SOME VARIETIES OF GROUPS

Dedicated to the memory of Hanna Neumann

M. F. NEWMAN

(Received 30 June 1972)

Communicated by G. E. Wall

1. Introduction

For positive integers n, c the class of groups all of whose n-generator subgroups are nilpotent of class (at most) c is a variety, here denoted $[n \to c]$. Hanna Neumann in her book ([15] pp. 93–98) reported on the first stage of the investigation of these varieties. The main result was that $[n \to c]$ is nilpotent if and only if $c \leqq n \geqq 2$ ([15] 34.33 and 34.54).

Heineken ([7] Remark) has observed that $[n \to 2n-2]$ is soluble. Recently Bachmuth and Mochizuki ([1] Theorem 2) have shown that $[n \to 2n-1]$ is always insoluble. This note is concerned with giving detailed information about the non-nilpotent soluble varieties of type $[n \to c]$, that is those with $n < c \leqq 2n-2$.

Heineken (ibid.) has also noted that $[n \to 2n-2]$ is contained in the product of a variety of finite (2-power ?) exponent and a nilpotent variety. Rather more is true.

The variety $[n \to 2n-2]$ *is the join of a nilpotent variety and a variety of locally finite 2-groups.*

To see this observe first that for distinct primes p, q the product variety $\mathfrak{A}_p\mathfrak{A}_q$ (all unexplained notation follows Hanna Neumann's book [15]) is not contained in $[n \to 2n-2]$ because $\mathfrak{A}_p\mathfrak{A}_q$ contains 2-generator non-nilpotent groups. Since $[n \to 2n-2]$ is soluble, it follows from a theorem of Groves ([4] Theorem A) that $[n \to 2n-2]$ is the join of a nilpotent variety and a variety $\mathfrak{P}$ of locally finite groups. A finite group in $[n \to 2n-2]$ is nilpotent ([8] Satz III.6.3). Hence $\mathfrak{P}$ is locally nilpotent and, therefore, the join of a finite number of soluble varieties of p-groups for certain primes p. A soluble variety of p-groups either contains $\mathfrak{A}_p\mathfrak{A}_p$ or is nilpotent ([10] Theorem 5). Now the wreath product of a cyclic group of order p by the direct product of $n-1$ cyclic groups of order p is an n-generator group in $\mathfrak{A}_p\mathfrak{A}_p$ and has class $(n-1)(p-1)+1$ ([12] Theorem 5.1.). So $[n \to 2n-2]$ can contain $\mathfrak{A}_p\mathfrak{A}_p$ only if p is 2 and the result is proved. Since $\mathfrak{A}_2\mathfrak{A}_2$

is a subvariety of $[n \to n+1]$ for all n([15] 34.53), the 2-groups are an essential part of the scene.

In another direction Gupta ([5] Remark 3) has noted that $[n \to 2n-2]$ is contained in the product of a nilpotent variety and the the variety of all abelian groups.

A more precise version of these results can be obtained. Before doing so it will be convenient to have another concept. Recall that a group is 2-torsion-free if it has no element of order 2. The *2-torsion-free core* of a variety $\mathfrak{B}$ is the subvariety of $\mathfrak{B}$ generated by all the 2-torsion-free groups in $\mathfrak{B}$. The 2-torsion-free core $\mathfrak{T}$ of $\mathfrak{B}$ has the property that, if $\mathfrak{B} = \mathfrak{U} \vee \mathfrak{P}$ where $\mathfrak{P}$ is a variety of 2-groups, then $\mathfrak{T}$ is a subvariety of $\mathfrak{U}$ (this is easy to see by considering the corresponding verbal subgroups in a free group of $\mathfrak{B}$ of infinite rank). A variety need not be the join of its 2-torsion-free core and a variety of 2-groups; for example $\mathfrak{A}_2\mathfrak{A}_3$ is not. A nilpotent variety always has such a join decomposition (see 2.5). It follows from the result proved earlier that $[n \to 2n-2]$ also has such a join decomposition. The following more precise result is proved in section 2.

THEOREM A. *For integers n,k such* $0 < k \leqq n-2$ *let* $\mathfrak{N}(n,k)$ *be the 2-torsion-free core of* $[n \to n+k]$. *There is a variety* $\mathfrak{X}$ *of nilpotent 2-groups (possibly depending on n and k) such that*

$$[n \to n+k] = \mathfrak{N}(n,k) \vee ([n \to n+k] \wedge \mathfrak{X}\mathfrak{A}_2).$$

Moreover $\mathfrak{N}(n,k)$ *is a subvariety of* $\mathfrak{N}_{n+2k}$.

In particular a 2-torsion-free group in $[n \to 2n-2]$ is nilpotent of class at most $3n-4$ (cf.[7] Theorem). The last assertion of the theorem is sharp in the sense that $\mathfrak{N}(n,k)$ is not a subvariety of $\mathfrak{N}_{n+2k-1}$ for there is a torsion-free group n $[n \to n+k]$ which has class precisely $n+2k$ (see 5.4).

Join-continuity of the lattice of varieties (equivalent to meet-continuity of the lattice of fully-invariant subgroups of a free group of infinite rank—cf. [2] p. 187) guarantees, via Zorn's Lemma, that there is a subvariety $\mathfrak{W}$ of $\mathfrak{X}\mathfrak{A}_2$ which is minimal with respect to $\mathfrak{N}(n,k) \vee \mathfrak{W}$ containing $[n \to n+k]$. Such a variety $\mathfrak{W}$ cannot be too small. It contains $\mathfrak{A}_2\mathfrak{A}_2$ because it is a soluble variety of 2-groups which is not nilpotent. Examples, given in section 5, point to further limitations. Specifically $\mathfrak{W}$ is not a subvariety of $\mathfrak{B}_{2^k}$, nor of $\mathfrak{B}_2\mathfrak{N}_i\mathfrak{A}_{2^{r-1}}$ for all positive integers i,r such that $2(i+r)-1 \leqq k$, nor, when $k \geqq 2$, of $\mathfrak{A}\mathfrak{N}_i$ for all i.

When k is 1 a more positive result can be obtained (proof in section 4).

THEOREM B. *For every integer n greater than or equal to 3*

$$[n \to n+1] = \mathfrak{A}_2\mathfrak{A}_2 \vee ([n \to n+1] \wedge \mathfrak{N}_{n+2}).$$

This has as a consequence that $[n+1 \to n+2]$ is the join of $[n \to n+1]$ and a nilpotent variety. An important step (2.17) in the proof of Theorem A shows that

$[n \to n+k]$ is a subvariety of $[n+1 \to n+k+1]$. These two facts encourage the following hope.

CONJECTURE. *For all integers n,k such that $0 < k \leqq n-2$, the variety $[n+1 \to n+k+1]$ is the join of $[n \to n+k]$ and a nilpotent variety.*

Finally there is a quite different join decomposition for $[n \to n+k]$ (which confirms a conjecture of N. D. Gupta—the case $k=1$ is essentially Theorem 6.1* (ii) of [17]). A proof is given in section 3.

THEOREM C. *For integers n,k such that $0 < k \leqq n-2$*

$$[n \to n+k] = \mathfrak{N}_{n+k} \vee ([n \to n+k] \wedge \mathfrak{N}_k\mathfrak{N}_2).$$

Some of the ideas and techniques reported in this paper had their origins in the work which lead to the original version of [6]. I am indebted to Professor N. D. Gupta for keeping me informed about his progress with this circle of ideas. I am also indebted to Dr L. G. Kovács for helpful and stimulating discussions.

2. Proof of Theorem A

Throughout this section n,k are integers such that $0 < k \leqq n-2$.

Commutator calculations later in this section will establish

2.1 $$[n \to n+k] \subseteq \mathfrak{N}_{n+2k}\mathfrak{A}_2$$

2.2 $$[n \to n+k] \subseteq \mathfrak{U}_k\mathfrak{N}_{n+2k}$$

where $\mathfrak{U}_k = \bigwedge_{i=0}^{k} \mathfrak{B}_{2^i}\mathfrak{N}_{k-i}$.

From these two results Lemma A of Groves' paper [4] yields

2.3 $$[n \to n+k] = \mathfrak{N} \vee \mathfrak{L}$$

where $\mathfrak{N}$ is a variety which is nilpotent of class $n+2k$ and $\mathfrak{L}$ is a locally finite variety.

Groves' Lemma A hinges on his Lemma 4. It is straightforward to check that the proof of the latter will adapt to prove:

2.4 *Let N be a nilpotent group whose Sylow 2-subgroup has finite exponent, then there is a positive integer s such that $\mathfrak{B}_{2^s}(N)$ contains no element of order 2.*

Modifying the proof of Groves' Lemma A then yields that $\mathfrak{L}$ in 2.3 may be taken to consist of 2-groups.

From 2.1 and 2.3

$$[n \to n+k] = (\mathfrak{N} \vee \mathfrak{L}) \wedge \mathfrak{N}_{n+2k}\mathfrak{A}_2.$$

Since $\mathfrak{N}$ is a subvariety of $\mathfrak{N}_{n+2k}$, the modular law gives

$$[n \to n+k] = \mathfrak{N} \vee (\mathfrak{L} \wedge \mathfrak{N}_{n+2k}\mathfrak{A}_2).$$

Hence

$$[n \to n+k] = \mathfrak{N} \vee \mathfrak{P}$$

with $\mathfrak{P}$ a subvariety of $(\mathfrak{L} \wedge \mathfrak{N}_{n+2k})\mathfrak{A}_2$.
Theorem A will then be proved once the next result is established.

2.5 *A nilpotent variety is the join of its 2-torsion-free core and a variety of 2-groups.*

PROOF. Let $\mathfrak{V}$ be a nilpotent variety and $\mathfrak{T}$ its 2-torsion-free core. Let F be a free group of $\mathfrak{V}$ of infinite rank. Let $T = \mathfrak{T}(F)$. Clearly T is the Sylow 2-subgroup of F. Let T_j be the subgroup T generated by all elements of order dividing 2^j, then $T_1 \leqq T_2 \leqq \cdots$ is an ascending chain of fully-invariant subgroups of F whose union is T. Since the finite basis theorem holds for nilpotent varieties ([15] 34.14), the chain breaks off and $T = T_j$ for some j. Hence T has finite exponent and the result follows from 2.4.

This argument is "known" but has not, as far as I know, appeared in print.

Some preparation is needed for the proofs of 2.1 and 2.2 The simpler commutator identities ([15] 33.34) are often used without explicit reference. If r is positive integer, then $[u,rv] = [u,(r-1)v,v]$.

Clearly $[x_1,\cdots,x_{n-k-1},2x_{n-k},\cdots,2x_n]$ is a law of $[n \to n+k]$. This makes it useful to have information about elements b of a group G such that $[b,2g] = e$ (the identity element) for all g in G. Such elements are called right Engel elements of length 2 in G; the set of them will be denoted $L(G)$. Kappe ([9] section 2) has proved

2.6 *$L(G)$ is a characteristic subgroup of G;*

for b in $L(G)$ and f,g,h in G

2.7 $$[b,g,b] = e;$$

2.8 $$[b,g,h] = [b,h,g]^{-1};$$

2.9 $$[b,[g,h]] = [b,g,h]^2.$$

Macdonald and Neumann ([13] proof of Theorem 3) deduced

2.10 $$[b,f,g,h]^2 = e.$$

It follows from 2.6 that $[b,f] \in L(G)$ and thus from 2.9 and 2.10 that

2.11 $$[b,f,[g,h]] = e.$$

Hence, using 2.8 and 2.9,

2.12 $$[b,[f,g,h]] = e.$$

These last two facts were drawn to my attention by N. D. Gupta; cf. Lemma 5.4.

https://doi.org/10.1017/S1446788700015469 Published online by Cambridge University Press

of [17].)* Clearly $[b,g^2] = [b,g]^2$ and $[b^2,g] = [b,g]^2$ by 2.7. Repeated use of these and 2.10 gives

2.13 $$[b,f^2,g,h] = e.$$

Since $[x_1, 2x_2, \cdots, 2x_n]$ is a law of $[n \to 2n-2]$. An easy induction using 2.11 gives that $[x_1, x_2, [x_3, x_4], x_5, \cdots, x_{3n-4}, [x_{3n-3}, x_{3n-2}]]$ is a law of $[n \to 2n-2]$. Replacing $x_5, \cdots, x_{3n-4}$ by commutators of weight 2 yields that $[n \to 2n-2]$ is nilpotent-(of class $2n-3$)-by-abelian (cf. [5] Remark 3).

A similar argument applying 2.13 to $[x_1, \cdots, x_{n-k-1}, 2x_{n-k}, \cdots, 2x_n]$ yields that

$$[x_1, \cdots, x_{n-k-1}, x_{n-k}^2, x_{n-k+1}, \cdots, x_{n+2k}^2, x_{n+2k+1}, x_{n+2k+2}]$$

is a law of $[n \to n+k]$ and hence that $[n \to n+k]$ is a subvariety of $\mathfrak{N}_{n+2k+1}\mathfrak{A}_2$. This combined with 2.2 would already yield a proof of Theorem A along the lines of that given at the beginning of this section. It will cost no more to prove 2.1 because the arguments involved are also used in the proof of 2.2.

One of the fundamental commutator identities ([15] 33.34(3)) can be written

2.14 $$[x_1, x_2, x_3] = [x_1, x_3, x_2][x_1, [x_2, x_3]]\, u$$

where u is a product of commutators in $\{x_1, x_2, x_3\}$ of weight at least 4.

Two variations of this will be needed. The first is obtained by replacing x_1 by $[x_1, \cdots, x_{i-1}]$, x_2 by x_i and x_3 by x_{i+1} in 2.14 and commuting the result with $x_{i+2}, \cdots, x_s$ in turn.

2.15 $$[x_1, \cdots, x_i, x_{i+1}, \cdots, x_s] = [x_1, \cdots, x_{i+1}, x_i, \cdots, x_s][x_1, \cdots, [x_i, x_{i+1}], \cdots, x_s]\; u$$

where $i \geqq 2$, $s \geqq 3$ and u is a product of commutators in $\{x_1, \cdots, x_s\}$ of weight at least $s+1$.

In what follows each u_j is a product of commutators in $\{x_1, x_2, x_3, x_4\}$ of weight at least 5. Taking 2.15 with $i = 3$, $s = 4$ and rewriting gives

$$[x_1, x_2, x_3, x_4][x_1, x_2, x_4, x_3]^{-1}[x_1, x_2, [x_3, x_4]]^{-1} = u_1.$$

Hence

$$[x_1, x_2, x_3, x_4][x_2, x_1, x_4, x_3][x_3, x_4, [x_1, x_2]] = u_2.$$

Using 2.15 again gives

2.16 $$[x_1, x_2, x_3, x_4][x_2, x_1, x_4, x_3][x_3, x_4, x_1, x_2][x_4, x_3, x_2, x_1] = u_3.$$

The next result provides information which makes it possible to dispose of commutators of 'higher' weight in some of the later calculations. The case $n \geqq 3k+2$ has been proved by Gupta, Levin and Rhemtulla ([17], Theorem 7.1).

2.17 $$[n \to n+k] \subseteq [n+1 \to n+k+1].$$

* (Added February 1973.) The above results are collected as Theorem 7.13 in [18].

PROOF. Let G be a group in $[n \to n+k]$ which can be generated by $n+1$ elements. Since $[n \to n+k]$ is soluble and therefore, by the argument used in the introduction, locally nilpotent, G is nilpotent. There is no harm in assuming G has class at most $n+k+2$. Let $\{a_1, \cdots, a_{n+1}\}$ be a set of generators of G. It suffices to show $[a_{\lambda(1)}, \cdots, a_{\lambda(n+k+2)}] = e$ for all $\lambda(1), \cdots, \lambda(n+k+2)$ in $\{1, \cdots, n+1\}$. This is clearly so if for some j in $\{1, \cdots, n+1\}$ none of $\lambda(1), \cdots, \lambda(n+k+2)$ is j. If each of $\{1, \cdots, n+1\}$ occurs among $\lambda(1), \cdots, \lambda(n+k+2)$, then at least two of $\{1, \cdots, n+1\}$ occur exactly once among $\lambda(1), \cdots, \lambda(n+k+2)$. Hence associated with each $(n+k+2)$- tuplet $(\lambda(1), \cdots, \lambda(n+k+2))$ which contains each of $\{1, \cdots, n+1\}$ there is a positive integer s such that $\lambda(s)$ is different from all the other entries but each of $\lambda(s+1), \cdots, \lambda(n+k+2)$ occurs at least twice. If $s = n+k+2$, then clearly $[a_{\lambda(1)}, \cdots, a_{\lambda(n+k+2)}] = e$. If $s < n+k+2$, assume inductively that $[a_{\mu(1)}, \cdots, a_{\mu(n+k+2)}] = e$ whenever $a_{\mu(s+1)}$ differs from all other entries. Let r be an element of $\{1, \cdots, s-1\}$ such that $a_{\lambda(r)}$ is different from all the other $a_{\lambda(j)}$. Because G is nilpotent of class $n+k+2$, using the inductive assumption and 2.15 gives

$$[\cdots, a_{\lambda(r)}, \cdots, a_{\lambda(s)}, a_{\lambda(s+1)}, \cdots] = [\cdots, a_{\lambda(r)}, \cdots, [a_{\lambda(s)}, a_{\lambda(s+1)}], \cdots].$$

Because G is in $[n \to n+k]$

$$[\cdots, a_{\lambda(r)}[a_{\lambda(s)}, a_{\lambda(s+1)}], \cdots, a_{\lambda(r)}[a_{\lambda(s)}, a_{\lambda(s+1)}], \cdots] = e$$

which on expansion gives

$$[\cdots, a_{\lambda(r)}, \cdots, [a_{\lambda(s)}, a_{\lambda(s+1)}], \cdots] = [\cdots, [a_{\lambda(s)}, a_{\lambda(s+1)}], \cdots, a_{\lambda(r)}, \cdots]^{-1}.$$

The inductive hypothesis and 2.15 yield that this last commutator is trivial. Therefore $[a_{\lambda(1)}, \cdots, a_{\lambda(n+k+2)}] = e$ as required.

It is now possible to prove the refinement which leads to 2.1 (cf. [17], Lemma 5.7).

2.18 *For* $\lambda(4), \cdots, \lambda(n+k+1)$ *in* $\{4, \cdots, n+1\}$, *the word*

$$[x_1, x_2, x_3, x_{\lambda(4)}, \cdots, x_{\lambda(n+k+1)}]$$

is a law of $[n \to n+k]$.

PROOF. The argument is exhibited by the case $n = 3$, $k = 1$. Since $[3 \to 4]$ is a subvariety of $[4 \to 5]$, all commutators in $\{x_1, \cdots, x_4\}$ of weight at least 6 are laws of $[3 \to 4]$. Hence, commuting 2.16 with x_4 gives

$$[x_1, x_2, x_3, x_4, x_4][x_2, x_1, x_4, x_3, x_4][x_3, x_4, x_1, x_2, x_4][x_4, x_3, x_2, x_1, x_4]$$

is a law of $[3 \to 4]$. Now $[x_2x_3, x_4, x_1, x_2x_3, x_4]$ is a law of $[3 \to 4]$ which on expanding gives $[x_2, x_4, x_1, x_3, x_4][x_3, x_4, x_1, x_2, x_4]$ is a law of $[3 \to 4]$. Similarly

$[x_4, x_1, x_2, x_3, x_4][x_4, x_3, x_2, x_1, x_4]$ is a law of $[3 \to 4]$. Commuting a suitably rewritten version of 2.14 with x_3 and x_4 gives

$$[x_1, x_2, x_4, x_3, x_4][x_2, x_4, x_1, x_3, x_4][x_4, x_1, x_2, x_3, x_4,]$$

is a law of $[3 \to 4]$. Combining these four laws of $[3 \to 4]$ gives that $[x_1, x_2, x_3, x_4, x_4]$ is a law of $[3 \to 4]$ as required.

In particular $[x_1, \cdots, x_{n-k+1}, 2x_{n-k+2}, \cdots, 2x_{n+1}]$ is a law of $[n \to n+k]$. The earlier inductive proof using 2.13 now yields 2.1.

Equation 2.2 can be proved in a conceptually similar, but technically more unpleasent, manner. One more preliminary is needed.

2.19 *If a group H has a generating set B such that $[b_1, \cdots, b_{i+1}]^{2^{k-i}} = e$ for all i in $\{0, \cdots, k\}$ and all $b_1, \cdots, b_{i+1}$ in B, then H is in $\mathfrak{U}_k$.*

PROOF. Certainly H is in $\mathfrak{N}_k$ because $[b_1, \cdots, b_{k+1}] = e$ for all $b_1, \cdots, b_{k+1}$ in B. Suppose, inductively, H is in $\mathfrak{B}_{2^{i-1}}\mathfrak{N}_{k-i+1} \wedge \cdots \wedge \mathfrak{N}_k$. Every element of $\mathfrak{N}_{k-i}(H)$ can be written $\prod_{j=1}^{s} h_j$ where each h_j is a commutator of weight at least $k-i+1$ with entries from B. It follows from a result of P. Hall (see [8] Satz III.9.4) that

$$\left(\prod_{j=1}^{s} h_j\right)^{2^i} = \prod_{j=1}^{s} h_j^{2^i} \cdot \prod_{r=2}^{2^i} u_r^{\eta(r)}$$

where u_r is a product of commutators of weight at least r with entries from $\{h_1, \cdots, h_s\}$ and $\eta(r) = \binom{2^i}{r}$. It follows that u_r is in $\mathfrak{N}_{r(k-i+1)-1}(H)$ and so, by the inductive hypothesis, $u_r^{\eta(r)} = e$. By assumption each $h_j^{2^i} = e$. Therefore H is in $\mathfrak{B}_{2^i}\mathfrak{N}_{k-i}$. The case $i = k$ gives the required result.

PROOF OF 2.2. Let G belong to $[n \to n+k]$. For i in $\{0, \cdots, k\}$ let V_i be the verbal subgroup of G corresponding to the variety $[n-k+i \to n-k+2i]$; except that if $n = k+2$, then V_0 is to be $\mathfrak{N}_2(G)$. Observe that V_i/V_{i+1} is generated by right Engel elements of length 2 of G/V_{i+1}; in the exceptional case this relies on 2.18. It follows from 2.6 that $[b, 2g]$ is in V_{i+1} for all b in V_i and all g in G. To proceed further it will be convenient to have some more notation. Let $v_m = [x_1, \cdots, x_m]$. For i in $\{0, \cdots, k\}$ put $w(i, 0) = v_{n-k+1+3i}$. For j in $\{1, \cdots, i\}$ put

$$w(i,j) = [w(i-1, j-1), [x_{n-k+3i-1}, x_{n-k+3i}, x_{n-k+3i+1}]].$$

The next step is to prove for i in $\{0, \cdots, k\}$ that for j in $\{0, \cdots, i\}$ $w(i,j)^{2^{i-j}}$ is a law in G/V_i. For $i = 0$ this is trivial. For $i \geqq 1$, it can be assumed, inductively, that $w(i-1,j)^{2^{i-1-j}}$ is a law in G/V_{i-1} for all j in $\{0, \cdots, i-1\}$, Hence

2.20 $$[w(i-1,j)^{2^{i-1-j}}, x_{n-k+3i-1}, x_{n-k+3i-1}] \text{ is a law in } G/V_i.$$

It follows from 2.12 that $[w(i-1,j)^{2^{i-1-j}}, [x_{n-k+3i-1}, x_{n-k+3i}, x_{n-k+3i+1}]]$ is a

https://doi.org/10.1017/S1446788700015469 Published online by Cambridge University Press

law in G/V_i. Since $w(i-1,j)$ involves $n-k+3i-2$ x's and, by 2.17, $[n-k+i \to n-k+2i]$ is a subvariety of $[n-k+3i-1 \to 2(n-k+3i-2)]$

2.21 $$[w(i-1,j), x_{n-k+3i-1}, w(i-1,j)] \text{ is a law in } G/V_i.$$

Hence $[w(i-1,j), [x_{n-k+3i-1}, x_{n-k+3i}, x_{n-k+3i+1}]]^{2^{i-1-j}}$ is a law in G/V_i for j in $\{0,\cdots,i-1\}$. That is, $w(i,j+1)^{2^{i-(j+1)}}$ is a law in G/V_i. The remaining case, that of $w(i,0)^{2^i}$, is similar. It follows from 2.20 with $j=0$, using 2.10, that $[w(i-1,0)^{2^{i-1}}, x_{n-k+3i-1}, x_{n-k+3i}, x_{n-k+3i+1}]^2$ is a law in G/V_i. Hence by repeated use of 2.21, $w(i,0)^{2^i}$ is a law in G/V_i. Since $V_k = E$ (the identity subgroup), the case $i=k$ gives that $w(k,j)^{2^{k-j}}$ is a law in G for j in $\{0,\cdots,k\}$. It follows that $\mathfrak{N}_{n+2k}(G)$ has a generating set B (all commutators of weight at least $n+2k+1$) such that $[b_1,\cdots,b_{j+1}]^{2^{k-j}} = e$ for all j in $\{0,\cdots,k\}$ and all $b_1,\cdots,b_{j+1}$ in B. Therefore, by 2.19, $\mathfrak{N}_{n+2k}(G)$ is in $\mathfrak{U}_k$ and the proof is complete.

3. Proof of Theorem C

The core of the argument is a continuation of the calculations in section 2 to derive further laws of $[n \to n+k]$.

Observe first that it suffices to deal with the case $n \geqq 2k+2$. When $n \leqq 2k+2$ Theorem C claims $[n \to n+k]$ is a subvariety of $\mathfrak{N}_k\mathfrak{N}_2$; so once it is established that $[2k+2 \to 3k+2]$ is a subvariety of $\mathfrak{N}_k\mathfrak{N}_2$ the rest follow from 2.17.

For the rest of this section $n \geqq 2k+2$.

3.1 (cf. [17], Theorem 7.2) *For i in $\{1,\cdots,k+1\}$ and $\lambda(3i+1),\cdots,\lambda(n+k+1)$ in $\{3i+1,\cdots,n+i\}$, if $y_1 = [x_1,x_2,x_3]$ and $y_2,\cdots,y_{n+k+1-2i}$ are $[x_4,x_5,x_6]$, $\cdots,[x_{3i-2},x_{3i-1},x_{3i}], x_{\lambda(3i+1)},\cdots,x_{\lambda(n+k+1)}$ in some order, then $[y_1,\cdots,y_{n+k+1-2i}]$ is a law of $[n \to n+k]$.*

PROOF. The case $i=1$ has been established in 2.18. For $i \geqq 1$ suppose $[x_{3i-2},x_{3i-1},x_{3i}]$ is y_j. A commutator which has $y_1,\cdots,y_{j-1},y_{j+1},\cdots,y_{n+k+1-2i}$ as entries and at least four entries from x_{3i-2},x_{3i-1},x_{3i} is, by the inductive hypothesis (for $i-1$), a law of $[n+1 \to n+k+1]$ and hence, by 2.17, of $[n \to n+k]$. Therefore by repeated use of the obvious variation of 2.14

$$[y_1,\cdots,y_j,\cdots,y_{n+k+1-2i}] =$$
$$[\cdots,y_{j-1},x_{3i-2},x_{3i-1},x_{3i},y_{j+1},\cdots][\cdots,y_{j-1},x_{3i},x_{3i-1},x_{3i-2},y_{j+1},\cdots]\times$$
$$[\cdots,y_{j-1},x_{3i-1},x_{3i-2},x_{3i},y_{j+1},\cdots]^{-1}[\cdots,y_{j-1},x_{3i},x_{3i-2},x_{3i-1},y_{j+1},\cdots]^{-1}u$$

where u is a law of $[n \to n+k]$. Now, by the inductive hypothesis followed by a suitable substitution, $[\cdots,y_{j-1},x_{3i-2}x_{3i},x_{3i-1},x_{3i-2}x_{3i},y_{j+1},\cdots]$ is a law of $[n \to n+k]$. Hence

https://doi.org/10.1017/S1446788700015469 Published online by Cambridge University Press

$$[\cdots, y_{j-1}, x_{3i-2}, x_{3i-1}, x_{3i}, y_{j+1}, \cdots][\cdots, y_{j-1}, x_{3i}, x_{3i-1}, x_{3i-2}, y_{j+1}, \cdots]$$

is a law of $[n \to n+k]$. The result now follows easily.

The case $i = k+1$ is the starting point for the next lemma.

3.2 *Let* $\mu(1), \cdots, \mu(k+1)$ *be integers which are at least* 3 *and such that* $v = \mu(1) + \cdots + \mu(k+1) \leqq n+k+1$; *let* $\lambda(v+1), \cdots, \lambda(n+k+1)$ *be in* $\{v+1, \cdots, n+k+1\}$; *and let* $v(j) = \mu(1) + \cdots + \mu(j)$ *for* j *in* $\{0, \cdots, k+1\}$. *If* $y_1 = [x_1, \cdots, x_{v(1)}]$ *and* $y_2, \cdots, y_{n+2k+2-v}$ *are* $[x_{v(1)+1}, \cdots, x_{v(2)}], \cdots, [x_{v(k)+1}, \cdots, x_v]$, $x_{\lambda(v+1)}, \cdots, x_{\lambda(n+k+1)}$ *in some order, then* $[y_1, \cdots, y_{n+2k+2-v}]$ *is a law of* $[n \to n+k]$.

PROOF. Induction on v. The case $v = 3k+3$ has been proved in 3.1. For $v > 3k+3$ there is a j such that $\mu(j) > 3$. Suppose $y_m = [y'_m, x_{\mu(j)}]$, then $[\cdots, y_m, \cdots] = [\cdots, [y'_m, x_{\mu(j)}], \cdots]$ can, by 2.14, be written as a product of commutators with entries $\cdots, y_{m-1}, y'_m, y_{m+1}, \cdots, y_{n+2k+2-v}, x_{\mu(j)}$ which are, by the inductive hypothesis, laws of $[n \to n+k]$.

The case $v = n+k+1$ gives

3.3 *If* $\mu(1), \cdots, \mu(k+1) \geqq 3$ *and* $\mu(1) + \cdots + \mu(k+1) = n+k+1$, *then*

$$[n \to n+k] \subseteq [\mathfrak{N}_{\mu(1)-1}, \cdots, \mathfrak{N}_{\mu(k+1)-1}].$$

Comparing Theorem 16.2 and 17.1 of Ward's paper [16] using for K any sequence beginning $3, k+2$ and the function ϕ given by $\phi(0) = n+k+1$, $\phi(1) = k+1$ and $\phi(j) = 0$ otherwise, gives

3.4 $$\wedge [\mathfrak{N}_{\mu(1)-1}, \cdots, \mathfrak{N}_{\mu(k+1)-1}] = \mathfrak{N}_{n+k} \vee \mathfrak{N}_k\mathfrak{N}_2$$

where the intersection is taken over all $(\mu(1), \cdots, \mu(k+1))$ *such that* $\mu(1), \cdots, \mu(k+1) \geqq 3$ *and* $\mu(1) + \cdots + \mu(k+1) = n+k+1$.

Combining 3.3 and 3.4 and using modularity completes the proof of Theorem C.

4. Proof of Theorem B

In this section $k = 1$ and $n \geqq 3$.

Since $\mathfrak{A}_2\mathfrak{A}_2$ is a subvariety of $[n \to n+1]$ the result follows by modularity once it is established that $[n \to n+1]$ is a subvariety of $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_{n+2}$. To this end it is enough to show that a basis for the laws of $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_{n+2}$ consists of laws of $[n \to n+1]$. It is well-known that $\{x_1^4, [x_1^2, x_2^2]\}$ is a basis for the laws of $\mathfrak{A}_2\mathfrak{A}_2$ ([15] p. 92). A result of Bryant ([3] Proposition 1) guarantees that $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_{n+2}$ has a finite basis for its laws and in theory enables a finite basis to be computed. In practice ad hoc methods seem easier.

Before stating the key lemma some notation is needed. Call a commutator *simple* if it has weight 1 or is the commutator of a simple commutator and a commutator of weight 1, and *complex* otherwise. Following Lamberth [11] the complex commutator $[[x_1, \cdots, x_r], [x_{r+1}, \cdots, x_m]]$ will be abbreviated to

$$[x_1, \cdots, x_{r,}, x_{r+1}, \cdots, x_m].$$

4.1 *A basis for the laws of* $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_m$ *is* B_m *where*

$$B_0 = \{x_1^4, [x_1^2, x_2^2]\}$$

$$B_1 = \{[x_1, x_2]^2, [x_1^2, x_2^2]\}$$

$$B_2 = \{[x_1, x_2, x_3]^2, [x_1, x_2, x_3^2], [x_1, x_2^2, x_1]\}$$

and for $m \geqq 3$

$$B_m = \{[x_1, \cdots, x_{m+1}]^2, [x_1, x_2, \cdots, x_{m-1}, x_m, x_m], [x_1, x_2, x_2, x_1, x_3, \cdots, x_{m-1}],$$
$$[x_1, \cdots, x_{r,}, x_{r+1}, \cdots, x_{m+1}] \textit{ for } r \textit{ in } \{2, \cdots, m-1\}\}.$$

Theorem B will follow easily from this using earlier results. All the complex commutators in B_{n+2} except those with r in $\{2, n+1\}$ are laws of $[n \to n+1]$ by 3.3. The two exceptions (which are equivalent) follow by 2.18 and 2.11. The third word is clearly a law of $[n \to n+1]$; the second is by 2.18; and the first by 2.18 and 2.10.

Proof of 4.1. It is clear that the words listed are laws of $\mathfrak{N}_m$. With perhaps the two following exceptions, it is equally clear that they are laws of $\mathfrak{A}_2\mathfrak{A}_2$. Since $[x_1, \cdots, x_{m-1}, x_m, x_m] = [x_1, \cdots, x_{m-1}, x_m]^{-2}[x_1, \cdots, x_m^2]$, it is a law of $\mathfrak{A}_2\mathfrak{A}_2$. Expanding $[x_1^2, x_2^2, x_3, \cdots, x_{m-1}]$ gives first that $[x_1, x_2^2, x_1, \cdots]$ and then that $[x_1, x_2, x_2, x_1, x_3, \cdots, x_{m-1}]$ is a law of $\mathfrak{A}_2\mathfrak{A}_2$. Thus B_m consists of laws of $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_m$.

The proof that B_m is a basis for the laws of $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_m$ is by induction on m. The case $m = 0$ has already been mentioned. For $m > 0$,

$$\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_m = (\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_{m-1}) \vee \mathfrak{N}_m$$

and the inductive hypothesis gives B_{m-1} as a basis for the laws of $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_{m-1}$.

Let $b_1, \cdots, b_s$ be the distinct (left-normed) simple basic commutators of weight m in some prescribed order. The key to the proof is the following statement:

4.2 *for* w *in* B_{m-1} *and* θ *an endomorphism of the word group,* $w\theta$ *is equivalent modulo consequences of* B_m *to* $\prod_{j=1}^s b_j^{\eta(j)} v$ *where the* $\eta(j)$ *are integers and* v *is a product of complex commutators of weight* m.

Let X_∞ denote the word group. Since every law of $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_{m-1}$ can be written $\Pi(w_i\theta_i)^{\varepsilon(i)}$ where w_i is in B_{m-1}, θ_i is an endomorphism of X_∞ and $\varepsilon(i)$ is -1 or 1, it follows from 4.2 that every law of $\mathfrak{A}_2\mathfrak{A}_2 \vee \mathfrak{N}_{m-1}$ is equivalent modulo consequences of B_m to $\prod_{j=1}^s b_j^{\zeta(j)} v'$ where the $\zeta(j)$ are integers and v' is a product of complex commutators of weight m. If this is also a law of $\mathfrak{N}_m$, then by 36.32 of [15] each $\zeta(j)=0$ and v' is a law of $\mathfrak{N}_m$, that is, $v' \in \mathfrak{N}_m(X_\infty) \wedge \mathfrak{A}^2(X_\infty)$. By Theorem 17.2 of [16] it follows that $v' \in \prod_{i=1}^{m-2}[\mathfrak{N}_i(X_\infty), \mathfrak{N}_{m-i-1}(X_\infty)]$ and so is a consequence of B_m.

Proof of 4.2. The result is clear if w is complex. For the remaining w three sample proofs will be given. The other proofs are similar in spirit and will be omitted. In each case it will be convenient to have certain consequences of B_m available.

(i) $m \geqq 2$, $w = [x_1, \cdots, x_m]^2$

If u is in $\mathfrak{N}_m(X_\infty)$, then u is a product of values of $[x_1, \cdots, x_{m+1}]$, and so u^2 is a product of squares of values of $[x_1, \cdots, x_{m+1}]$. Therefore u^2 is a consequence of B_m.

If θ is an endomorphism of X_∞, then $[x_1, \cdots, x_m]\theta$ can be written $\prod_{j=1}^{s} b_j^{\xi(j)} v''u''$ where the $\xi(j)$ are integers, v'' is a product of complex commutators of weight m, and u'' is in $\mathfrak{N}_m(X_\infty)$. It follows readily that w has the claimed form.

(ii) $m = 4$, $w = [x_1, x_2, x_3, x_3]$

First note that in 2.14 and 2.15 u is a product of complex commutators. It follows that B_4 has a consequences $[x_1, x_2, x_4, x_3, x_4]$ and $[x_1, x_2, x_4, x_4, x_3]$.

The aim is to prove for every endomorphism θ of X_∞ that $[x_1, x_2, x_3, x_3]\theta$ is equivalent modulo consequences of B_4 to $\prod_{j=1}^{s} b_j^{\eta(j)} v$ where $b_1, \cdots, b_s$ are the distinct simple basic commutators of weight 4, the $\eta(j)$ are integers, and v is a product of complex commutators of weight 4. The proof is by induction on the sum of the lengths (as words in X_∞) of $x_1\theta, x_2\theta, x_3\theta$. If any one of $x_1\theta, x_2\theta, x_3\theta$ has length 0, there is nothing to prove. If $x_1\theta$ has length exceeding 1, then $x_1\theta = x_1\psi . x_1\phi$ with ψ, ϕ endomorphisms of X_∞ and $x_1\psi, x_1\phi$ of shorter length. Hence

$$[x_1, x_2, x_3, x_3]\theta \equiv [x_1\psi, x_2\theta, x_3\theta, x_3\theta][x_1\phi, x_2\theta, x_3\theta, x_3\theta]$$

(here and in what follows $\equiv$ denotes equivalence modulo consequences of B_4). By the inductive hypothesis both commutators on the right are equivalent modulo consequences of B_4 to an expression of the required type, and then, clearly, so is their product. The case $x_2\theta$ has length exceeding 1 goes similarly. If both $x_1\theta$ and $x_2\theta$ have length 1, it is enough to consider $[x_{\lambda(1)}, x_{\lambda(2)}, x_3\theta, x_3\theta]$; because, for example, $[x_{\lambda(1)}^{-1}, x_{\lambda(2)}, x_3\theta, x_3\theta] \equiv [x_{\lambda(1)}, x_{\lambda(2)}, x_3\theta, x_3\theta]^{-1}$. If $x_3\theta$ has length exceeding 1, then $x_3\theta = x_3\psi . x_3\phi$ and

$$[x_{\lambda(1)}, x_{\lambda(2)}, x_3\theta, x_3\theta] \equiv [x_{\lambda(1)}, x_{\lambda(2)}, x_3\psi, x_3\psi][x_{\lambda(1)}, x_{\lambda(2)}, x_3\phi, x_3\phi] \cdot [x_{\lambda(1)}, x_{\lambda(2)}, x_3\psi, x_3\phi][x_{\lambda(1)}, x_{\lambda(2)}, x_3\phi, x_3\psi].$$

By the inductive hypothesis the first two commutators on the right are equivalent to expressions of the required kind. The other pair are equivalent, using 2.15, to $[x_{\lambda(1)}, x_{\lambda(2)}, x_3\psi, x_3\phi]^2 [x_{\lambda(1)}, x_{\lambda(2)}, x_3\psi, x_3\phi]$; both of these have been shown earlier to be equivalent to expressions of the required kind; so this case follows. If $x_3\theta$ also has length 1, then as before it can be taken as $x_{\lambda(3)}$.

If $[x_{\lambda(1)}, x_{\lambda(2)}, x_{\lambda(3)}, x_{\lambda(3)}]$ is basic, the proof is complete. If not, repeated use of 2.15 gives the required result.

(iii) $m = 2, w = [x_1^2, x_2^2]$

Since $[x_1^2, x_2^2] = [x_1, x_2^2][x_1, x_2^2, x_1][x_1, x_2^2]$, it is equivalent modulo B_3 to $[x_1, x_2^2]^2$ and the result follows from (i).

5. Examples

The various examples promised in the introduction are described here. The minor examples are disposed of first.

5.1 *The wreath product W of a cyclic group of order 2^k by a countably infinite elementary abelian 2-group B is in $[n \to n + k]$.*

To see this observe that every n-generator subgroup of W can be regarded as a subgroup of $W_1 = A$ wr B_1 where A is the direct product of n cyclic groups of order 2^k and B_1 is the direct product of n cyclic groups of order 2. Since W_1 has class $n + k$ ([12] Theorem 5.1.), it follows that W is in $[n \to n + k]$.

Since $[x_1, \cdots, x_{s+1}]^{2^{k-1}}$ is a law of $\mathfrak{B}_{2^{k-1}}\mathfrak{N}_s$ but is not a law in W, it follows that $[n \to n + k]$ is not a subvariety of $\mathfrak{B}_{2^{k-1}}\mathfrak{N}_s$ for all s. Similarly consideration of the word $[(x_1x_2)^{2^k} x_2^{-2^k} x_1^{-2^k}, x_3, \cdots, x_{s+1}]$ shows that $[n \to n + k]$ is not a subvariety of $\mathfrak{N}_s \vee \mathfrak{B}_{2^k}$ for all s.

Essentially the same argument as after 5.1 justifies the next claim.

5.2 *Let D be the direct product of i cyclic groups of order 4 and B (of 5.1). The wreath product of a cyclic group of order 2^r by D is in $[n \to n + 2(r + i) - 1]$.*

Hence, by considering the word $[x_1, x_2, x_3^2, \cdots, x_{i+2}^2, x_{i+3}, \cdots, x_{s+1}]^{2^{r-1}}$, $[n \to n + 2(r + i) - 1]$ is not a subvariety of $\mathfrak{N}_s \vee \mathfrak{B}_{2^{r-1}}\mathfrak{N}_i\mathfrak{A}_2$ for all s.

5.3 *The (standard restricted) crown product (see [14] section 8) of a dihedral group of order 8 by B is in $[n \to n + 2]$.*

To see this it suffices to observe that the central factor group is in $\mathfrak{A}_2\mathfrak{A}_2$ and hence in $[n \to n + 1]$.

This example shows that $[n \to n + 2]$ is not a subvariety of $\mathfrak{A}\mathfrak{N}_i$ for all i.

The final example is more intricate.

5.4 *For each positive integer m there is a torsion-free nilpotent group of class precisely m which belongs to $[m - 2r \to m - r]$ for each non-negative integer r (less than $m/2$).*

Proof. Nothing interesting is claimed unless m exceeds 3, so assume this to avoid degenerate cases. Put $M = \{1, \cdots, m - 1\}$. Let A be a free abelian group of rank 2^{m-1} freely generated by $\{a_S : S \subseteq M\}$. For i,j in M, let

$$\varepsilon(i,j) = \begin{cases} 0 & \text{when} \quad i = j \\ 1 & \text{when} \quad i < j \\ -1 & \text{when} \quad i > j. \end{cases}$$

For every subset S of M, put $\varepsilon(S,j) = \prod_{i\in S}\varepsilon(i,j)$. Let $\beta_1,\cdots,\beta_{m-1}$ be the automorphisms of A defined by $a_S\beta_i = a_S a_{S\cup\{i\}}^{\varepsilon(S,i)}$. Let B be the subgroup of the automorphism group of A generated by $\{\beta_1,\cdots,\beta_{m-1}\}$. Let G be the splitting extension of A by B. It will be fairly easy to see that G has the required properties after certain information about the relevant part of the endomorphism ring, end A, of A has been obtained. Let ι be the identity of end A and R the subring of end A generated by all $\beta - \iota$ with β in B. Let $\beta_i^* = \beta_i - \iota$. It is easy to check from the definition of the β_i that $\beta_i^*\beta_i^* = 0$ and $\beta_i^*\beta_j^* + \beta_j^*\beta_i^* = 0$ for all i,j in M. The rest of the (somewhat sketchy) discussion depends only on these relations. It is straightforward to check that every element of R is an integral linear combination of products of the β_i^*. It follows that $R^m = 0$. Let R^ι be the subring of end A generated by R and ι, then $\beta_i^*\rho\beta_i^* = 0$ and $\beta_i^*\rho\beta_j^* + \beta_j^*\rho\beta_i^* = 0$ for all i,j in M and all ρ in R^ι. Hence

5.5 $$\gamma\rho\gamma = 0$$

for all integral linear combinations γ of the β_i^* and all ρ in R. Let $\lambda(1),\cdots,\lambda(m-r)$ be elements of $\{1,\cdots,m-2r\}$. The next step is to show

5.6 $$\prod_{j=1}^{m-r} \rho_{\lambda(j)} = 0 \text{ for all } \rho_{\lambda(j)} \text{ in } R.$$

Now $\rho_{\lambda(j)}$ can be written $\gamma_{\lambda(j)} + \delta_{\lambda(j)}$ where $\gamma_{\lambda(j)}$ is an integral linear combination of the β_i^* and $\delta_{\lambda(j)}$ is in R^2. Since the left-hand-side of 5.6 can be written as a sum of products $\prod_{j=1}^{m-r}\omega_{\lambda(h)}$ where ω is γ or δ, it suffices to show such products are 0. If at least r of the ω's are δ's, this follows from $R^m = 0$; while if less than r of the ω's are δ's, the product contains the same γ twice and the result follows from 5.5.

Every element of G can be written ab with a in A and b in B. Since A is abelian it follows that every commutator of weight $m-r+1$ in elements $g_1,\cdots,g_{m-2r}$ of G can be written as a product of commutators of the form $[a, b_{\mu(1)},\cdots,b_{\mu(s)}]$ where a is in A, the b_μ are in B, the $\mu(j)$ belong to $\{1,\cdots,m-2r\}$ and s is at least $m-r$. By 5.6 each such commutator is trivial. Therefore G belongs to $[m-2r \to m-r]$ for all non-negative r. In particular G has class m. Since $[a_\phi,\beta_1,\cdots,\beta_{m-1}] = a_M$, the class of G is precisely m.

It is routine to check that $[\beta_i,\beta_j] = \iota + 2\beta_i^*\beta_j^*$ and hence that B is nilpotent of class 2. Every element of B can therefore be written in the form $\prod_{i=1}^{m-1}\beta_i^{\kappa(i)}\prod_{i>j}[\beta_i,\beta_j]^{\nu(i,j)}$. By considering the image of a_ϕ under this mapping, it is straightforward to see that the expression is unique. Hence B is free nilpotent of class 2 of rank $m-1$. In particular B is torsion-free. Therefore G is torsion-free and the proof is complete.

References

[1] S. Bachmuth and H. Y. Mochizuki, 'Third Engel groups and the Macdonald-Neumann conjecture', *Bull. Austral. Math. Soc.* 5 (1971), 379–386; MR to appear.

[2] Garrett Birkhoff, *Lattice Theory*, Third (New) Edition (Amer. Math. Soc. Colloq. Publ., vol. 25. Amer. Math. Soc., Providence, 1967); MR 37 # 2638.

[3] Roger M. Bryant, 'On some varieties of groups', *Bull. London. Math. Soc.* 1 (1969), 60–64; MR 39 # 2848.

[4] J. R. J. Groves, 'Varieties of soluble groups and a dichotomy of P. Hall', *Bull. Austral. Math. Soc.* 5 (1971), 391–410; MR to appear.

[5] Narain Gupta, 'Third-Engel 2-groups are soluble', *Canad. Math. Bull.* 15 (1972), 523–524; MR to appear.

[6] C. K. Gupta, N. D. Gupta and M. F. Newman, 'Some finite nilpotent p-groups', *J. Austral. Math. Soc.* 9 (1969), 278–288; MR 39 # 2877.

[7] H. Heineken, 'Bounds for the nilpotency class of a group', *J. London. Math. Soc.* 37 (1962), 456–458; MR 27 # 1512.

[8] Bertram Huppert, *Endliche Gruppen I*, (Die Grundlehren der mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, Heidelberg, New York, 1967); MR 37 # 302.

[9] Wolfagng Kappe, 'Die A-Norm einer Gruppe', *Illinois J. Math.* 5 (1961), 187–197; MR 22 # 12137.

[10] L. G. Kovács and M. F. Newman, 'On non-Cross varieties of groups', *J. Austral. Math. Soc.* 12 (1971), 129–144; MR 45 # 1966.

[11] Walter J. Lamberth, *Automatic collection of group commutators and expressions*, M.Sc. Thesis, (Australian National University, 1968).

[12] Hans Liebeck, 'Concerning nilpotent wreath products', *Proc. Cambridge Philos. Soc.* 58 (1962), 443–451; MR 25 # 3087.

[13] I. D. Macdonald and B. H. Neumann, 'A third-Engel 5-group', *J. Austral. Math. Soc.* 7 (1967), 555–569; MR 36 # 5223.

[14] B. H. Neumann, 'Ascending derived series', *Compositio Math.* 13 (1956), 47–64; MR 19, 632.

[15] Hanna Neumann, *Varieties of groups*, (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967); MR 35 # 6734.

[16] M. A. Ward, 'Basic commutators', *Philos. Trans. Roy. Soc. London Ser.* A. 264 (1969), 343–412; MR 40 # 4379.

[17] Narain Gupta, Frank Levin and Akbar Rhemtulla, 'Chains of varieties', *Canad. J. Math.*, submitted.

[18] Derek J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Part 2, (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 63. Springer-Verlag, Berlin, Heidelberg. New York, 1972); MR to appear.

Australian National University
Canberra, ACT, 2600