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ON THE SET OF KRONECKER NUMBERS

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Abstract

A positive even number is said to be a Maillet number if it can be written as the difference between two primes, and a Kronecker number if it can be written in infinitely many ways as the difference between two primes. It is believed that all even numbers are Kronecker numbers. We study the division and multiplication of Kronecker numbers and show that these numbers are rather abundant. We prove that there is a computable constant k and a set D consisting of at most 720 computable Maillet numbers such that, for any integer n, kn can be expressed as a product of a Kronecker number and a Maillet number in D. We also prove that every positive rational number can be written as a ratio of two Kronecker numbers.

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1. Introduction

The distribution of the differences of primes is a recurring theme in number theory. It is generally believed that every even number is the difference of two primes.

DEFINITION 1.1 (Maillet number, Kronecker number). An even number n is called a Maillet number if it can be written as the difference of two primes and a Kronecker number if it can be written in infinitely many ways as the difference of two primes.

For a given positive even number, we can check directly if it is a Maillet number by finding a pair of primes. However, no concrete Kronecker number is known. Whether 2 is a Kronecker number is the well-known twin prime conjecture.

CONJECTURE 1.2 (Kronecker [11]). Every even number can be written in infinitely many ways as the difference of two primes.

This conjecture is currently out of reach. However, recent breakthroughs towards the twin prime conjecture indicate that there is a Kronecker number not exceeding 246 (see [2, 12, 14, 15]).

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Let $\mathcal{K}$ be the set of all Kronecker numbers. We investigate how 'large' this set might be. A set $S \subset \mathbb{N}$ is called *syndetic* if there exists an integer k such that $\{a+1, a+2, \ldots, a+k\} \cap S \neq \emptyset$ for any $a \in \mathbb{N}$. Pintz [13] proved that $\mathcal{K}$ is a syndetic set and Granville *et al.* [4] gave a different proof. Unfortunately, the integer k is not determined effectively in [4, 13].

DEFINITION 1.3 (Δ_r-set and Δ_r^*-set). Let r be a given positive integer.

- A Δ_r-set is a difference set of a set $S \subset \mathbb{N}$ with $|S| \geq r$, that is,

$$\Delta(S) = (S - S) \cap \mathbb{N} = \{a - b : a, b \in S, a > b\}.$$

- A set $S \subset \mathbb{N}$ is a Δ_r^*-set if the intersection of S with any Δ_r-set is not empty.

Clearly, every Δ_r^*-set is syndetic.

THEOREM 1.4 (Huang and Wu [10]). *$\mathcal{K}$ is a Δ_r^*-set for any $r \geq 721$.*

It is also mentioned in [10] that the number 721 can be sharpened to 19 if the primes have level of distribution θ for every $\theta < 1$. Given $\theta > 0$, we say the primes have 'level of distribution θ' if, for any $W > 0$,

$$\sum_{q \leq x^\theta} \max_{(a,q)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_W \frac{x}{(\log x)^W}.$$

The numerical bound for r in Theorem 1.4 gives an effective lower bound for the density α of $\mathcal{K}$ among even numbers. As shown in the Appendix, one may deduce from Theorem 1.4 that

$$\alpha \geq \frac{1}{360} \prod_{p \leq 720} \left(1 - \frac{1}{p}\right).$$

We try to obtain more information on how large $\mathcal{K}$ is by studying the division and multiplication of Kronecker numbers. Our first result about the representation of integers by products of differences of primes is motivated by the following question.

QUESTION 1.5 (Fish [1]). For a given infinite set $E \subset \mathbb{Z}$, how much structure does the set $(E - E) \cdot (E - E)$ possess?

Fish [1] considered the question when E is a subset of $\mathbb{Z}$ of positive density. Using Furstenberg's correspondence principle, he proved that there exist k_0 (depending on the densities of E_1 and E_2) and $k \leq k_0$ such that

$$k\mathbb{Z} \subset (E_1 - E_1) \cdot (E_2 - E_2).$$

It is natural to consider the question when E is the set of primes. This case is not covered by Fish's work since the set of primes is an infinite set of $\mathbb{Z}$ but does not have positive upper Banach density. In [3], Goswami used Theorem 1.4 to extend Fish's result to the case of primes, showing that

$$k\mathbb{Z} \subset (\mathbb{P} - \mathbb{P}) \cdot (\mathbb{P} - \mathbb{P}). \tag{1.1}$$

We can say more. On the right-hand side of (1.1), one factor can be restricted to a finite subset of $\mathbb{P} - \mathbb{P}$, consisting of 720 Maillet numbers, while the other factor takes values among Kronecker numbers.

THEOREM 1.6. *There exist a computable constant k and a set D, consisting of at most 720 computable Maillet numbers, such that $k\mathbb{Z} \subset D \cdot \mathcal{K}$.*

The proof of Theorem 1.6 is based on Theorem 1.4 as well as recent work on linear equations in primes by Green *et al.* [7]. The number 720 could be sharpened to 18 if the primes have level of distribution θ for every $\theta < 1$.

Seeking further evidence on the size of $\mathcal{K}$, we also consider the ratio of two Kronecker numbers and prove the following result.

THEOREM 1.7. *Every positive rational number can be written as a ratio of two elements from $\mathcal{K}$.*

2. Representation of integers

In this section, we give the proof of Theorem 1.6.

2.1. Linear equations in primes. We outline some of the work of Green and Tao on linear equations in primes. More details can be found in [5]. Let d, t be integers. A system of affine-linear forms on $\mathbb{Z}^d$ is a collection $\Psi = \{\psi_1, \ldots, \psi_t\}$ with $\psi_i : \mathbb{Z}^d \to \mathbb{Z}$ being affine-linear forms. If $N > 0$, the *size* $\|\Psi\|_N$ of Ψ relative to the scale N is

$$\|\Psi\|_N := \sum_{i=1}^{t}\sum_{j=1}^{d} |\dot{\psi}_i(e_j)| + \sum_{i=1}^{t}\left|\frac{\psi_i(0)}{N}\right|,$$

where

$$\dot{\psi}_i(e_j) = \psi_i(e_j) - \psi_i(0)$$

with $e_1, e_2, \ldots, e_d$ being the standard basis for $\mathbb{Z}^d$. For a system Ψ, its *local factor* β_p for a prime p is

$$\beta_p := \frac{1}{p^d}\sum_{n\in\mathbb{Z}_p^d}\prod_{i=1}^{t}\Lambda_{\mathbb{Z}_p}(\psi_i(n)), \tag{2.1}$$

where $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is the set of residue classes of integers modulo p and $\Lambda_{\mathbb{Z}_p}(n)$ is the *local von Mangoldt function* defined by

$$\Lambda_{\mathbb{Z}_p}(n) = \begin{cases} \dfrac{p}{p-1} & \text{if } (n, p) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For a convex body $K \subset [-N, N]^d$, the *archimedean factor* is

$$\beta_\infty := \mathrm{vol}_d(K \cap \Psi^{-1}(\mathbb{R}^+)^t).$$

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DEFINITION 2.1 (Complexity). The complexity of Ψ is the least integer s such that, for each ψ_i, one can partition the remaining $t-1$ forms $\{\psi_j : j \neq i\}$ into $s+1$ groups, such that ψ_i does not lie in the affine-linear span of any group; if no such s exists, we say that the complexity is ∞ (see [5, Definition 1.5]).

The main results of [5–7] can be summarised as follows.

THEOREM 2.2 (Green, Tao and Ziegler). *Let N, d, t, L be positive integers and let $\Psi = \{\psi_1, \ldots, \psi_t\}$ be a system of affine-linear forms with size $\|\Psi\|_N \leq L$. Let $K \subset [-N, N]^d$ be a convex body. If Ψ has finite complexity, then*

$$\#\{n \in K \cap \mathbb{Z}^d : \psi_1(n), \ldots, \psi_t(n) \ \text{prime}\} = (1 + o_{t,d,L}(1)) \frac{\beta_\infty}{\log^t N} \prod_p \beta_p + o_{t,d,L}\Big(\frac{N^d}{\log^t N}\Big),$$

where β_∞ is typically of size N^d and the singular product $\prod_p \beta_p$ is always convergent.

Theorem 2.2 was first proved by Green and Tao assuming that the inverse Gowers-norm conjecture and the Möbius and nilsequences conjecture are true (see [5, Main Theorem]). These two conjectures were already known for the case $s \leq 2$ (see [5, Corollary 1.7]). In their following papers, they proved the Möbius and nilsequences conjecture for any s (see [6, Main Theorem]) and, in combination with Ziegler, proved the inverse Gowers-norm conjecture (see [7, Theorem 1.3]).

It was pointed out in [5] that the singular product $\prod_p \beta_p$ is always convergent, but it may still vanish since $\beta_p = 0$ is possible for small p. So, the theorem only works for a system of affine-linear forms with $\beta_p \neq 0$, for all p. (It is enough to consider small $p = O_{t,d,L}(1)$ here.)

2.2. Proof of Theorem 1.6. To prove our theorem, we will appeal to a special case of Theorem 2.2, which we provide in the following lemma.

LEMMA 2.3. *Let $\Psi = \{\psi_1, \ldots, \psi_t\}$ be a system of affine-linear forms of finite complexity with $\psi_i : \mathbb{Z}^{+d} \to \mathbb{Z}^+$, $1 \leq i \leq t$, and $\beta_p \neq 0$ for any prime p. Then there are infinitely many lattice points $n \in \mathbb{Z}^{+d}$, which make all $\psi_i(n)$ prime.*

PROOF. In Theorem 2.2, we can take $K = [-N, N]^d$. For $\psi_i : \mathbb{Z}^{+d} \to \mathbb{Z}^+$,

$$\beta_\infty := \mathrm{vol}_d(K \cap \Psi^{-1}((\mathbb{R}^+)^t) \geq \mathrm{vol}_d([-N, N]^d \cap \mathbb{Z}^{+d}) \geq N^d.$$

Also, the singular product $\prod_p \beta_p$ does not vanish since $\beta_p \neq 0$ for all p. Thus, the asymptotic formula in Theorem 2.2 has a dominant main term, and the lemma follows immediately. □

For $j = 1, 2, \ldots, 720$, we define $a_j = 720!/j$. We consider the system of affine-linear forms $\Psi = \{\psi_1, \ldots, \psi_{1440}\}$ defined by

$$\psi_{2j-1}(n_1, \ldots, n_{720}, m) = n_j,$$
$$\psi_{2j}(n_1, \ldots, n_{720}, m) = n_j + a_j m,$$

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for $j = 1, \ldots, 720$. Note that each form $\psi_i \in \Psi$ lies in the affine span of all the other forms, but we can partition the remaining forms into two groups, such that ψ_i does not lie in the affine span of either group. For example, for $i = 2j - 1$, we may take ψ_{2j} as one group and the rest as the other group, and for $i = 2j$, we take ψ_{2j-1} as one group and the rest as the other group. Thus, Ψ is a special system of affine-linear forms with complexity $s = 1$. It is obvious that $\psi_i : \mathbb{Z}^{+d} \to \mathbb{Z}^+$ since all coefficients are positive integers.

To apply Lemma 2.3, we also need $\beta_p \neq 0$. By (2.1), this is the case if, for each p, we can find a lattice point $n \in \mathbb{Z}_p^d$ such that $(\psi_i(n), p) = 1$ for all i. Obviously, the lattice point $n = (1, 1, \ldots, 1, 0) \in \mathbb{Z}_p^{721}$ has this property. Thus, by Lemma 2.3, there are infinitely many lattice points $(n_1, \ldots, n_{720}, m) \in \mathbb{Z}^{+721}$, which make all ψ_i prime. That is to say, for each m in this set of lattice points, the set $\{a_1 m, a_2 m, \ldots, a_{720} m\}$ consists of Maillet numbers. We choose m' as the least one of these m; it is a computable number since the system of affine-linear forms is specific.

To prove the theorem, we take the constant $k = 720!\, m'$ and the set to be

$$D = \{a_1 m', a_2 m', \ldots, a_{720} m'\}.$$

For any integer $b > 0$, Theorem 1.4 shows that there is at least one Kronecker number in the set $\{b, 2b, \ldots, 720b\}$. If jb with $1 \leq j \leq 720$ is a Kronecker number, then

$$kb = a_j m' \cdot jb \in D \cdot \mathcal{K},$$

which establishes Theorem 1.6.

3. Representation of rationals

In this section, we will use arguments from Ramsey theory to prove Theorem 1.7. First we will prove three lemmas and then a more general Theorem 3.6 which implies Theorem 1.7 as a corollary. The lemmas are standard and can be found in [9]. We include the short proofs for the sake of completeness.

The notions of *IP* sets and IP_r sets are well studied in Ramsey theory (see [9]). Let $\mathcal{P}_f(\mathbb{N})$ denote the collection of nonempty finite subsets of $\mathbb{N}$.

Definition 3.1. A set $A \subset \mathbb{N}$ is said to be an *IP* set if there exists a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ such that $A = FS(\langle x_n \rangle_{n\in\mathbb{N}}) = \{\sum_{t\in H} x_t : H \in \mathcal{P}_f(\mathbb{N})\}$. Similarly, a set $A \subset \mathbb{N}$ is said to be an IP_r set for some $r \in \mathbb{N}$ if there exists a sequence $\langle x_n \rangle_{n=1}^r$ such that $A = FS(\langle x_n \rangle_{n=1}^r)$.

A set is said to be an $IP^\star$ set if it intersects every *IP* set and an $IP_r^\star$ set if it intersects every IP_r set. Note that every IP_r set contains a Δ_r set. To check this, let $FS(\langle x_n \rangle_{n=1}^r)$ be an IP_r set and let

$$S = \{x_1, x_1 + x_2, \ldots, x_1 + x_2 + \cdots + x_n\}.$$

Then $FS(\langle x_n \rangle_{n=1}^r)$ contains all elements of the form $\{s - t : s > t \text{ and } s, t \in S\}$. Hence, every $\Delta_r^\star$ set is $IP_r^\star$. Again, every *IP* set contains an IP_r set for some $r \in \mathbb{N}$ and hence every $IP_r^\star$ set is $IP^\star$. In particular, $\mathcal{K}$ is $IP_{721}^\star$ and hence an $IP^\star$ set.

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A sub-*IP* set of $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ is a set of the form $FS(\langle y_n\rangle_{n\in\mathbb{N}}) \subseteq FS(\langle x_n\rangle_{n\in\mathbb{N}})$, where $y_t = \sum_{s\in H_t} x_s$ for each $t \in \mathbb{N}$ and $(H_i)_{i\in\mathbb{N}}$ is a sequence in $\mathcal{P}_f(\mathbb{N})$ such that $H_i \cap H_j = \emptyset$ for each $i \neq j$.

The following lemma is a direct corollary of Hindman's theorem [8]. Here, 'colouring' means a disjoint partition and a set is 'monochromatic' if it lies in one part of the partition.

LEMMA 3.2. *For every finite colouring of an IP set, there exists a monochromatic sub-IP set.*

The next lemma says that dilation of an $IP^\star$ set by a number is again an $IP^\star$ set.

LEMMA 3.3. *Let $A \subseteq \mathbb{N}$ be an $IP^\star$ set. Then for any $m \in \mathbb{N}$, $m \cdot A = \{mx : x \in A\}$ is again an $IP^\star$ set.*

PROOF. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be any sequence and for each $i \in \mathbb{N}$, let $x_i \equiv j(i) \bmod m$, where $j(i) \in \{0, 1, \ldots, m-1\}$. As $\mathbb{Z}_m$ is finite, there exists $k \in \{0, 1, \ldots, m-1\}$ and an infinite sequence $(n_i)_{i\in\mathbb{N}}$ such that $x_{n_i} \equiv k \bmod m$. Let H_1 be a set of m terms from the sequence (n_i) so that $m \mid \sum_{t\in H_1} x_t$. Continue to choose further sets of terms in this way to obtain disjoint finite subsets of H_n of $\mathbb{N}$ such that $m \mid \sum_{t\in H_n} x_t$ for each $n \in \mathbb{N}$. Choose a new sequence $\langle y_n\rangle_{n\in\mathbb{N}}$ such that $y_n = \sum_{t\in H_n} x_t/m$ for each $n \in \mathbb{N}$. Then $A \cap FS(\langle y_n\rangle_{n\in\mathbb{N}}) \neq \emptyset$ and this implies $m \cdot A \cap FS(\langle x_n\rangle_{n\in\mathbb{N}}) \neq \emptyset$, finishing the proof. □

The next lemma says that any $IP^\star$ set contains an *IP* set. In fact, it contains a sub-*IP* set of any given *IP* set.

LEMMA 3.4. *Let $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ be any IP set and let A be any $IP^\star$ set. Then there exists a sub-IP set $FS(\langle y_n\rangle_{n\in\mathbb{N}})$ of $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ such that A contains $FS(\langle y_n\rangle_{n\in\mathbb{N}})$.*

PROOF. Partition $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ by

$$FS(\langle x_n\rangle_{n\in\mathbb{N}}) = (A \cap FS(\langle x_n\rangle_{n\in\mathbb{N}})) \cup (FS(\langle x_n\rangle_{n\in\mathbb{N}}) \setminus A).$$

From Lemma 3.2, there is a sub-*IP* set $FS(\langle y_n\rangle_{n\in\mathbb{N}})) \subseteq FS(\langle x_n\rangle_{n\in\mathbb{N}}))$ such that either $FS(\langle y_n\rangle_{n\in\mathbb{N}})) \subseteq A \cap FS(\langle x_n\rangle_{n\in\mathbb{N}})$ or $FS(\langle y_n\rangle_{n\in\mathbb{N}})) \subseteq FS(\langle x_n\rangle_{n\in\mathbb{N}}) \setminus A$. Since A is an $IP^\star$ set, $FS(\langle y_n\rangle_{n\in\mathbb{N}})) \cap A$ must be nonempty. This immediately implies that the second case is not possible. So, $FS(\langle y_n\rangle_{n\in\mathbb{N}})) \subseteq A$ and the lemma follows. □

The next lemma is the final ingredient for our proof.

LEMMA 3.5. *The intersection of finitely many $IP^\star$ sets is again an $IP^\star$ set.*

PROOF. Let $A_1, A_2, \ldots, A_n$ be $IP^\star$ sets and let $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ be any *IP* set. From Lemma 3.4, there is a sub-*IP* set $FS(\langle y_n\rangle_{n\in\mathbb{N}})) \subseteq A_1 \cap FS(\langle x_n\rangle_{n\in\mathbb{N}})$ and so $FS(\langle y_n\rangle_{n\in\mathbb{N}})) \subseteq A_1$. Applying Lemma 3.4 again gives a sub-*IP* set $FS(\langle z_n\rangle_{n\in\mathbb{N}})) \subseteq A_2 \cap FS(\langle y_n\rangle_{n\in\mathbb{N}})$ and so $FS(\langle z_n\rangle_{n\in\mathbb{N}})) \subseteq A_2$. Hence, $FS(\langle z_n\rangle_{n\in\mathbb{N}})) \subseteq A_1 \cap A_2$. Iterating this argument produces

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a sub-*IP* set $FS(\langle a_n\rangle_{n\in\mathbb{N}})$ of $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ such that $FS(\langle a_n\rangle_{n\in\mathbb{N}}) \subseteq A_1 \cap A_2 \cap \cdots \cap A_n$. As $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ is any *IP* set, arbitrarily chosen, $A_1 \cap A_2 \cap \cdots \cap A_n$ is an $IP^\star$ set. This completes the proof. □

The following theorem is an abstract formulation which strengthens Theorem 1.7.

THEOREM 3.6. *If A is any $IP^\star$ set and B is an IP set, then*

$$\mathbb{Q}_{>0} = \frac{A}{B} = \Big\{\frac{a}{b} : a \in A, b \in B\Big\}.$$

PROOF. Let $m/n \in \mathbb{Q}_{>0}$. Now $m \cdot B$ is an *IP* set and, from Lemma 3.3, $n \cdot A$ is an $IP^\star$ set. Hence, $n \cdot A \cap m \cdot B \neq \emptyset$. Let $x = na = mb$, where $a \in A, b \in B$. Then $m/n = a/b \in A/B$. This completes the proof. □

PROOF OF THEOREM 1.7. As $\mathcal{K}$ is an $IP^\star_{721}$ set, it is an $IP^\star$ set. Again, by Lemma 3.4, $\mathcal{K}$ contains an *IP* set. So the desired result follows from Theorem 3.6. □

The proof of Theorem 1.7 gives the following more powerful result.

COROLLARY 3.7. *For any IP set $\mathcal{D} \subset \mathcal{K}$, we have $\mathbb{Q}_{>0} = \mathcal{K}/\mathcal{D}$ and hence also $\mathbb{Q}_{>0} = \mathcal{D}/\mathcal{K}$.*

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Appendix. The density of Kronecker numbers

In this section, we provide a simple proof for

$$\alpha \geq \frac{1}{360} \prod_{p \leq 720} \Big(1 - \frac{1}{p}\Big) \tag{A.1}$$

by providing a lower bound for the density of a general Δ_r^*-set. With $S = \{a, 2a, \ldots, ra\}$ in the definition of Δ_r^*-set, the following fact is obvious.

LEMMA A.1. *Let H be a Δ_r^*-set and $A_r(a) = \{a, 2a, \ldots, (r-1)a\}$. Then, $H \cap A_r(a) \neq \emptyset$ for any integer $a > 0$,*

THEOREM A.2. *If H is a Δ_r^*-set, then*

$$\frac{|H \cap [1, N]|}{N} \geq \prod_{p \leq r-1} \Big(1 - \frac{1}{p}\Big) + o(1).$$

PROOF. By Lemma A.1, every $A_r(a)$ contains at least one element of H. We obtain a lower bound for the cardinality of H by counting the number of the sets $A_r(a)$ which are mutually disjoint. If $a < b$ are two integers with $A_r(a) \cap A_r(b) \neq \emptyset$, then there are integers i, j with $(i, j) = 1$ and $1 \leq i < j \leq r-1$ such that $a/b = i/j$. The sets $A_r(a)$

where a is not divisible by any prime less than $r-1$ are therefore mutually disjoint. For sufficiently large N,

$$\left|\left\{a : \left(a, \prod_{p\leq r-1} p\right) = 1, A_r(a) \subset [1, N]\right\}\right| = \frac{N}{r} \prod_{p\leq r-1} \left(1 - \frac{1}{p}\right) + O(1).$$

The theorem follows immediately. □

Finally, (A.1) follows immediately from Theorems 1.4 and A.2 by taking $r = 721$.

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