)_v \subset D^b(\mathscr{C}(d)) \tag{2.20}$$

with objects $\mathcal{E}$ such that $i_*\mathcal{E}$ is in $\widetilde{\mathbb{T}}(d)_v$. In [PTa, Lemma 4.8], we showed that the Hall product restricts to functors

$$m : \mathbb{T}(d_1)_{v_1} \otimes \mathbb{T}(d_2)_{v_2} \to \mathbb{T}(d)_v$$

for $(d, v) = (d_1, v_1) + (d_2, v_2)$ and $\frac{v_1}{d_1} = \frac{v_2}{d_2}$. Also, there is a semiorthogonal decomposition; see [Păd23, Corollary 3.3]:

$$D^b(\mathscr{C}(d)) = \left\langle \mathbb{T}(d_1)_{v_1} \boxtimes \cdots \boxtimes \mathbb{T}(d_k)_{v_k} \;\middle|\; \begin{matrix} v_1/d_1 < \cdots < v_k/d_k \\ d_1 + \cdots + d_k = d \end{matrix} \right\rangle. \tag{2.21}$$

In the above, each fully faithful functor

$$\mathbb{T}(d_1)_{v_1} \boxtimes \cdots \boxtimes \mathbb{T}(d_k)_{v_k} \hookrightarrow D^b(\mathscr{C}(d))$$

is given by the categorical Hall product (2.17).

Consider the grading induced by the action of $\mathbb{C}^*$ on $\mathcal{X}(d)$ scaling the linear map corresponding to $Z$ with weight 2. The Koszul duality equivalence, also called dimensional reduction in the literature, gives the following equivalence [Isi13, Hir17, Toda]:

$$\Phi\colon D^b(\mathscr{C}(d)) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d), \operatorname{Tr} W). \tag{2.22}$$

Under this equivalence, we have that $\Phi\colon \mathbb{T}(d)_v \xrightarrow{\sim} \mathbb{S}^{\mathrm{gr}}(d)_v$.

### *2.9. Constructions of objects in* $\mathbb{T}(d)_v$

Here, we review the construction of objects $\mathcal{E}_{d,v} \in \mathbb{T}(d)_v$ (which also produces an object in $\mathbb{T}_T(d)_v$) following [PTa, Subsection 4.3]. Let $\mathcal{Z} \subset \mathscr{C}(1, 1, \ldots, 1)$ be the closed substack defined as follows. Let $\lambda$ be the cocharacter

$$\lambda\colon \mathbb{C}^* \to GL(V),\ t \mapsto (t^d, t^{d-1}, \ldots, t). \tag{2.23}$$

The attracting stack of $\mathcal{Y}(d)$ with respect to $\lambda$ is given by

$$\mathcal{Y}(d)^{\lambda \geqslant 0} := \left(\mathfrak{g}^{\lambda \geqslant 0}\right)^{\oplus 2} / GL(V)^{\lambda \geqslant 0}, \tag{2.24}$$

where $GL(V)^{\lambda \geqslant 0} \subset GL(V)$ is the subgroup of upper triangular matrices. Then the morphism (2.13) restricts to the morphism

$$s^{\lambda \geqslant 0}\colon \mathcal{Y}(d)^{\lambda \geqslant 0} \to \mathfrak{g}^{\lambda \geqslant 0} \tag{2.25}$$

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whose derived zero locus $\mathscr{C}(d)^{\lambda\geqslant 0}$ is equivalent to $\mathscr{C}(1,\ldots,1)$. Let $X=(x_{i,j})$ and $Y=(y_{i,j})$ be elements of $\mathfrak{g}^{\lambda\geqslant 0}$ for $1\leqslant i,j\leqslant d$, where $x_{i,j}=y_{i,j}=0$ for $i>j$. Then the equation $s^{\lambda\geqslant 0}(X,Y)=0$ is equivalent to the equations

$$\sum_{i\leqslant a\leqslant j} x_{i,a}y_{a,j}=\sum_{i\leqslant a\leqslant j} y_{i,a}x_{a,j},$$

for each $(i,j)$ with $i\leqslant j$. We call the above equation $\mathbb{E}_{i,j}$. The equation $\mathbb{E}_{i,i}$ is $x_{i,i}y_{i,i}-y_{i,i}x_{i,i}=0$, which always holds but imposes a nontrivial derived structure on $\mathscr{C}(1,\ldots,1)$. The equation $\mathbb{E}_{i,i+1}$ is

$$(x_{i,i}-x_{i+1,i+1})y_{i,i+1}-(y_{i,i}-y_{i+1,i+1})x_{i,i+1}=0.$$

The above equation is satisfied if the following equation $\mathbb{F}_{i,i+1}$ is satisfied:

$$\{x_{i,i}-x_{i+1,i+1}=0, y_{i,i}-y_{i+1,i+1}=0\}.$$

We define the closed derived substack

$$\mathcal{Z}:=\mathcal{Z}(d)\subset\mathcal{Y}^{\lambda\geqslant 0}(d) \tag{2.26}$$

to be the derived zero locus of the equations $\mathbb{F}_{i,i+1}$ for all $i$ and $\mathbb{E}_{i,j}$ for all $i+2\leqslant j$. We usually drop $d$ from the notation if the dimension is clear from the context. Then $\mathcal{Z}$ is a closed substack of $\mathscr{C}(d)^{\lambda\geqslant 0}=\mathscr{C}(1,\ldots,1)$. Note that, set theoretically, the closed substack $\mathcal{Z}$ corresponds to filtrations (2.16) such that each $Q_i/Q_{i-1}$ is isomorphic to $\mathcal{O}_x$ for some $x\in\mathbb{C}^2$ independent of $i$.

We have the diagram of attracting loci

$$\mathscr{C}(1)^{\times d}=\mathscr{C}(d)^{\lambda}\xleftarrow{q}\mathscr{C}(d)^{\lambda\geqslant 0}\xrightarrow{p}\mathscr{C}(d),$$

where $p$ is a proper morphism. We set

$$m_i:=\left\lceil\frac{vi}{d}\right\rceil-\left\lceil\frac{v(i-1)}{d}\right\rceil+\delta_i^d-\delta_i^1\in\mathbb{Z}, \tag{2.27}$$

where $\delta_i^j$ is the Kronecker delta function defined by $\delta_i^j=1$ if $i=j$ and $\delta_i^j=0$ otherwise. For a weight $\chi=\sum_{i=1}^d n_i\beta_i$ with $n_i\in\mathbb{Z}$, we denote by $\mathbb{C}(\chi)$ the one-dimensional $GL(V)^{\lambda\geqslant 0}$-representation given by

$$GL(V)^{\lambda\geqslant 0}\to GL(V)^{\lambda}=T(d)\xrightarrow{\chi}\mathbb{C}^*,$$

where the first morphism is the projection.

**Definition 2.3** [PTa, Definition 4.2]. We define the complex $\mathcal{E}_{d,v}$ by

$$\mathcal{E}_{d,v}:=p_*(\mathcal{O}_{\mathcal{Z}}\otimes\mathbb{C}(m_1,\ldots,m_d))\in D^b(\mathscr{C}(d))_v. \tag{2.28}$$

The construction above is $T$-equivariant, so we also obtain an object $\mathcal{E}_{d,v}\in D^b_T(\mathscr{C}(d))_v$.

In [PTa, Lemma 4.3], we showed that $\mathcal{E}_{d,v}$ is an object of $\mathbb{T}(d)_v$ and $\mathbb{T}_T(d)_v$.

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## 2.10. Shuffle algebras

**2.10.1.**
Consider the $\mathbb{N}$-graded $\mathbb{K}$-module:

$$\mathcal{S}h := \bigoplus_{d\geqslant 0} \mathbb{K}\big[z_1^{\pm 1}, \dots, z_d^{\pm 1}\big]^{\mathfrak{S}_d}.$$

We define a shuffle product on $\mathcal{S}h$ as follows. Let $\xi(x)$ be defined by

$$\xi(x) := \frac{(1-q_1^{-1}x)(1-q_2^{-1}x)(1-q^{-1}x^{-1})}{1-x},$$

where $q := q_1 q_2$. For $f \in \mathbb{K}\big[z_1^{\pm}, \dots, z_a^{\pm}\big]$ and $g \in \mathbb{K}\big[z_{a+1}^{\pm}, \dots, z_{a+b}^{\pm}\big]$, we set

$$f * g := \frac{1}{a!b!}\mathrm{Sym}\left( fg \cdot \prod_{\substack{1\leqslant i\leqslant a,\\ a<j\leqslant a+b}} \xi(z_i z_j^{-1})\right), \tag{2.29}$$

where we denote by $\mathrm{Sym}(h(z_1, \dots, z_d))$ the sum of $h(z_{\sigma(1)}, \dots, z_{\sigma(d)})$ after all permutations $\sigma \in \mathfrak{S}_d$. Let $\mathcal{S} \subset \mathcal{S}h$ be the subalgebra generated by $z_1^l$ for $l \in \mathbb{Z}$. Let $\mathcal{S}_{\mathbb{F}} := \mathcal{S} \otimes_{\mathbb{K}} \mathbb{F}$. It is proved in [Neg23, Theorem 4.6] that there is an isomorphism

$$i_*\colon \bigoplus_{d\geqslant 0} G_T(\mathscr{C}(d)) \otimes_{\mathbb{K}} \mathbb{F} \xrightarrow{\cong} \mathcal{S}_{\mathbb{F}}. \tag{2.30}$$

The above isomorphism is induced by the algebra homomorphism which will be defined in Equation (2.37).

**2.10.2.**
Let

$$\mathcal{S}' \subset \bigoplus_{d\geqslant 0} \mathbb{K}(z_1, \dots, z_d)^{\mathfrak{S}_d}$$

be the $\mathbb{K}$-subalgebra generated by elements of the form

$$A'_{k_\bullet} := \mathrm{Sym}\left( \frac{z_1^{k_1}\cdots z_d^{k_d}}{(1-q^{-1}z_1^{-1}z_2)\cdots(1-q^{-1}z_{d-1}^{-1}z_d)} \cdot \prod_{j>i} w(z_i z_j^{-1})\right) \tag{2.31}$$

for various $(k_1, \dots, k_d) \in \mathbb{Z}^d$ and $d \geqslant 1$, for the shuffle product (2.29) where we replace $\xi(x)$ with $w(x)$ defined by

$$w(x) := \frac{(1-q_1^{-1}x)(1-q_2^{-1}x)}{(1-x)(1-q^{-1}x)}.$$

Let $\mathcal{S}'_{\mathbb{F}} := \mathcal{S}' \otimes_{\mathbb{K}} \mathbb{F}$. Consider the morphism

$$\bigoplus_{d\geqslant 0} \mathbb{F}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]^{\mathfrak{S}_d} \to \bigoplus_{d\geqslant 0} \mathbb{F}(z_1, \dots, z_d)^{\mathfrak{S}_d} \tag{2.32}$$

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defined by

$$f(z_1,\dots,z_d)\mapsto f(z_1,\dots,z_d)\cdot\prod_{i\neq j}(1-q^{-1}z_iz_j^{-1})^{-1}.$$

Then Equation (2.32) induces an algebra homomorphism $\mathcal{S}\to\mathcal{S}'$. There is an isomorphism

$$\mathcal{S}_{\mathbb{F}}\xrightarrow{\cong}\mathcal{S}'_{\mathbb{F}}, \tag{2.33}$$

see [PTa, Proof of Lemma 4.11]. For $(d,v)\in\mathbb{N}\times\mathbb{Z}$, we set $A'_{d,v}$ to be $A'_{m_\bullet}$ for the choice of $m_\bullet$ in Equation (2.27). By [Neg22, Equation (2.12)], we have the following isomorphism of $\mathbb{K}$-modules:

$$\mathcal{S}'=\bigoplus_{v_1/d_1\leqslant\cdots\leqslant v_k/d_k}\mathbb{K}\cdot A'_{d_1,v_1}*\cdots*A'_{d_k,v_k}, \tag{2.34}$$

where the tuples $(d_i,v_i)_{i=1}^k$ appearing above are unordered for subtuples $(d_i,v_i)_{i=a}^b$ with $v_a/d_a=\cdots=v_b/d_b$. We also define

$$A_{d,v}:=\mathrm{Sym}\left(\frac{z_1^{m_1}\cdots z_d^{m_d}}{(1-q^{-1}z_1^{-1}z_2)\cdots(1-q^{-1}z_{d-1}^{-1}z_d)}\cdot\prod_{j>i}\xi(z_iz_j^{-1})\right), \tag{2.35}$$

where the exponents $m_i$ for $1\leqslant i\leqslant d$ are given by Equation (2.27).

**2.10.3.**
The $T$-equivariant Hall product (2.18) induces an associative algebra structure

$$m\colon G_T(\mathscr{C}(d_1))\otimes_{\mathbb{K}}G_T(\mathscr{C}(d_2))\to G_T(\mathscr{C}(d)). \tag{2.36}$$

Let $i\colon\mathscr{C}(d)\hookrightarrow\mathcal{Y}(d)$ be the closed immersion. The pull-back via $\mathcal{Y}(d)\to BGL(d)$ gives the isomorphism

$$\bigoplus_{d\geqslant 0}K_T(BGL(d))=\bigoplus_{d\geqslant 0}\mathbb{K}\left[z_1^{\pm 1},\dots,z_d^{\pm 1}\right]^{\mathfrak{S}_d}\xrightarrow{\cong}\bigoplus_{d\geqslant 0}K_T(\mathcal{Y}(d)).$$

Therefore, the push-forward by $i$ induces a morphism

$$i_*\colon\bigoplus_{d\geqslant 0}G_T(\mathscr{C}(d))\to\bigoplus_{d\geqslant 0}\mathbb{K}\left[z_1^{\pm 1},\dots,z_d^{\pm 1}\right]^{\mathfrak{S}_d}. \tag{2.37}$$

The product (2.36) is compatible with a shuffle product defined on the right-hand side of Equation (2.37); see [PTa, Subsection 4.5]. In [PTa, Lemma 4.11], we showed that

$$i_*[\mathcal{E}_{d,v}]=(1-q_1^{-1})^{d-1}(1-q_2^{-1})^{d-1}A_{d,v}. \tag{2.38}$$

### *2.11. Compatibility of the Hall product under the Koszul equivalence*

In this subsection, we denote by $m$ the Hall product (2.17) and by $\widetilde{m}$ the Hall product for the quiver with potential $(Q,W)$ from Subsection 2.2.1. Using the results in [Toda, Section 2.4], Koszul duality equivalences are compatible with the Hall products by the following commutative diagram (see [Păd23, Proposition 3.1]):

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$$\begin{tikzcd} D^b(\mathscr{C}(d_1)) \boxtimes D^b(\mathscr{C}(d_2)) \arrow[r, "m"] \arrow[d, "\widetilde{\Phi}"] & D^b(\mathscr{C}(d)) \arrow[d, "\Phi"] \\ \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d_1), \mathrm{Tr}\, W_{d_1}) \boxtimes \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d_2), \mathrm{Tr}\, W_{d_2}) \arrow[r, "\widetilde{m}"] & \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d), \mathrm{Tr}\, W_d), \end{tikzcd} \tag{2.39}$$

where the left arrow $\widetilde{\Phi}$ is the composition of Koszul duality equivalences (2.22) with the tensor product of

$$\det((\mathfrak{g}^{\nu>0})^\vee(2))[-\dim \mathfrak{g}^{\nu>0}] = (\det V_1)^{-d_2} \otimes (\det V_2)^{d_1}[d_1 d_2]. \tag{2.40}$$

The cocharacter $\nu\colon \mathbb{C}^* \to T(d)$ is $\nu(t) = (\overbrace{t, \ldots, t}^{d_1}, \overbrace{1, \ldots, 1}^{d_2})$, the vector spaces $V_i$ have $\dim V_i = d_i$ for $i = 1, 2$ and $(1)$ is a twist by the weight one $\mathbb{C}^*$-character, which is isomorphic to the shift functor $[1]$ of the category of graded matrix factorizations.

### *2.12. Symmetric polynomials*

Let $\Lambda$ be the $\mathbb{Z}$-algebra of symmetric polynomials [Mac79, Chapter I, Section 2], [Sch12, Subsection 2.4]:

$$\Lambda \cong \varprojlim \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n},$$

with multiplication defined by

$$f(x_1, \ldots, x_a) \star g(x_{a+1}, \ldots, x_{a+b}) := \sum_{\mathfrak{S}_{a+b}/\mathfrak{S}_a \times \mathfrak{S}_b} w(f(x_1, \ldots, x_a) g(x_{a+1}, \ldots, x_{a+b}))$$

and comultiplication induced by the restriction map

$$\mathbb{Z}[x_1, \ldots, x_{a+b}]^{\mathfrak{S}_{a+b}} \to \mathbb{Z}[x_1, \ldots, x_a]^{\mathfrak{S}_a} \otimes \mathbb{Z}[x_{a+1}, \ldots, x_{a+b}]^{\mathfrak{S}_b}.$$

Alternatively, $\Lambda$ is isomorphic to the Grothendieck group of the monoidal category

$$\mathcal{R} := \bigoplus_{n \geqslant 0} \mathrm{Rep}(\mathfrak{S}_n),$$

where $\mathrm{Rep}(\mathfrak{S}_n)$ is the abelian category of finite-dimensional $\mathfrak{S}_n$-representations, multiplication is given by the induction functor

$$\mathrm{Ind} : \mathrm{Rep}(\mathfrak{S}_a \times \mathfrak{S}_b) \to \mathrm{Rep}(\mathfrak{S}_{a+b}),$$

and comultiplication is given by the restriction functor

$$\mathrm{Res} : \mathrm{Rep}(\mathfrak{S}_{a+b}) \to \mathrm{Rep}(\mathfrak{S}_a \times \mathfrak{S}_b).$$

The isomorphism $\mathcal{R} \xrightarrow{\cong} \Lambda$ is given by sending an irreducible $\mathfrak{S}_n$-representation $W_\lambda$ corresponding to a partition $\lambda$ of $n$ to the Schur function $s_\lambda$, see [Mac79, Chapter I, Equation (7.5)].

For $R$ a ring with a map $\mathbb{Z} \to R$, denote by $\Lambda_R := R \otimes_{\mathbb{Z}} \Lambda$ the $R$-algebra with multiplication and comultiplication induced from those of $\Lambda$. Consider the elementary symmetric functions

$$e_n := \sum_{i_1 < \ldots < i_n} x_{i_1} \ldots x_{i_n} \in \Lambda$$

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and the power sum functions

$$p_n := \sum_i x_i^n \in \Lambda.$$

We also denote by $e_n$ and $p_n$ the images of these symmetric functions in $\Lambda_R$. Let $t$ be a formal variable. These functions are connected via the identity

$$\sum_{n\geqslant 0} e_n t^n = \exp\left(\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} p_n t^n\right). \tag{2.41}$$

There are isomorphisms (see [Mac79, Chapter I, Equations (2.4), (2.14)]):

$$\Lambda_{\mathbb{Q}} \cong \mathbb{Q}[e_1, e_2, \ldots] \cong \mathbb{Q}[p_1, p_2, \ldots]. \tag{2.42}$$

For $n \geqslant 1$, let $P(n)$ be the free one-dimensional $\mathbb{Z}$-module with generator $p_n$. By Equation (2.42), there is an isomorphism of $\mathbb{N}$-graded $\mathbb{Q}$-vector spaces

$$\Lambda_{\mathbb{Q}} \cong \bigotimes_{n\geqslant 1} \mathrm{Sym}\big(P(n)_{\mathbb{Q}}\big). \tag{2.43}$$

## 3. The support of complexes in quasi-BPS categories

Recall the regular function (2.2). Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C}oh(\mathbb{C}^3, d) \arrow[r, equal] \arrow[d, "\pi"'] & \mathrm{Crit}(\mathrm{Tr}\, W_d) \arrow[r, hook] & \mathcal{X}(d) \arrow[d, "\pi"'] \arrow[dr, "\mathrm{Tr}\, W_d"] & \\
\mathrm{Sym}^d(\mathbb{C}^3) \arrow[rr, hook] & & X(d) \arrow[r] & \mathbb{C},
\end{tikzcd} \tag{3.1}$$

where the vertical arrows are good moduli space morphisms and the horizontal arrows are closed immersions. The left vertical arrow sends a zero-dimensional sheaf to its support. Let $\Delta\colon \mathbb{C}^3 \subset \mathrm{Sym}^d(\mathbb{C}^3)$ be the small diagonal

$$\mathbb{C}^3 \subset \mathrm{Sym}^d(\mathbb{C}^3),\ x \mapsto (x, \ldots, x).$$

We abuse notation and denote the image of $\Delta$ also by $\Delta \cong \mathbb{C}^3$. We consider the pull-back $\pi^{-1}(\Delta) \subset \mathcal{C}oh(\mathbb{C}^3, d)$, which is a closed substack of $\mathrm{Crit}(\mathrm{Tr}\, W_d) \subset \mathcal{X}(d)$. Davison [Davb, Theorem 5.1] showed that the BPS sheaf for the moduli stack of degree $d$ sheaves on $\mathbb{C}^3$ is

$$\mathcal{BPS}_d = \Delta_* \mathrm{IC}_{\mathbb{C}^3}.$$

Recall the notations involving formal completions from Subsection 2.1. The following is the main result of this section, which will be proved in Subsection 3.1:

**Theorem 3.1.** *Consider a pair $(d, w) \in \mathbb{N}\times\mathbb{Z}$, and let $\mathcal{F}$ be an object in $\mathbb{S}(d)_w$. Assume there exists $p \in \mathrm{Sym}^d(\mathbb{C}^3) \setminus \Delta$ such that the support of $\mathcal{F}$ intersects $\pi^{-1}(p)$. Write $p = \sum_{i=1}^{l} p^{(i)}$, $p^{(i)} = d^{(i)}x^{(i)}$, $d^{(i)} \in \mathbb{Z}_{>0}$, $x^{(i)} \neq x^{(i')}$ for $i \neq i'$ and $l \geqslant 2$. Then there exist nonzero objects $\mathcal{F}_i \in \mathrm{MF}(\mathcal{X}_{p^{(i)}}(d^{(i)}), \mathrm{Tr}\, W_{d^{(i)}})_{w^{(i)}}$ for $1 \leqslant i \leqslant l$ with*

$$\frac{w^{(1)}}{d^{(1)}} = \cdots = \frac{w^{(l)}}{d^{(l)}}$$

*and* $w^{(1)} + \cdots + w^{(l)} = w$*. In particular, if* $\gcd(d, w) = 1$*, then any object in* $\mathbb{S}(d)_w$ *is supported on* $\pi^{-1}(\Delta)$*. The same result holds for the categories* $\mathbb{S}^{\bullet}_{*}(d)_w$ *introduced in Subsection 2.6.2.*

In Subsection 4.1, we discuss divisibility properties of complexes supported on $\pi^{-1}(\Delta)$; see Lemma 4.1. In Subsection 4.2, we use Theorem 3.1 and Lemma 4.1 to show that $K_T(\mathbb{S}(d)_w)'$ is a free $\mathbb{K}$-module with generator $[\mathcal{E}_{d,w}]$ when $\gcd(d, w) = 1$.

### 3.1. Proof of the main result

*Proof of Theorem 3.1.* Suppose that an object $\mathcal{F} \in \mathbb{S}(d)_w$ has support not contained in $\pi^{-1}(\Delta)$. Then there is $p \in \mathrm{Sym}^d(\mathbb{C}^3) \setminus \Delta$ such that $\mathcal{F}|_{\mathcal{X}_p(d)} \neq 0$ in $\mathrm{MF}(\mathcal{X}_p(d), \mathrm{Tr}\, W_d)$, where $\mathcal{X}_p(d)$ is the formal fiber of $p$ along the good moduli space morphism $\mathcal{X}(d) \to X(d)$. Since $p \notin \Delta$, it is written as

$$p = \sum_{i=1}^{l} p^{(i)}, \ p^{(i)} = d^{(i)} x^{(i)}, \ d^{(i)} \in \mathbb{Z}_{>0}, \ x^{(i)} \neq x^{(i')} \text{ for } i \neq i', \ l \geqslant 2.$$

The unique closed point in $\pi^{-1}(p)$ corresponds to the semisimple $Q$-representation

$$R = \bigoplus_{i=1}^{l} V^{(i)} \otimes R^{(i)},$$

where $R^{(i)}$ is the one-dimensional $Q$-representation corresponding to $\mathcal{O}_{x^{(i)}}$ and $V^{(i)}$ is a $d^{(i)}$-dimensional vector space such that $d^{(1)} + \cdots + d^{(l)} = d$. Below we write a basis of $V^{(i)}$ as $\beta^{(i)}_1, \ldots, \beta^{(i)}_{d^{(i)}}$ and set

$$\{\beta_1, \ldots, \beta_d\} = \{\beta^{(1)}_1, \ldots, \beta^{(1)}_{d^{(1)}}, \beta^{(2)}_1, \ldots, \beta^{(2)}_{d^{(2)}}, \ldots\}.$$

The étale slice theorem implies that

$$\mathcal{X}_p(d) \cong \widehat{\mathrm{Ext}}^1_Q(R, R)/G_p, \tag{3.2}$$

where $G_p = \mathrm{Aut}(R) = \prod_{i=1}^{l} GL(V^{(i)})$ and $\widehat{\mathrm{Ext}}^1_Q(R, R)$ is the formal fiber of the origin along the morphism $\mathrm{Ext}^1_Q(R, R) \to \mathrm{Ext}^1_Q(R, R)/\!\!/G_p$. By Lemma 3.2, the Ext-group $\mathrm{Ext}^1_Q(R, R)$ is computed as follows:

$$\mathrm{Ext}^1_Q(R, R) = \bigoplus_{i=1}^{l} \mathrm{End}(V^{(i)}, V^{(i)})^{\oplus 3} \oplus \bigoplus_{i \neq j} \mathrm{Hom}(V^{(i)}, V^{(j)})^{\oplus 2}. \tag{3.3}$$

Note that the maximal torus $T(d) \subset GL(V)$ is contained in $G_p$.

We define

$$\mathbb{S}_p(d)_w \subset \mathrm{MF}(\mathcal{X}_p(d), \mathrm{Tr}\, W_d)$$

to be the full subcategory generated by matrix factorizations whose entries are of the form $\Gamma_{G_p}(\chi) \otimes \mathcal{O}$, where $\chi$ is a $G_p$-dominant $T(d)$-weight satisfying

$$\chi + \rho_p \in \mathbf{W}_p(d)_w. \tag{3.4}$$

Here, $\rho_p$ is half the sum of positive roots of $G_p$ and $\mathbf{W}_p(d)_w$ is defined as in Equation (2.9) for the $G_p$-representation $\mathrm{Ext}^1_Q(R, R)$

$$\mathbf{W}_p(d)_w := \frac{1}{2}\mathrm{sum}[0, \beta] + w\tau_d,$$

where the Minkowski sum is after all weights $\beta$ in $\mathrm{Ext}^1_Q(R,R)$. By Lemma 3.5, we have $\mathcal{F}|_{\mathcal{X}_p(d)} \in \mathbb{S}_p(d)_w$, in particular $\mathbb{S}_p(d)_w \neq 0$.

Below, in order to simplify the notation, we treat the case $l = 2$. Since any zero-dimensional sheaf $Q$ on $\mathbb{C}^3$ supported on $p$ decomposes into $Q^{(1)} \oplus Q^{(2)}$, where $Q^{(i)}$ is supported on $x^{(i)}$, we have

$$\widehat{Coh}_p(\mathbb{C}^3, d) = \mathrm{Crit}(\mathrm{Tr}\, W_d|_{\mathcal{X}_p(d)}) = \widehat{Coh}_{p^{(1)}}(\mathbb{C}^3, d^{(1)}) \times \widehat{Coh}_{p^{(2)}}(\mathbb{C}^3, d^{(2)}). \tag{3.5}$$

Here, $\widehat{Coh}_p(\mathbb{C}^3, d)$ is the formal fiber of the left vertical arrow in Equation (3.1) at $p$. Indeed, by Lemma 3.3, we can show that, by replacing the isomorphism (3.2) if necessary, the regular function $\mathrm{Tr}\, W_d$ restricted to $\mathcal{X}_p(d)$ is written as

$$\mathrm{Tr}\, W_d|_{\mathcal{X}_p(d)} = \mathrm{Tr}\, W_{d^{(1)}} \boxplus \mathrm{Tr}\, W_{d^{(2)}} \boxplus q. \tag{3.6}$$

Here, $\mathrm{Tr}\, W_{d^{(i)}}$ is the regular function (2.2) on $\mathcal{X}(d^{(i)})$ restricted to $\mathcal{X}_{p^{(i)}}(d^{(i)})$ and $q$ is a nondegenerate $G_p$-invariant quadratic form on $U \oplus U^\vee$ given by $q(u,v) = \langle u, v\rangle$, where $U$ is the following self-dual $G_p$-representation

$$U := \mathrm{Hom}(V^{(1)}, V^{(2)}) \oplus \mathrm{Hom}(V^{(2)}, V^{(1)}).$$

The decomposition (3.6) in particular implies Equation (3.5).

We have the following diagram

$$\begin{tikzcd}
\mathcal{U} \arrow[r, hook, "i"] \arrow[d, "p"'] & \mathcal{U} \oplus \mathcal{U}^\vee & \mathcal{X}_p(d) \arrow[l, "j"'] \\
\mathcal{X}_{p^{(1)}}(d^{(1)}) \times \mathcal{X}_{p^{(2)}}(d^{(2)}), & &
\end{tikzcd}$$

where $\mathcal{U}$ is the vector bundle on $\mathcal{X}_{p^{(1)}}(d^{(1)}) \times \mathcal{X}_{p^{(2)}}(d^{(2)})$ determined by the $G_p$-representation $U$, $i$ is the closed immersion $x \mapsto (x, 0)$, and $j$ is the natural morphism induced by the formal completion which induces the isomorphism on critical loci of $\mathrm{Tr}\, W_d$. We have the following functors

$$\begin{aligned}\Psi := j^* i_* p^* \colon \mathrm{MF}(\mathcal{X}_{p^{(1)}}(d^{(1)}), \mathrm{Tr}\, W_{d^{(1)}}) \boxtimes \mathrm{MF}(\mathcal{X}_{p^{(2)}}(d^{(2)}), \mathrm{Tr}\, W_{d^{(2)}}) \\ \xrightarrow{\sim} \mathrm{MF}(\mathcal{U} \oplus \mathcal{U}^\vee, \mathrm{Tr}\, W_d) \overset{j^*}{\hookrightarrow} \mathrm{MF}(\mathcal{X}_p(d), \mathrm{Tr}\, W_d).\end{aligned} \tag{3.7}$$

Here, the first arrow is an equivalence by Knörrer periodicity (see [Hir17, Theorem 4.2]), and the second arrow is fully faithful with dense image (see [Todb, Lemma 6.4]). By Lemma 3.4, the above functor restricts to the fully faithful functor

$$\bigoplus_{\substack{w^{(1)}+w^{(2)}=w \\ w^{(1)}/d^{(1)}=w^{(2)}/d^{(2)}}} \mathbb{S}_{p^{(1)}}(d^{(1)})_{w^{(1)}} \boxtimes \mathbb{S}_{p^{(2)}}(d^{(2)})_{w^{(2)}} \to \mathbb{S}_p(d)_w. \tag{3.8}$$

Below, we show that the above functor (3.8) has dense image, and thus the conclusion follows. By the semiorthogonal decomposition (2.21) together with Equation (2.39), we have

$$\begin{aligned}&\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W_d) \\ &= \left\langle *_{i=1}^k \mathbb{S}(d_i)_{v_i + d_i(\sum_{i>j} d_j - \sum_{j>i} d_j)} \,\middle|\, \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k}, d_1 + \cdots + d_k = d \right\rangle.\end{aligned} \tag{3.9}$$

The weights $w_i := v_i + d_i(\sum_{i>j} d_j - \sum_{j>i} d_j)$ are computed as in Equation (2.7):

$$\sum_{i=1}^{k} w_i \tau_{d_i} = \sum_{i=1}^{k} v_i \tau_{d_i} - \mathfrak{g}^{\lambda>0}, \tag{3.10}$$

where $\mathfrak{g}$ is the Lie algebra of $GL(d)$ and $\lambda$ is the antidominant cocharacter corresponding to the decomposition $d = d_1 + \cdots + d_k$:

$$\lambda(t) = (\overbrace{t^k, \ldots, t^k}^{d_1}, \overbrace{t^{k-1}, \ldots, t^{k-1}}^{d_2}, \ldots, \overbrace{t, \ldots, t}^{d_k}). \tag{3.11}$$

For $x \in \mathbb{C}^3$, let $q = d[x] := (x, \ldots, x) \in \mathrm{Sym}^d(\mathbb{C}^3)$. Since $G_q = GL(V)$ and $\mathcal{X}_q(d) \cong \mathfrak{g}_{d[0]}^{\oplus 3}/GL(V)$, the argument showing Equation (3.9) applies verbatim to show the semiorthogonal decomposition

$$\begin{aligned} &\mathrm{MF}(\mathcal{X}_q(d), \operatorname{Tr} W_d) \\ &= \left\langle *_{i=1}^{k} \mathbb{S}_q(d_i)_{v_i + d_i(\sum_{i>j} d_j - \sum_{j>i} d_j)} \,\middle|\, \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k}, d_1 + \cdots + d_k = d \right\rangle. \end{aligned} \tag{3.12}$$

From Equation (3.12), the left-hand side of Equation (3.7) admits a semiorthogonal decomposition S whose summands are of the form

$$*_{i=1}^{a} \mathbb{S}_{p_i^{(1)}}(d_i^{(1)})_{v_i^{(1)} + d_i^{(1)}(\sum_{i>j} d_j^{(1)} - \sum_{j>i} d_j^{(1)})} \boxtimes *_{i=1}^{b} \mathbb{S}_{p_i^{(2)}}(d_i^{(2)})_{v_i^{(2)} + d_i^{(2)}(\sum_{i>j} d_j^{(2)} - \sum_{j>i} d_j^{(2)})}. \tag{3.13}$$

Here, the left-hand side in Equation (3.8) comes to the rightmost part of the semiorthogonal decomposition S, $p_i^{(j)} = d_i^{(j)} x^{(j)}$, the integers $d_i^{(j)}$ satisfy

$$d_1^{(1)} + \cdots + d_a^{(1)} = d^{(1)}, \ d_1^{(2)} + \cdots + d_b^{(2)} = d^{(2)},$$

and we have the inequalities

$$\frac{v_1^{(1)}}{d_1^{(1)}} < \cdots < \frac{v_a^{(1)}}{d_a^{(1)}}, \ \frac{v_1^{(2)}}{d_1^{(2)}} < \cdots < \frac{v_b^{(2)}}{d_b^{(2)}}.$$

We write

$$\left\{ \frac{v_1^{(1)}}{d_1^{(1)}}, \ldots, \frac{v_a^{(1)}}{d_a^{(1)}}, \ldots, \frac{v_1^{(2)}}{d_1^{(2)}}, \ldots, \frac{v_b^{(2)}}{d_b^{(2)}} \right\} = \{\mu_1 < \cdots < \mu_k\},$$

where $k$ is the number of distinct elements in the left-hand side. For $1 \leqslant i \leqslant k$, we replace $(d_i^{(j)}, v_i^{(j)})$ by

$$(d_i^{(j)}, v_i^{(j)}) \mapsto \begin{cases} (d_l^{(j)}, v_l^{(j)}), & \text{if } \mu_i = v_l^{(j)}/d_l^{(j)} \text{ for some } l, \\ (0,0), & \text{otherwise}. \end{cases}$$

The subcategory (3.13) of Equation (3.8) is unchanged under the above replacement. Therefore, we may assume that $a = b = k$ and, by setting $(d_i, v_i) = (d_i^{(1)}, v_i^{(1)}) + (d_i^{(2)}, v_i^{(2)})$, we have

$$\frac{v_i}{d_i} = \frac{v_i^{(j)}}{d_i^{(j)}}, \ \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k}.$$

Here, the first identity holds whenever $(d_i^{(j)}, v_i^{(j)}) \neq (0,0)$.

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Let us take decompositions $V^{(j)} = \oplus_{i=1}^k V_i^{(j)}$ with $\dim V_i^{(j)} = d_i^{(j)}$ and a cocharacter as in Equation (3.11)

$$\lambda\colon \mathbb{C}^* \to GL(V^{(1)}) \times GL(V^{(2)}) \tag{3.14}$$

which acts on $V_i^{(j)}$ with weight $k+1-i$. The diagrams of attracting loci with respect to $\lambda$ give the commutative diagram; see [Todb, Proposition 2.5]:

$$\begin{tikzcd}
\boxtimes_{i=1}^k (\mathrm{MF}(\mathcal{X}_{p_i^{(1)}}(d_i^{(1)}), \operatorname{Tr} W_{d_i^{(1)}}) \boxtimes (\mathrm{MF}(\mathcal{X}_{p_i^{(2)}}(d_i^{(2)}), \operatorname{Tr} W_{d_i^{(2)}})) \arrow[r] \arrow[d, "*\boxtimes *"'] & \boxtimes_{i=1}^k \mathrm{MF}(\mathcal{X}_{p_i}(d_i), \operatorname{Tr} W_{d_i}) \arrow[d, "*"] \\
\mathrm{MF}(\mathcal{X}_{p^{(1)}}(d^{(1)}), \operatorname{Tr} W_{d^{(1)}}) \boxtimes \mathrm{MF}(\mathcal{X}_{p^{(2)}}(d^{(2)}), \operatorname{Tr} W_{d^{(2)}}) \arrow[r] & \mathrm{MF}(\mathcal{X}_p(d), \operatorname{Tr} W_d).
\end{tikzcd}$$

Here, the vertical arrows are categorical Hall products determined by $\lambda$, $p_i := p_i^{(1)} + p_i^{(2)}$, the bottom horizontal arrow is Equation (3.7), and the top horizontal arrow is the composition of the Knörrer periodicity equivalences with the tensor product of (a shift of)

$$\det\left((U^{\lambda>0})^\vee\right) = \bigotimes_{i=1}^k \left((\det V_i^{(1)})^{\sum_{i>j} d_j^{(2)} - \sum_{j>i} d_j^{(2)}} \otimes (\det V_i^{(2)})^{\sum_{i>j} d_j^{(1)} - \sum_{j>i} d_j^{(1)}}\right).$$

By the above commutative diagram together with the fact that Equation (3.7) restricts to the functor (3.8), the functor (3.7) sends Equation (3.13) to

$$*_{i=1}^k \mathbb{S}_{p_i}(d_i)_{v_i + d_i(\sum_{i>j} d_j - \sum_{j>i} d_j)} \subset \mathrm{MF}(\mathcal{X}_p(d), \operatorname{Tr} W_d). \tag{3.15}$$

Let us take a decomposition

$$w = v_1 + \cdots + v_k = w_1 + \cdots + w_k,$$

where $w_i$ is given in Equation (3.10). Then by Lemma 3.6, the subcategory (3.15) for $k \geqslant 2$ is right orthogonal to $\mathbb{S}_p(d)_w$. Together with the semiorthogonal decomposition S with summands (3.13), we conclude that the functor (3.8) has dense image. Indeed, for an object $A \in \mathbb{S}_p(d)_w$, there is a distinguished triangle

$$A_1 \xrightarrow{\alpha} A \xrightarrow{\beta} A_2, \tag{3.16}$$

where $A_1$ is a direct summand of an object $\Psi(B_1)$ for some $B_1$ in the left-hand side of Equation (3.8) and $A_2$ is a direct summand of an object $\Psi(B_2)$ for some $B_2$ in Equation (3.13) for $a = b \geqslant 2$. Then $\Psi(B_2)$ is an object of Equation (3.15) for $k \geqslant 2$, hence $\beta$ is a zero map by Lemma 3.6. So $\alpha$ is an isomorphism, hence the functor (3.8) has dense image. □

We have postponed several lemmas, which are given below:

**Lemma 3.2.** *The* Ext*-group* $\operatorname{Ext}^1_Q(R,R)$ *is computed as Equation (3.3).*

*Proof.* By the Euler pairing computation, we have $\chi_Q(R^{(i)}, R^{(j)}) = -2$. Since $\operatorname{Hom}(R^{(i)}, R^{(j)}) = \mathbb{C}\delta_{ij}$, we have

$$\operatorname{Ext}^1_Q(R^{(i)}, R^{(j)}) = \begin{cases} \mathbb{C}^2, & i \neq j, \\ \mathbb{C}^3, & i = j. \end{cases}$$

Therefore, Equation (3.3) holds. □

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**Lemma 3.3.** *By replacing the isomorphism (3.2) if necessary, the identity (3.6) holds.*

*Proof.* We may assume $x^{(1)} = (0,0,1)$ and $x^{(2)} = (0,0,0)$. We write an element of $\mathrm{Ext}^1_Q(R,R)$ as

$$(X^{(1)}, Y^{(1)}, Z^{(1)}, X^{(2)}, Y^{(2)}, Z^{(2)}, A^{(12)}, A^{(21)}, B^{(12)}, B^{(21)}), \tag{3.17}$$

where $X^{(i)} \in \mathrm{End}(V^{(i)})$ and $A^{(ij)} \in \mathrm{Hom}(V^{(i)}, V^{(j)})$. We have the morphism of algebraic stacks

$$\nu\colon \widehat{\mathrm{Ext}}^1_Q(R,R)/G_p \to \mathcal{X}_p(d)$$

which sends Equation (3.17) to

$$\left(\begin{pmatrix} X^{(1)} & A^{(12)} \\ A^{(21)} & X^{(2)} \end{pmatrix}, \begin{pmatrix} Y^{(1)} & B^{(12)} \\ B^{(21)} & Y^{(2)} \end{pmatrix}, \begin{pmatrix} Z^{(1)} + I & 0 \\ 0 & Z^{(2)} \end{pmatrix}\right) \in \mathfrak{g}^{\oplus 3}.$$

Indeed, the above correspondence is $GL(V)$-equivariant using the embedding $G_p \subset GL(V)$, so it determines a morphism $\nu$. Note that $\nu(0)$ corresponds to the polystable $Q$-representation $R = (V^{(1)} \otimes R^{(1)}) \oplus (V^{(2)} \otimes R^{(2)})$, where $R^{(i)}$ corresponds to $\mathcal{O}_{x^{(i)}}$.

We now explain that the morphism $\nu$ is étale at $\nu(0)$. The tangent complex of $\mathcal{X}(d)$ at $\nu(0)$ is

$$\mathbb{T}_{\mathcal{X}(d)}|_{\nu(0)} = \left(\mathrm{End}(V) \to \mathrm{End}(V)^{\oplus 3}\right), \ \alpha \mapsto (0,0,[\alpha,u]), \ u = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

The kernel of the above map is $\mathrm{End}(V^{(1)}) \oplus \mathrm{End}(V^{(2)})$, and the cokernel is $\mathrm{Ext}^1_Q(R,R)$, so the morphism $\nu$ induces a quasi-isomorphism on tangent complexes at $\nu(0)$.

A straightforward computation shows that

$$\begin{aligned} \nu^* \operatorname{Tr} W_d = {} & \operatorname{Tr}(Z^{(1)}[X^{(1)}, Y^{(1)}]) + \operatorname{Tr}(Z^{(2)}[X^{(2)}, Y^{(2)}]) \\ & + \operatorname{Tr}(A^{(12)}(B^{(21)} + B^{(21)}Z^{(1)} - Z^{(2)}B^{(21)})) \\ & + \operatorname{Tr}(B^{(12)}(Z^{(2)}A^{(21)} - A^{(21)}Z^{(1)} - A^{(21)})). \end{aligned}$$

By the following $G_p$-equivariant variable change

$$A^{(21)} \mapsto Z^{(2)}A^{(21)} - A^{(21)}Z^{(1)} - A^{(21)}, \ B^{(21)} \mapsto B^{(21)} + B^{(21)}Z^{(1)} - Z^{(2)}B^{(21)},$$

we obtain the identity (3.6). □

**Lemma 3.4.** *The functor (3.7) restricts to the functor (3.8).*

*Proof* (cf. the proof of [KT21, Theorem 2.7]). For a cocharacter $\lambda\colon \mathbb{C}^* \to T(d)$, let

$$n_{\lambda,p} := \left\langle \lambda, \mathbb{L}^{\lambda>0}_{\mathcal{X}_p(d)}|_0 \right\rangle, \ n'_{\lambda,p} := \left\langle \lambda, \mathbb{L}^{\lambda>0}_{\mathcal{X}_{p^{(1)}}(d^{(1)}) \times \mathcal{X}_{p^{(2)}}(d^{(2)})}|_0 \right\rangle.$$

Define the sets of weights

$$\begin{aligned} \nabla_{p,w} &:= \left\{\chi \in M_{\mathbb{R}} \,\middle|\, -\frac{1}{2} n_{\lambda,p} \leqslant \langle \lambda, \chi \rangle \leqslant \frac{1}{2} n_{\lambda,p} \text{ for all } \lambda \right\} + w\tau_d, \\ \nabla'_{p,w} &:= \left\{\chi \in M_{\mathbb{R}} \,\middle|\, -\frac{1}{2} n'_{\lambda,p} \leqslant \langle \lambda, \chi \rangle \leqslant \frac{1}{2} n'_{\lambda,p} \text{ for all } \lambda \right\} + w\tau_d. \end{aligned}$$

Then an object in the right- (resp. left-) hand side of Equation (3.7) lies in the right- (resp. left-) hand side of Equation (3.8) if and only if its $T(d)$-weights are contained in $\nabla_{p,w}$ (resp. $\nabla'_{p,w}$); see [HLS20, Lemma 2.9]. Let $\mathcal{E}$ be an object in the left-hand side of Equation (3.8). Using a Koszul resolution, the complex $\Psi(\mathcal{E})$ is generated by $p^*(\mathcal{E}) \otimes \wedge^\bullet U$. Let $\chi$ be a $T(d)$-weight of $p^*(\mathcal{E}) \otimes \wedge^\bullet U$. We then have that

$$-\frac{1}{2}n'_{\lambda,p} + \langle \lambda, U^{\lambda<0} \rangle \leqslant \langle \lambda, \chi \rangle \leqslant \frac{1}{2}n'_{\lambda,p} + \langle \lambda, U^{\lambda>0} \rangle.$$

Since $U$ is self-dual, we have $\langle \lambda, U^{\lambda<0} \rangle = -\langle \lambda, U^{\lambda>0} \rangle$. Moreover, from Equation (3.3), it is easy to see that

$$\frac{1}{2}n'_{\lambda,p} + \langle \lambda, U^{\lambda>0} \rangle = \frac{1}{2}n_{\lambda,p}.$$

Therefore, $\Psi(\mathcal{E})$ is an object of the right-hand side of Equation (3.8). □

**Lemma 3.5.** *If a $T(d)$-weight $\chi$ satisfies Equation (3.4), then*

$$\chi + \rho \in \mathbf{W}(d)_w. \tag{3.18}$$

*Conversely if a $GL(V)$-dominant $T(d)$-weight $\chi$ satisfies Equation (3.18), then it satisfies Equation (3.4).*

*Proof.* If a $T(d)$-weight $\chi$ satisfies Equation (3.4), it is written as

$$\chi + \rho_P = \sum_{i,j,a} c_{ij}^{(a)} (\beta_i^{(a)} - \beta_j^{(a)}) + \sum_{i,j,a \neq b} c_{ij}^{(ab)} (\beta_i^{(a)} - \beta_j^{(b)}), \tag{3.19}$$

where $0 \leqslant c_{ij}^{(a)} \leqslant 3/2$ and $0 \leqslant c_{ij}^{(ab)} \leqslant 1$. Since we have

$$\rho - \rho_P = \frac{1}{2} \sum_{i,j,a<b} (\beta_i^{(b)} - \beta_j^{(a)}), \tag{3.20}$$

the weight $\chi + \rho$ satisfies Equation (3.18). Conversely, if $\chi$ is a $GL(V)$-dominant $T(d)$-weight satisfying Equation (3.18), from [PTa, Proposition 3.5] it is written as

$$\chi + \rho = \sum_{i>j} c_{ij} (\beta_i - \beta_j)$$

for $0 \leqslant c_{ij} \leqslant 3/2$. Therefore, from Equation (3.20), the weight $\chi + \rho_P$ is written as Equation (3.19). □

**Lemma 3.6.** *The subcategory (3.15) for $k \geqslant 2$ is right orthogonal to $\mathbb{S}_P(d)_w$ for $w = w_1 + \cdots + w_k$, where $w_i$ is given in Equation (3.10).*

*Proof.* Recall that $w_i = v_i + d_i(\sum_{i>j} d_j - \sum_{j>i} d_j)$. Choose weights $\psi_i \in \mathbf{W}_{p_i}(d_i)_0$ for $1 \leqslant i \leqslant k$. From the proof of semiorthogonality in [Păda, Proposition 4.3], it is enough to show that the $r$-invariant of the weight

$$\sum_{i=1}^k \psi_i + \sum_{i=1}^k w_i \tau_{d_i} - \sum_{i=1}^k \rho_{p_i} + \rho_P \tag{3.21}$$

with respect to the polytope $\mathbf{W}_p(d)$ is bigger than $1/2$. Here, the polytope $\mathbf{W}_p(d)$ is defined as follows:

$$\mathbf{W}_p(d) := \frac{1}{2}\mathrm{sum}[0, \beta] + \mathbb{R}\tau_d,$$

where the sum is after all weights in $\mathrm{Ext}^1_Q(R, R)$.

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Suppose the contrary, that is, the weight (3.21) lies in $\mathbf{W}_p(d)_w$. Let $\lambda$ be the cocharacter as in Equation (3.14), and write it as $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, where $\lambda^{(j)}$ is the cocharacter of $GL(V^{(j)})$. We set $\mathfrak{g} = \mathrm{End}(V)$ and $\mathfrak{g}^{(j)} = \mathrm{End}(V^{(j)})$. The weight (3.21) is written as

$$\sum_{i=1}^{k} \psi_i + \sum_{i=1}^{k} v_i \tau_{d_i} - \mathfrak{g}^{\lambda>0} - \frac{1}{2}(\mathfrak{g}^{(1)})^{\lambda^{(1)}>0} - \frac{1}{2}(\mathfrak{g}^{(2)})^{\lambda^{(2)}>0}.$$

By the assumption, the above weight is an element of $\mathbf{W}_p(d)_w$. Then using an argument as in the proof of Lemma 3.5, we have that

$$\sum_{i=1}^{k} \psi_i + \sum_{i=1}^{k} v_i \tau_{d_i} - \frac{3}{2}\mathfrak{g}^{\lambda>0} \in \mathbf{W}(d)_w. \tag{3.22}$$

Note that we have

$$\left\langle \lambda, \sum_{i=1}^{k} \psi_i + \sum_{i=1}^{k} v_i \tau_{d_i} - \frac{3}{2}\mathfrak{g}^{\lambda>0} - w\tau_d \right\rangle = \sum_{i=1}^{k} (k+1-i) d_i \left( \frac{v_i}{d_i} - \frac{w}{d} \right) - \left\langle \lambda, \frac{3}{2}\mathfrak{g}^{\lambda>0} \right\rangle.$$

For $1 \leqslant i \leqslant k$, define

$$\tilde{v}_i := d_i \left( \frac{v_i}{d_i} - \frac{w}{d} \right).$$

Then $\tilde{v}_1 + \cdots + \tilde{v}_k = 0$ and $\tilde{v}_1 + \cdots + \tilde{v}_l < 0$ for $1 \leqslant l < k$. Therefore,

$$\sum_{i=1}^{k} (k+1-i) d_i \left( \frac{v_i}{d_i} - \frac{w}{d} \right) = \sum_{l=1}^{k} \left( \sum_{i=1}^{l} \tilde{v}_i \right) < 0.$$

It follows that

$$\left\langle \lambda, \sum_{i=1}^{k} \psi_i + \sum_{i=1}^{k} v_i \tau_{d_i} - \frac{3}{2}\mathfrak{g}^{\lambda>0} - w\tau_d \right\rangle < -\left\langle \lambda, \frac{3}{2}\mathfrak{g}^{\lambda>0} \right\rangle. \tag{3.23}$$

On the other hand, we claim that for any weight $\chi \in \mathbf{W}(d)_0$, we have that

$$\langle \lambda, \chi \rangle \geqslant -\left\langle \lambda, \frac{3}{2}\mathfrak{g}^{\lambda>0} \right\rangle. \tag{3.24}$$

We thus obtain a contradiction with Equations (3.23) and (3.22). To prove Equation (3.24), write $\chi = \sum_{i,j} c_{ij}(\beta_i - \beta_j)$ with $0 \leqslant c_{ij} \leqslant 3/2$. Then

$$\langle \lambda, \chi \rangle \geqslant \sum_{\langle \lambda, \beta_i - \beta_j \rangle < 0} c_{ij} \langle \lambda, \beta_i - \beta_j \rangle \geqslant \frac{3}{2} \sum_{\langle \lambda, \beta_i - \beta_j \rangle < 0} \langle \lambda, \beta_i - \beta_j \rangle = -\left\langle \lambda, \frac{3}{2}\mathfrak{g}^{\lambda>0} \right\rangle. \qquad \square$$

## 4. Integral generator of equivariant K-theory of quasi-BPS categories

Recall the graded quasi-BPS category $\mathbb{S}^{\mathrm{gr}}(d)_v$, which is equivalent via Koszul duality to

$$\mathbb{T}(d)_v \subset D^b(\mathscr{C}(d)).$$

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In [PTa], we proved that the monomials $[\mathcal{E}_{d_1,v_1}]*\cdots*[\mathcal{E}_{d_k,v_k}]$ for $v_i/d_i = v/d$ give a basis of the $\mathbb{F}$-vector space $K_T(\mathbb{T}(d)_v)\otimes_{\mathbb{K}}\mathbb{F}$. In this section, we consider the torsion-free equivariant K-theory defined by

$$K_T(\mathbb{T}(d)_v)' := K_T(\mathbb{T}(d)_v)/(\mathbb{K}\text{-torsion}).$$

It is conjectured by Schiffmann–Vasserot [SV13, Conjecture 7.13] that $K_T(\mathscr{C}(d))$ is torsion free as a $\mathbb{K}$-module, so conjecturally $K_T(\mathbb{T}(d)_v)'$ is isomorphic to $K_T(\mathbb{T}(d)_v)$. We compute $K_T(\mathbb{T}(d)_v)'$ when $\gcd(d,v)=1$, and when $(d,v)=(2,0)$. Theorem 3.1 plays a key role in the case of $\gcd(d,v)=1$.

### *4.1. Sheaves supported over the small diagonal*

Let

$$\Omega_{\mathscr{C}(d)}[-1] := \operatorname{Spec}\operatorname{Sym}\big(\mathbb{T}_{\mathscr{C}(d)}[1]\big)$$

be the $(-1)$-shifted cotangent of $\mathscr{C}(d)$. An object $\mathcal{E}\in D^b(\mathscr{C}(d))$ has singular support [AG15]:

$$\operatorname{Supp}^{\mathrm{sg}}(\mathcal{E})\subset(\Omega_{\mathscr{C}(d)}[-1])^{\mathrm{cl}}.$$

We identify the right-hand side with $\operatorname{Crit}(\operatorname{Tr}W_d)\subset\mathcal{X}(d)$, which is isomorphic to the moduli stack of zero-dimensional coherent sheaves on $\mathbb{C}^3$ of length $d$. Under the Koszul duality equivalence (2.22), we have by [Toda, Proposition 2.3.9] that

$$\operatorname{Supp}^{\mathrm{sg}}(\mathcal{E})=\operatorname{Supp}(\Phi(\mathcal{E})).$$

In the following lemma, we show that the K-theory class of an object in $D^b_T(\mathscr{C}(d))$ with singular support contained in $\pi^{-1}(\Delta)$ has a certain divisibility property. The proof is inspired by the proof of wheel conditions in [Zha, Theorem 2.9, Corollary 2.10], [Neg23, Proposition 2.11]

**Lemma 4.1.** *For any $\mathcal{E}\in D^b_T(\mathscr{C}(d))$ whose singular support is contained in $\pi^{-1}(\Delta)$, the element $i_*[\mathcal{E}]\in K_T(\mathcal{Y}(d))=\mathbb{K}[z_1^{\pm},\ldots,z_d^{\pm1}]^{\mathfrak{S}_d}$ is divisible by*

$$(q_1-1)^{d-1}(q_2-1)^{d-1}(q_1q_2-1)^{d-1}\in\mathbb{K}. \tag{4.1}$$

*Proof.* By [PTa, Lemma 4.9], it is enough to show that, for $\mathcal{F}\in\mathrm{MF}_T(\mathcal{X}(d),\operatorname{Tr}W)$ supported on $\pi^{-1}(\Delta)$, its image under the forget-the-potential map (1.8):

$$\Theta([\mathcal{F}]) := [\mathcal{F}^0]-[\mathcal{F}^1]\in K_T(\mathrm{MF}(\mathcal{X}(d),0))=\mathbb{K}[z_1^{\pm1},\ldots,z_d^{\pm d}]^{\mathfrak{S}_d} \tag{4.2}$$

is divisible by $(q_1-1)^{d-1}(q_2-1)^{d-1}(q_1q_2-1)^{d-1}$. We consider the morphism

$$\begin{aligned} h\colon \mathbb{C}^{d-1}\setminus\{0\} &\to \operatorname{Crit}(\operatorname{Tr}W_d)\subset\mathfrak{g}^{\oplus 3}\\ (t_1,\ldots,t_{d-1}) &\mapsto (0,0,(t_1,\ldots,t_{d-1},0)),\end{aligned}$$

where $(t_1,\ldots,t_{d-1},0)$ is the diagonal matrix. Let $T=(\mathbb{C}^*)^2$ act on $\mathbb{C}^{d-1}$ with weight $q_1^{-1}q_2^{-1}$, and let the maximal torus $T(d)\subset GL(d)$ act on $\mathbb{C}^{d-1}$ trivially. Then the above morphism $h$ is $T\times T(d)$-equivariant, so it induces a morphism

$$h\colon (\mathbb{C}^{d-1}\setminus\{0\})/(T\times T(d))\to\mathfrak{g}^{\oplus 3}/(T\times T(d))\to\mathfrak{g}^{\oplus 3}/(T\times GL(d)).$$

By the localization sequence

$$K_{T\times T(d)}(\{0\})\to K_{T\times T(d)}(\mathbb{C}^{d-1})\to K_{T\times T(d)}(\mathbb{C}^{d-1}\setminus\{0\})\to 0,$$

we have an isomorphism

$$K_{T\times T(d)}(\mathbb{C}^{d-1}\setminus\{0\})\cong\mathbb{K}[z_1^{\pm1},\dots,z_d^{\pm1}]/(1-q_1^{-1}q_2^{-1})^{d-1}.$$

Note that we have

$$\pi\circ h(t_1,\dots,t_{d-1})=\{(0,0,t_1),\dots,(0,0,t_{d-1}),(0,0,0)\}\in\mathrm{Sym}^d(\mathbb{C}^3),$$

in particular the image of $h$ does not intersect $\pi^{-1}(\Delta)\subset\mathrm{Crit}(\mathrm{Tr}\,W_d)$. We thus have that

$$h^*(\Theta([\mathcal{F}]))=0\text{ in }K_{T\times T(d)}(\mathbb{C}^{d-1}\setminus\{0\}).$$

Therefore, the element (4.2) is divisible by $(q_1q_2-1)^{d-1}$. By replacing $h$ with

$$(t_1,\dots,t_{d-1})\mapsto(0,(t_1,\dots,t_{d-1},0),0),$$
$$(t_1,\dots,t_{d-1})\mapsto((t_1,\dots,t_{d-1},0),0,0),$$

the element (4.2) is also divisible by $(q_1-1)^{d-1}$, $(q_2-1)^{d-1}$, respectively. □

By combining the above lemma with Theorem 3.1, we obtain the following:

**Corollary 4.2.** *For any object $\mathcal{E}\in\mathbb{T}_T(d)_v$ with $\gcd(d,v)=1$, the element*

$$i_*[\mathcal{E}]\in K_T(\mathcal{Y}(d))=\mathbb{K}[z_1^{\pm1},\dots,z_d^{\pm1}]^{\mathfrak{S}_d}$$

*is divisible by Equation (4.1). In particular, the element $i_*[\mathcal{E}_{d,v}]$ in Equation (2.38) for $\gcd(d,v)=1$ is divisible by Equation (4.1).*

**Remark 4.3.** It is not clear from the expression (2.38) that $i_*[\mathcal{E}_{d,v}]$ is divisible by Equation (4.1) when $\gcd(d,v)=1$. Further, the condition $\gcd(d,v)=1$ is necessary for the above divisibility. Indeed, for $d=2$, a direct computation shows that

$$i_*[\mathcal{E}_{2,0}]=(1-q_1^{-1})(1-q_2^{-1})(1-q_1^{-1}-q_2^{-1}-q_1^{-1}q_2^{-1}+z_1^{-1}z_2+z_1z_2^{-1}),$$
$$i_*[\mathcal{E}_{2,1}]=(1-q_1^{-1})(1-q_2^{-1})(1-q_1^{-1}q_2^{-1})(z_1+z_2).$$

The element $i_*[\mathcal{E}_{2,0}]$ is not divisible by Equation (4.1). In particular, the singular support of $\mathcal{E}_{2,0}$ is not included in $\pi^{-1}(\Delta)$.

### 4.2. *Integral generator for the coprime case*

Recall that $K_T(\mathbb{T}(d)_v)$ is expected to be torsion free as a $\mathbb{K}$-module. We also expect that $K_T(\mathbb{T}(d)_v)$ is freely generated by $[\mathcal{E}_{d,v}]$ if $\gcd(d,v)=1$. The following theorem, which is an application of Corollary 4.2, gives evidence towards this expectation.

**Theorem 4.4.** *Consider a pair $(d,v)\in\mathbb{N}\times\mathbb{Z}$ with $\gcd(d,v)=1$. There is an isomorphism of $\mathbb{K}$-modules:*

$$K_T(\mathbb{T}(d)_v)'\cong\mathbb{K}[\mathcal{E}_{d,v}].$$

*Proof.* By the isomorphism (2.30), the $\mathbb{K}$-module $K_T(\mathbb{T}(d)_v)'$ is isomorphic to the image of

$$i_*\colon K_T(\mathbb{T}(d)_v)\to\mathbb{K}[z_1^{\pm1},\dots,z_d^{\pm}]^{\mathfrak{S}_d}.$$

It is enough to show that the image of the above morphism is generated by $i_*[\mathcal{E}_{d,v}]$ as a $\mathbb{K}$-module. By Corollary 4.2, we have

$$\mathrm{Im}(i_*)\subset(q_1-1)^{d-1}(q_2-1)^{d-1}(q_1q_2-1)^{d-1}\mathbb{K}[z_1^{\pm1},\dots,z_d^{\pm1}]^{\mathfrak{S}_d}.$$

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Moreover, $\mathrm{Im}(i_*) \otimes_{\mathbb{K}} \mathbb{F}$ is generated by $i_*[\mathcal{E}_{d,v}]$ over $\mathbb{F}$ by [PTa, Theorem 4.12]. It is thus enough to show that $i_*[\mathcal{E}_{d,v}]$ is written as

$$i_*[\mathcal{E}_{d,v}] = (q_1-1)^{d-1}(q_2-1)^{d-1}(q_1q_2-1)^{d-1}\cdot E,$$

where $E$ is not divisible by a nonunit element in $\mathbb{K}$.

By Equation (2.38), we have that

$$\mathrm{Sym}\left(\frac{z_1^{m_1}\cdots z_d^{m_d}}{(1-q^{-1}z_1^{-1}z_2)\cdots(1-q^{-1}z_{d-1}^{-1}z_d)}\cdot\prod_{j>i}\xi(z_iz_j^{-1})\right) = (q_1q_2-1)^{d-1}\cdot E. \tag{4.3}$$

By setting

$$\begin{aligned} f_d(z_1,\ldots,z_d) := \prod_{i=1}^{d-1}(z_{i+1}-q_1^{-1}z_i)(z_{i+1}-q_2^{-1}z_i)\cdot \\ \prod_{j>i+1}(z_j-q_1^{-1}z_i)(z_j-q_2^{-1}z_i)(z_j-qz_i), \end{aligned} \tag{4.4}$$

we can write Equation (4.3) as

$$(-q)^{-\frac{1}{2}(d-1)(d-2)}(z_1\cdots z_d)^{2-d}\cdot\mathrm{Sym}\left(\frac{z_1^{m_1}\cdots z_{d-1}^{m_{d-1}}z_d^{m_d-1}f_d(z_1,\ldots,z_d)}{\prod_{j>i}(z_j-z_i)}\right). \tag{4.5}$$

Plug in $z_i = q_1^i$ for $1\leqslant i\leqslant d$ in the formula (4.5). The only nonzero term in the sum above corresponds to the identity permutation. The factors of this term not in $\mathbb{K}^*$ divide

$$q_1^{a+1}-1,\ q_1^a-q_2^{-1},\ \text{ or } q_1^a-q_2 \text{ for some } a\geqslant 1. \tag{4.6}$$

Next, plug in $z_i = q_2^i$ for $1\leqslant i\leqslant d$ in the formula (4.5). The only nonzero term in the sum of Equation (4.5) corresponds to the identity permutation. The factors of this term not in $\mathbb{K}^*$ divide

$$q_2^{a+1}-1,\ q_2^a-q_1^{-1},\ \text{ or } q_2^a-q_1 \text{ for some } a\geqslant 1. \tag{4.7}$$

The only factors which divide terms in both sets (4.6) and (4.7) are $q_1-q_2$ and $q_1q_2-1$. The factor $q_1q_2-1$ appears with multiplicity $d-1$ as it corresponds to $z_{i+1}-q_2^{-1}z_i$ for $1\leqslant i\leqslant d-1$.

It suffices to show that $q_1-q_2$ does not divide Equation (4.5). We will be using computations from [Neg22, Section 2]. Note that $q_1, q_2, q$ from our paper correspond to $q_1^{-1}, q_2^{-1}, q^{-1}$ in loc. cit. Further, the weight $v\in\mathbb{Z}$ corresponds to $k\in\mathbb{Z}$ in loc. cit. By [Neg22, Equation (2.35)], the equality $P_{d,k}=E_{d,k}$ for $\gcd(d,k)=1$ in loc. cit. (see [Neg22, Equations (2.6) and (2.35)]) and the isomorphism between shuffle algebras $\mathcal{S}_{\mathbb{F}} \xrightarrow{\sim} \mathcal{S}'_{\mathbb{F}}$, we can write

$$(q_1q_2-1)^{d-1}\cdot E = y+t \text{ in } \mathcal{S}\otimes_{\mathbb{K}}\mathbb{K}\left[\frac{1}{1-q_2^{-1}},\frac{1}{1-q^{-1}}\right]$$

for $t$ a $\mathbb{K}$-torsion element and

$$y := \frac{(1-q_2^{-1})(1-q)^v}{(1-q_2^{-1})^v(1-q^{-1})}\cdot\mathrm{Sym}\left(\frac{z_1^{m_1}\cdots z_d^{m_d}}{(1-q_2z_1^{-1}z_2)\cdots(1-q_2z_{d-1}^{-1}z_d)}\cdot\prod_{j>i}\xi(z_iz_j^{-1})\right). \tag{4.8}$$

It suffices to show that $q_1 - q_2$ does not divide $y$. By setting

$$g_d(z_1, \dots, z_d) := \prod_{i=1}^{d-1} (z_{i+1} - q_1^{-1} z_i)(z_{i+1} - q z_i) \cdot \prod_{j>i+1} (z_j - q_1^{-1} z_i)(z_j - q_2^{-1} z_i)(z_j - q z_i), \tag{4.9}$$

we can write the $\mathrm{Sym}(-)$ term of $y$ as

$$(-q_2)^{-\frac{1}{2}(d-1)(d-2)} (z_1 \cdots z_d)^{2-d} \cdot \mathrm{Sym}\left( \frac{z_1^{m_1} \dots z_{d-1}^{m_{d-1}} z_d^{m_d - 1} g_d(z_1, \dots, z_d)}{\prod_{j>i} (z_j - z_i)} \right). \tag{4.10}$$

It suffices to show that $q_1 - q_2$ does not divide Equation (4.10). Let $z_i = q^{-i}$ for $1 \leqslant i \leqslant d$. The only nonzero term in the sum of Equation (4.10) corresponds to the identity permutation. The factors of this term not in $\mathbb{K}^*$ divide

$$q^{-a-1} - 1, q^{-a} - q_1^{-1}, q^{-a} - q_1 \text{ for some } a \geqslant 1.$$

None of these polynomials is divisible by $q_1 - q_2$, and the conclusion thus follows. □

### 4.3. *Integral generator for* $K_T(\mathbb{T}(2)_0)'$

The computation of $K_T(\mathbb{T}(d)_v)'$ is more subtle when $\gcd(d, v) > 1$, since the monomials $[\mathcal{E}_{d_1,v_1}] * \cdots * [\mathcal{E}_{d_k,v_k}]$ for $v_i/d_i = v/d$ do not generate it over $\mathbb{K}$; see Remark 4.8. We need to find other objects giving $\mathbb{K}$-basis of $K_T(\mathbb{T}(d)_v)'$. Here, we give a computation for $(d, v) = (2, 0)$.

Let $V$ be a two-dimensional vector space, and let $\mathfrak{sl} \subset \mathfrak{g} = \mathrm{End}(V)$ be its traceless part. Note that

$$[\mathfrak{sl}] = 1 + z_1^{-1} z_2 + z_1 z_2^{-1} \in K(BGL(2)) = \mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}]^{\mathfrak{S}_2}.$$

The structure sheaf of the classical truncation $\mathscr{C}(2)^{\mathrm{cl}}$ fits into the exact sequence

$$0 \to \mathcal{O}_{\mathcal{Y}(2)}(q_1^{-1} q_2^{-2}) \oplus \mathcal{O}_{\mathcal{Y}(2)}(q_1^{-2} q_2^{-1}) \xrightarrow{A} \mathfrak{sl} \otimes \mathcal{O}_{\mathcal{Y}(2)}(q_1^{-1} q_2^{-1}) \xrightarrow{B} \mathcal{O}_{\mathcal{Y}(2)} \to \mathcal{O}_{\mathscr{C}(2)^{\mathrm{cl}}} \to 0. \tag{4.11}$$

Here, over $(X, Y) \in \mathfrak{g}^{\oplus 2}$, the maps $A, B$ are given by

$$A|_{(X,Y)} = (2X - \mathrm{Tr}\, X \cdot I, 2Y - \mathrm{Tr}\, Y \cdot I), \ B|_{(X,Y)}(Z) = \mathrm{Tr}(Z[X, Y]).$$

We set $M_1 := \mathcal{O}_{\mathscr{C}(2)^{\mathrm{cl}}}(q_1 q_2)$. We also define a coherent sheaf $M_2$ on $\mathscr{C}(2)^{\mathrm{cl}}$ by the exact sequence

$$0 \to \mathcal{O}_{\mathcal{Y}(2)}(q_1^{-1} q_2^{-1}) \xrightarrow{B^\vee} \mathfrak{sl} \otimes \mathcal{O}_{\mathcal{Y}(2)} \xrightarrow{A^\vee} \mathcal{O}_{\mathcal{Y}(2)}(q_1) \oplus \mathcal{O}_{\mathcal{Y}(2)}(q_2) \to M_2 \to 0. \tag{4.12}$$

The sequences (4.11), (4.12) are the Eagon–Northcott complex and the Buchsbaum–Rim complex associated with $A^\vee : \mathfrak{sl} \otimes \mathcal{O}_{\mathcal{Y}(2)} \to \mathcal{O}_{\mathcal{Y}(2)}(q_1) \oplus \mathcal{O}_{\mathcal{Y}(2)}(q_2)$, respectively. In particular they are exact; see [Eis95, Theorem A2.10]. From the exact sequences (4.11), (4.12), we have $M_1, M_2 \in \mathbb{T}(2)_0$.

**Proposition 4.5.** *As a $\mathbb{K}$-module, we have*

$$K_T(\mathbb{T}(2)_0)' = \mathbb{K}[M_1] \oplus \mathbb{K}[M_2]. \tag{4.13}$$

*Proof.* It is enough to show that the image of

$$i_* : K_T(\mathbb{T}(2)_0) \to \mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}]^{\mathfrak{S}_2} \tag{4.14}$$

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is generated by $i_*M_1, i_*M_2$ and that $i_*M_1, i_*M_2$ are linearly independent over $\mathbb{F}$. Below, we omit $i_*$ from the notation $i_*M_j$. From the exact sequences (4.11), (4.12), we have

$$[M_1] = q_1q_2 + q_1^{-1} + q_2^{-1} - \mathfrak{sl},\ [M_2] = q_1^{-1}q_2^{-1} + q_1 + q_2 - \mathfrak{sl}. \tag{4.15}$$

A direct computation shows that

$$\begin{aligned}[\mathcal{E}_{1,0} * \mathcal{E}_{1,0}] &= 1 + q_1^{-1} + q_2^{-1} + q_1^{-1}q_2^{-2} + q_1^{-2}q_2^{-1} + q_1^{-2}q_2^{-2} - 2\mathfrak{sl}q_1^{-1}q_2^{-1} \\ &= q_1^{-1}q_2^{-1}([M_1] + [M_2]), \\ [\mathcal{E}_{2,0}] &= (1 - q_1^{-1})(1 - q_2^{-1})(\mathfrak{sl} - q_1^{-1} - q_2^{-1} - q_1^{-1}q_2^{-1}) \\ &= (q_1^{-1} + q_2^{-1})[M_1] - (q_1^{-1}q_2^{-1} + 1)[M_2].\end{aligned} \tag{4.16}$$

Here, $[\mathcal{E}_{1,0} * \mathcal{E}_{1,0}]$ is computed from Equation (2.29) and $[\mathcal{E}_{2,0}]$ is computed from Equation (2.38). Therefore, we have

$$\mathbb{K}[\mathcal{E}_{2,0}] \oplus \mathbb{K}[\mathcal{E}_{1,0} * \mathcal{E}_{1,0}] \subset \mathbb{K}[M_1] \oplus \mathbb{K}[M_2]. \tag{4.17}$$

Since we have

$$\det\begin{pmatrix} q_1^{-1}q_2^{-1} & q_1^{-1}q_2^{-1} \\ q_1^{-1} + q_2^{-1} & -(q_1^{-1}q_2^{-1} + 1)\end{pmatrix} = -q_1^{-1}q_2^{-1}(1 + q_1^{-1})(1 + q_2^{-1}) \in \mathbb{K} \setminus \{0\}, \tag{4.18}$$

the embedding (4.17) is an isomorphism after taking $\otimes_{\mathbb{K}}\mathbb{F}$. In particular, $[M_1]$, $[M_2]$ are linearly independent over $\mathbb{F}$.

By the above argument, we have

$$K_T(\mathbb{T}(2)_0) \otimes_{\mathbb{K}} \mathbb{F} = \mathbb{F}[\mathcal{E}_{2,0}] \oplus \mathbb{F}[\mathcal{E}_{1,0} * \mathcal{E}_{1,0}] = \mathbb{F}[M_1] \oplus \mathbb{F}[M_2],$$

where the first isomorphism is proved in [PTa, Theorem 4.12]. It follows that any element in the image of Equation (4.14) is written as $a_1[M_1] + a_2[M_2]$ for $a_1, a_2 \in \mathbb{F}$. As it lies in $\mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}]$, from Equation (4.15) we have

$$a_1(q_1q_2 + q_1^{-1} + q_2^{-1}) + a_2(q_1^{-1}q_2^{-1} + q_1 + q_2) \in \mathbb{K},\ a_1 + a_2 \in \mathbb{K}.$$

By solving the above equation, there exist $b, c \in \mathbb{K}$ such that

$$a_1 = \frac{b}{\Delta},\ a_2 = -\frac{b}{\Delta} + c, \tag{4.19}$$

where $\Delta$ is given by

$$\begin{aligned}\Delta &:= q_1^{-1} + q_2^{-1} + q_1q_2 - q_1 - q_2 - q_1^{-1}q_2^{-1} \\ &= q_1^{-1}q_2^{-1}(q_1 - 1)(q_2 - 1)(q_1q_2 - 1).\end{aligned}$$

It is enough to show that $b$ is divisible by $\Delta$. If $a_1, a_2$ are given by Equation (4.19), then $a_1[M_1]+a_2[M_2] = b + c[M_2]$. As we assumed that $a_1[M_1] + a_2[M_2]$ lies in the image of Equation (4.14), by the wheel condition [Neg23, Proposition 2.11] we have

$$(b + c[M_2])\big|_{z_i=q_1^{-1}z_j=q_1^{-1}q_2^{-1}z_k} = (b + c[M_2])\big|_{z_i=q_2^{-1}z_j=q_1^{-1}q_2^{-1}z_k} = 0 \tag{4.20}$$

unless $i = j = k$; see Remark 4.6. By setting $i = k = 1$ and $j = 2$, we obtain $b|_{q_1q_2=1} = 0$. Similarly, we also obtain $b|_{q_1=1} = b|_{q_2=1} = 0$, so $b$ is divisible by $\Delta$. □

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**Remark 4.6.** In [Neg23, Proposition 2.11], it is assumed that $i$, $j$ and $k$ are pairwise distinct for the identity (4.20), but the proof in loc. cit. works unless $i = j = k$. Indeed, using the notation in loc. cit., let $\phi_e = (x_{ij})$ for $x_{bc} \neq 0$, $x_{ij} = 0$ for $(i, j) \neq (b, c)$, $\phi_e^* = (y_{ij})$ for $y_{ab} \neq 0$, $y_{ij} = 0$ for $(i, j) \neq (a, b)$. Then $\phi_e^*\phi_e$ is concentrated on $(a, c)$ with value $y_{ab}x_{bc} \neq 0$, $\phi_e\phi_e^*$ is concentrated on $(b, b)$ (either zero or nonzero). Therefore, if $[\phi_e, \phi_e^*] = 0$ holds, then we must have $a = b = c$. Unless $a = b = c$, we have $\mu_n^{-1}(0) \cap V_e = \emptyset$ in the notation of the proof of [Neg23, Proposition 2.11]. The rest of the argument is verbatim.

**Remark 4.7.** Note that

$$\mathscr{C}(2)^{\mathrm{cl}} \cong [R/GL(2)] \times \mathbb{C}^2,$$

where $R$ is the determinantal variety of $(3\times 2)$-matrices, and $M_1 \oplus M_2$ gives a noncommutative resolution of $R \times \mathbb{C}^2$ by [BLdB10, Theorem A].

**Remark 4.8.** Since Equation (4.18) is not invertible in $\mathbb{K}$, the inclusion (4.17) is not an isomorphism if we do not take $\otimes_{\mathbb{K}}\mathbb{F}$.

**Remark 4.9.** For $(d, v) \in \mathbb{N} \times \mathbb{Z}$ with $\gcd(d, v) = 1$ and $n \geqslant 1$, let $P_{nd,nv}$ be defined as in [Neg14, Equation (1.2)]:

$$P_{nd,nv} := \frac{(q_1^{-1}-1)^{nd}(q_2^{-1}-1)^{nd}}{(q_1^{-n}-1)(q_2^{-n}-1)} \cdot \\ \mathrm{Sym}\left(\frac{\prod_{i=1}^{nd} z_i^{\left\lfloor \frac{iv}{d}\right\rfloor - \left\lfloor \frac{(i-1)v}{d}\right\rfloor}}{\prod_{i=1}^{nd-1}\left(1-q^{-1}z_{i+1}z_i^{-1}\right)} \sum_{s=0}^{n-1} q^{-s}\frac{z_{d(n-1)+1}\cdots z_{d(n-s)+1}}{z_{d(n-1)}\cdots z_{d(n-s)}} \prod_{i<j}\xi\left(\frac{z_i}{z_j}\right)\right). \tag{4.21}$$

Then we have

$$P_{2,0} = q_1^{-2}q_2^{-2}(q_1-1)(q_2-1)(q_1q_2-1) = q_1^{-1}q_2^{-1}(M_1 - M_2). \tag{4.22}$$

Together with (4.16), we have

$$\mathbb{K}[P_{1,0} * P_{1,0}] \oplus \mathbb{K}[P_{2,0}] \subsetneq \mathbb{K}[M_1] \oplus \mathbb{K}[M_2] \tag{4.23}$$

with cokernel $\mathbb{Z}/2$. The inclusion (4.23) is an isomorphism after $\otimes_{\mathbb{Z}}\mathbb{Q}$.

More generally, we expect that

$$K_T(\mathbb{S}(nd)_{nv})'_{\mathbb{Q}} = \bigoplus_{n_1+\cdots+n_k=n} \mathbb{K}_{\mathbb{Q}}[P_{n_1d,n_1v} * \cdots * P_{n_kd,n_kv}].$$

More details will be discussed in [PTb].

## 5. The coproduct on quasi-BPS categories and K-theoretic BPS spaces

Recall that $Q$ is the quiver with one vertex and three edges $x, y, z$. Recall the regular function (2.2) induced from the superpotential $W := x[y, z]$. We will denote by $\widetilde{m}$ the Hall product (2.5) of $(Q, W)$ and by $m$ the Hall product (2.17).

In Subsection 5.1, we define the coproduct $\widetilde{\Delta}$ for the categories $\mathbb{S}_*^\bullet(d)_w$. In Subsection 5.2, we prove the compatibility of the product and coproduct for the quasi-BPS categories $\mathbb{S}_T^\bullet(d)_w$; see Theorem 5.2. In Subsection 5.3, we use the Koszul equivalence to define the coproduct $\Delta$ for the categories $\mathbb{T}_*(d)_v$ and to check the compatibility between the product and the coproduct for the quasi-BPS categories $\mathbb{T}_T(d)_v$.

Recall the definition of $\mathcal{D}_{d,v}$ for $(d, v) \in \mathbb{N} \times \mathbb{Z}$ for $\gcd(d, v) = 1$ from Equation (1.10). In Subsection 5.4, we show that $K(\mathcal{D}_{d,v})_{\mathbb{F}}$ is isomorphic to the $\mathbb{F}$-algebra of symmetric polynomials; see Proposition 5.11. Further, we consider the space of primitive elements

$$\mathrm{P}(nd, nv) \subset K_T(\mathbb{T}(nd)_{nv})$$

which we regard as an analogue of cohomological BPS spaces in K-theory; see Proposition 5.11 and the equality (1.5).

Note that the definition of Hall multiplication involves attracting stacks of antidominant cocharacters. The definition of the coproduct is through attracting stacks of dominant cocharacters. In this section, we will use dominant cocharacters.

### *5.1. Preliminaries*

Let $d \in \mathbb{N}$. Recall the stack of representation of $Q$ of dimension $d$:

$$\mathcal{X}(d) := R(d)/G(d) := \mathfrak{g}^{\oplus 3}/GL(d).$$

In this section, we allow partitions which have terms equal to zero. For $w \in \mathbb{Z}$, let $H_{d,w}$ be the set of partitions $(d_i, w_i)_{i=1}^k$ of $(d, w)$ with $d_i \geqslant 1$ for $i \in \{1, \ldots, k\}$ such that, for $v_i := w_i - d_i\left(\sum_{i>j} d_j - \sum_{j>i} d_j\right)$ for $i \in \{1, \ldots, k\}$, we have

$$\frac{v_1}{d_1} = \ldots = \frac{v_k}{d_k}. \tag{5.1}$$

For a partition $A = (d_i, w_i)_{i=1}^k$ with terms possibly equal to zero, let $I \subset \{1, \ldots, k\}$ be the subset of $i$ with $d_i \geqslant 1$, and define the partition of $(d, w)$ with nonzero terms $\overline{A} := (d_i, w_i)_{i \in I}$.

**5.1.1.**
For $\lambda$ and $\mu$ two cocharacters, let $A_\lambda$ be the set of $(T(d) \times T)$-weights $\beta$ of $R(d)$ such that $\langle \lambda, \beta \rangle > 0$, let $I_\lambda^\mu \subset A_\lambda$ be the set of weights such that $\langle \mu, \beta \rangle < 0$ and let $A_\lambda^\mu \subset A_\lambda$ be the set of weights such that $\langle \mu, \beta \rangle = 0$. Let $J_\lambda^\mu$ be the set of weights $\beta$ of $\mathfrak{g}$ such that $\langle \lambda, \beta \rangle > 0$ and $\langle \mu, \beta \rangle < 0$. Define the weights

$$N_\lambda^\mu := \sum_{I_\lambda^\mu} \beta, \ \mathfrak{g}_\lambda^\mu := \sum_{J_\lambda^\mu} \beta.$$

**5.1.2.**
Let $\lambda$ and $\mu$ be dominant cocharacters of $T(d)$ with associated partitions $A = (d_i)_{i=1}^k$ and $B = (e_i)_{i=1}^s$, respectively. Let $W \cong \mathfrak{S}_d$ be the Weyl group of $GL(d)$, let $W^\lambda \cong \times_{i=1}^k \mathfrak{S}_{d_i}$ be the Weyl group of $GL(d)^\lambda$ and let $W^\mu$ be the Weyl group of $GL(d)^\mu$. Define the set of cosets

$$S_\lambda^\mu := W^\mu \backslash W / W^\lambda.$$

A coset $C \in S_\lambda^\mu$ corresponds to partitions $(f_{ij})$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant s$ such that

$$\sum_{j=1}^s f_{ij} = d_i \text{ for } 1 \leqslant i \leqslant k, \ \sum_{i=1}^k f_{ij} = e_j \text{ for } 1 \leqslant j \leqslant s.$$

Let $v$ be a dominant cocharacter corresponding to the partition

$$(f_{11}, \cdots, f_{1s}, f_{21}, \cdots, f_{2s}, \cdots, f_{k1}, \cdots, f_{ks}), \tag{5.2}$$

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and let $\kappa$ be a dominant cocharacter corresponding to the partition

$$(f_{11}, \cdots, f_{k1}, f_{12}, \cdots, f_{k2}, \cdots, f_{1s}, \cdots, f_{ks}). \tag{5.3}$$

Consider the permutation $w$ of $\mathfrak{S}_d$ of minimal length which permutes blocks of consecutive integers

$$\begin{aligned} w = w_C\colon (f_{11}, \cdots, f_{1s}, f_{21}, \cdots, f_{2s}, \cdots, f_{k1}, \cdots, f_{ks}) \mapsto& \\ (f_{11}, \cdots, f_{k1}, f_{12}, \cdots, f_{k2}, \cdots, f_{1s}, \cdots, f_{ks}).& \end{aligned} \tag{5.4}$$

Consider the partition

$$D = (f_{ij}, u_{ij}), \tag{5.5}$$

where $f_{ij}$ are ordered as on Equation (5.2), and assume that $\overline{D}$ is in $H_{d,w}$. Then there also exists a partition $E$ with terms $(f_{ij}, \widetilde{u}_{ij})$ where $f_{ij}$ are ordered as on Equation (5.3) such that $\overline{E}$ is in $H_{d,w}$. Consider the order $1 \leqslant l \leqslant ks$ of the pairs $(i, j)$ as on the first line on Equation (5.4). Define the functor

$$\begin{aligned} \mathrm{sw}_C : \boxtimes_{l=1}^{ks} \mathbb{M}(f_l)_{u_l} &\to \boxtimes_{l=1}^{ks} \mathbb{M}(f_{w(l)})_{\widetilde{u}_{w(l)}}, \\ \boxtimes_{l=1}^{ks} \mathcal{A}_l &\mapsto \boxtimes_{l=1}^{ks} \mathcal{A}_{w(l)} \end{aligned} \tag{5.6}$$

which permutes the factors as in Equation (5.4). Define

$$\begin{aligned} \widetilde{\mathrm{sw}}_C\colon\ \boxtimes_{i=1}^{k} \boxtimes_{j=1}^{s} \mathbb{M}(f_{ij})_{u_{ij}} &\to \boxtimes_{j=1}^{s} \boxtimes_{i=1}^{k} \mathbb{M}(f_{ij})_{\widetilde{u}_{ij}}, \\ \mathcal{A} &\mapsto \mathrm{sw}_C\Big(\mathcal{A} \otimes \mathcal{O}(-N_\lambda^{w^{-1}\mu} + \mathfrak{g}_\lambda^{w^{-1}\mu})[|I_\lambda^{w^{-1}\mu}| - |J_\lambda^{w^{-1}\mu}|]\Big), \end{aligned}$$

where $\mathcal{O}(-N_\lambda^{w^{-1}\mu} + \mathfrak{g}_\lambda^{w^{-1}\mu})$ is a one-dimensional representation of $G^\nu \cong G^\kappa$. Consider the maps

$$\mathcal{X}(d)^\kappa \xleftarrow{q_{\kappa^{-1}\mu}} (\mathcal{X}(d)^\mu)^{\kappa^{-1}\geqslant 0} \xrightarrow{p_{\kappa^{-1}\mu}} \mathcal{X}(d)^\mu.$$

Finally, define

$$\widehat{m}_{DE} = p_{\kappa^{-1}\mu*} q^*_{\kappa^{-1}\mu} \widetilde{\mathrm{sw}}_C\colon \mathbb{M}_C \to \mathbb{M}_B.$$

There are analogous such functors for categories of (equivariant and/ or graded) matrix factorizations

$$\widehat{m}_{CB}\colon \mathbb{S}^\bullet_{*,C} \to \mathbb{S}^\bullet_{*,B}.$$

**5.1.3.**
We introduce some more maps and functors needed in the rest of this section. Let $\lambda$ and $\mu$ be dominant cocharacters, and let $\nu$ be a dominant cocharacter corresponding to a partition in $S^\mu_\lambda$. The map $p_{\lambda^{-1}}$ factors as $p_{\lambda^{-1}} = \pi_{\lambda^{-1}} \iota_{\lambda^{-1}}$, where

$$\begin{aligned} \pi_{\lambda^{-1}} &: R(d)/G(d)^{\lambda^{-1}\geqslant 0} \to R(d)/G(d), \\ \iota_{\lambda^{-1}} &: R(d)^{\lambda^{-1}\geqslant 0}/G(d)^{\lambda^{-1}\geqslant 0} \to R(d)/G(d)^{\lambda^{-1}\geqslant 0}. \end{aligned}$$

There are similarly defined maps

$$\begin{aligned} \pi_{\kappa^{-1}\mu} &: R(d)^\mu/(G(d)^\mu)^{\kappa^{-1}\geqslant 0} \to R(d)^\mu/G(d)^\mu, \\ \iota_{\kappa^{-1}\mu} &: (R(d)^\mu)^{\kappa^{-1}\geqslant 0}/(G(d)^\mu)^{\kappa^{-1}\geqslant 0} \to R(d)^\mu/(G(d)^\mu)^{\kappa^{-1}\geqslant 0}. \end{aligned}$$

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**5.1.4.**
Let $\mu$ be a dominant cocharacter of $T(d)$, let $b \in \mathbb{Z}$ and let $D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)_{\leqslant b}$ be the subcategory of $D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)$ generated by complexes $q_\mu^*\mathcal{A}$ for $\mathcal{A} \in D^b(\mathcal{X}(d)^\mu)_i$ and $i \leqslant b$. There is a semiorthogonal decomposition

$$D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)_{\leqslant b} = \left\langle D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)_{\leqslant b-1}, D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)_b \right\rangle,$$

and there are equivalences $q_\mu^* : D^b(\mathcal{X}(d)^\mu)_b \xrightarrow{\sim} D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)_b$. We define the functor

$$\beta_b : D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)_{\leqslant b} \to D^b\left(\mathcal{X}(d)^{\mu\geqslant 0}\right)_b$$

to be the projection with respect to the above semiorthogonal decomposition.

**5.1.5.**
Let $B = (d_i, w_i)_{i=1}^k$ be a partition of $(d, w)$ in $H_{d,w}$. Let $\mu$ be a dominant cocharacter for the partition $(d_i)_{i=1}^k$, and let $b := \frac{n_\mu}{2} = \langle \mu, \mathfrak{g}^{\mu>0}\rangle$; see Equation (2.3) for the definition of $n_\lambda$.

**Lemma 5.1.** *There is a functor*

$$\widetilde{\Delta}_B := \left(q_\mu^*\right)^{-1} \beta_b\, p_\mu^* : \mathbb{M}(d)_w \to \mathbb{M}_B := \boxtimes_{i=1}^k \mathbb{M}(d_i)_{w_i}. \tag{5.7}$$

*Proof.* First, note that the image of $p_\mu^*(\mathbb{M}(d)_w)$ is in $D^b(\mathcal{X}(d)^{\mu\geqslant 0})_{\leqslant b}$ from the description of the category $\mathbb{M}(d)_w$ in Lemma 2.2. Thus, the image of $\widetilde{\Delta}_B$ is in $D^b(\mathcal{X}(d)^\mu)_b$. Also, note that the category $\mathbb{M}_B$ is a subcategory of $D^b(\mathcal{X}(d)^\mu)_b$ because $B = (d_i, w_i)_{i=1}^k$ is in $H_{d,w}$.

Let $\chi$ be a dominant weight of $T(d)$ such that $\chi + \rho \in \mathbf{W}(d)_w$. If $\chi + \rho$ is not on the face $F(\mu)$ of the polytope $\mathbf{W}(d)$, then

$$\beta_b p_\mu^*\left(\mathcal{O}_{\mathcal{X}(d)} \otimes \Gamma_{GL(d)}(\chi)\right) = 0.$$

If $\chi + \rho$ is on the face $F(\mu)$, then by [Păda, Corollary 3.4] we can write

$$\chi + \rho - w\tau_d = \frac{1}{2}N^{\mu>0} + \sum_{i=1}^k (\psi_i + \rho_i), \tag{5.8}$$

where $\psi_i \in M(d_i)$ and $\psi_i + \rho_i \in \mathbf{W}(d_i)_0$. In particular, we have that

$$\chi = \sum_{i=1}^k \psi_i + \mathfrak{g}^{\mu>0} + w\tau_d.$$

Write $\chi = \sum_{i=1}^k \chi_i$. We have

$$\langle 1_{d_i}, \chi_i\rangle = \frac{d_i w}{d} + \langle 1_{d_i}, \mathfrak{g}^{\mu>0}\rangle = v_i + \langle 1_{d_i}, \mathfrak{g}^{\mu>0}\rangle = w_i.$$

Further, we have that $\chi_i = \psi_i + w_i\tau_{d_i}$, so $\chi_i + \rho_i \in \mathbf{W}(d_i)_{w_i}$. Therefore, we have

$$\left(q_\mu^*\right)^{-1} \beta_b\, p_\mu^*\left(\mathcal{O}_{\mathcal{X}(d)} \otimes \Gamma_{GL(d)}(\chi)\right) = \boxtimes_{i=1}^k \left(\mathcal{O}_{\mathcal{X}(d_i)} \otimes \Gamma_{GL(d_i)}(\chi_i)\right) \in \mathbb{M}_B.$$

□

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Let $\chi$ be a dominant weight with $\chi + \rho \in \mathbf{W}(d)$. Note that

$$\widetilde{\Delta}_B\big(\mathcal{O}_{\mathcal{X}(d)} \otimes \Gamma_{GL(d)}(\chi)\big) = \begin{cases} \mathcal{O}_{\mathcal{X}(d)^\mu} \otimes \Gamma_{GL(d)^\mu}(\chi), \text{ if } \chi \in F(\mu), \\ 0, \text{ otherwise.} \end{cases} \tag{5.9}$$

The functor (5.7) induces a functor

$$\widetilde{\Delta}_B := \left(q_\mu^*\right)^{-1} \beta_b\, p_\mu^* : \mathbb{S}^\bullet_*(d)_w \to \mathbb{S}^\bullet_{*,B}. \tag{5.10}$$

Let $A = (d_i, w_i)_{i=1}^k$ and $C = (f_i, u_i)_{i=1}^l$ be partitions in $H_{d,w}$ such that $C$ is a refinement of $A$; see Subsection 2.2.6. One can analogously define functors

$$\widetilde{\Delta}_{AC} : \mathbb{M}_A \to \mathbb{M}_C,\ \widetilde{\Delta}_{AC} : \mathbb{S}^\bullet_{*,A} \to \mathbb{S}^\bullet_{*,C}$$

for categories $\mathbb{S}^\bullet_*$ as in Subsection 2.6.2.

### *5.2. Compatibility between the product and the coproduct*

In this section, we show that $\widetilde{m}$ and $\widetilde{\Delta}$ are compatible. Recall the forget-the-potential map (1.8):

$$\Theta\colon K_T(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W)) \to K_T(\mathrm{MF}(\mathcal{X}(d), 0)).$$

Recall that $K_T(\mathbb{S}_A)_{\mathbb{F}} \hookrightarrow K_T(\mathbb{M}_A)_{\mathbb{F}}$; see [PTa, Theorem 4.12 and Equation (4.36)]. Let

$$K_T(\mathbb{S}_A)' := K_T(\mathbb{S}_A)/(\mathbb{K}\text{-torsion}) \xrightarrow{\cong} \mathrm{image}(\Theta\colon K_T(\mathbb{S}_A) \to K_T(\mathbb{M}_A)).$$

**Theorem 5.2.** *Consider a pair $(d, w) \in \mathbb{N} \times \mathbb{Z}$. Let $\lambda$ and $\mu$ be dominant cocharacters with associated partitions $A = (d_i, w_i)_{i=1}^k$ and $B = (e_i, v_i)_{i=1}^s$ in $H_{d,w}$. Let $S_A^B \subset S_\lambda^\mu$ be the set of partitions $C = (f_i, u_i)_{i=1}^l$ with $l = ks$ with $\overline{C}$ in $H_{d,w}$ and such that*

$$\begin{aligned} f_{(i-1)s+1} + f_{(i-1)s+2} + \ldots + f_{is} &= d_i \textit{ for } 1 \leqslant i \leqslant k, \\ f_j + f_{s+j} + \ldots + f_{(k-1)s+j} &= e_j \textit{ for } 1 \leqslant j \leqslant s. \end{aligned}$$

*Then the following diagram commutes:*

$$\begin{tikzcd} K_T(\mathbb{S}_A)' \arrow[r, "\widetilde{m}_A"] \arrow[d, "\bigoplus \widetilde{\Delta}_{AC}"] & K_T(\mathbb{S}(d)_w)' \arrow[d, "\widetilde{\Delta}_B"] \\ \bigoplus_{C \in S_A^B} K_T(\mathbb{S}_C)' \arrow[r, "\bigoplus \widehat{m}_{CB}"] & K_T(\mathbb{S}_B)'. \end{tikzcd}$$

Theorem 5.2 is a $T$-equivariant version of [Pădа, Theorem 5.2], and the same proof works to show this statement. However, we present an alternative proof which first proves a categorical statement about complexes in $D^b(\mathcal{X}(d)^\mu)$ which is stronger than the results in loc. cit. and is of independent interest for computations in categorical Hall algebras.

Note that the compatibility of the product and coproduct for localized K-theory, either in the above setting or in the setting of Corollary 5.6, follows by a direct computation and Propositions 5.7 and 5.8.

**Proposition 5.3.** *Let $A, B$ be as in Theorem 5.2. For $1 \leqslant i \leqslant k$, let $\chi_i$ be a dominant weight of $T(d_i)$ for $1 \leqslant i \leqslant k$ such that $\chi_i + \rho_i \in \mathbf{W}(d_i)$. Let $\chi := \sum_{i=1}^k \chi_i$. For any $C \in S_A^B$, there are natural maps,*

$$\widehat{m}_{CB} \widetilde{\Delta}_{AC}\big(\mathcal{O}_{\mathcal{X}(d)^\lambda} \otimes \Gamma_{GL(d)^\lambda}(\chi)\big) \to \widetilde{\Delta}_B \widetilde{m}_A\big(\mathcal{O}_{\mathcal{X}(d)^\lambda} \otimes \Gamma_{GL(d)^\lambda}(\chi)\big) \tag{5.11}$$

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*such that there is an isomorphism*

$$\bigoplus_{C\in S^B_A} \widehat{m}_{CB}\widetilde{\Delta}_{AC}\big(\mathcal{O}_{\mathcal{X}(d)^\lambda}\otimes\Gamma_{GL(d)^\lambda}(\chi)\big)\xrightarrow{\sim}\widetilde{\Delta}_B\widetilde{m}_A\big(\mathcal{O}_{\mathcal{X}(d)^\lambda}\otimes\Gamma_{GL(d)^\lambda}(\chi)\big). \tag{5.12}$$

*Proof of Theorem 5.2.* By Proposition 5.3, the following diagram commutes:

$$\begin{tikzcd} K_T(\mathbb{M}_A) \arrow[r, "\widetilde{m}_A"] \arrow[d, "\bigoplus\widetilde{\Delta}_{AC}"] & K_T(\mathbb{M}(d)_w) \arrow[d, "\widetilde{\Delta}_B"] \\ \bigoplus_{C\in S^B_A} K_T(\mathbb{M}_C) \arrow[r, "\bigoplus\widehat{m}_{CB}"] & K_T(\mathbb{M}_B). \end{tikzcd}$$

The maps $\widetilde{m}$ and $\widetilde{\Delta}$ are compatible with the forget-the-potential map (1.8) by [Păd22, Proposition 3.6] and [Păda, Proposition 5.1], respectively. The conclusion thus follows. □

*Proof of Proposition 5.3.* The argument follows closely the proof of [Pădb, Theorem 5.2]. We give an overview of the proof. We use a Koszul resolution to compute $\widetilde{m}_A\big(\mathcal{O}_{\mathcal{X}(d)^\lambda}\otimes\Gamma_{GL(d)^\lambda}(\chi)\big)$ in terms of $\mathcal{O}\otimes\Gamma_{GL(d)}(\theta)$, where $\theta=(\chi-\sigma_I)^+$ for $\sigma_I$ a partial sum of weights pairing positively with $\lambda$. Let $\mathcal{O}\otimes\Gamma_{GL(d)}(\theta)$ be a vector bundle appearing in the Koszul resolution with nonzero $\widetilde{\Delta}_B$. Then the weight $\theta$ is on a face of $\mathbf{W}(d)$; see Equation (5.9). We use Proposition 5.4 to characterize the highest weights $\theta$ on a face of $\mathbf{W}(d)$ in terms of partitions $C\in S^B_A$. The proof then follows from a direct comparison with the right-hand side of Equation (5.12). Let $\sigma_I$ be a sum such that $\chi-\sigma_I$ is on a face of $\mathbf{W}(d)$ corresponding to $C\in S^B_A$ with associated permutation $w$; see Equation (5.4). The swap morphism appears because conjugating $\chi-\sigma_I$ to $\theta=(\chi-\sigma_I)^+$ first requires to act by $w$.

The multiplication $\widetilde{m}_A$ is defined as $\widetilde{m}_A=p_{\lambda^{-1}*}q^*_{\lambda^{-1}}$. Let $\chi:=\sum_{i=1}^k\chi_i$. Consider the Koszul resolution

$$\begin{aligned}\widetilde{m}_A\big(\mathcal{O}_{\mathcal{X}(d)^\lambda}\otimes\Gamma_{GL(d)^\lambda}(\chi)\big)&=\pi_{\lambda^{-1}*}\iota_{\lambda^{-1}*}q^*_{\lambda^{-1}}\big(\mathcal{O}_{\mathcal{X}(d)^\lambda}\otimes\Gamma_{GL(d)^\lambda}(\chi)\big)\\ &\cong\pi_{\lambda^{-1}*}\left(\bigoplus_{I\subset A_\lambda}\mathcal{O}_{R(d)}\otimes\Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi-\sigma_I)[|I|],d\right);\end{aligned}$$

see Proposition 2.1 for the notation. The differential $d$ is induced by multiplication with generators $(e_i)_{i=1}^t$ of the polynomial ring $\mathbb{C}\big[R(d)^{\lambda<0}\big]\cong\mathbb{C}[e_1,\ldots,e_t]$. Fix $C\in S^B_A$, consider the associated dominant cocharacters $\nu$ and $\kappa$ and let $w=w_C\in\mathfrak{S}_d$ as in Equation (5.4). Let $\varphi:=w^{-1}\mu$. There are natural maps of complexes

$$\begin{aligned}&\pi_{\lambda^{-1}*}\left(\bigoplus_{J\subset A^\varphi_\lambda}\mathcal{O}_{R(d)}\otimes\Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi)\otimes\mathcal{O}\big(-N^\varphi_\lambda-\sigma_J\big)[|I^\varphi_\lambda|+|J|],d\right)\to\\ &\qquad\pi_{\lambda^{-1}*}\left(\bigoplus_{I\subset A_\lambda}\mathcal{O}_{R(d)}\otimes\Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi)\otimes\mathcal{O}(-\sigma_I)[|I|],d\right)\end{aligned} \tag{5.13}$$

induced by the inclusion of sets

$$I_C:=\{N^\varphi_\lambda+\sigma_J\mid J\subset A^\varphi_\lambda\}\subset\{\sigma_I\mid I\subset A_\lambda\}.$$

The differential $d$ of the complex on the first line is induced by multiplication with generators of the polynomial ring $\mathbb{C}\big[(R(d)^\varphi)^{\lambda<0}\big]$. For $C,C'$ different elements in $S^B_A$, we have that $I_C\cap I_{C'}=\emptyset$. If $\sigma\in\{\sigma_I\mid I\subset A_\lambda\}\setminus\Big(\bigcup_{C\in S^B_A}I_C\Big)$, then $\mathcal{O}_{R(d)}\otimes\Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi)\otimes\mathcal{O}(-\sigma_I)$ has $u\mu$-weights strictly less than $\frac{n_\mu}{2} = \langle \mu, \mathfrak{g}^{\mu>0} \rangle$ for all $u \in \mathfrak{S}_d$; see Proposition 5.4. It follows that

$$\widetilde{\Delta}_B \pi_{\lambda^{-1}*}\Big(\mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi) \otimes \mathcal{O}(-\sigma_I)\Big) = 0 \tag{5.14}$$

for such sums $\sigma$; see Equation (5.9). It suffices to show that, for $C \in S_A^B$, we have natural isomorphisms of complexes

$$\begin{aligned}\widetilde{\Delta}_B \pi_{\lambda^{-1}*}\Bigg(\bigoplus_{J \subset A_\lambda^\varphi} \mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi) \otimes \mathcal{O}\big(-N_\lambda^\varphi - \sigma_J\big)[|I_\lambda^\varphi| + |J|], d\Bigg)& \\ \cong \widehat{m}_{CB} \widetilde{\Delta}_{AC}\big(\mathcal{O}_{\mathcal{X}(d)^\lambda} \otimes \Gamma_{GL(d)^\lambda}(\chi)\big).&\end{aligned} \tag{5.15}$$

Using Equation (5.13), we obtain the natural maps (5.11), and further we obtain the isomorphism (5.12) using the vanishing (5.14).

For $J \subset A_\lambda^\varphi$, we have that

$$w * \big(\chi - N_\lambda^\varphi - \sigma_J\big) + \rho \in F(\mu) \subset \mathbf{W}(d);$$

see Proposition 5.4. Let $\widetilde{w}_J \in \mathfrak{S}_d$ be the element of minimal length such that $\widetilde{w}_J * \big(\chi - N_\lambda^\varphi - \sigma_J\big)$ is dominant or zero. Observe that $\widetilde{w}_\emptyset = w$. However, for general $J$, we have that $\widetilde{w}_J = u_J \circ w$ for a permutation $u_J$ in $\times_{i=1}^s \mathfrak{S}_{e_i} \cong W^\mu$ and $\ell(\widetilde{w}_J) = \ell(w) + \ell(u_J)$. By the Borel–Bott–Weyl theorem, there is a natural isomorphism

$$\begin{aligned}&\pi_{\lambda^{-1}*}\Big(\mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}\big(\chi - N_\lambda^\varphi - \sigma_J\big)\Big) \cong \\ &\pi_{\lambda^{-1}*}\Big(\mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}\big(w * (\chi - N_\lambda^\varphi - \sigma_J)\big)[-|J_\lambda^\varphi|]\Big)\end{aligned}$$

and further there are natural isomorphisms

$$\begin{aligned}&\widetilde{\Delta}_B \pi_{\lambda^{-1}*}\Bigg(\bigoplus_{J \subset A_\lambda^\varphi} \mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}\big(\chi - N_\lambda^\varphi - \sigma_J\big)[|I_\lambda^\varphi| + |J|], d\Bigg) \\ &\cong \widetilde{\Delta}_B \pi_{\lambda^{-1}*}\Bigg(\bigoplus_{J \subset A_\lambda^\varphi} \mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}\big(w * (\chi - N_\lambda^\varphi - \sigma_J)\big)[|I_\lambda^\varphi| - |J_\lambda^\varphi| + |J|], d\Bigg) \\ &\cong \pi_{\kappa^{-1}\mu*}\Bigg(\bigoplus_{J \subset A_\lambda^\varphi} \mathcal{O}_{R(d)^\mu} \otimes \Gamma_{(GL(d)^\mu)^{\kappa\leqslant 0}}\big(w * (\chi - N_\lambda^\varphi - \sigma_J)\big)[|I_\lambda^\varphi| - |J_\lambda^\varphi| + |J|], d\Bigg).\end{aligned} \tag{5.16}$$

We have that

$$\mathcal{X}(d)^\nu = \mathcal{X}(d)^\kappa \text{ and } G(d)^\nu = G(d)^\kappa. \tag{5.17}$$

For a weight $\theta$ of $T(d)$, denote by

$$\Psi\big(\mathcal{O}_{\mathcal{X}(d)^\kappa} \otimes \Gamma_{G(d)^\kappa}(\theta)\big) := \mathcal{O}_{\mathcal{X}(d)^\kappa} \otimes \Gamma_{G(d)^\kappa}(w * \theta). \tag{5.18}$$

For a subset $J \subset A_\lambda^\varphi$, let $J' = \{w\beta \mid \beta \in J\}$ be the corresponding subset of $I_{\kappa\mu} := \{\beta \text{ weight of } R(d)^\mu \mid \langle \kappa, \beta \rangle > 0\}$; see Proposition 5.5. There is an isomorphism

$$\mathcal{O}_{(\mathcal{X}(d)^{\mu})^{\kappa\leqslant 0}} \otimes \Gamma_{(GL(d)^{\mu})^{\kappa\leqslant 0}}\big(w*(\chi - N^{\varphi}_{\lambda} - \sigma_J)\big)[-|J^{\varphi}_{\lambda}|] \cong \tag{5.19}$$
$$q^{*}_{\kappa^{-1}\mu}\big(\Psi\big(\mathcal{O}_{\mathcal{X}(d)^{\nu}} \otimes \Gamma_{GL(d)^{\nu}}(\chi - N^{\varphi}_{\lambda})[-|J^{\varphi}_{\lambda}|]\big) \otimes \mathcal{O}(-\sigma_{J'})\big).$$

For $\mathcal{B}$ a complex in $D^b(\mathcal{X}(d)^{\nu})$, by the definition of $\widetilde{\mathrm{sw}}_C$ we have that

$$\widetilde{\mathrm{sw}}_C(\mathcal{B}) := \Psi\big(\mathcal{B} \otimes \mathcal{O}(-N^{\varphi}_{\lambda})\big)[|I^{\varphi}_{\lambda}| - |J^{\varphi}_{\lambda}|]. \tag{5.20}$$

We next want to use Proposition 2.1 for the map $\iota_{\kappa^{-1}\mu}$. For this, it is convenient to use the following notation

$$\mathrm{F}\Big(\mathcal{O}_{(R(d)^{\mu})^{\kappa\leqslant 0}} \otimes \Gamma_{(G(d)^{\mu})^{\kappa\leqslant 0}}(\theta)\Big) := \mathcal{O}_{R(d)^{\mu}} \otimes \Gamma_{(G(d)^{\mu})^{\kappa\leqslant 0}}(\theta)$$

for $\theta$ a dominant weight of $T(d)$. We consider the Koszul resolution

$$\iota_{\kappa^{-1}\mu *} q^{*}_{\kappa^{-1}\mu}\big(\widetilde{\mathrm{sw}}_C\big(\mathcal{O}_{\mathcal{X}(d)^{\kappa}} \otimes \Gamma_{GL(d)^{\kappa}}(\chi)\big)\big) \cong \tag{5.21}$$
$$\left(\bigoplus_{J' \subset I_{\kappa\mu}} \mathrm{F} q^{*}_{\kappa^{-1}\mu}\big(\widetilde{\mathrm{sw}}_C\big(\mathcal{O}_{\mathcal{X}(d)^{\kappa}} \otimes \Gamma_{GL(d)^{\kappa}}(\chi)\big)\big) \otimes \mathcal{O}(-\sigma_{J'})[|J'|], d\right),$$

where the differential $d$ is induced by multiplication with generators of the polynomial ring $\mathbb{C}\big[(R(d)^{\mu})^{\kappa<0}\big]$. Rewrite Equation (5.16) using Equations (5.18), (5.19), (5.20):

$$\widetilde{\Delta}_B \pi_{\lambda^{-1}*}\left(\bigoplus_{J \subset A^{\varphi}_{\lambda}} \mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi - N^{\varphi}_{\lambda} - \sigma_J)[|I^{\varphi}_{\lambda}| + |J|], d\right) \tag{5.22}$$
$$\cong \pi_{\kappa^{-1}\mu *}\left(\bigoplus_{J' \subset I_{\kappa\mu}} \mathrm{F} q^{*}_{\kappa^{-1}\mu}\big(\widetilde{\mathrm{sw}}_C\big(\mathcal{O}_{\mathcal{X}(d)^{\kappa}} \otimes \Gamma_{GL(d)^{\kappa}}(\chi)\big)\big) \otimes \mathcal{O}(-\sigma_{J'})[|J'|], d)\right).$$

There are isomorphisms

$$\begin{aligned}
&\widetilde{\Delta}_B \pi_{\lambda^{-1}*}\left(\bigoplus_{J \subset A^{\varphi}_{\lambda}} \mathcal{O}_{R(d)} \otimes \Gamma_{GL(d)^{\lambda\leqslant 0}}(\chi) \otimes \mathcal{O}(-N^{\varphi}_{\lambda} - \sigma_J)[|I^{\varphi}_{\lambda}| + |J|], d\right)\\
&\overset{(1)}{\cong} \pi_{\kappa^{-1}\mu *}\left(\bigoplus_{J' \subset I_{\kappa\mu}} \mathrm{F} q^{*}_{\kappa^{-1}\mu}\big(\widetilde{\mathrm{sw}}_C\big(\mathcal{O}_{\mathcal{X}(d)^{\kappa}} \otimes \Gamma_{GL(d)^{\kappa}}(\chi)\big)\big) \otimes \mathcal{O}(-\sigma_{J'})[|J'|], d\right)\\
&\overset{(2)}{\cong} \pi_{\kappa^{-1}\mu *}\iota_{\kappa^{-1}\mu *} q^{*}_{\kappa^{-1}\mu}\big(\widetilde{\mathrm{sw}}_C\big(\mathcal{O}_{\mathcal{X}(d)^{\kappa}} \otimes \Gamma_{GL(d)^{\kappa}}(\chi)\big)\big)\\
&\overset{(3)}{\cong} \widetilde{m}_{CB}\big(\widetilde{\mathrm{sw}}_C\big(\mathcal{O}_{\mathcal{X}(d)^{\kappa}} \otimes \Gamma_{GL(d)^{\kappa}}(\chi)\big)\big)\\
&\overset{(4)}{\cong} \widetilde{m}_{CB}\Big(\widetilde{\mathrm{sw}}_C\Big(\widetilde{\Delta}_{AC}\big(\mathcal{O}_{\mathcal{X}(d)^{\lambda}} \otimes \Gamma_{GL(d)^{\lambda}}(\chi)\big)\Big)\Big).
\end{aligned}$$

Recall that $\widetilde{m}_{\kappa\mu} = \widetilde{m}_{CB}$ and $\widetilde{\Delta}_{\nu\lambda} = \widetilde{\Delta}_{AC}$. The isomorphism (1) is the isomorphism (5.22), and it respects the differentials. The isomorphism (2) follows from Equation (5.21). The isomorphism (3) follows from the definition of $\widetilde{m}_{CB}$. The isomorphism (4) follows from Equation (5.17) and the equality

$$\widetilde{\Delta}_{AC}\big(\mathcal{O}_{\mathcal{X}(d)^{\lambda}} \otimes \Gamma_{GL(d)^{\lambda}}(\chi)\big) = \mathcal{O}_{\mathcal{X}(d)^{\nu}} \otimes \Gamma_{GL(d)^{\nu}}(\chi) = \mathcal{O}_{\mathcal{X}(d)^{\kappa}} \otimes \Gamma_{GL(d)^{\kappa}}(\chi). \qquad \square$$

**Proposition 5.4.** *Let $\lambda$ and $\mu$ be dominant cocharacters of $GL(d)$, and let $w \in W$. Assume that $\chi$ is a dominant weight with $\chi + \rho \in F(\lambda)$ and that $I \subset A_{\lambda}$ such that $w * (\chi - \sigma_I) + \rho \in F(\mu)$. Let $\varphi := w^{-1}\mu$. Then*

$$I = \{\beta \in A_\lambda \mid \langle \varphi, \beta \rangle < 0\} \sqcup J \tag{5.23}$$

*for a subset $J \subset A_\lambda^\varphi$.*

*Conversely, for all $I \subset A_\lambda$ as in Equation (5.23), we have $w * (\chi - \sigma_I) + \rho \in F(\mu)$.*

*Proof.* For two cocharacters $\tau$ and $\tau'$, we use the notations

$$N^{\tau>0} := \sum_{A_\tau} \beta,\ N^{\tau'=0,\tau>0} = \sum_{A_\tau^{\tau'}} \beta.$$

Let $v = \langle 1_d, \chi \rangle$. Write

$$\begin{aligned} \chi - v\tau_d + \rho &= \frac{1}{2} N^{\lambda>0} + \psi, \\ w * (\chi - v\tau_d - \sigma_I) + \rho &= \frac{1}{2} N^{\mu>0} + \phi', \\ \chi - v\tau_d - \sigma_I + \rho &= \frac{1}{2} N^{\varphi>0} + \phi, \end{aligned}$$

where $\psi \in \mathbf{W}(\lambda)_0$ and $\phi \in \mathbf{W}(\varphi)_0$; see [Păd23, Proposition 3.4]. Then

$$\sigma_I = N_\lambda^\varphi + \left( \frac{1}{2} N^{\varphi=0,\lambda>0} - \phi \right) + \left( \psi - \frac{1}{2} N^{\lambda=0,\varphi>0} \right). \tag{5.24}$$

The weight $\widetilde{\phi} := \frac{1}{2} N^{\varphi=0,\lambda>0} - \phi$ is a sum with nonnegative coefficients of weights $\beta$ such that $\langle \varphi, \beta \rangle = 0$ and $\langle \lambda, \beta \rangle > 0$ and weights $\beta'$ such that $\langle \varphi, \beta' \rangle = \langle \lambda, \beta' \rangle = 0$. The weight $\widetilde{\psi} := \psi - \frac{1}{2} N^{\lambda=0,\varphi>0}$ is a sum with nonnegative coefficients of weights $\beta$ such that $\langle \lambda, \beta \rangle = 0$ and $\langle \varphi, \beta \rangle < 0$ and weights $\beta'$ such that $\langle \varphi, \beta' \rangle = \langle \lambda, \beta' \rangle = 0$. We denote by $\sigma_{\varphi 0}^{\lambda+}$ a sum with nonnegative coefficients of weights $\beta$ such that $\langle \lambda, \beta \rangle > 0$ and $\langle \varphi, \beta \rangle = 0$ etc. Then we can write

$$\sigma_I - N_\lambda^\varphi = \sigma_{\varphi 0}^{\lambda+} + \sigma_{\varphi+}^{\lambda+} - n_{\varphi-}^{\lambda+}, \tag{5.25}$$

where all the sums on the right-hand side are further partial sums of weights in $A_\lambda$. We can rewrite Equation (5.24) as

$$\sigma_{\varphi 0}^{\lambda+} + \sigma_{\varphi+}^{\lambda+} = n_{\varphi-}^{\lambda+} + \widetilde{\phi}_{\varphi 0}^{\lambda+} + \widetilde{\phi}_{\varphi 0}^{\lambda 0} + \widetilde{\psi}_{\varphi-}^{\lambda 0} + \widetilde{\psi}_{\varphi 0}^{\lambda 0}. \tag{5.26}$$

The $\varphi$-weight of the left-hand side is nonnegative, while the $\varphi$-weight of the right-hand side is nonpositive. Thus, $\sigma_{\varphi+}^{\lambda+} = n_{\varphi-}^{\lambda+} = \widetilde{\psi}_{\varphi-}^{\lambda 0} = 0$. In particular, Equation (5.25) becomes

$$\sigma_I = N_\lambda^\varphi + \sigma_{\varphi 0}^{\lambda+},$$

which implies the first direction. The converse follows in a similar way. □

**Proposition 5.5.** *For a subset $J \subset A_\lambda^\varphi$, the set $J' = \{w\beta \mid \beta \in J\}$ is a subset of $I_{\kappa\mu} := \{\beta$ weight of $R(d)^\mu \mid \langle \kappa, \beta \rangle > 0\}$. This transformation induces a bijection of sets $A_\lambda^\varphi \xrightarrow{\sim} I_{\kappa\mu}$.*

*Proof.* It suffices to check the first claim. To construct an inverse of this transformation, send $L \subset I_{\kappa\mu}$ to $L^\circ = \{w^{-1}\beta \mid \beta \in L\}$.

Let $\beta \in A_\lambda^\varphi$. Recall that $\varphi = w^{-1}\mu$, so we have that $\langle w^{-1}\mu, \beta \rangle = 0$, and thus $\langle \mu, w\beta \rangle = 0$.

It suffices to show that if a weight $\beta_i - \beta_j$ is in $A_\lambda^\varphi$, then $\langle \kappa, \beta_{w(i)} - \beta_{w(j)} \rangle > 0$. For simplicity, we discuss the case when $k = s = 2$. Rename $f_{11} = f_1$, $f_{12} = f_2$, $f_{21} = f_3$, $f_{22} = f_4$. The permutation $w$ is

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$$w(i) = \begin{cases} i + f_3, \text{ if } f_1 + 1 \leqslant i \leqslant f_1 + f_2, \\ i - f_2, \text{ if } f_1 + f_2 + 1 \leqslant i \leqslant f_1 + f_2 + f_3, \\ i, \text{ otherwise.} \end{cases}$$

We have that

$$\langle \lambda, \beta_i - \beta_j \rangle > 0 \text{ and } \langle \mu, \beta_{w(i)} - \beta_{w(j)} \rangle = 0,$$

so one of two possibilities happens:

(1) $f_1 + f_2 + f_3 + 1 \leqslant i \leqslant f_1 + f_2 + f_3 + f_4$ and $f_1 + 1 \leqslant j \leqslant f_1 + f_2$,
(2) $f_1 + f_2 + 1 \leqslant i \leqslant f_1 + f_2 + f_3$ and $1 \leqslant j \leqslant f_1$.

In case (1), we have that $i = w(i)$ and $f_1 + f_3 + 1 \leqslant w(j) \leqslant f_1 + f_3 + f_2$, and then $\langle \kappa, \beta_{w(i)} - \beta_{w(j)} \rangle > 0$. In case (2), we have that $f_1 + 1 \leqslant w(i) \leqslant f_1 + f_3$ and $j = w(j)$, and then $\langle \kappa, \beta_{w(i)} - \beta_{w(j)} \rangle > 0$. □

### *5.3. The bialgebra structure under the Koszul equivalence*

Let $(d, v) \in \mathbb{N} \times \mathbb{Z}$ be coprime integers. For $n \in \mathbb{N}$, denote by $R_n$ the set of ordered partitions $A = (n_i)_{i=1}^k$ of $n$ with $n_i \geqslant 1$. For each such partition $A$ of $n$, denote also by $A$ the partition $(n_i d, n_i v)_{i=1}^k$ of $(nd, nv)$. Let $\gamma_i$ be the weights of $T$ for $i \in \{1, 2\}$ with $q^{\gamma_i} = q_i \in \mathbb{K}$. Let $\gamma = \gamma_1 + \gamma_2$, and let $q^\gamma = q_1 q_2$. For simplicity, we will denote $q^\gamma$ just by $q$.

**5.3.1.**
The coproduct (5.10) induces coproduct maps

$$\Delta_{AC} \colon \mathbb{T}_A \to \mathbb{T}_C$$

for $C$ a refinement of $A$ as follows. Recall the Koszul equivalence

$$\Phi : \mathbb{T}(e)_v \cong \mathbb{S}^{\mathrm{gr}}(e)_v \tag{5.27}$$

for all pairs $(e, v) \in \mathbb{N} \times \mathbb{Z}$. Let $A = (n_i d, w_i)_{i=1}^k$ be a partition of $(nd, nv)$ with associated prime partition $A' = (n_i d, n_i v)_{i=1}^k$; see Subsection 2.2.7. Let $\mathbb{T}_A := \otimes_{i=1}^k \mathbb{T}(n_i d)_{n_i v}$. There is a Koszul equivalence

$$\Phi_A \colon \mathbb{T}_A \xrightarrow{\sim} \mathbb{S}^{\mathrm{gr}}_A.$$

Let $\Phi_A^{-1}$ be its inverse. For $V$ a vector space of dimension $nd$, let $\mathrm{I} := \mathrm{End}(V)^{\lambda_A}$. Let $\lambda_C$ be the antidominant cocharacter of $T(nd) \subset GL(nd)^{\lambda_A}$ corresponding to $C$; let $\omega_{AC} := \det\big((\mathrm{I}^{\lambda_C > 0})^\vee\big)[-\dim \mathrm{I}^{\lambda_C > 0}]$, where $T$ acts on $\mathrm{I}$ with weight $\gamma := \gamma_1 + \gamma_2$. Note that $\det\big((\mathrm{I}^{\lambda_C > 0})^\vee\big)$ is a character of $GL(d)^{\lambda_C}$; hence, it determines a line bundle on $\mathcal{X}(nd)^{\lambda_C}$. We define the functor

$$\Delta_{AC} \colon \mathbb{T}_A \to \mathbb{T}_C$$

by the commutative diagram

$$\begin{tikzcd} \mathbb{T}_A \arrow[r, "\Delta_{AC}"] \arrow[d, "\Phi_A"', "\sim"] & \mathbb{T}_C \arrow[d, "\sim"', "\Phi_C"] \\ \mathbb{S}^{\mathrm{gr}}_A \arrow[r, "\widetilde{\Delta}_{AC}(-) \otimes \omega_{AC}^{-1}"'] & \mathbb{S}^{\mathrm{gr}}_C. \end{tikzcd}$$

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When $A = (nd, nv)$ and $C$ is a two term partition $(n_i d, n_i v)_{i=1}^2$, the term $\omega_{n_1,n_2} := \omega_{AC}$ is Equation (2.40) for $\dim V_i = n_i d$ for $i \in \{1,2\}$. For these partitions, we let $\Delta_{n_1 n_2} := \Delta_{AC}$. For such partitions $A$ and $C$, consider the $(T \times T(d))$-weight

$$\nu_{n_1,n_2} := \sum_{i > n_1 d \geqslant j} (\beta_i - \beta_j - \gamma).$$

Then $\omega_{n_1,n_2} = (-1)^{(n_1 d)\cdot(n_2 d)} q^{\nu_{n_1,n_2}}$. Alternatively, $\omega_{n_1,n_2}$ measures a ratio (called renormalized twist in [VV22, Proof of Lemma 2.3.7]) constructed from the shuffle product for $\widetilde{m}$ with kernel

$$\xi'(x) := \frac{(1-q_1^{-1}x)(1-q_2^{-1}x)(1-qx)}{1-x},$$

and the shuffle product for $m$ with kernel $\xi(x)$; see the computation in [PTa, Proof of Lemma 4.9].

**5.3.2.**

Let $(d, v) \in \mathbb{N} \times \mathbb{Z}$ be coprime integers, and let $n \in \mathbb{N}$. Consider partitions $A = (d_i)_{i=1}^k$ and $B = (e_i)_{i=1}^s$ of $n$. Let $S_A^B$ be the set of partitions $C = (f_i)_{i=1}^l$ of $n$ with $l = ks$ such that

$$\begin{aligned} f_{(i-1)s+1} + f_{(i-1)s+2} + \ldots + f_{is} &= d_i \text{ for } 1 \leqslant i \leqslant k, \\ f_j + f_{s+j} + \ldots + f_{(k-1)s+j} &= e_j \text{ for } 1 \leqslant j \leqslant s. \end{aligned}$$

Let $D$ be the partition on $n$ constructed as in Equation (5.5). Define $m'_{BC} := m_{BC} \circ \mathrm{sw}_C$, for $\mathrm{sw}_C$ as in Equation (5.6). Theorem 5.2 implies the following:

**Corollary 5.6.** *In the above setting, the following diagram commutes:*

$$\begin{tikzcd} K_T(\mathbb{T}_A)' \arrow[r, "\widetilde{m}_A"] \arrow[d, "\bigoplus \widetilde{\Delta}_{AC}"] & K_T(\mathbb{T}(d)_w)' \arrow[d, "\widetilde{\Delta}_B"] \\ \bigoplus_{C \in S_A^B} K_T(\mathbb{T}_C)' \arrow[r, "\bigoplus \widehat{m}_{CB}"] & K_T(\mathbb{T}_B)'. \end{tikzcd}$$

*Proof.* For simplicity of notation, we assume that $k = l = 2$, that $A$ is the partition $a + b = n$, and that $B$ is the partition $c + e = n$. Then $S := S_A^B$ is the set of partitions $C = (f_i)_{i=1}^4 \in \mathbb{N}^4$ such that

$$f_1 + f_2 = a,\ f_3 + f_4 = b,\ f_1 + f_3 = c,\ f_2 + f_4 = e.$$

Note that, for such a partition $C$, the partition $D$ is $(f_1, f_3, f_2, f_4)$. The swap morphism is

$$\begin{aligned} \mathrm{sw}_{ab} : K_T(\mathbb{T}(ad)_{av})' \otimes K_T(\mathbb{T}(bd)_{bv})' &\to K_T(\mathbb{T}(bd)_{bv})' \otimes K_T(\mathbb{T}(ad)_{av})', \\ x \otimes y &\mapsto y \otimes x. \end{aligned}$$

We abuse notation and write $m_{ab}$ instead of $m_{ad,bd}$ and so on. We then need to show that the following diagram commutes:

$$\begin{tikzcd} K_T(\mathbb{T}(ad)_{av})' \otimes K_T(\mathbb{T}(bd)_{bv})' \arrow[r, "m_{ab}"] \arrow[d, "\Delta"] & K_T(\mathbb{T}(nd)_{nv})' \arrow[d, "\Delta_{ce}"] \\ \bigoplus_S \bigotimes_{i=1}^4 K_T(\mathbb{T}(f_i d)_{f_i v})' \arrow[r, "m'"] & K_T(\mathbb{T}(cd)_{cv})' \otimes K_T(\mathbb{T}(ed)_{ev})', \end{tikzcd}$$

where $m' := \bigoplus_S (m_{f_1 f_3} \otimes m_{f_2 f_4})(1 \otimes \mathrm{sw}_{f_2 f_3} \otimes 1)$ and $\Delta := \bigoplus_S \Delta_{f_1 f_2} \otimes \Delta_{f_3 f_4}$.

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For $m \in \mathbb{N}$, recall that $\sigma_m := m\tau_m = \sum_{i=1}^m \beta_i$. Then $\sigma_{ad} = \sigma_{f_1 d} + \sigma_{f_2 d}$ and so on. In this proof, we will use the notation $z := q^{-\gamma}$ instead of $q^{-1}$ to reduce the use of the letter $q$. We use the notation $\Phi_n$ for the Koszul equivalence (5.27) for $(nd, nv)$. Then

$$q^{\nu_{a,b}} = q^{-bd\sigma_{ad}} q^{ad\sigma_{bd}} z^{abd^2}, \; \omega_{a,b} = q^{-bd\sigma_{ad}} q^{ad\sigma_{bd}} (-z)^{abd^2}.$$

Fix a tuplet $C = (f_i)_{i=1}^4$ as above, and let $w = w_C \in \mathfrak{S}_{nd}$ be its corresponding Weyl element as in Subsection 5.1.2. Let $x_m \in K_T(\mathbb{T}(md)_{mv})$ for $m \in \{f_i, a, b, n \mid 1 \leqslant i \leqslant 4\}$. By the discussions in Subsections 2.11 and 5.3.1, we have that

$$\begin{aligned} m_{ab}(x_a \boxtimes x_b) &= \Phi_n^{-1} \widetilde{m}_{ab}\left(\Phi_a(x_a) q^{-bd\sigma_{ad}} \boxtimes \Phi_b(x_b) q^{ad\sigma_{bd}} (-z)^{d^2ab}\right), \\ \Delta_{ab}(x_n) &= \Phi_a^{-1} \boxtimes \Phi_b^{-1}\left(\left(\widetilde{\Delta}_{ab}\Phi_n(x_n)\right) q^{bd\sigma_{ad}} q^{-ad\sigma_{bd}} (-z)^{-d^2ab}\right), \\ \widetilde{\mathrm{sw}}_C(x_{f_1} \boxtimes x_{f_2} \boxtimes x_{f_3} \boxtimes x_{f_4}) &= x_{f_1} \boxtimes \left(x_{f_3} q^{2f_3\sigma_{f_2}}\right) \boxtimes \left(x_{f_2} q^{-2f_2\sigma_{f_3}}\right) \boxtimes x_{f_4}. \end{aligned}$$

We use the shorthand notations $\mathrm{sw}_{23} = 1 \boxtimes \mathrm{sw}_{f_2 f_3} \boxtimes 1$, $\widetilde{m}_{13} = \widetilde{m}_{f_1 f_3}$, $\widetilde{m}_C := \widetilde{m}_{13} \boxtimes \widetilde{m}_{24}$ and so on in what follows. We compute, by ignoring the $z$ factor for simplicity of notation,

$$\begin{aligned} &\Delta_{ce} m_{ab}(x_a \boxtimes x_b) \qquad (5.28)\\ &= \Phi_c^{-1} \boxtimes \Phi_e^{-1}\left(\widetilde{\Delta}_{cf}\left(\widetilde{m}_{ab}(\Phi_a(x_a) q^{-bd\sigma_{ad}} \boxtimes \Phi_b(x_b) q^{ad\sigma_{bd}})\right) q^{ed\sigma_{cd}} q^{-cd\sigma_{ed}}\right) \\ &= \Phi_c^{-1} \boxtimes \Phi_e^{-1} \sum_S \widetilde{m}_C\left(\widetilde{\mathrm{sw}}_C\left(\widetilde{\Delta}_{12}(\Phi_a(x_a) q^{-bd\sigma_{ad}}) \boxtimes \widetilde{\Delta}_{34}(\Phi_b(x_b) q^{ad\sigma_{bd}})\right)\right) q^{ed\sigma_{cd}} q^{-cd\sigma_{ed}}. \end{aligned}$$

We next compute

$$\begin{aligned} &\sum_S m' \Delta(x_a \boxtimes x_b) \qquad (5.29)\\ &= \Phi_c^{-1} \boxtimes \Phi_e^{-1} \sum_S (\widetilde{m}_{13} \boxtimes \widetilde{m}_{24})\left(q^{-f_3 d\sigma_{f_1 d}} \boxtimes q^{f_1 d\sigma_{f_3 d}} \boxtimes q^{-f_4 d\sigma_{f_2 d}} \boxtimes q^{f_2 d\sigma_{f_4 d}}\right) \\ &\quad \times \mathrm{sw}_{23}\left(\left(\widetilde{\Delta}_{12}(\Phi_a(x_a)) q^{f_2 d\sigma_{f_1 d}} q^{-f_1 d\sigma_{f_2 d}}\right) \boxtimes \left(\widetilde{\Delta}_{34}(\Phi_b(x_b)) q^{f_4 d\sigma_{f_3 d}} q^{-f_3 d\sigma_{f_4 d}}\right)\right). \end{aligned}$$

We claim that the expressions (5.28) and (5.29) are equal. We only match the coefficients in $\mathbb{K}$ corresponding to $f_1$ and $f_2$ as the computations for $f_3$ and $f_4$ are similar:

$$\begin{aligned} q^{(f_2+f_4)d\sigma_{f_1 d}} q^{-(f_3+f_4)d\sigma_{f_1 d}} &= q^{f_2 d\sigma_{f_1 d}} q^{-f_3 d\sigma_{f_1 d}}, \\ q^{-(f_1+f_3)d\sigma_{f_2 d}} q^{-(f_3+f_4)d\sigma_{f_2 d}} \cdot q^{2f_3 d\sigma_{f_2 d}} &= q^{-f_1 d\sigma_{f_2 d}} q^{-f_4 d\sigma_{f_2 d}}. \end{aligned}$$

Finally, the $z$ factors in $\Delta_{ce} m_{ab}$ and $m'\Delta$ are equal because

$$(-z)^{d^2ab - d^2ce} = (-z)^{-d^2 f_1 f_2 - d^2 f_3 f_4 + d^2 f_1 f_3 + d^2 f_2 f_4}. \qquad \square$$

### 5.4. *The localized bialgebra*

Recall that for $V$ a $\mathbb{K}$-module, we let $V_{\mathbb{F}} := V \otimes_{\mathbb{K}} \mathbb{F}$. Recall that $(d, v) \in \mathbb{N} \times \mathbb{Z}$ are coprime. Consider the $\mathbb{N}$-graded $\mathbb{F}$-vector space

$$V := K(\mathcal{D}_{d,v})_{\mathbb{F}} = \bigoplus_{n \geqslant 0} K_T(\mathbb{T}(nd)_{nv})_{\mathbb{F}}. \tag{5.30}$$

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We next explain that the operations $m$ and $\Delta$ endow $V$ with the structure of a commutative and cocommutative $\mathbb{F}$-bialgebra. We show this by an explicit computation using the generators of these vector spaces; see Equations (2.34) and (2.33). Recall the complex $\mathcal{E}_{e,v}$ from Definition 2.3 and the shuffle elements $A'_{e,v}$ and $A_{e,v}$ from Equations (2.31) and (2.35) for a pair $(e,v) \in \mathbb{N} \times \mathbb{Z}$.

**Proposition 5.7.** *Let $a, b \in \mathbb{N}$ with $a+b=n$. Then $[\mathcal{E}_{ad,av}] \cdot [\mathcal{E}_{bd,bv}] = [\mathcal{E}_{bd,bv}] \cdot [\mathcal{E}_{ad,av}]$ in $K_T(\mathbb{T}(nd)_{nv})$.*

*Proof.* By Equations (2.34) and (2.33), it suffices to check the statement for $A_{ad,aw}$ and $A_{bd,bw}$ or, alternatively, for $A'_{ad,aw}$ and $A'_{bd,bw}$ for $a, b \in \mathbb{N}$. Any two such elements commute because they are in the subalgebra of $\mathcal{S}'_{\mathbb{F}} \cong \mathcal{S}_{\mathbb{F}}$ generated by elements $A'_{e,v}$ of fixed slope $\frac{v}{e}$, and such algebra are commutative; see, for example, [Neg19, Subsection 3.2]. □

For the following proposition, it is convenient to introduce the element

$$\hat{A}_{nd,nv} := \left(-q^{-1}\right)^{n-1} A_{nd,nv} = \frac{\left(-q^{-1}\right)^{n-1}}{(1-q_1^{-1})^{nd-1}(1-q_2^{-1})^{nd-1}}[\mathcal{E}_{nd,nv}] \in K_T(\mathbb{T}(nd)_{nv})_{\mathbb{F}}.$$

**Proposition 5.8.** *Let $a, b, n \in \mathbb{N}$ be such that $a+b=n$. Then*

$$\Delta_{ab}\left(\hat{A}_{nd,nv}\right) = \hat{A}_{ad,av} \boxtimes \hat{A}_{bd,bv}.$$

For $d, v, n$ as above, let $O_n$ and $L_n$ be the following sets of $(T \times T(d))$-weights:

$$\begin{aligned} O_n &:= \{\beta_j - \beta_i + \gamma \mid i > j+1\}, \\ L_n &:= \{\beta_i - \beta_j + \gamma_l \mid i > j,\ 1 \leqslant l \leqslant 2\} \sqcup O_n. \end{aligned}$$

For a subset $I \subset L_n$, let $\sigma_I$ be the sum of the corresponding $T(d)$-weights in $I$, let $\gamma_I$ be the sum of the weights of the corresponding $T$-weights in $I$ and let $q_I := q^{\gamma_I}$. Let $\ell(I)$ be the length of the minimal Weyl element such that $w * (\chi - \sigma_I)$ is dominant or zero. Let

$$\chi_n := \sum_{i=1}^{nd-1}\left(\frac{vi}{d} + 1 - \left\lceil \frac{vi}{d} \right\rceil\right)(\beta_{i+1} - \beta_i) + \frac{v}{d}\sum_{i=1}^{nd}\beta_i \in M(nd).$$

The element $A_{nd,nv} \in K_T(\mathcal{X}(nd))$ from Equation (2.35) can be written as

$$A_{nd,nv} := \sum_{I \subset L_n} (-1)^{|I| - \ell(I)} q_I^{-1} \left[\Gamma_{GL(nd)}\left((\chi_n - \sigma_I)^+\right) \otimes \mathcal{O}_{\mathcal{X}(nd)}\right].$$

Recall the framework on Subsection 5.3.1. Consider the composition of the Koszul equivalence (2.22) and the forget-the-potential map (1.8):

$$\Psi : K_T(\mathbb{T}(nd)_{nv})_{\mathbb{F}} \xrightarrow{\sim} K_T(\mathbb{S}(nd)_{nv})_{\mathbb{F}} \xrightarrow{\Theta} K_T(\mathrm{MF}(\mathcal{X}(d), 0))_{\mathbb{F}} \xrightarrow{\sim} K_T(\mathcal{X}(nd))_{\mathbb{F}}.$$

Using Equation (2.38) and the computation in [PTa, Lemma 4.3], we have that

$$\Psi\left((-q)^{n-1}\hat{A}_{nd,nv}\right) = A_{nd,nv}.$$

**Remark 5.9.** Note that $\Psi\left(\hat{A}_{nd,nv}\right)$ equals the shuffle element $E_{k,d}$ considered by Neguţ in [Neg18, Equation 2.10], where $q$, $k$, $d$ in loc. cit. correspond to $q^{-1}$, $nd$ and $nv$, respectively, in our paper.

*Proof of Proposition 5.8.* By the definition of $\Delta$ from Subsection 5.3.1, it suffices to show that

$$\widetilde{\Delta}_{ab}\left(A_{nd,nv}\right) = \left(A_{ad,av} \boxtimes A_{bd,bv}\right) \otimes (-q)(-1)^{abd^2} q^{\nu_{a,b}}, \tag{5.31}$$

where we are using $\widetilde{\Delta}_{ab}$ instead of $\widetilde{\Delta}_{ad,bd}$. Let $I \subset L_n$. Then

$$\chi_n - \sigma_I + \rho = \sum_{i>j} d_{ij}(\beta_i - \beta_j) + \frac{v}{d}\sum_{i=1}^{nd} \beta_i,$$

where

$$-\frac{3}{2} \leqslant d_{ij} \leqslant \frac{3}{2},\ -\frac{3}{2} < d_{j+1,j} \leqslant \frac{3}{2}.$$

Let $\lambda$ be the antidominant cocharacter associated to the partition $(ad, bd)$ of $nd$. Assume such a weight is on a wall $F(w\lambda)$ for some $w \in \mathfrak{S}_{nd}$. Then there exists a partition $E \sqcup C = \{1, \cdots, nd\}$ with $|C| = ad$ and

$$d_{ij} = \begin{cases} \frac{3}{2} \text{ for } i \in E, j \in C \text{ and } i > j, \\ -\frac{3}{2} \text{ for } i \in E, j \in C \text{ and } i < j; \end{cases} \tag{5.32}$$

see [HLS20, Lemma 3.12], [Păda, Proposition 3.2]. We claim that $C = \{1, \ldots, ad\}$. Otherwise, there exists $1 \leqslant j \leqslant ad$ with $j \in E$ and $j+1 \in C$. Then $d_{j+1,j} > -\frac{3}{2}$, and this contradicts Equation (5.32). Then $d_{ij} = -\frac{3}{2}$ for $i > ad \geqslant j$. Further, we have

$$I \subset Q := \{\beta_j - \beta_i + \gamma \mid i > ad \geqslant j, i-1 > j\}$$

and $I$ does not contain any weights $\beta_i - \beta_j - \gamma_l$ for $i > ad \geqslant j$ and $l \in \{1, 2\}$. Define $L_a$ and $L_b$ similarly to $L_n$, using the weights $\beta_i$ with $1 \leqslant i \leqslant ad$ for $L_a$ and $\beta_i$ with $ad < i \leqslant nd$ for $L_b$. We can thus write $I = Q \sqcup I_a \sqcup I_b$, where $I_a \subset L_a$ and $I_b \subset L_b$. Let $\mathcal{I}$ be the set of such sets $I \subset L_n$. We have that (see Equation (5.9))

$$\widetilde{\Delta}_{ab}\left(A_{nd,nv}\right) = \widetilde{\Delta}_{ab}\left(\sum_{I\in\mathcal{I}} (-1)^{|I|-\ell(I)} q_I^{-1}\left[\Gamma_{GL(nd)}\left((\chi_n - \sigma_I)^+\right) \otimes \mathcal{O}_{\mathcal{X}(nd)}\right]\right). \tag{5.33}$$

Write $I = Q \sqcup I_a \sqcup I_b$ with $I_a \subset L_a$ and $I_b \subset L_b$. A direct computation shows that

$$\chi_n - \sigma_I + \rho = \sum_{i>ad\geqslant j} \frac{3}{2}(\beta_i - \beta_j) - (d^2ab - 1)\gamma + (\chi_a - \sigma_{I_a} + \rho_a) + (\chi_b - \sigma_{I_b} + \rho_b).$$

Let $w \in \mathfrak{S}_{nd}$ be the Weyl element of minimal length such that $w * (\chi_n - \sigma_I)$ is dominant, and let $w_a \in \mathfrak{S}_{na}$ and $w_b \in \mathfrak{S}_{nb}$ be the Weyl elements of minimal length such that $w_a * (\chi_a - \sigma_{I_a})$ and $w_b * (\chi_b - \sigma_{I_b})$ are dominant. Then $w = w_a w_b$, therefore

$$(\chi_n - \sigma_I)^+ = \nu_{a,b} + \gamma + (\chi_a - \sigma_{I_a})^+ + (\chi_b - \sigma_{I_b})^+.$$

We have that $|Q| = d^2ab - 1$, and

$$|I| = d^2ab - 1 + |I_a| + |I_b|,\ \ell(I) = \ell(I_a) + \ell(I_b).$$

For $m \in \{a, b, n\}$, let

$$E_m := (-1)^{|I_m|-\ell(I_m)}\left[\Gamma_{GL(md)}\left((\chi_m - \sigma_{I_m})^+\right) \otimes \mathcal{O}_{\mathcal{X}(md)}\right].$$

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The coproduct (5.33) thus simplifies to

$$\widetilde{\Delta}_{ab}(E_n) = (-q)(-1)^{d^2ab} q^{\nu_{a,b}} (E_a \boxtimes E_b).$$

The conclusion thus follows. $\square$

**Corollary 5.10.** *The operations m and $\Delta$ endow V with the structure of an $\mathbb{N}$-graded commutative and cocommutative $\mathbb{F}$-bialgebra.*

### 5.5. *Primitive elements*

Let $R'_n \subset R_n$ be the complement of the trivial partition $n$. For $A = (n_i)_{i=1}^k$, let $\mathbb{T}_A := \otimes_{i=1}^k \mathbb{T}(n_i d)_{n_i v}$. Define

$$\mathrm{P}(nd)_{nv} := \ker\left(\bigoplus_{A \in R'_n} \Delta_A : K_T(\mathbb{T}(nd)_{nv}) \to \bigoplus_{A \in R'_n} K_T(\mathbb{T}_A)\right).$$

Let $\mathrm{P}(nd)_{nv,\mathbb{F}} := \mathrm{P}(nd)_{nv} \otimes_{\mathbb{K}} \mathbb{F}$. Corollary 5.10 can be rephrased as follows; see the isomorphism (2.43):

**Corollary 5.11.** *Recall the bialgebra $\Lambda_{\mathbb{F}} := \lambda \otimes_{\mathbb{Z}} \mathbb{F}$ from Subsection (2.12). There exists an isomorphism of bialgebras*

$$\begin{aligned} \Phi \colon \Lambda_{\mathbb{F}} &\cong V, \\ e_n &\mapsto \widehat{A}_{nd,nv}. \end{aligned}$$

*In particular, the $\mathbb{F}$-vector space $\mathrm{P}(nd)_{nv,\mathbb{F}}$ is one-dimensional.*

*Proof.* Both $\Lambda_{\mathbb{F}}$ and $V$ are commutative and cocommutative, and $\Phi$ is a morphism of algebras by construction. The coproduct is respected by Proposition 5.8. Finally, $\Phi$ is an isomorphism of $\mathbb{N}$-graded vector spaces by [PTa, Theorem 4.12], so the conclusion follows. $\square$

**Remark 5.12.** The isomorphism $\Phi$ sends $e_n$ to $\widehat{A}_{nd,nv}$. Since $\widehat{A}_{nd,nv}$ is not contained in the integral part $K_T(\mathbb{T}(nd)_{nv})$, it does not restrict to a morphism $\Lambda_{\mathbb{K}} \to K_T(\mathcal{D}_{d,v})$. On the other hand, we expect the existence of a McKay-type functor $\mathrm{MF}^{\mathrm{gr}}([\mathbb{C}^{3d}/\mathfrak{S}_d], 0) \to \mathbb{T}(nd)_{nv}$ which may induce an isomorphism

$$\Lambda_{\mathbb{K}} \xrightarrow{\cong} K_T(\mathcal{D}_{d,v}).$$

For a full understanding of $K_T(\mathbb{T}(nd)_{nv})$, we thus need to construct an isomorphism different from the one in Corollary 5.11. Such an isomorphism will be discussed in [PTb].

Finally, we prove Corollary 1.5:

**Corollary 5.13.** *There is an isomorphism of $\mathbb{N}$-graded $\mathbb{F}$-vector spaces*

$$\bigoplus_{d \geqslant 0} K_T(\mathcal{DT}(d))_{\mathbb{F}} \cong \bigotimes_{\substack{0 \leqslant v < d \\ \gcd(d,v)=1}} \left(\bigotimes_{n \geqslant 1} \mathrm{Sym}\big(\mathrm{P}(nd)_{nv,\mathbb{F}}\big)\right).$$

*Proof.* For $\bullet \in \{\emptyset, \mathrm{gr}\}$, there are equivalences $\mathbb{S}_T^\bullet(d)_w \xrightarrow{\sim} \mathbb{S}_T^\bullet(d)_{d+w}$. By [PTa, Theorem 1.1], there is an isomorphism

$$K_T(\mathcal{DT}(d))_{\mathbb{F}} \cong \bigoplus_{\substack{0 \leqslant v_1/d_1 < \ldots < v_k/d_k < 1 \\ d_1 + \ldots + d_k = d}} \bigotimes_{i=1}^{k} K_T\big(\mathbb{S}(d_i)_{v_i}\big)_{\mathbb{F}}. \tag{5.34}$$

Further, there are isomorphisms of $\mathbb{N}$-graded $\mathbb{F}$-vector spaces

$$\bigoplus_{0\leqslant v_1/d_1<\ldots<v_k/d_k<1} \bigotimes_{i=1}^{k} K_T\left(\mathbb{S}(d_i)_{v_i}\right)_{\mathbb{F}} \cong \bigotimes_{\substack{0\leqslant\mu<1\\ \mu=a/b,\ \gcd(a,b)=1}} \left(\bigoplus_{n\geq 1} K_T\left(\mathbb{S}(nb)_{na}\right)_{\mathbb{F}}\right). \tag{5.35}$$

For each coprime $(a,b)\in\mathbb{Z}\times\mathbb{N}$, by Corollary 5.11 and the isomorphism (2.43), we obtain the isomorphism

$$\bigoplus_{n\geq 1} K_T\left(\mathbb{S}(nb)_{na}\right)_{\mathbb{F}} \cong \bigotimes_{n\geqslant 1} \mathrm{Sym}\left(\mathrm{P}(nb)_{na,\mathbb{F}}\right). \tag{5.36}$$

We obtain the conclusion by combining the isomorphisms (5.34), (5.35), (5.36). □

We explain how the above isomorphism categorifies Equation (1.6) up to a sign. We use the same computation as in [PTa, Subsection 4.7]. Let $a_d := \dim_{\mathbb{F}} K_T(\mathcal{DT}(d))_{\mathbb{F}}$. We have that $\dim_{\mathbb{F}} \mathrm{P}(nb)_{na,\mathbb{F}} = 1$ for any $(a,b)$ with $\gcd(a,b)=1$ and $n\in\mathbb{Z}_{\geqslant 1}$. For each $d\in\mathbb{Z}_{\geqslant 1}$, there is a bijection

$$\{(n,a,b)\in\mathbb{Z}^3_{\geqslant 0} : d=bn, \gcd(a,b)=1, 0\leqslant a<b\} \xrightarrow{\cong} \{0,1,\ldots,d-1\}$$

given by $(n,a,b)\mapsto na$. In particular, the number of the elements of the left-hand side equals to $d$. We compute

$$\sum_{d\geqslant 0} a_d q^d = \prod_{\substack{0\leqslant\mu<1\\ \mu=a/b,\ \gcd(a,b)=1}} \prod_{n\geqslant 1} \frac{1}{1-q^{bn}} = \prod_{d\geqslant 1} \frac{1}{\left(1-q^d\right)^d}.$$

Compare with the wall-crossing formula for DT invariants (1.6).

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