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Restriction Operators Acting on Radial Functions on Vector Spaces over Finite Fields

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Abstract. We study $L^p \to L^r$ restriction estimates for algebraic varieties V in the case when restriction operators act on radial functions in the finite field setting. We show that if the varieties V lie in odd dimensional vector spaces over finite fields, then the conjectured restriction estimates are possible for all radial test functions. In addition, assuming that the varieties V are defined in even dimensional spaces and have few intersection points with the sphere of zero radius, we also obtain the conjectured exponents for all radial test functions.

1 Introduction

Let V be a subset of $\mathbb{R}^d$, $d \geq 2$, and $d\sigma$ a positive measure supported on V. One may ask for which values of p and r does the following inequality hold:

$$\|\widehat{f}\|_{L^r(V,d\sigma)} \leq C_{p,r,d}\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d).$$

This problem is known as the restriction problem in Euclidean space and it was first posed by E. M. Stein in 1967. The restriction problem for the circle and the parabola in the plane was completely solved by Zygmund [21]. The problem for cones in three and four dimensions was also established by Barcelo [1] and Wolff [19], respectively. However, this problem is still open in other higher dimensions and it has been considered as one of the most important, challenging problems in harmonic analysis. We refer the reader to [2, 4, 16–18, 20] for further discussion and recent progress on the Euclidean restriction problem.

As an analog of the Euclidean restriction problem, Tao and Mockenhaupt [15] recently reformulated and studied the restriction problem for various algebraic varieties in the finite field setting. In the introduction we review the definitions, conjectures, and known results on the restriction problem for algebraic varieties in d-dimensional vector spaces over finite fields. Let $\mathbb{F}_q^d$ be a d-dimensional vector space

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over the finite field $\mathbb{F}_q$ with q elements. We endow this space with a counting measure dm. Thus, if $f\colon \mathbb{F}_q^d \to \mathbb{C}$, then its integral over $\mathbb{F}_q^d$ is given by

$$\int_{\mathbb{F}_q^d} f(m)\,dm = \sum_{m\in\mathbb{F}_q^d} f(m).$$

We denote by $\mathbb{F}_{q*}^d$ the dual space of $\mathbb{F}_q^d$. We endow the dual space $\mathbb{F}_{q*}^d$ with a normalized counting measure dx. Hence, given a function $g\colon \mathbb{F}_{q*}^d \to \mathbb{C}$, we define its integral

$$\int_{\mathbb{F}_{q*}^d} g(x)\,dx = \frac{1}{q^d}\sum_{x\in\mathbb{F}_{q*}^d} g(x).$$

Recall that the space $\mathbb{F}_q^d$ is isomorphic to its dual space $\mathbb{F}_{q*}^d$ as an abstract group. Also recall that if f is a complex-valued function on $\mathbb{F}_q^d$, its Fourier transform, denoted by $\widehat{f}$, is actually defined on its dual space $\mathbb{F}_{q^*}^d$:

$$\widehat{f}(x) = \int_{\mathbb{F}_q^d} \chi(-m\cdot x) f(m)\,dm = \sum_{m\in\mathbb{F}_q^d} \chi(-m\cdot x) f(m),$$

where χ denotes a nontrivial additive character of $\mathbb{F}_q$. Let V be an algebraic variety in the dual space $\mathbb{F}_{q*}^d$. Throughout the paper we always assume that $|V| \sim q^{d-1}$. Namely, we view the variety V as a hypersurface in $\mathbb{F}_{q*}^d$. Recall that a normalized surface measure on V, denoted by $d\sigma$, is defined by the relation

$$\int g(x)\,d\sigma(x) = \frac{1}{|V|}\sum_{x\in V} g(x),$$

where $g\colon \mathbb{F}_{q*}^d \to \mathbb{C}$.

With notation above, the restriction problem for the variety V is to determine $1 \le p, r \le \infty$ such that the following restriction estimate holds:

$$\|\widehat{f}\|_{L^r(V,d\sigma)} \le C\|f\|_{L^p(\mathbb{F}_q^d,dm)} \quad \text{for all functions } f\colon \mathbb{F}_q^d \to \mathbb{C}, \tag{1.1}$$

where the constant $C > 0$ is independent of functions f and the size of the underlying finite field $\mathbb{F}_q$. We shall use the notation $R(p \to r) \lesssim 1$ to indicate that the restriction estimate (1.1) holds. By duality, inequality (1.1) is same as the following extension estimate:

$$\|(gd\sigma)^\vee\|_{L^{p'}(\mathbb{F}_q^d,dm)} \le C\|g\|_{L^{r'}(V,d\sigma)}.$$

Mockenhaupt and Tao [15] observed[1] that necessary conditions for inequality (1.1) take

$$1 \le p, r \le \infty, \quad \frac{1}{p} \ge \frac{d+1}{2d} \quad \text{and} \quad \frac{d}{p} + \frac{d-1}{r} \ge d. \tag{1.2}$$

Namely, $R(p \to r) \lesssim 1$ only if $(1/p, 1/r)$ lies in the convex hull of points

$$(1,0), \quad (1,1), \quad \left(\frac{d+1}{2d}, 1\right), \quad \left(\frac{d+1}{2d}, \frac{1}{2}\right). \tag{1.3}$$

[1] In [15] Mockenhaupt and Tao actually stated inequalities in (1.2) in terms of dual exponents.

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They also showed that these necessary conditions can be improved in the case when an affine subspace is contained in the variety $V \subset \mathbb{F}_q^d$ with $|V| \sim q^{d-1}$. However, it has been conjectured that if no affine subspaces lie in V, then necessary conditions in (1.2) are also sufficient conditions for $R(p \to r) \lesssim 1$. Indeed, Mockenhaupt and Tao [15] proved that necessary conditions (1.3) are in fact sufficient conditions for $A(p \to r) \lesssim 1$ if V is the parabola in $\mathbb{F}_{q*}^2$. In [11], Koh and Shen generalized the result to general algebraic curves in two dimensions. However, in higher dimensions the restriction problem has not been solved, and the known results are even weaker than those in Euclidean space. The currently best known results on restriction problems for paraboloids in $\mathbb{F}_{q*}^d$ are due to A. Lewko and M. Lewko [13]. They established certain endpoint restriction estimates for paraboloids that slightly improve on the previously known results by Mockenhaupt and Tao [15] in three dimensions and those by Iosevich and Koh [7] in higher dimensions. More precisely, the following restriction results were essentially proved by them.

Proposition 1.1 *Let $V = \{x \in \mathbb{F}_{q^*}^d : x_1^2 + \cdots + x_{d-1}^2 = x_d\}$ be a paraboloid. If $d \geq 4$ is even or if $d = 4k + 3$ for some $k \in \mathbb{N}$ and $-1 \in \mathbb{F}_{q*}$ is not a square, then $A(p \to r) \lesssim 1$ whenever $(1/p, 1/r)$ is contained in the convex hull of the points*

$$(1,0), \quad (1,1), \quad \Big(\frac{d^2+2d-2}{2d^2}, 1\Big), \quad \Big(\frac{d^2+2d-2}{2d^2}, \frac{1}{2}\Big), \quad \Big(\frac{3}{4}, \frac{d+2}{4d}\Big).$$

In the case that $d = 3$ and $-1 \in \mathbb{F}_{q*}$ is not a square, Bennett, Carbery, Carrigos, and Wright first proved[2] that $A(p \to r) \lesssim 1$ whenever $(1/p, 1/r)$ lies in the convex hull of points

$$(1,0), (1,1), (13/18, 1), (13/18, 1/2), \quad \text{and} \quad (3/4, 3/8).$$

This result has been recently improved by M. Lewko [12], but the improvement is not enough to establish the conjectured exponents given in (1.3).

Our main results below imply that the restriction conjecture (1.2) holds for paraboloids in $\mathbb{F}_{q*}^3$ with $-1 \in \mathbb{F}_{q^*}$ not square if the restriction operator acts on radial functions. The main purpose of this paper is to address general properties of varieties for which the restriction conjecture holds for all radial test functions (see Theorem 2.3 below).

Remark 1.2 In the Euclidean case, it is well known that the spherical restriction conjecture is valid for the class of radial functions (see [5, p. 2]). More generally, De Carli and Grafakos [5] proved it for the class of functions spanned by products of radial functions and spherical harmonics.

2 Statement of Main Results

While we do not know how to improve Theorem 1.1, we are able to show that if the test functions are radial functions, then the $L^p \to L^r$ restriction estimates for paraboloids hold for the exponents given in (1.3). In fact, we shall prove more general results. In order to clearly state our main theorems, let us introduce certain

[2]This result was not published, but it can be found in Carbery's lecture notes [3].

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definitions and notation. For each $m = (m_1, \ldots, m_d) \in \mathbb{F}_q^d$, define

$$\|m\| = m_1^2 + \cdots + m_d^2.$$

We say that a function $f \colon \mathbb{F}_q^d \to \mathbb{C}$ is a radial function if

$$f(m) = f(n) \quad \text{whenever } \|m\| = \|n\|.$$

For each $j \in \mathbb{F}_q$, we define

$$S_j^{d-1} = \{m = (m_1, \ldots, m_d) \in \mathbb{F}_q^d : m_1^2 + \cdots + m_d^2 = j\} \tag{2.1}$$

which will be named as the sphere with j radius.

Definition 2.1 We write $R_{\mathrm{rad}}(p \to r) \lesssim 1$ if the restriction estimate (1.1) holds for all radial functions $f : \mathbb{F}_q^d \to \mathbb{C}$.

2.1 Restriction Results on Radial Functions

Our first result below shows that the restriction operators acting on radial functions have quite good mapping properties.

Theorem 2.2 *Let $d\sigma$ be the normalized surface measure on an algebraic variety $V \subset \mathbb{F}_{q*}^d$ with $|V| \sim q^{d-1}$. Then we have*

$$R_{\mathrm{rad}}\Big(\frac{2d}{d+1} \to 2\Big) \lesssim 1 \quad \text{for } d \geq 3 \text{ odd} \tag{2.2}$$

and

$$R_{\mathrm{rad}}\Big(\frac{2d-2}{d} \to \frac{2(d-1)^2}{d^2-2d}\Big) \lesssim 1 \quad \text{for } d \geq 4 \text{ even.} \tag{2.3}$$

Using the nesting properties of L^p-norms and interpolating (2.2) with the trivial $L^1 \to L^\infty$, we see that necessary conditions (1.2) for $A(p \to r) \lesssim 1$ are sufficient conditions for $R_{\mathrm{rad}}(p \to r) \lesssim 1$ if the variety V with $|V| \sim q^{d-1}$ lies in odd dimensional vector spaces over finite fields. Notice that the result of Theorem 2.2 in even dimensions is weaker than that in odd dimensions. However, the following theorem shows that if the variety V does not contain a lot of elements in the sphere with zero radius, then the result in even dimensions can be improved to that in odd dimensions.

Theorem 2.3 *Let $d\sigma$ be the normalized surface measure on an algebraic variety $V \subset \mathbb{F}_{q*}^d$ with $|V| \sim q^{d-1}$. Suppose that $|V \cap S_0^{d-1}| \lesssim q^{\frac{d^2-d-1}{d}}$. Then*

$$R_{\mathrm{rad}}\Big(\frac{2d}{d+1} \to 2\Big) \lesssim 1 \quad \text{for } d \geq 3.$$

It seems that if the algebraic variety V does not contain S_0^{d-1}, the sphere of zero radius, then the conclusion of Theorem 2.3 holds. For example, if

$$V = \{x \in \mathbb{F}_{q*}^d : x_1^2 + \cdots + x_{d-1}^2 = x_d\}$$

is the paraboloid or

$$V = \{x \in \mathbb{F}_{q*}^d : x_1 + \cdots + x_d = 0\}$$

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is the plane, then $|V \cap S_0^{d-1}| \lesssim q^{d-2} < q^{(d^2-d-1)/d}$. In this case, we therefore obtain the conclusion of Theorem 2.3. This fact is very interesting in that the Fourier transform of radial functions can be meaningfully restricted to the plane.

Remark 2.4 When studying the finite field analogue of Euclidean problems, we often find dichotomic results between even dimensions and odd dimensions. Theorem 2.2 is one of examples showing such an unusual phenomena. Authors in [6] also addressed it on the Erdös–Falconer distance conjecture in finite fields. One of the main reasons for this can be explained in terms of the maximal dimension of affine subspaces lying in the algebraic variety $V \subset \mathbb{F}_q^d$. For example, let us assume that $-1 \subset \mathbb{F}_q$ is a square number. It is well known in [9] that if the dimension d is *odd*, then the sphere $S_j^{d-1} \subset \mathbb{F}_q^d$ contains the maximal affine subspace H with $|H| = q^{(d-1)/2}$. However, this never happens if the dimension d is *even*, because $(d-1)/2$ is not an integer for d even. In fact, if d is even, then we can show that $|H| = q^{(d-2)/2}$ for $j \neq 0$, and $|H| = q^{d/2}$ for $j = 0$, where H denotes the maximal affine subspace lying in $S_j^{d-1} \subset \mathbb{F}_q^d$.

3 Fourier Decay Estimates on Spheres

Since the Fourier transform of a radial function can be written as a linear combination of the Fourier transforms on spheres, the Fourier decay estimates on spheres will play a crucial role in proving our results. In this section, we go over the decay properties of the Fourier transform on spheres $S_j^{d-1} \subset \mathbb{F}_q^d$. We begin with reviewing the classical exponential sum estimates. For each $a \in \mathbb{F}_q^*$, the Gauss sum is defined by

$$G_a := \sum_{s \in \mathbb{F}_q^*} \eta(s)\chi(as),$$

where η denotes the quadratic character of $\mathbb{F}_q^*$. In particular, we will write G for the Gauss sum G_1. The Kloosterman sum is given by

$$K(a,b) := \sum_{s \in \mathbb{F}_q^*} \chi(as + bs^{-1}) \quad \text{for } a, b \in \mathbb{F}_q.$$

In addition, recall that the Salié sum is the exponential sum given by

$$S(a,b) := \sum_{s \in \mathbb{F}_q^*} \eta(s)\chi(as + bs^{-1}) \quad \text{for } a, b \in \mathbb{F}_q.$$

It is well known that $|G_a| = \sqrt{q}$ for $a \in \mathbb{F}_q^*$, $|K(a,b)| \leq 2\sqrt{q}$ for $a \neq 0$ or $b \neq 0$, and $|S(a,b)| \leq 2\sqrt{q}$ for $a, b \in \mathbb{F}_q$ (see [14, p. 193] and [10, pp. 322–323]). In terms of the aforementioned exponential sums, the Fourier transform on the spheres in $\mathbb{F}_q^d$ can be explicitly expressed. Modifying a normalizing factor, one can deduce the following result from Lemma 4 in [8].

Lemma 3.1 *Let S_j^{d-1} be the sphere in $\mathbb{F}_q^d$, defined as in (2.1). Then for any $x \in \mathbb{F}_{q*}^d$, we have*

$$\widehat{S_j^{d-1}}(x) = \begin{cases} q^{d-1}\delta_0(x) + q^{-1}G^d K(-j, -4^{-1}\|x\|) & \text{for } d \geq 2 \text{ even}, \\ q^{d-1}\delta_0(x) + q^{-1}G^d S(-j, -4^{-1}\|x\|) & \text{for } d \geq 3 \text{ odd}, \end{cases}$$

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where $\delta_0(x) = 1$ *if* $x = (0, \dots, 0)$ *and* $\delta_0(x) = 0$ *otherwise.*

Proof By the definition of the Fourier transform of a function on $\mathbb{F}_q^d$, we see that if $S_j^{d-1} \subset \mathbb{F}_q^d$ and $x \in \mathbb{F}_{q^*}^d$, then

$$\widehat{S_j^{d-1}}(x) = \sum_{m \in S_j^{d-1}} \chi(-x \cdot m) = \sum_{m \in \mathbb{F}_q^d} \chi(-x \cdot m) \delta_0(\|m\| - j).$$

Applying the orthogonality relation of χ, we can write

$$\delta_0(\|m\| - j) = q^{-1} \sum_{s \in \mathbb{F}_q} \chi\big(s(\|m\| - j)\big) \quad \text{for } x \in \mathbb{F}_{q^*}^d.$$

It therefore follows that

$$\widehat{S_j^{d-1}}(x) = q^{-1} \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) + q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(-js) \Big(\sum_{m \in \mathbb{F}_q^d} \chi(s\|m\| - x \cdot m) \Big) \tag{3.1}$$

$$= q^{d-1} \delta_0(x) + q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(-js) \prod_{k=1}^{d} \sum_{m_k \in \mathbb{F}_q} \chi(s m_k^2 - x_k m_k).$$

Completing the square and using a change of variables, it follows that

$$\sum_{m_k \in \mathbb{F}_q} \chi(s m_k^2 - x_k m_k) = \chi\big(-x_k^2/(4s)\big) \sum_{m_k \in \mathbb{F}_q} \chi(s m_k^2) \quad \text{for } k = 1, 2, \dots, d. \tag{3.2}$$

Let $A = \{t \in \mathbb{F}_q^* : t \text{ is a square number}\}$ and observe that, for each $s \in \mathbb{F}_q^*$,

$$\sum_{t \in \mathbb{F}_q} \chi(st^2) = 1 + \sum_{t \in \mathbb{F}_q^*} \chi(st^2) = 1 + \sum_{t \in A} 2\chi(st)$$

$$= 1 + \sum_{t \in \mathbb{F}_q^*} \big(1 + \eta(t)\big) \chi(st) = \sum_{t \in \mathbb{F}_q^*} \eta(t) \chi(st) = \eta(s) G_1.$$

Applying this equality to (3.2), it follows from (3.1) that

$$\widehat{S_j^{d-1}}(x) = q^{d-1} \delta_0(x) + q^{-1} G^d \sum_{s \in \mathbb{F}_q^*} \eta^d(s) \chi\Big(-js + \frac{\|x\|}{-4s}\Big).$$

Since $\eta^d = 1$ for $d \geq 2$ even, and $\eta^d = \eta$ for $d \geq 3$ odd, the statement of Lemma 3.1 follows immediately from the definitions of the Kloosterman sum and the Salié sum. ■

The following corollary can be obtained by applying the estimates of the Gauss sum G, the Kloosterman $K(a, b)$, and the Salié sum $S(a, b)$ to Lemma 3.1.

Corollary 3.2 *Let $d \geq 3$ be an integer. Then,*

$$\widehat{S_j^{d-1}}(0, \dots, 0) = |S_j^{d-1}| \sim q^{d-1} \quad \textit{for } j \in \mathbb{F}_q. \tag{3.3}$$

If $d \geq 2$ and $x \in \mathbb{F}_{q*}^d \setminus \{(0, \dots, 0)\}$ then

$$|\widehat{S_j^{d-1}}(x)| \lesssim \begin{cases} q^{\frac{d-1}{2}} & \textit{for } d \textit{ odd}, \ j \in \mathbb{F}_q \\ q^{\frac{d-1}{2}} & \textit{for } d \textit{ even}, \ j \neq 0 \\ q^{\frac{d}{2}} & \textit{for } d \textit{ even}, \ j = 0. \end{cases} \tag{3.4}$$

In particular, if $\|x\| \neq 0$ and $d \geq 4$ is even, then

$$|\widehat{S_0^{d-1}}(x)| = q^{\frac{d-2}{2}}. \tag{3.5}$$

Proof First, let us prove (3.3). It is clear from the definition of the Fourier transform that

$$\widehat{S_j^{d-1}}(0,\ldots,0) = \sum_{m\in\mathbb{F}_q^d} \chi\big(m\cdot(0,\ldots,0)\big)\, S_j^{d-1}(m) = |S_j^{d-1}|.$$

On the other hand, it follows from Lemma 3.1 that

$$\widehat{S_j^{d-1}}(0,\ldots,0) = \begin{cases} q^{d-1} + q^{-1}G^dK(-j,0) & \text{for } d \geq 2 \text{ even} \\ q^{d-1} + q^{-1}G^dS(-j,0) & \text{for } d \geq 3 \text{ odd.} \end{cases}$$

Since $|G| = \sqrt{q}$, $|K(-j,0)| \leq q$ for $j \in \mathbb{F}_q$, and $|S(-j,0)| \leq 2\sqrt{q}$ for $j \in \mathbb{F}_q$, we see that if $d \geq 3$, then $q^{d-1}+q^{-1}G^dK(-j,0) \sim q^{d-1}+q^{-1}G^dS(-j,0) \sim q^{d-1}$. Thus, (3.3) holds. Next, using Lemma 3.1, the conclusion (3.4) is an immediate consequence from facts that $|G| = \sqrt{q}$, $|K(a,b)| \leq 2\sqrt{q}$ if $a \neq 0$ or $b \neq 0$, $|K(a,b)| = q-1$ if $a = b = 0$, and $|S(a,b)| \leq 2\sqrt{q}$ if $a, b \in \mathbb{F}_q$. Finally, equality (3.5) follows from Lemma 3.1 and the observations that $|G| = \sqrt{q}$, $|K(0,b)| = 1$ for $b \neq 0$. ∎

4 Proofs of Theorem 2.2 and Theorem 2.3

In this section we shall complete the proofs of main results on restriction operators acting on radial functions. First, we shall derive sufficient conditions for $R_{\text{rad}}(p \to r) \lesssim 1$. We aim to find certain conditions on $1 \leq p, r \leq \infty$ such that

$$\|\widehat{f}\|_{L^r(V,d\sigma)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d,dm)} \quad \text{for all radial functions } f\colon \mathbb{F}_q^d \to \mathbb{C}.$$

Without loss of generality, we may assume that f is a nonnegative, radial function on $\mathbb{F}_q^d$. Therefore, we can write

$$f(m) = M_j \geq 0 \quad \text{if } m \in S_j^{d-1} \text{ for some } j \in \mathbb{F}_q.$$

By multiplying a normalizing constant, we may also assume that

$$\|f\|_{L^p(\mathbb{F}_q^d,dm)} = 1.$$

It therefore follows that

$$\begin{aligned} 1 = \|f\|^p_{L^p(\mathbb{F}_q^d,dm)} &= \sum_{m\in\mathbb{F}_q^d} |f(m)|^p \\ &= \sum_{j\in\mathbb{F}_q} \sum_{m\in S_j^{d-1}} M_j^p = \sum_{j\in\mathbb{F}_q} M_j^p |S_j^{d-1}|. \end{aligned}$$

Since $|S_j^{d-1}| \sim q^{d-1}$ for $j \in \mathbb{F}_q$ and $d \geq 3$, we have

$$\sum_{j\in\mathbb{F}_q} M_j^p \sim q^{1-d}. \tag{4.1}$$

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Under the above assumptions on the radial function f, it suffices to find certain conditions on $1 \le p, r \le \infty$ such that

$$\|\widehat{f}\|^r_{L^r(V,d\sigma)} = \frac{1}{|V|}\sum_{x\in V} |\widehat{f}(x)|^r \lesssim 1. \tag{4.2}$$

Since $\widehat{f}(x) = \sum_{m\in\mathbb{F}_q^d} \chi(-m\cdot x) f(m) = \sum_{j\in\mathbb{F}_q} \sum_{m\in S_j^{d-1}} \chi(-m\cdot x) M_j$, it follows that

(4.3)
$$\begin{aligned}\|\widehat{f}\|^r_{L^r(V,d\sigma)} &= \frac{1}{|V|}\sum_{x\in V}\Big|\sum_{j\in\mathbb{F}_q} M_j \widehat{S_j^{d-1}}(x)\Big|^r\\ &= \frac{1}{|V|}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\in\mathbb{F}_q} M_j \widehat{S_j^{d-1}}(x)\Big|^r + \frac{1}{|V|}\Big|\sum_{j\in\mathbb{F}_q} M_j \widehat{S_j^{d-1}}(0,\dots,0)\Big|^r.\end{aligned}$$

Since $M_j \ge 0$, $|V| \sim q^{d-1}$, and $\widehat{S_j^{d-1}}(0,\dots,0) = |S_j^{d-1}| \sim q^{d-1}$ for $d \ge 3$, we see that

$$\|\widehat{f}\|^r_{L^r(V,d\sigma)} \sim \frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\in\mathbb{F}_q} M_j \widehat{S_j^{d-1}}(x)\Big|^r + \frac{q^{r(d-1)}}{q^{d-1}}\Big(\sum_{j\in\mathbb{F}_q} M_j\Big)^r.$$

From Hölder's inequality and (4.1), observe that

$$\Big(\sum_{j\in\mathbb{F}_q} M_j\Big)^r \le \Big(\sum_{j\in\mathbb{F}_q} 1^{p'}\Big)^{\frac{r}{p'}}\Big(\sum_{j\in\mathbb{F}_q} M_j^p\Big)^{\frac{r}{p}} \sim q^{r(1-\frac{d}{p})}, \tag{4.4}$$

where p' denotes the Hölder conjugate of p, that is $p' = p/(p-1)$. It follows that

$$\|\widehat{f}\|^r_{L^r(V,d\sigma)} \lesssim \frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\in\mathbb{F}_q} M_j \widehat{S_j^{d-1}}(x)\Big|^r + q^{rd(1-\frac{1}{p})-d+1}.$$

Combining this with (4.2), the sufficient conditions on $1 \le p, r \le \infty$ for $R_{\mathrm{rad}}(p \to r) \lesssim 1$ are given by

$$\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\in\mathbb{F}_q} M_j \widehat{S_j^{d-1}}(x)\Big|^r \lesssim 1 \tag{4.5}$$

and

$$rd\left(1-\frac{1}{p}\right) - d + 1 \le 0. \tag{4.6}$$

Remark 4.1 Observe that if $(0,\dots,0) \notin V$, then the second term in (4.3) must disappear. In this case, we therefore see that the sufficient condition for $R_{\mathrm{rad}}(p \to r) \lesssim 1$ only takes the condition (4.5).

4.1 Proof of the First Part of the Conclusions in Theorem 2.2

We prove (2.2) of Theorem 2.2. Namely, we shall prove that if $d \ge 3$ is odd, then

$$\|\widehat{f}\|_{L^2(V,d\sigma)} \lesssim \|f\|_{L^{\frac{2d}{d+1}}(\mathbb{F}_q^d,dm)} \quad \text{for all radial functions } f\colon \mathbb{F}_q^d \to \mathbb{C}.$$

With $p = 2d/(d+1)$ and $r = 2$, it is enough to show that (4.5) and (4.6) hold. Now, (4.6) follows immediately from a direct observation that if $p = 2d/(d+1)$ and $r = 2$,

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then $rd(1-1/p)-d+1=0$. To prove (4.5), we recall from (3.4) in Corollary 3.2 that if $d\geq 3$ is odd, then

$$|\widehat{S_j^{d-1}}(x)|\lesssim q^{\frac{d-1}{2}}\quad \text{for } j\in\mathbb{F}_q, x\neq(0,\dots,0).$$

From this fact and (4.4), it is not hard to obtain (4.5) for $p=2d/(d+1)$ and $r=2$. Indeed, we have

$$(4.7)\quad \frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(x)\Big|^2\lesssim\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big(\sum_{j\in\mathbb{F}_q}M_j\Big)^2$$
$$\lesssim|V|\Big(\sum_{j\in\mathbb{F}_q}M_j\Big)^2\lesssim q^{d-1}q^{2(1-\frac{d+1}{2})}=1,$$

where we also used that $M_j\geq 0$ and $|V|\sim q^{d-1}$.

4.2 Proof of the Second Part of the Conclusions in Theorem 2.2

We prove (2.3) of Theorem 2.2. When $d\geq 4$ is even, we must prove $R_{\mathrm{rad}}(p\to r)\lesssim 1$ for $p=2d-2/d$ and $r=2(d-1)^2/(d^2-2d)$. As mentioned before, it suffices to prove both (4.5) and (4.6) for $p=(2d-2)/d$ and $r=2(d-1)^2/(d^2-2d)$. In this case, (4.6) is clearly true because $rd(1-1/p)-d+1=0$. To prove (4.5), we recall from (3.4) in Corollary 3.2 that if $d\geq 4$ is even and $x\neq(0,\dots,0)$, then

$$|\widehat{S_j^{d-1}}(x)|\lesssim\begin{cases}q^{\frac{d-1}{2}} & \text{for } j\neq 0\\ q^{\frac{d}{2}} & \text{for } j=0.\end{cases}$$

From this fact, the left part of (4.5) can be estimated as follows:

$$\begin{aligned}&\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(x)\Big|^r\\
&\lesssim\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|M_0\widehat{S_0^{d-1}}(x)\Big|^r+\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\neq 0}M_j\widehat{S_j^{d-1}}(x)\Big|^r\\
&\lesssim\frac{1}{q^{d-1}}q^{\frac{rd}{2}}M_0^r\Big(\sum_{x\in V\setminus\{(0,\dots,0)\}}1\Big)+\frac{1}{q^{d-1}}q^{\frac{r(d-1)}{2}}\Big(\sum_{j\neq 0}M_j\Big)^r\Big(\sum_{x\in V\setminus\{(0,\dots,0)\}}1\Big)\\
&\lesssim q^{\frac{rd}{2}}M_0^r+q^{\frac{r(d-1)}{2}}\Big(\sum_{j\in\mathbb{F}_q}M_j\Big)^r\lesssim q^{\frac{rd}{2}}M_0^r+q^{\frac{r(d-1)}{2}}q^{r(1-\frac{d}{p})},\end{aligned}$$

where the last inequality follows from (4.4). Since $M_j\geq 0$ for $j\in\mathbb{F}_q$, it is clear from (4.1) that $M_0\lesssim q^{\frac{1-d}{p}}$. Thus, we have

$$(4.8)\qquad M_0^r\lesssim q^{\frac{r(1-d)}{p}}.$$

We therefore see that if $p=(2d-2)/d$ and $r=2(d-1)^2/(d^2-2d)$, then

$$\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\Big|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(x)\Big|^r\lesssim q^{\frac{rd}{2}+\frac{r(1-d)}{p}}+q^{\frac{r(d-1)}{2}+r(1-\frac{d}{p})}$$
$$=q^0+q^{-\frac{d-1}{d(d-2)}}\lesssim 1,$$

which proves (4.5) and we complete the proof.

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4.3 Proof of Theorem 2.3

Let $d\sigma$ be the normalized surface measure on an algebraic variety $V \subset \mathbb{F}_{q*}^d$ with $|V| \sim q^{d-1}$. Assuming that $|V \cap S_0^{d-1}| \lesssim q^{(d^2-d-1)/d}$, we aim to prove that

$$R_{\mathrm{rad}}\Big(\frac{2d}{d+1} \to 2\Big) \lesssim 1 \quad \text{for } d \geq 3.$$

In the case when $d \geq 3$ is odd, this statement was already proved in the first part of Theorem 2.2 with much weaker assumptions. Thus, we may assume that $d \geq 4$ is even. Suppose that

$$|V \cap S_0^{d-1}| \lesssim q^{\frac{d^2-d-1}{d}}.$$

As before, our task is to prove that both (4.5) and (4.6) hold for $p = 2d/(d+1)$ and $r = 2$. As mentioned before, (4.6) clearly holds for $p = 2d/(d+1)$ and $r = 2$. To prove (4.5), we set

$$\mathrm{L} := \frac{1}{q^{d-1}} \sum_{x \in V \setminus \{(0,\dots,0)\}} \Big| \sum_{j \in \mathbb{F}_q} M_j \widehat{S_j^{d-1}}(x) \Big|^r$$

and show that $\mathrm{L} \lesssim 1$. It follows that

$$\begin{aligned}\mathrm{L} &\lesssim \frac{1}{q^{d-1}} \sum_{x \in V \setminus \{(0,\dots,0)\}} M_0^r |\widehat{S_0^{d-1}}(x)|^r + \frac{1}{q^{d-1}} \sum_{x \in V \setminus \{(0,\dots,0)\}} \Big| \sum_{j \neq 0} M_j \widehat{S_j^{d-1}}(x) \Big|^r \\ &:= \mathrm{R} + \mathrm{M}.\end{aligned}$$

It suffices to prove that for $p = \frac{2d}{d+1}$ and $r = 2$,

$$\mathrm{R} = \frac{1}{q^{d-1}} \sum_{x \in V \setminus \{(0,\dots,0)\}} M_0^r |\widehat{S_0^{d-1}}(x)|^r \lesssim 1 \tag{4.9}$$

and

$$\mathrm{M} = \frac{1}{q^{d-1}} \sum_{x \in V \setminus \{(0,\dots,0)\}} \Big| \sum_{j \neq 0} M_j \widehat{S_j^{d-1}}(x) \Big|^r \lesssim 1. \tag{4.10}$$

Notice that (4.10) follows immediately from the same argument as in (4.7). To prove (4.9), we write

$$\mathrm{R} = \frac{1}{q^{d-1}} \sum_{x \in V \setminus \{(0,\dots,0)\} : \|x\| = 0} M_0^r |\widehat{S_0^{d-1}}(x)|^r + \frac{1}{q^{d-1}} \sum_{x \in V \setminus \{(0,\dots,0)\} : \|x\| \neq 0} M_0^r |\widehat{S_0^{d-1}}(x)|^r.$$

Since $d \geq 4$ is even, the application of (3.4) and (3.5) in Corollary 3.2 yields that

$$\mathrm{R} \lesssim \frac{M_0^r}{q^{d-1}} q^{\frac{rd}{2}} |V \cap S_0^{d-1}| + \frac{M_0^r}{q^{d-1}} q^{\frac{r(d-2)}{2}} |V|.$$

By (4.8) and our assumption that $|V \cap S_0^{d-1}| \lesssim q^{\frac{d^2-d-1}{d}}$, we see that if $p = \frac{2d}{d+1}$ and $r = 2$, then

$$\mathrm{R} \lesssim q^{\frac{r(1-d)}{p} + \frac{rd}{2} - \frac{1}{d}} + q^{\frac{r(1-d)}{p} + \frac{r(d-2)}{2}} = q^0 + q^{\frac{1-2d}{d}} \lesssim 1.$$

The proof of Theorem 2.3 is complete.

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