



Amenability and Covariant Injectivity of Locally Compact Quantum Groups II

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Abstract. Building on our previous work, we study the non-relative homology of quantum group convolution algebras. Our main result establishes the equivalence of amenability of a locally compact quantum group \mathbb{G} and 1-injectivity of $L^\infty(\widehat{\mathbb{G}})$ as an operator $L^1(\widehat{\mathbb{G}})$ -module. In particular, a locally compact group G is amenable if and only if its group von Neumann algebra $VN(G)$ is 1-injective as an operator module over the Fourier algebra $A(G)$. As an application, we provide a decomposability result for completely bounded $L^1(\widehat{\mathbb{G}})$ -module maps on $L^\infty(\widehat{\mathbb{G}})$, and give a simplified proof that amenable discrete quantum groups have co-amenable compact duals, which avoids the use of modular theory and the Powers–Størmer inequality, suggesting that our homological techniques may yield a new approach to the open problem of duality between amenability and co-amenable.

1 Introduction

The connection between amenability of a locally compact group G and injectivity of its group von Neumann algebra $VN(G)$ has been a topic of interest in abstract harmonic analysis for decades. Amenability of G entails injectivity of $VN(G)$ [16]. However, the converse is not true, *e.g.*, if $G = SL(n, \mathbb{R})$ for $n \geq 2$. Indeed, a result of Connes [5, Corollary 7], attributed to Dixmier, states that $VN(G)$ is injective for any separable connected locally compact group.

In [7], we clarified this connection by exploiting the $\mathcal{T}(L^2(G))$ -module structure of $\mathcal{B}(L^2(G))$, showing the equivalence of amenability of a locally compact group G and covariant injectivity of $VN(G)$, meaning the existence of a conditional expectation $E: \mathcal{B}(L^2(G)) \rightarrow VN(G)$ commuting with the canonical $\mathcal{T}(L^2(G))$ -action [7, Theorem 4.2]. We also established a corresponding result at the level of locally compact quantum groups \mathbb{G} and studied the relationship between amenability of \mathbb{G} and relative 1-injectivity of its various operator modules over $\mathcal{T}(L^2(\mathbb{G}))$.

In this paper we build on results from [7], focusing on the non-relative homology of operator modules over $L^1(\mathbb{G})$ and $\mathcal{T}(L^2(\mathbb{G}))$. Our main result states that a locally compact quantum group \mathbb{G} is amenable if and only if $L^\infty(\widehat{\mathbb{G}})$ is 1-injective as an operator $L^1(\widehat{\mathbb{G}})$ -module. This new homological manifestation of quantum group duality shows that in order to recover properties of \mathbb{G} , one should not only consider the von Neumann algebraic structure of $L^\infty(\widehat{\mathbb{G}})$, but rather its *operator module* structure.

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Our main result has already been used to establish a fundamental hereditary property of amenability [6], providing the main tool in the proof that amenability passes to closed quantum subgroups. As a further application, we provide a decomposability result in the spirit of Haagerup [17] for completely bounded $L^1(\mathbb{G})$ -module maps on $L^\infty(\mathbb{G})$. Specifically, if $\widehat{\mathbb{G}}$ is amenable then

$$\begin{aligned} \mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})) &= \text{span } \mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})), \\ \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G})) &= \text{span } \mathcal{CP}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G})). \end{aligned}$$

Using the latter equality, we show that amenability of $\widehat{\mathbb{G}}$ entails

$$C_u(\mathbb{G})^* = M_{cb}^r(L^1(\mathbb{G})),$$

set theoretically. Similar techniques also lead to a characterization of the predual $Q_{cb}^r(L^1(\mathbb{G}))$ of the completely bounded (right) multipliers $M_{cb}^r(L^1(\mathbb{G}))$ for an arbitrary locally compact quantum group \mathbb{G} as $Q_{cb}^r(L^1(\mathbb{G})) \cong C_0(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G})$, which is new even in the group setting.

Arguably the biggest open problem in abstract harmonic analysis on locally compact quantum groups is the duality between amenability and co-amenability. In the group setting, this is Leptin’s theorem [30], which states that a locally compact group G is amenable if and only if its Fourier algebra $A(G)$ has a bounded approximate identity. In the quantum group setting, many partial results have been obtained over the years. Ruan showed that a compact Kac algebra \mathbb{G} is co-amenable if and only if its discrete dual $\widehat{\mathbb{G}}$ is amenable [35, Theorem 4.5]. This equivalence was later generalized by Tomatsu (and, independently by Blanchard and Vaes) to arbitrary compact quantum groups [40, Theorem 3.8]. Tomatsu’s argument relies on the specific modular theory of discrete quantum groups in order to apply the Powers–Størmer inequality in a crucial step. As another application of our main result, we give a considerably simplified proof of Tomatsu’s theorem which avoids the use of modular theory and the Powers–Størmer inequality, suggesting that our homological techniques may provide a new approach to the general duality problem of amenability and co-amenability.

For regular quantum groups \mathbb{G} , we obtain a version of our main result at the predual level, showing the equivalence of discreteness of \mathbb{G} and 1-projectivity of $L^1(\mathbb{G})$ as an operator module over itself.

The paper is structured as follows. We begin in Section 2 with some preliminaries on the homology of operator modules, and include some new results on the relationship between relative and non-relative homology. Section 3 is devoted to a brief overview of the relevant machinery from locally compact quantum groups, their associated operator modules, and completely bounded multipliers. Section 4 outlines the operator module structure of $\mathcal{B}(L^2(\mathbb{G}))$ over $\mathcal{T}(L^2(\mathbb{G}))$ and contains new results which are used in the proof of the main theorem. Section 5 contains the main result of the paper along with its aforementioned applications.

2 Preliminaries

Let \mathcal{A} be a completely contractive Banach algebra. We say that an operator space X is a right *operator \mathcal{A} -module* if it is a right Banach \mathcal{A} -module such that the module map $m_X: X \widehat{\otimes} \mathcal{A} \rightarrow X$ is completely contractive, where $\widehat{\otimes}$ denotes the operator space

projective tensor product. We say that X is *faithful* if for every non-zero $x \in X$, there is $a \in \mathcal{A}$ such that $x \cdot a \neq 0$, and we say that X is *essential* if $\langle X \cdot \mathcal{A} \rangle = X$, where $\langle \cdot \rangle$ denotes the closed linear span. Note that our definition of faithfulness, the standard notion in operator modules, is opposite in nature to the usual definition in algebra. We denote by $\mathbf{mod}\text{-}\mathcal{A}$ the category of right operator \mathcal{A} -modules with morphisms given by completely bounded module homomorphisms. Left operator \mathcal{A} -modules and operator \mathcal{A} -bimodules are defined similarly, and we denote the respective categories by $\mathcal{A}\text{-}\mathbf{mod}$ and $\mathcal{A}\text{-}\mathbf{mod}\text{-}\mathcal{A}$.

Remark 2.1 Regarding terminology, in what follows we will often omit the term “operator” when discussing homological properties of operator modules as we will be working exclusively in the operator space category.

Let \mathcal{A} be a completely contractive Banach algebra, $X \in \mathbf{mod}\text{-}\mathcal{A}$ and $Y \in \mathcal{A}\text{-}\mathbf{mod}$. The \mathcal{A} -module tensor product of X and Y is the quotient space $X \widehat{\otimes}_{\mathcal{A}} Y := X \widehat{\otimes} Y / N$, where $N = \langle x \cdot a \otimes y - x \otimes a \cdot y \mid x \in X, y \in Y, a \in \mathcal{A} \rangle$, and, again, $\langle \cdot \rangle$ denotes the closed linear span. It follows that (see [3, Corollary 3.5.10]) $\mathcal{CB}_{\mathcal{A}}(X, Y^*) \cong N^\perp \cong (X \widehat{\otimes}_{\mathcal{A}} Y)^*$, where $\mathcal{CB}_{\mathcal{A}}(X, Y^*)$ is the space of completely bounded right \mathcal{A} -module maps $\Phi: X \rightarrow Y^*$. If $Y = \mathcal{A}$, then clearly $N \subseteq \text{Ker}(m_X)$ where $m_X: X \widehat{\otimes} \mathcal{A} \rightarrow X$ is the module map. If the induced mapping $\widetilde{m}_X: X \widehat{\otimes}_{\mathcal{A}} \mathcal{A} \rightarrow X$ is a completely isometric isomorphism, we say that X is an *induced \mathcal{A} -module*. A similar definition applies for left modules. In particular, we say that \mathcal{A} is *self-induced* if $\widetilde{m}_{\mathcal{A}}: \mathcal{A} \widehat{\otimes}_{\mathcal{A}} \mathcal{A} \cong \mathcal{A}$ completely isometrically.

Let \mathcal{A} be a completely contractive Banach algebra and $X \in \mathbf{mod}\text{-}\mathcal{A}$. The identification $\mathcal{A}^+ = \mathcal{A} \oplus_1 \mathbb{C}$ turns the unitization of \mathcal{A} into a unital completely contractive Banach algebra, and it follows that X becomes a right operator \mathcal{A}^+ -module via the extended action $x \cdot (a + \lambda e) = x \cdot a + \lambda x$, for $a \in \mathcal{A}^+, \lambda \in \mathbb{C}, x \in X$. Let $C \geq 1$. Then X is *relatively C -projective* if there exists a morphism $\Phi^+: X \rightarrow X \widehat{\otimes} \mathcal{A}^+$ that is a right inverse to the extended module map $m_X^+: X \widehat{\otimes} \mathcal{A}^+ \rightarrow X$ such that $\|\Phi^+\|_{cb} \leq C$. When X is essential, then X is relatively C -projective if and only if there exists a morphism $\Phi: X \rightarrow X \widehat{\otimes} \mathcal{A}$ satisfying $\|\Phi\|_{cb} \leq C$ and $m_X \circ \Phi = \text{id}_X$ by the operator analogue of [9, Proposition 1.2]. We say that X is *C -projective* if for every $Y, Z \in \mathbf{mod}\text{-}\mathcal{A}$, every complete quotient morphism $\Psi: Y \twoheadrightarrow Z$, every morphism $\Phi: X \rightarrow Z$, and every $\varepsilon > 0$, there exists a morphism $\widetilde{\Phi}_\varepsilon: X \rightarrow Y$ such that $\|\widetilde{\Phi}_\varepsilon\|_{cb} < C\|\Phi\|_{cb} + \varepsilon$ and $\Psi \circ \widetilde{\Phi}_\varepsilon = \Phi$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
 & & Y \\
 & \nearrow \widetilde{\Phi}_\varepsilon & \downarrow \Psi \\
 X & \xrightarrow{\Phi} & Z
 \end{array}$$

When $\mathcal{A} = \mathbb{C}$, the definition of C -projectivity coincides with that of a C -projective operator space [2, Definition 3.3]. In general, it is not immediately clear whether C -projectivity implies relative C -projectivity, as one might expect. To give a sense of the potential distinction, the Fourier algebra $A(G)$ of a locally compact group G is relatively 1-projective in $\mathbf{mod}\text{-}A(G)$ if and only if G is an IN group [8, Proposition 3.8], while it is 1-projective in $\mathbf{mod}\text{-}A(G)$ if and only if G is compact by Theorem 5.15.

The following relationship between relative projectivity and projectivity appears to be new, and will be used to characterize the 1-projectivity of quantum group convolution algebras.

Proposition 2.2 *Let \mathcal{A} be a completely contractive Banach algebra and $X \in \mathbf{mod}\text{-}\mathcal{A}$. If X is C_1 -projective in $\mathbf{mod}\text{-}\mathbb{C}$ and is relatively C_2 -projective in $\mathbf{mod}\text{-}\mathcal{A}$, then X is C_1C_2 -projective in $\mathbf{mod}\text{-}\mathcal{A}$.*

Proof Let $Y, Z \in \mathbf{mod}\text{-}\mathcal{A}$, let $\Psi: Y \twoheadrightarrow Z$ be a complete quotient morphism and let $\Phi: X \rightarrow Z$ be a morphism. By relative C_2 -projectivity, there exists a morphism $\alpha^+: X \rightarrow X \widehat{\otimes} \mathcal{A}^+$ satisfying $m_X^+ \circ \alpha^+ = \text{id}_X$ and $\|\alpha^+\|_{cb} \leq C_2$. Since X is a C_1 -projective operator space, for every $\varepsilon > 0$, there exists a lifting $\Phi_\varepsilon: X \rightarrow Y$ satisfying $\Psi \circ \Phi_\varepsilon = \Phi$ and $\|\Phi_\varepsilon\|_{cb} < C_1\|\Phi\|_{cb} + \varepsilon/C_2$. The morphism $(\Phi_\varepsilon \otimes \text{id}): X \widehat{\otimes} \mathcal{A}^+ \rightarrow Y \widehat{\otimes} \mathcal{A}^+$ then satisfies $\|\Phi_\varepsilon \otimes \text{id}\|_{cb} < C_1\|\Phi\|_{cb} + \varepsilon/C_2$, and composing with α^+ together with the multiplication $m_Y^+: Y \widehat{\otimes} \mathcal{A}^+ \rightarrow Y$, we obtain a morphism $\tilde{\Phi}_\varepsilon := m_Y^+ \circ (\Phi_\varepsilon \otimes \text{id}) \circ \alpha^+: X \rightarrow Y$ satisfying $\|\tilde{\Phi}_\varepsilon\|_{cb} < C_1C_2\|\Phi\|_{cb} + \varepsilon$. Moreover, using the module properties of the relevant morphisms we have

$$\begin{aligned} \Psi \circ \tilde{\Phi}_\varepsilon &= \Psi \circ m_Y^+ \circ (\Phi_\varepsilon \otimes \text{id}) \circ \alpha^+ = m_Z^+ \circ (\Psi \otimes \text{id}) \circ (\Phi_\varepsilon \otimes \text{id}) \circ \alpha^+ \\ &= m_Z^+ \circ (\Phi \otimes \text{id}) \circ \alpha^+ = \Phi \circ m_X^+ \circ \alpha^+ = \Phi. \end{aligned}$$

Hence, X is C_1C_2 -projective. ■

Note that the converse of Proposition 2.2 (when $C_1 = C_2 = 1$) is not true in general, as \mathcal{A} is both 1-projective and relatively 1-projective in $\mathbf{mod}\text{-}\mathcal{A}$ for any unital C^* -algebra. However, the only C^* -algebra which is a 1-projective operator space is \mathbb{C} by [2, Theorem 3.4].

Given a completely contractive Banach algebra \mathcal{A} and $X \in \mathbf{mod}\text{-}\mathcal{A}$, there is a canonical completely contractive morphism $\Delta^+: X \rightarrow \mathcal{CB}(\mathcal{A}^+, X)$ given by

$$\Delta^+(x)(a) = x \cdot a, \quad x \in X, a \in \mathcal{A}^+,$$

where the right \mathcal{A} -module structure on $\mathcal{CB}(\mathcal{A}^+, X)$ is defined by

$$(\Psi \cdot a)(b) = \Psi(ab), \quad a \in \mathcal{A}, \Psi \in \mathcal{CB}(\mathcal{A}^+, X), b \in \mathcal{A}^+.$$

An analogous construction exists for objects in $\mathcal{A}\text{-mod}$. Let $C \geq 1$. Then X is *relatively C -injective* if there exists a morphism $\Phi^+: \mathcal{CB}(\mathcal{A}^+, X) \rightarrow X$ such that $\Phi^+ \circ \Delta^+ = \text{id}_X$ and $\|\Phi^+\|_{cb} \leq C$. When X is faithful, then X is relatively C -injective if and only if there exists a morphism $\Phi: \mathcal{CB}(\mathcal{A}, X) \rightarrow X$ such that $\Phi \circ \Delta = \text{id}_X$ and $\|\Phi\|_{cb} \leq C$ by the operator analogue of [9, Proposition 1.7], where $\Delta(x)(a) := \Delta^+(x)(a)$ for $x \in X$ and $a \in \mathcal{A}$. We say that X is *C -injective* if for every $Y, Z \in \mathbf{mod}\text{-}\mathcal{A}$, every completely isometric morphism $\Psi: Y \hookrightarrow Z$, and every morphism $\Phi: Y \rightarrow X$, there exists a morphism $\tilde{\Phi}: Z \rightarrow X$ such that $\|\tilde{\Phi}\|_{cb} \leq C\|\Phi\|_{cb}$ and $\tilde{\Phi} \circ \Psi = \Phi$, that is, the following diagram commutes:

$$\begin{array}{ccc} & Z & \\ & \uparrow & \searrow \tilde{\Phi} \\ Y & \xrightarrow{\Phi} & X \end{array}$$

Clearly, when $\mathcal{A} = \mathbb{C}$, the definition of C -injectivity coincides with that of a C -injective operator space [33, §24]. For general \mathcal{A} , the dual X^* of any $X \in \mathbf{mod}\text{-}\mathcal{A}$ has a canonical left \mathcal{A} -module structure, and it follows that X^* is C -injective in $\mathcal{A}\text{-mod}$ whenever X is C -projective in $\mathbf{mod}\text{-}\mathcal{A}$ by an operator module version of [2, Theorem 3.5]. The next proposition also appears to be new and will be used in the proof of our main result.

Proposition 2.3 *Let \mathcal{A} be a completely contractive Banach algebra and $X \in \mathbf{mod}\text{-}\mathcal{A}$. If X is C_1 -injective in $\mathbf{mod}\text{-}\mathbb{C}$ and is relatively C_2 -injective in $\mathbf{mod}\text{-}\mathcal{A}$, then X is C_1C_2 -injective in $\mathbf{mod}\text{-}\mathcal{A}$.*

Proof We first show that $\mathcal{CB}(\mathcal{A}^+, X)$ is C_1 -injective in $\mathbf{mod}\text{-}\mathcal{A}$ using the standard argument. To this end, let $Y, Z \in \mathbf{mod}\text{-}\mathcal{A}$, let $\kappa: Y \hookrightarrow Z$ be a completely isometric morphism, and let $\alpha: Y \rightarrow \mathcal{CB}(\mathcal{A}^+, X)$ be a morphism. Define $\beta: Y \rightarrow X$ by $\beta(y) = \alpha(y)(e)$, $y \in Y$, where e is the identity in \mathcal{A}^+ . Then $\|\beta\|_{cb} \leq \|\alpha\|_{cb}$, and by C_1 -injectivity of X in $\mathbf{mod}\text{-}\mathbb{C}$, there exists an extension $\tilde{\beta}: Z \rightarrow X$ satisfying $\beta = \tilde{\beta} \circ \kappa$ and $\|\tilde{\beta}\|_{cb} \leq C_1\|\beta\|_{cb} \leq C_1\|\alpha\|_{cb}$. Define $\tilde{\alpha}: Z \rightarrow \mathcal{CB}(\mathcal{A}^+, X)$ by $\tilde{\alpha}(z)(a) = \tilde{\beta}(z \cdot a)$, for $z \in Z$, $a \in \mathcal{A}^+$. Then $(\tilde{\alpha}(z) \cdot a)(b) = \tilde{\alpha}(z)(ab) = \tilde{\beta}(z \cdot ab) = \tilde{\alpha}(z \cdot a)(b)$ for all $z \in Z$ and $a, b \in \mathcal{A}$. Thus, $\tilde{\alpha}$ is a module map extending α such that $\|\tilde{\alpha}\|_{cb} \leq C_1\|\alpha\|_{cb}$.

By relative C_2 -injectivity of X in $\mathbf{mod}\text{-}\mathcal{A}$ there is a morphism $\Phi^+: \mathcal{CB}(\mathcal{A}^+, X) \rightarrow X$ satisfying $\Phi^+ \circ \Delta^+ = \text{id}_X$ and $\|\Phi^+\|_{cb} \leq C_2$. Thus, if $Y, Z \in \mathbf{mod}\text{-}\mathcal{A}$ with $\kappa: Y \hookrightarrow Z$ a completely isometric morphism, and $\alpha: Y \rightarrow X$ is a morphism, then we may extend the morphism $\Delta^+ \circ \alpha: Y \rightarrow \mathcal{CB}(\mathcal{A}^+, X)$ to a morphism $\tilde{\alpha}: Z \rightarrow \mathcal{CB}(\mathcal{A}^+, X)$ with $\|\tilde{\alpha}\|_{cb} \leq C_1\|\Delta^+ \circ \alpha\|_{cb} \leq C_1\|\alpha\|_{cb}$. The morphism $\Phi^+ \circ \tilde{\alpha}: Z \rightarrow X$ satisfies $\Phi^+ \circ \tilde{\alpha} \circ \kappa = \alpha$ and $\|\Phi^+ \circ \tilde{\alpha}\|_{cb} \leq C_1C_2\|\alpha\|_{cb}$, and is therefore the desired extension. ■

The converse of Proposition 2.3 is not true in general (when $C_1 = C_2 = 1$). Indeed, for any unital completely contractive Banach algebra \mathcal{A} and any 1-injective operator space X , it follows from the proof of Proposition 2.3 that $\mathcal{CB}(\mathcal{A}, X)$ is 1-injective in $\mathbf{mod}\text{-}\mathcal{A}$. This clearly implies relative 1-injectivity in $\mathbf{mod}\text{-}\mathcal{A}$. However, consider $\mathcal{A} = B(G)$ and $X = \mathbb{C}$, where $B(G)$ is the Fourier–Stieltjes algebra of a non-amenable discrete group G . Since $B(G)$ is the operator dual of the full group C^* -algebra $C^*(G)$, we have $\mathcal{CB}(\mathcal{A}, X) = B(G)^* = C^*(G)^{**}$. If this were a 1-injective operator space, the group C^* -algebra $C^*(G)$ would be nuclear [4], forcing G to be amenable by [29, Theorem 4.2].

Remark 2.4 Our notions of projectivity and injectivity are closer in spirit to the approach taken in operator space theory [2] and the recent approach of Helemskii [19] rather than Banach algebra homology, where the related notions are usually studied solely from the relative perspective. In particular, we caution the reader that “injectivity”, as defined in our previous work [7], coincides with relative 1-injectivity as defined above.

3 Locally Compact Quantum Groups

A *locally compact quantum group* is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Gamma, \varphi, \psi)$, where $L^\infty(\mathbb{G})$ is a Hopf-von Neumann algebra with co-multiplication

$$\Gamma: L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G}),$$

and φ and ψ are fixed left and right Haar weights on $L^\infty(\mathbb{G})$, respectively [28, 41]. For every locally compact quantum group \mathbb{G} , there exists a *left fundamental unitary operator* W on $L_2(\mathbb{G}, \varphi) \otimes L_2(\mathbb{G}, \varphi)$ and a *right fundamental unitary operator* V on $L_2(\mathbb{G}, \psi) \otimes L_2(\mathbb{G}, \psi)$ implementing the co-multiplication Γ via

$$\Gamma(x) = W^*(1 \otimes x)W = V(x \otimes 1)V^*, \quad x \in L^\infty(\mathbb{G}).$$

Both unitaries satisfy the *pentagonal relation*, that is,

$$W_{12}W_{13}W_{23} = W_{23}W_{12} \quad \text{and} \quad V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

By [28, Proposition 2.11], we may identify $L_2(\mathbb{G}, \varphi)$ and $L_2(\mathbb{G}, \psi)$, so we will simply use $L^2(\mathbb{G})$ for this Hilbert space throughout the paper. We denote by R the unitary antipode of \mathbb{G} .

Let $L^1(\mathbb{G})$ denote the predual of $L^\infty(\mathbb{G})$. Then the pre-adjoint of Γ induces an associative completely contractive multiplication on $L^1(\mathbb{G})$, defined by

$$\star: L^1(\mathbb{G}) \overline{\otimes} L^1(\mathbb{G}) \ni f \otimes g \mapsto f \star g = \Gamma_*(f \otimes g) \in L^1(\mathbb{G}).$$

The multiplication \star is a complete quotient map from $L^1(\mathbb{G}) \overline{\otimes} L^1(\mathbb{G})$ onto $L^1(\mathbb{G})$, implying $\langle L^1(\mathbb{G}) \star L^1(\mathbb{G}) \rangle = L^1(\mathbb{G})$. Moreover, $L^1(\mathbb{G})$ is always self-induced. The proof is a simple application of [42, Theorem 2.7], but we provide the details for the convenience of the reader.

Proposition 3.1 *Let \mathbb{G} be a locally compact quantum group. Then $L^1(\mathbb{G})$ is a self-induced completely contractive Banach algebra.*

Proof Let $\tilde{m}: L^1(\mathbb{G}) \overline{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G})$ be the induced multiplication map. Then $\tilde{m}^*: L^\infty(\mathbb{G}) \rightarrow (L^1(\mathbb{G}) \overline{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}))^*$ is nothing but the co-multiplication Γ . Since $(L^1(\mathbb{G}) \overline{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}))^* \cong N^\perp$, where $N \subseteq L^1(\mathbb{G}) \overline{\otimes} L^1(\mathbb{G})$ is the closed linear span of $\{f \star g \otimes h - f \otimes g \star h \mid f, g, h \in L^1(\mathbb{G})\}$, given $X \in (L^1(\mathbb{G}) \overline{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}))^* \subseteq L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$, it follows that $(\Gamma \otimes \text{id})(X) = (\text{id} \otimes \Gamma)(X)$. Hence, $X \in \Gamma(L^\infty(\mathbb{G}))$ by [42, Theorem 2.7], and \tilde{m}^* is surjective. Since $\tilde{m}^* = \Gamma$ is also a complete isometry, the result follows. ■

For any locally compact quantum group \mathbb{G} , the canonical $L^1(\mathbb{G})$ -bimodule structure on $L^\infty(\mathbb{G})$ is given by $f \star x = (\text{id} \otimes f)\Gamma(x)$ and $x \star f = (f \otimes \text{id})\Gamma(x)$, for $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$. A *left invariant mean* on $L^\infty(\mathbb{G})$ is a state $m \in L^\infty(\mathbb{G})^*$ satisfying $\langle m, x \star f \rangle = \langle f, 1 \rangle \langle m, x \rangle$, for $x \in L^\infty(\mathbb{G})$, $f \in L^1(\mathbb{G})$. Right and two-sided invariant means are defined similarly. A locally compact quantum group \mathbb{G} is said to be *amenable* if there exists a left invariant mean on $L^\infty(\mathbb{G})$. It is known that \mathbb{G} is amenable if and only if there exists a right (equivalently, two-sided) invariant mean (cf. [13, Proposition 3]). We say that \mathbb{G} is *co-amenable* if $L^1(\mathbb{G})$ has a bounded left (equivalently, right or two-sided) approximate identity (cf. [1, Theorem 3.1]).

The *left regular representation* $\lambda: L^1(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ of \mathbb{G} , defined by

$$\lambda(f) = (f \otimes \text{id})(W), \quad f \in L^1(\mathbb{G}),$$

is an injective, completely contractive homomorphism from $L^1(\mathbb{G})$ into $\mathcal{B}(L^2(\mathbb{G}))$. Then $L^\infty(\widehat{\mathbb{G}}) := \{\lambda(f) : f \in L^1(\mathbb{G})\}''$ is the von Neumann algebra associated with the dual quantum group $\widehat{\mathbb{G}}$. Analogously, we have the *right regular representation* $\rho: L^1(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ defined by $\rho(f) = (\text{id} \otimes f)(V)$, for $f \in L^1(\mathbb{G})$, which is also an injective, completely contractive homomorphism from $L^1(\mathbb{G})$ into $\mathcal{B}(L^2(\mathbb{G}))$. Then $L^\infty(\widehat{\mathbb{G}}') := \{\rho(f) : f \in L^1(\mathbb{G})\}''$ is the von Neumann algebra associated with the quantum group $\widehat{\mathbb{G}}'$. It follows that $L^\infty(\widehat{\mathbb{G}}') = L^\infty(\widehat{\mathbb{G}})'$, and the left and right fundamental unitaries satisfy $W \in L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$ and $V \in L^\infty(\widehat{\mathbb{G}}') \overline{\otimes} L^\infty(\mathbb{G})$ [28, Proposition 2.15]. Moreover, dual quantum groups always satisfy [44, Proposition 3.4]

$$L^\infty(\mathbb{G}) \cap L^\infty(\widehat{\mathbb{G}}) = L^\infty(\mathbb{G}) \cap L^\infty(\widehat{\mathbb{G}}') = \mathbb{C}1.$$

If G is a locally compact group, we let $\mathbb{G}_a = (L^\infty(G), \Gamma_a, \varphi_a, \psi_a)$ denote the *commutative* quantum group associated with the commutative von Neumann algebra $L^\infty(G)$, where the co-multiplication is given by $\Gamma_a(f)(s, t) = f(st)$, and φ_a and ψ_a are integration with respect to a left and right Haar measure, respectively. The dual $\widehat{\mathbb{G}}_a$ of \mathbb{G}_a is the *co-commutative* quantum group $\mathbb{G}_s = (\text{VN}(G), \Gamma_s, \varphi_s, \psi_s)$, where $\text{VN}(G)$ is the left group von Neumann algebra with co-multiplication $\Gamma_s(\lambda(t)) = \lambda(t) \otimes \lambda(t)$, and $\varphi_s = \psi_s$ is Haagerup's Plancherel weight (cf. [38, §VII.3]). Then $L^1(\mathbb{G}_a)$ is the usual group convolution algebra $L^1(G)$, and $L^1(\mathbb{G}_s)$ is the Fourier algebra $A(G)$. It is known that every commutative locally compact quantum group is of the form \mathbb{G}_a [37, 43, Theorem 2; §2]. Therefore, every commutative locally compact quantum group is co-amenable, and is amenable if and only if the underlying locally compact group is amenable. By duality, every co-commutative locally compact quantum group is of the form \mathbb{G}_s , which is always amenable [34, Theorem 4], and is co-amenable if and only if the underlying locally compact group is amenable, by Leptin's theorem [30].

For a locally compact quantum group \mathbb{G} , we let $C_0(\mathbb{G}) := \overline{\widehat{\lambda}(L^1(\widehat{\mathbb{G}}))}^{\|\cdot\|}$ denote the *reduced quantum group C^* -algebra* of \mathbb{G} . We say that \mathbb{G} is *compact* if $C_0(\mathbb{G})$ is a unital C^* -algebra, in which case we denote $C_0(\mathbb{G})$ by $C(\mathbb{G})$. We say that \mathbb{G} is *discrete* if $L^1(\mathbb{G})$ is unital. It is well known that \mathbb{G} is compact if and only if $\widehat{\mathbb{G}}$ is discrete, and in that case, $L^1(\widehat{\mathbb{G}}) \cong \bigoplus_1 \{T_{n_\alpha}(\mathbb{C}) \mid \alpha \in \text{Irr}(\mathbb{G})\}$, where $T_{n_\alpha}(\mathbb{C})$ is the space of $n_\alpha \times n_\alpha$ trace-class operators, and $\text{Irr}(\mathbb{G})$ denotes the set of (equivalence classes of) irreducible co-representations of the compact quantum group \mathbb{G} [45]. For general \mathbb{G} , the operator dual $M(\mathbb{G}) := C_0(\mathbb{G})^*$ is a completely contractive Banach algebra containing $L^1(\mathbb{G})$ as a norm closed two-sided ideal via the map $L^1(\mathbb{G}) \ni f \mapsto f|_{C_0(\mathbb{G})} \in M(\mathbb{G})$ [21].

We let $C_u(\mathbb{G})$ be the *universal quantum group C^* -algebra* of \mathbb{G} , and denote the canonical surjective $*$ -homomorphism onto $C_0(\mathbb{G})$ by $\pi_u: C_u(\mathbb{G}) \rightarrow C_0(\mathbb{G})$ [27]. The space $C_u(\mathbb{G})^*$ then has the structure of a unital completely contractive Banach algebra such that the map $L^1(\mathbb{G}) \rightarrow C_u(\mathbb{G})^*$ given by the composition of the inclusion $L^1(\mathbb{G}) \subseteq M(\mathbb{G})$ and $\pi_u^*: M(\mathbb{G}) \rightarrow C_u(\mathbb{G})^*$ is a completely isometric homomorphism, and it follows that $L^1(\mathbb{G})$ is a norm closed two-sided ideal in $C_u(\mathbb{G})^*$ [27, Proposition 8.3].

Let \mathbb{G} be a locally compact quantum group. An element $\widehat{b}' \in L^\infty(\widehat{\mathbb{G}})'$ is said to be a *completely bounded right multiplier* of $L^1(\mathbb{G})$ if $\rho(f)\widehat{b}' \in \rho(L^1(\mathbb{G}))$ for all $f \in L^1(\mathbb{G})$ and the induced map $m_{\widehat{b}'}^r: L^1(\mathbb{G}) \ni f \mapsto \rho^{-1}(\rho(f)\widehat{b}') \in L^1(\mathbb{G})$ is completely bounded on $L^1(\mathbb{G})$. We let $M_{cb}^r(L^1(\mathbb{G}))$ denote the space of all completely bounded right multipliers of $L^1(\mathbb{G})$, which is a completely contractive Banach algebra with respect to the norm

$$\|[\widehat{b}'_{ij}]\|_{M_n(M_{cb}^r(L^1(\mathbb{G})))} = \|[m_{\widehat{b}'_{ij}}^r]\|_{cb}.$$

Completely bounded left multipliers are defined analogously, and we denote the corresponding completely contractive Banach algebra by $M_{cb}^l(L^1(\mathbb{G}))$. We now review the relevant properties of completely bounded multipliers, adopting the notation of [23].

There is a canonical, injective, completely contractive homomorphism

$$\widetilde{\rho}: C_u(\mathbb{G})^* \rightarrow M_{cb}^r(L^1(\mathbb{G})),$$

extending $\widetilde{\rho}$ from [21, §4], which maps $\mu \in C_u(\mathbb{G})^*$ to the operator of right multiplication by μ on $L^1(\mathbb{G})$. In general, given $\widehat{b}' \in M_{cb}^r(L^1(\mathbb{G}))$, the adjoint $\Theta^r(\widehat{b}') := (m_{\widehat{b}'}^r)^*$ defines a normal completely bounded right $L^1(\mathbb{G})$ -module map on $L^\infty(\mathbb{G})$. The restriction $\Theta^r(\widehat{b}')|_{C_0(\mathbb{G})}$ leaves $C_0(\mathbb{G})$ invariant by [23, Proposition 4.1], and, together with [23, Proposition 4.2], we have the completely isometric identifications

$$\Theta^r: M_{cb}^r(L^1(\mathbb{G})) \cong \mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G})) \cong \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G})).$$

It is known that $M_{cb}^r(L^1(\mathbb{G}))$ is a dual operator space [21, Theorem 3.5], with predual $Q_{cb}^r(L^1(\mathbb{G}))$. When $\mathbb{G} = \mathbb{G}_s$ is co-commutative, Haagerup and Kraus gave a representation for elements of $Q_{cb}(L^1(\mathbb{G}_s)) = Q_{cb}(A(G))$ as $\Omega_{A,\rho}$ for $A \in C_\lambda^*(G) \otimes_{\min} K_\infty$ and $\rho \in A(G) \widehat{\otimes} T_\infty$ [18, Proposition 1.5], where $\langle \varphi, \Omega_{A,\rho} \rangle = \langle (\Theta^r(\varphi) \otimes \text{id}_{K_\infty})(A), \rho \rangle$, for $\varphi \in M_{cb}A(G)$; $C_\lambda^*(G)$ is the reduced C^* -algebra of G ; the spaces K_∞ and T_∞ denote the compact and trace-class operators on a countably infinite-dimensional Hilbert space, respectively, and \otimes_{\min} denotes the minimum tensor product of C^* -algebras. This was later generalized to the setting of Kac algebras by Kraus and Ruan [26, Theorem 3.3]. Relying upon the general result [18, Lemma 1.6], their argument readily extends to arbitrary locally compact quantum groups.

Proposition 3.2 *Let \mathbb{G} be a locally compact quantum group. Then*

$$Q_{cb}^r(L^1(\mathbb{G})) = \{\Omega_{A,\rho} \mid A \in C_0(\mathbb{G}) \otimes_{\min} K_\infty, \rho \in L^1(\mathbb{G}) \widehat{\otimes} T_\infty\},$$

where $\langle \widehat{b}', \Omega_{A,\rho} \rangle = \langle (\Theta^r(\widehat{b}') \otimes \text{id}_{K_\infty})(A), \rho \rangle$, $\widehat{b}' \in M_{cb}^r(L^1(\mathbb{G}))$.

4 $\mathcal{J}(L^2(\mathbb{G})) \simeq \mathcal{B}(L^2(\mathbb{G}))$

Let \mathbb{G} be a locally compact quantum group. The right fundamental unitary V of \mathbb{G} induces a co-associative co-multiplication

$$\Gamma^r: \mathcal{B}(L^2(\mathbb{G})) \ni T \mapsto V(T \otimes 1)V^* \in \mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G})),$$

and the restriction of Γ^r to $L^\infty(\mathbb{G})$ yields the original co-multiplication Γ on $L^\infty(\mathbb{G})$. The pre-adjoint of Γ^r induces an associative completely contractive multiplication on the space of trace-class operators $\mathcal{T}(L^2(\mathbb{G}))$, defined by

$$\triangleright: \mathcal{T}(L^2(\mathbb{G})) \widehat{\otimes} \mathcal{T}(L^2(\mathbb{G})) \ni \omega \otimes \tau \mapsto \omega \triangleright \tau = \Gamma_*^r(\omega \otimes \tau) \in \mathcal{T}(L^2(\mathbb{G})).$$

Since Γ^r is a complete isometry, it follows that Γ_*^r is a complete quotient map, so we have

$$(4.1) \quad \mathcal{T}(L^2(\mathbb{G})) = \langle \mathcal{T}(L^2(\mathbb{G})) \triangleright \mathcal{T}(L^2(\mathbb{G})) \rangle.$$

Analogously, the left fundamental unitary W of \mathbb{G} induces a co-associative co-multiplication $\Gamma^l: \mathcal{B}(L^2(\mathbb{G})) \ni T \mapsto W^*(1 \otimes T)W \in \mathcal{B}(L^2(\mathbb{G})) \widehat{\otimes} \mathcal{B}(L^2(\mathbb{G}))$, and the restriction of Γ^l to $L^\infty(\mathbb{G})$ is also equal to Γ . The pre-adjoint of Γ^l induces another associative completely contractive multiplication

$$\triangleleft: \mathcal{T}(L^2(\mathbb{G})) \widehat{\otimes} \mathcal{T}(L^2(\mathbb{G})) \ni \omega \otimes \tau \mapsto \omega \triangleleft \tau = \Gamma_*^l(\omega \otimes \tau) \in \mathcal{T}(L^2(\mathbb{G})).$$

It was shown in [21, Lemma 5.2] that the pre-annihilator $L^\infty(\mathbb{G})_\perp$ of $L^\infty(\mathbb{G})$ in $\mathcal{T}(L^2(\mathbb{G}))$ is a norm closed two-sided ideal in $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$ and $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$, respectively, and the complete quotient map

$$\pi: \mathcal{T}(L^2(\mathbb{G})) \ni \omega \mapsto f = \omega|_{L^\infty(\mathbb{G})} \in L^1(\mathbb{G})$$

is an algebra homomorphism from $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$, respectively, $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$, onto $L^1(\mathbb{G})$.

By [28, Proposition 2.1] the unitary antipode R satisfies $R(x) = \widehat{J}x^*\widehat{J}$, for $x \in L^\infty(\mathbb{G})$, where \widehat{J} is the modular conjugation associated to the dual left Haar weight $\widehat{\varphi}$. It therefore extends to a $*$ -anti-automorphism (still denoted)

$$R: \mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathcal{B}(L^2(\mathbb{G})),$$

via $R(T) = \widehat{J}T^*\widehat{J}$, $T \in \mathcal{B}(L^2(\mathbb{G}))$. The extended antipode maps $L^\infty(\mathbb{G})$ and $L^\infty(\widehat{\mathbb{G}})$ onto $L^\infty(\mathbb{G})$ and $L^\infty(\widehat{\mathbb{G}}')$, respectively, and satisfies the generalized antipode relations, *i.e.*,

$$(4.2) \quad (R \otimes R) \circ \Gamma^r = \Sigma \circ \Gamma^l \circ R \quad \text{and} \quad (R \otimes R) \circ \Gamma^l = \Sigma \circ \Gamma^r \circ R,$$

where Σ is the flip map on $\mathcal{B}(L^2(\mathbb{G})) \widehat{\otimes} \mathcal{B}(L^2(\mathbb{G}))$. At the level of $\mathcal{T}(L^2(\mathbb{G}))$, the relations (4.2) mean

$$R_*(\omega \triangleright \tau) = R_*(\tau) \triangleleft R_*(\omega) \quad \text{and} \quad R_*(\omega \triangleleft \tau) = R_*(\tau) \triangleright R_*(\omega)$$

for all $\omega, \tau \in \mathcal{T}(L^2(\mathbb{G}))$. We may therefore pass between the left and right products using R , and as a result, we will often focus on the right product \triangleright throughout the article.

Since $L^2(\mathbb{G}) \cong L^2(\widehat{\mathbb{G}})$ for any locally compact quantum group \mathbb{G} , applying the above construction to the co-multiplication $\widehat{\Gamma}$ on $L^\infty(\widehat{\mathbb{G}})$ yields two *dual* products

$$\begin{aligned} \widehat{\triangleright}: \mathcal{T}(L^2(\mathbb{G})) \widehat{\otimes} \mathcal{T}(L^2(\mathbb{G})) \ni \omega \otimes \tau \mapsto \omega \widehat{\triangleright} \tau &= \widehat{\Gamma}_*^r(\omega \otimes \tau) \in \mathcal{T}(L^2(\mathbb{G})), \\ \widehat{\triangleleft}: \mathcal{T}(L^2(\mathbb{G})) \widehat{\otimes} \mathcal{T}(L^2(\mathbb{G})) \ni \omega \otimes \tau \mapsto \omega \widehat{\triangleleft} \tau &= \widehat{\Gamma}_*^l(\omega \otimes \tau) \in \mathcal{T}(L^2(\mathbb{G})). \end{aligned}$$

This lifting of quantum group convolution to $\mathcal{T}(L^2(\mathbb{G}))$ allows one to study properties of \mathbb{G} and $\widehat{\mathbb{G}}$, as well as their interactions on a *single space*. One such interaction was obtained in [25], and states that the dual products anti-commute.

Theorem 4.1 ([25, Theorem 3.3]) *Let \mathbb{G} be a locally compact quantum group. Then for every $\rho, \omega, \tau \in \mathcal{T}(L^2(\mathbb{G}))$ we have*

$$(4.3) \quad (\rho \triangleright \omega) \widehat{\triangleright} \tau = (\rho \widehat{\triangleright} \tau) \triangleright \omega.$$

Equation (4.3) has an important consequence (Proposition 4.2) that will be used in the proof of the main result.

For a locally compact quantum group \mathbb{G} , the multiplications \triangleright and \triangleleft define operator $\mathcal{T}(L^2(\mathbb{G}))$ -bimodule structures on $\mathcal{B}(L^2(\mathbb{G}))$ such that for $x \in L^\infty(\mathbb{G})$ and $f = \omega|_{L^\infty(\mathbb{G})}$ with $\omega \in \mathcal{T}(L^2(\mathbb{G}))$, we have

$$x \triangleright \omega = x \triangleleft \omega = x \star f \quad \text{and} \quad \omega \triangleleft x = \omega \triangleright x = f \star x.$$

The bimodule actions of $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$ and $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$ on $\mathcal{B}(L^2(\mathbb{G}))$ are therefore liftings of the usual bimodule action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$. For details on these bimodules we refer the reader to [7, 21]. In what follows we denote the algebra of completely bounded right $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$ -module (respectively, left $(\mathcal{T}(L^2(\mathbb{G})), \widehat{\triangleright})$ -module) maps by $\mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G})))$ (respectively, ${}_{\mathcal{T}_\widehat{\triangleright}}\mathcal{CB}(\mathcal{B}(L^2(\mathbb{G})))$).

In [21, Remark 7.4], the authors observed that for co-amenable \mathbb{G} we have

$$\mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G}))) \subseteq \mathcal{CB}_{L^\infty(\widehat{\mathbb{G}})}^{L^\infty(\mathbb{G})}(\mathcal{B}(L_2(\mathbb{G}))),$$

where the right-hand side is the algebra of completely bounded $L^\infty(\widehat{\mathbb{G}})$ -bimodule maps on $\mathcal{B}(L^2(\mathbb{G}))$ that leave $L^\infty(\mathbb{G})$ globally invariant. As a corollary to the commutation relation (4.3), we can remove the co-amenable hypothesis in the above inclusion using the following “automatic” module property.

Proposition 4.2 *Let \mathbb{G} be a locally compact quantum group. Then*

$$\mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G}))) \subseteq {}_{\mathcal{T}_\widehat{\triangleright}}\mathcal{CB}(\mathcal{B}(L^2(\mathbb{G}))).$$

Proof Let $\Phi \in \mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G})))$, and fix $\rho \in \mathcal{T}(L^2(\mathbb{G}))$ and $T \in \mathcal{B}(L^2(\mathbb{G}))$. Then for any $\omega, \tau \in \mathcal{T}(L^2(\mathbb{G}))$, we have

$$\begin{aligned} \langle (\rho \widehat{\triangleright} T) \triangleright \tau, \omega \rangle &= \langle \rho \widehat{\triangleright} T, \tau \triangleright \omega \rangle = \langle T, (\tau \triangleright \omega) \widehat{\triangleright} \rho \rangle \\ &= \langle T, (\tau \widehat{\triangleright} \rho) \triangleright \omega \rangle = \langle T \triangleright (\tau \widehat{\triangleright} \rho), \omega \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \langle \Phi(\rho \widehat{\triangleright} T), \tau \triangleright \omega \rangle &= \langle \Phi(\rho \widehat{\triangleright} T) \triangleright \tau, \omega \rangle = \langle \Phi((\rho \widehat{\triangleright} T) \triangleright \tau), \omega \rangle \\ &= \langle \Phi(T \triangleright (\tau \widehat{\triangleright} \rho)), \omega \rangle = \langle \Phi(T) \triangleright (\tau \widehat{\triangleright} \rho), \omega \rangle \\ &= \langle \Phi(T), (\tau \widehat{\triangleright} \rho) \triangleright \omega \rangle = \langle \Phi(T), (\tau \triangleright \omega) \widehat{\triangleright} \rho \rangle \\ &= \langle \rho \widehat{\triangleright} \Phi(T), \tau \triangleright \omega \rangle. \end{aligned}$$

By (4.1) it follows that $\Phi(\rho \widehat{\triangleright} T) = \rho \widehat{\triangleright} \Phi(T)$, as required. ■

Corollary 4.3 For any locally compact quantum group \mathbb{G} , we have

$$\mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G}))) \subseteq \mathcal{CB}_{L^\infty(\widehat{\mathbb{G}})}^{L^\infty(\mathbb{G})}(\mathcal{B}(L_2(\mathbb{G}))).$$

Proof Let $\Phi \in \mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G})))$, and $\widehat{x}, \widehat{y} \in L^\infty(\widehat{\mathbb{G}})$. Then for any $\rho \in \mathcal{T}(L^2(\mathbb{G}))$ and $T \in \mathcal{B}(L^2(\mathbb{G}))$ we have

$$\begin{aligned} (\widehat{xT}\widehat{y}) \triangleright \rho &= (\rho \otimes \text{id})V(\widehat{xT}\widehat{y} \otimes 1)V^* = (\rho \otimes \text{id})((\widehat{x} \otimes 1)V(T \otimes 1)V^*(\widehat{y} \otimes 1)) \\ &= (\widehat{y} \cdot \rho \cdot \widehat{x} \otimes \text{id})V(T \otimes 1)V^* = T \triangleright (\widehat{y} \cdot \rho \cdot \widehat{x}). \end{aligned}$$

Thus, for any $\omega \in \mathcal{T}(L^2(\mathbb{G}))$ we obtain

$$\begin{aligned} \langle \Phi(\widehat{xT}\widehat{y}), \rho \triangleright \omega \rangle &= \langle \Phi(\widehat{xT}\widehat{y}) \triangleright \rho, \omega \rangle = \langle \Phi((\widehat{xT}\widehat{y}) \triangleright \rho), \omega \rangle \\ &= \langle \Phi(T \triangleright (\widehat{y} \cdot \rho \cdot \widehat{x})), \omega \rangle = \langle \Phi(T) \triangleright (\widehat{y} \cdot \rho \cdot \widehat{x}), \omega \rangle \\ &= \langle (\widehat{x}\Phi(T)\widehat{y}) \triangleright \rho, \omega \rangle = \langle \widehat{x}\Phi(T)\widehat{y}, \rho \triangleright \omega \rangle. \end{aligned}$$

Again by (4.1), it follows that Φ is an $L^\infty(\widehat{\mathbb{G}})$ -bimodule map on $\mathcal{B}(L^2(\mathbb{G}))$.

By Proposition 4.2 $\Phi \in_{\mathcal{T}_\triangleright} \mathcal{CB}(\mathcal{B}(L^2(\mathbb{G})))$, and since $\widehat{V} \in L^\infty(\mathbb{G})' \otimes L^\infty(\widehat{\mathbb{G}})$, for any $x \in L^\infty(\mathbb{G})$ and $\rho \in \mathcal{T}(L^2(\mathbb{G}))$ we have

$$(\text{id} \otimes \rho)\widehat{V}(\Phi(x) \otimes 1)\widehat{V}^* = \rho \widehat{\triangleright} \Phi(x) = \Phi(\rho \widehat{\triangleright} x) = \langle \rho, 1 \rangle \Phi(x) = (\text{id} \otimes \rho)(\Phi(x) \otimes 1).$$

It follows that $\widehat{V}(\Phi(x) \otimes 1)\widehat{V}^* = \Phi(x) \otimes 1$, which implies that $\widehat{\rho}(\widehat{f})\Phi(x) = \Phi(x)\widehat{\rho}(\widehat{f})$ for every $\widehat{f} \in L^1(\widehat{\mathbb{G}})$. Since $\widehat{\rho}(L^1(\widehat{\mathbb{G}}))$ is weak* dense in $L^\infty(\mathbb{G})'$, we have $\Phi(x) \in L^\infty(\mathbb{G})'' = L^\infty(\mathbb{G})$. Thus, Φ leaves $L^\infty(\mathbb{G})$ globally invariant, and the claim follows. ■

In [7], we studied the existence of conditional expectations

$$E: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\widehat{\mathbb{G}})$$

commuting with the four $\mathcal{T}(L^2(\mathbb{G}))$ -module structures on $\mathcal{B}(L^2(\mathbb{G}))$ arising from \mathbb{G} . We now complete this picture by studying the four remaining $\mathcal{T}(L^2(\mathbb{G}))$ -module structures on $\mathcal{B}(L^2(\mathbb{G}))$ arising from $\widehat{\mathbb{G}}$. We denote by

$$\mathcal{T}_\triangleleft \mathcal{CB}(\mathcal{B}(L^2(\mathbb{G}))) \quad \text{and} \quad \mathcal{CB}_{\mathcal{T}_\triangleleft}(\mathcal{B}(L^2(\mathbb{G})))$$

the algebra of completely bounded left and right $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$ -module maps on $\mathcal{B}(L^2(\mathbb{G}))$, respectively, and similarly for $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$.

Proposition 4.4 Let \mathbb{G} be a locally compact quantum group. There exists a conditional expectation $E: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\widehat{\mathbb{G}})$ in $\mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G})))$ if and only if $\mathbb{G} = \mathbb{C}1$.

Proof For any $T \in \mathcal{B}(L^2(\mathbb{G}))$ and $\rho \in \mathcal{T}(L^2(\mathbb{G}))$, we have $T \widehat{\triangleright} \rho \in L^\infty(\widehat{\mathbb{G}})$, so if such a conditional expectation E exists, then $T \widehat{\triangleright} \rho = E(T \widehat{\triangleright} \rho) = E(T) \widehat{\triangleright} \rho$. By density of products (4.1), it follows that $E(T) = T$. In particular $\mathcal{B}(L^2(\mathbb{G})) \subseteq L^\infty(\widehat{\mathbb{G}})$, which entails that $\mathbb{G} = \widehat{\mathbb{G}} = \mathbb{C}1$. The converse is trivial. ■

Proposition 4.5 Let \mathbb{G} be a locally compact quantum group. There exists a conditional expectation $E: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\widehat{\mathbb{G}})$ in $\mathcal{T}_\triangleleft \mathcal{CB}(\mathcal{B}(L^2(\mathbb{G})))$ if and only if $\mathbb{G} = \mathbb{C}1$.

Proof Using the extended unitary antipode of $\widehat{\mathbb{G}}$, denoted by \widehat{R} , it follows that $\widehat{R} \circ E \circ \widehat{R}$ is a conditional expectation onto $L^\infty(\widehat{\mathbb{G}})$ in $\mathcal{CB}_{\mathcal{T}_{\widehat{\mathbb{F}}}}(\mathcal{B}(L^2(\mathbb{G})))$, so the result follows from Proposition 4.4. ■

Proposition 4.6 *Let \mathbb{G} be a locally compact quantum group. There exists a conditional expectation $E: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\widehat{\mathbb{G}})$ in $\mathcal{CB}_{\mathcal{T}_{\widehat{\mathbb{F}}}}(\mathcal{B}(L^2(\mathbb{G})))$ if and only if \mathbb{G} is amenable.*

Proof If \mathbb{G} is amenable, then by [7, Theorem 4.2] there exists a conditional expectation onto $L^\infty(\widehat{\mathbb{G}})$ in $\mathcal{CB}_{\mathcal{T}_{\triangleright}}(\mathcal{B}(L^2(\mathbb{G})))$, which, thanks to Proposition 4.2, lies in $\mathcal{CB}_{\mathcal{T}_{\widehat{\mathbb{F}}}}(\mathcal{B}(L^2(\mathbb{G})))$.

On the other hand, if there exists such a conditional expectation E , then for any $x \in L^\infty(\mathbb{G})$ and $\rho \in \mathcal{T}(L^2(\mathbb{G}))$ we have $\rho \triangleright E(x) = E(\rho \triangleright x) = \langle \rho, 1 \rangle E(x)$. As in the proof of Corollary 4.3, this implies $E(x) \in L^\infty(\mathbb{G}) \cap L^\infty(\widehat{\mathbb{G}}) = \mathbb{C}1$. Hence, \mathbb{G} is amenable by [36, Theorem 3]. ■

Proposition 4.7 *Let \mathbb{G} be a locally compact quantum group. There exists a conditional expectation $E: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\widehat{\mathbb{G}})$ in $\mathcal{CB}_{\mathcal{T}_{\triangleleft}}(\mathcal{B}(L^2(\mathbb{G})))$ if and only if \mathbb{G} is amenable.*

Proof This follows from Proposition 4.6 using the extended unitary antipode \widehat{R} . ■

We record the normal version of Proposition 4.7 for later use. The proof is left to the reader.

Proposition 4.8 *Let \mathbb{G} be a locally compact quantum group. There exists a normal conditional expectation $E: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\widehat{\mathbb{G}})$ in $\mathcal{CB}_{\mathcal{T}_{\triangleleft}}(\mathcal{B}(L^2(\mathbb{G})))$ if and only if \mathbb{G} is compact.*

5 Main Result and Applications

We are now in position to prove the main result of the paper: the equivalence of amenability of $\widehat{\mathbb{G}}$ and 1-injectivity of $L^\infty(\mathbb{G})$ as an operator $L^1(\mathbb{G})$ -module. The theorem also reveals a duality between the left and right $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$ -module structures on $\mathcal{B}(L^2(\mathbb{G}))$: amenability of \mathbb{G} is captured by left injectivity of $\mathcal{B}(L^2(\mathbb{G}))$ [7, Theorem 5.5], while amenability of $\widehat{\mathbb{G}}$ is captured by right injectivity of $\mathcal{B}(L^2(\mathbb{G}))$.

Theorem 5.1 *Let \mathbb{G} be a locally compact quantum group. The following conditions are equivalent:*

- (i) $\widehat{\mathbb{G}}$ is amenable;
- (ii) $\mathcal{B}(L^2(\mathbb{G}))$ is 1-injective in $\mathbf{mod}\text{-}(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$;
- (iii) $L^\infty(\mathbb{G})$ is 1-injective in $\mathbf{mod}\text{-}L^1(\mathbb{G})$.

Proof (i) \Rightarrow (ii). By [7, Proposition 5.8] amenability of $\widehat{\mathbb{G}}$ implies that $\mathcal{B}(L^2(\mathbb{G}))$ is relatively 1-injective in $\mathbf{mod}\text{-}(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$. Since $\mathcal{B}(L^2(\mathbb{G}))$ is a 1-injective operator space, the implication follows from Proposition 2.3.

(ii) \Rightarrow (iii). If $\mathcal{B}(L^2(\mathbb{G}))$ is 1-injective in $\mathbf{mod}\text{-}(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$, there exists a completely contractive morphism $\Phi: \mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ which is a left

inverse to Γ^r , where the pertinent right $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$ -module structure on

$$\mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}))$$

is defined by

$$X \blacktriangleright \rho = (\rho \otimes \text{id} \otimes \text{id})(\Gamma^r \otimes \text{id})(X), \quad X \in \mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G})), \rho \in \mathcal{T}(L^2(\mathbb{G})).$$

By the proof of Proposition 4.2 it follows that $\Phi(\rho \blacktriangleright X) = \rho \blacktriangleright \Phi(X)$, where the left module action \blacktriangleright is given by

$$\rho \blacktriangleright X = (\text{id} \otimes \rho \otimes \text{id})(\widehat{\Gamma}^r \otimes \text{id})(X), \quad X \in \mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G})), \rho \in \mathcal{T}(L^2(\mathbb{G})).$$

Furthermore, the proof of Corollary 4.3 entails the invariance

$$\Phi(L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}))) \subseteq L^\infty(\mathbb{G}).$$

Since $\Gamma^l: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}))$, the composition $\Phi \circ \Gamma^l$ therefore maps into $L^\infty(\mathbb{G})$. Moreover, if $x \in L^\infty(\mathbb{G})$, then $\Phi \circ \Gamma^l(x) = \Phi \circ \Gamma^r(x) = x$, so that $\Phi \circ \Gamma^l$ is a projection of norm one from $\mathcal{B}(L^2(\mathbb{G}))$ onto $L^\infty(\mathbb{G})$. Thus, $L^\infty(\mathbb{G})$ is a 1-injective operator space.

Next consider the map $\Psi = \Phi|_{L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})}: L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$. Since the right $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$ -module action on $\mathcal{B}(L^2(\mathbb{G}))$ restricts to the canonical right $L^1(\mathbb{G})$ -module action on $L^\infty(\mathbb{G})$, it follows that Ψ is a completely contractive right $L^1(\mathbb{G})$ -module map such that $\Psi \circ \Gamma = \text{id}_{L^\infty(\mathbb{G})}$. Under the completely isometric identification $L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G}) \cong \mathcal{CB}(L^1(\mathbb{G}), L^\infty(\mathbb{G}))$, $\Gamma = \Delta$, and since $L^\infty(\mathbb{G})$ is faithful in $\mathbf{mod}\text{-}L^1(\mathbb{G})$, the existence of Ψ entails the relative 1-injectivity of $L^\infty(\mathbb{G})$ in $\mathbf{mod}\text{-}L^1(\mathbb{G})$. By Proposition 2.3, $L^\infty(\mathbb{G})$ is 1-injective in $\mathbf{mod}\text{-}L^1(\mathbb{G})$.

(3) \Rightarrow (1). Viewing $\mathcal{B}(L^2(\mathbb{G}))$ as a right operator $L^1(\mathbb{G})$ -module via

$$T \triangleleft f = (f \otimes \text{id})\Gamma^l(T) = (f \otimes \text{id})W^*(1 \otimes T)W, \quad f \in L^1(\mathbb{G}), T \in \mathcal{B}(L^2(\mathbb{G})),$$

1-injectivity of $L^\infty(\mathbb{G})$ in $\mathbf{mod}\text{-}L^1(\mathbb{G})$ gives an extension of $\text{id}_{L^\infty(\mathbb{G})}$ to a completely contractive morphism $E: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\mathbb{G})$. Proposition 4.7 then entails the amenability of $\widehat{\mathbb{G}}$. ■

Analogously, there is a left module version of Theorem 5.1 involving the left product \triangleleft (which relies on the corresponding left $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$ -module version of [7, Proposition 5.8]). The proof follows similarly.

Theorem 5.2 *Let \mathbb{G} be a locally compact quantum group. The following conditions are equivalent:*

- (i) $\widehat{\mathbb{G}}$ is amenable;
- (ii) $\mathcal{B}(L^2(\mathbb{G}))$ is 1-injective in $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)\text{-mod}$;
- (iii) $L^\infty(\mathbb{G})$ is 1-injective in $L^1(\mathbb{G})\text{-mod}$.

In the co-commutative setting, we obtain a new characterization of amenable locally compact groups.

Corollary 5.3 *Let G be a locally compact group. The following conditions are equivalent:*

- (i) G is amenable;

- (ii) $VN(G)$ is 1-injective in $\mathbf{mod}\text{-}A(G)$;
- (iii) $VN(G)$ is 1-injective in $A(G)\text{-}\mathbf{mod}$.

Remark 5.4 Corollary 5.3 highlights the significance of the non-relative homology of $VN(G)$ as an operator $A(G)$ -module. In fact, relative 1-injectivity of $VN(G)$ in $\mathbf{mod}\text{-}A(G)$ is equivalent to inner amenability of G [8, Theorem 3.4]. Related results at the level of quantum groups will appear in subsequent work.

In [17] Haagerup provided an elegant characterization of injective von Neumann algebras via decomposability of completely bounded maps. More specifically, a von Neumann algebra M is injective if and only if $\mathcal{CB}(M) = \text{span } \mathcal{CP}(M)$, where $\mathcal{CP}(M)$ is the set of completely positive maps $\Phi: M \rightarrow M$. The next result provides a similar decomposition for $L^1(\mathbb{G})$ -module maps on $L^\infty(\mathbb{G})$ when $L^\infty(\mathbb{G})$ is 1-injective in $\mathbf{mod}\text{-}L^1(\mathbb{G})$.

Proposition 5.5 Let \mathbb{G} be a locally compact quantum group. If $L^\infty(\mathbb{G})$ is 1-injective in $\mathbf{mod}\text{-}L^1(\mathbb{G})$ (equivalently, $\widehat{\mathbb{G}}$ is amenable), then

$$\mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})) = \text{span } \mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})).$$

Proof Viewing $M_n(L^\infty(\mathbb{G}))$ as an operator $L^1(\mathbb{G})$ -module under the amplified action: $[x_{ij}] \star f = [x_{ij} \star f]$, for $[x_{ij}] \in M_n(L^\infty(\mathbb{G}))$, $f \in L^1(\mathbb{G})$, we claim that $M_n(L^\infty(\mathbb{G}))$ is 1-injective in $\mathbf{mod}\text{-}L^1(\mathbb{G})$ for any $n \in \mathbb{N}$. Indeed, the canonical morphism

$$\Delta_n: M_n(L^\infty(\mathbb{G})) \rightarrow \mathcal{CB}(L^1(\mathbb{G}), M_n(L^\infty(\mathbb{G}))) = M_n(\mathcal{CB}(L^1(\mathbb{G}), L^\infty(\mathbb{G})))$$

is nothing but the n -th amplification of $\Delta: L^\infty(\mathbb{G}) \rightarrow \mathcal{CB}(L^1(\mathbb{G}), L^\infty(\mathbb{G}))$, so the n -th amplification of a completely contractive module left inverse of Δ (which exists by 1-injectivity of $L^\infty(\mathbb{G})$) provides a completely contractive module left inverse to Δ_n . Since $M_n(L^\infty(\mathbb{G}))$ is 1-injective in $\mathbf{mod}\text{-}\mathbb{C}$ [39, Proposition XV.3.2], the claim follows from Proposition 2.3.

Now let $\Phi \in \mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ be a complete contraction, and consider the Paulsen system $\mathcal{S} \subseteq M_2(L^\infty(\mathbb{G}))$ defined by

$$\mathcal{S} = \left\{ \begin{pmatrix} \alpha 1 & x \\ y & \beta 1 \end{pmatrix} \mid x, y \in L^\infty(\mathbb{G}), \alpha, \beta \in \mathbb{C} \right\}.$$

Then \mathcal{S} is an $L^1(\mathbb{G})$ -submodule of $M_2(L^\infty(\mathbb{G}))$ and Φ gives rise to a unital completely positive $L^1(\mathbb{G})$ -module map $\Phi_{\mathcal{S}}: \mathcal{S} \rightarrow M_2(L^\infty(\mathbb{G}))$ via off-diagonalization [32]:

$$\Phi_{\mathcal{S}} \left(\begin{pmatrix} \alpha 1 & x \\ y & \beta 1 \end{pmatrix} \right) = \begin{pmatrix} \alpha 1 & \Phi(x) \\ \Phi^*(y) & \beta 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha 1 & x \\ y & \beta 1 \end{pmatrix} \in \mathcal{S},$$

where $\Phi^*(y) = \Phi(y^*)^*$, $y \in L^\infty(\mathbb{G})$. By 1-injectivity of $M_2(L^\infty(\mathbb{G}))$ in $\mathbf{mod}\text{-}L^1(\mathbb{G})$, the map $\Phi_{\mathcal{S}}$ extends to a completely contractive $L^1(\mathbb{G})$ -module map

$$\widetilde{\Phi}: M_2(L^\infty(\mathbb{G})) \rightarrow M_2(L^\infty(\mathbb{G}))$$

such that

$$\tilde{\Phi} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, $\tilde{\Phi}$ is completely positive and is of the form

$$\tilde{\Phi} \left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right) = \begin{pmatrix} \Psi_1(x_{11}) & \Phi(x_{12}) \\ \Phi^*(x_{21}) & \Psi_2(x_{22}) \end{pmatrix}, \quad [x_{ij}] \in M_2(L^\infty(\mathbb{G})),$$

where $\Psi_i \in \mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ is associated to $P_{ii} \circ \tilde{\Phi} \circ P_{ii}$, and

$$P_{ii} \in \mathcal{CP}_{L^1(\mathbb{G})}(M_2(L^\infty(\mathbb{G})))$$

is the diagonal projection onto the (i, i) -th entry for $i = 1, 2$. By [15, Proposition 5.4.2], it follows that the map

$$\tilde{\Phi}|_{L^\infty(\mathbb{G})}: L^\infty(\mathbb{G}) \ni x \mapsto \begin{pmatrix} \Psi_1(x) & \Phi(x) \\ \Phi^*(x) & \Psi_2(x) \end{pmatrix} \in M_2(L^\infty(\mathbb{G}))$$

is a completely positive $L^1(\mathbb{G})$ -module map. Thus, via polarization (as in [15, Proposition 5.4.1]), it follows that $\Phi \in \text{span } \mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$. ■

Remark 5.6 As the proof of Proposition 5.5 shows, when $L^\infty(\mathbb{G})$ is 1-injective in **mod**- $L^1(\mathbb{G})$ we can decompose any $\Phi \in \mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ into a linear combination of four completely positive $L^1(\mathbb{G})$ -module maps.

For a locally compact group G , it is well known that $B(G) = M_{cb}A(G)$ whenever G is amenable [12, Corollary 1.8]. Using Proposition 5.5 together with [11, Theorem 5.2], we can now generalize this implication to arbitrary locally compact quantum groups.

Lemma 5.7 *Let \mathbb{G} be a locally compact quantum group. Then there exists a completely contractive projection*

$$\mathcal{P}^\sigma: \mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})) \rightarrow \mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$$

mapping $\mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ onto $\mathcal{CP}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$.

Proof First we claim that $\mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G})) = \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}))$. One inclusion is obvious, so let $\Phi \in \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G}))$. Then the restriction of its adjoint $\Phi^*|_{L^1(\mathbb{G})} \in {}_{L^1(\mathbb{G})}\mathcal{CB}(L^1(\mathbb{G}), M(\mathbb{G})) = {}_{L^1(\mathbb{G})}\mathcal{CB}(L^1(\mathbb{G}))$, noting that $L^1(\mathbb{G}) = \langle L^1(\mathbb{G}) \star L^1(\mathbb{G}) \rangle$ is a closed ideal in $M(\mathbb{G})$. Hence, $(\Phi^*|_{L^1(\mathbb{G})})^* \in \mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G})) = \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}))$ by [23, Proposition 4.1]. But

$$\langle (\Phi^*|_{L^1(\mathbb{G})})^*(x), f \rangle = \langle x, \Phi^*|_{L^1(\mathbb{G})}(f) \rangle = \langle x, \Phi^*(f) \rangle = \langle \Phi(x), f \rangle$$

for all $x \in C_0(\mathbb{G})$ and $f \in L^1(\mathbb{G})$, so $(\Phi^*|_{L^1(\mathbb{G})})^*$ is an extension of Φ which leaves $C_0(\mathbb{G})$ invariant, hence so, too, does Φ .

Letting \mathcal{J} denote the complete isometry

$$\mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G})) \ni \Phi \mapsto (\Phi^*|_{L^1(\mathbb{G})})^* \in \mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$$

and $\mathcal{R}: \mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})) \rightarrow \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G}))$ the completely contractive restriction map, it follows that $\mathcal{P}^\sigma := \mathcal{J} \circ \mathcal{R}: \mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})) \rightarrow \mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$

is a completely contractive projection onto $\mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$. Moreover, \mathcal{P}^σ maps $\mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ onto $\mathcal{CP}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$. ■

The observations in the proof of Lemma 5.7 lead to the following new characterization of the predual of $M_{cb}^r(L^1(\mathbb{G}))$.

Proposition 5.8 *Let \mathbb{G} be a locally compact quantum group. Then*

$$Q_{cb}^r(L^1(\mathbb{G})) \cong C_0(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G})$$

completely isometrically.

Proof As noted in the proof of Lemma 5.7, we have

$$\mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G})) = \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G})).$$

Thus, $\Theta^r: M_{cb}^r(L^1(\mathbb{G})) \cong \mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G}))$ completely isometrically ([23, Proposition 4.1]). We need to show that Θ^r is a weak*-weak* homeomorphism. Since $\mathcal{CB}_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G})) \cong (C_0(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}))^*$ weak* homeomorphically, and Θ^r is a completely isometric isomorphism, it suffices to show that Θ^r is weak* continuous on bounded sets (see [10, Lemma 10.1]). Let $(\widehat{b}'_i)_{i \in I}$ be a bounded net in $M_{cb}^r(L^1(\mathbb{G}))$ converging weak* to \widehat{b}' . By Proposition 3.2, for any $A \in C_0(\mathbb{G}) \otimes_{\min} K_\infty$ and $\rho \in L^1(\mathbb{G}) \widehat{\otimes} T_\infty$ we have $\Omega_{A,\rho} \in Q_{cb}^r(L^1(\mathbb{G}))$, where

$$\langle \widehat{a}', \Omega_{A,\rho} \rangle = \langle (\Theta^r(\widehat{a}') \otimes \text{id}_{K_\infty})(A), \rho \rangle, \quad \widehat{a}' \in M_{cb}^r(L^1(\mathbb{G})).$$

Then $(\Theta^r(\widehat{b}'_i))_{i \in I}$ converges point weak* to $\Theta^r(\widehat{b}')$ in $\mathcal{CB}(C_0(\mathbb{G}), L^\infty(\mathbb{G}))$. Letting

$$q: C_0(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G}) \rightarrow C_0(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G})$$

be the quotient map, and viewing $\Theta^r(\widehat{b}'_i) \in (C_0(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}))^*$, the density of the image $q(C_0(\mathbb{G}) \otimes L^1(\mathbb{G}))$ of the algebraic tensor product $C_0(\mathbb{G}) \otimes L^1(\mathbb{G})$ in $C_0(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G})$, together with the boundedness of $(\Theta^r(\widehat{b}'_i))_{i \in I}$ imply that $(\Theta^r(\widehat{b}'_i))_{i \in I}$ converges weak* to $\Theta^r(\widehat{b}')$ in $(C_0(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}))^*$. ■

Remark 5.9 The identification of $Q_{cb}^r(L^1(\mathbb{G}))$ in Proposition 5.8 is new even in the co-commutative case, that is, for any locally compact group G we have

$$Q_{cb}(G) \cong C_\lambda^*(G) \widehat{\otimes}_{A(G)} A(G)$$

completely isometrically.

Proposition 5.10 *Let \mathbb{G} be a locally compact quantum group. If $\widehat{\mathbb{G}}$ is amenable, then $\widehat{\rho}: C_u(\mathbb{G})^* \rightarrow M_{cb}^r(L^1(\mathbb{G}))$ is surjective.*

Proof Since $\widehat{\mathbb{G}}$ is amenable, $\mathcal{CB}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})) = \text{span } \mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ by Proposition 5.5. So given $\Phi \in \mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$, there exist $\Phi_i \in \mathcal{CP}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ for $i = 1, \dots, 4$ such that $\Phi = \frac{1}{4}(\Phi_1 - \Phi_2 + i(\Phi_3 - \Phi_4))$. But then, by Lemma 5.7

$$\Phi = \mathcal{P}^\sigma(\Phi) = \frac{1}{4}(\mathcal{P}^\sigma(\Phi_1) - \mathcal{P}^\sigma(\Phi_2) + i(\mathcal{P}^\sigma(\Phi_3) - \mathcal{P}^\sigma(\Phi_4))),$$

and it follows that $\mathcal{CB}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G})) = \text{span } \mathcal{CP}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$. By [11, Theorem 5.2], $\text{span } \mathcal{CP}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G})) = \Theta^r(\tilde{\rho}(C_u(\mathbb{G})^*))$. It follows that

$$\tilde{\rho}(C_u(\mathbb{G})^*) = M_{cb}^r(L^1(\mathbb{G})). \quad \blacksquare$$

Remark 5.11 A natural question is whether amenability of $\widehat{\mathbb{G}}$ implies that $\tilde{\rho}$ is a weak*-weak* homeomorphic completely isometric isomorphism. This and similar questions will be pursued elsewhere.

For our final application, we now give a simplified proof of the fact that amenability of a discrete quantum group implies co-amenability of its compact dual.

Theorem 5.12 *A compact quantum group \mathbb{G} is co-amenable if and only if $\widehat{\mathbb{G}}$ is amenable.*

Proof Co-amenability of \mathbb{G} always implies amenability of $\widehat{\mathbb{G}}$ [1, Theorem 3.2] So assume that $\widehat{\mathbb{G}}$ is amenable. By Theorem 5.1 we know that $L^\infty(\mathbb{G})$ is 1-injective in **mod**- $L^1(\mathbb{G})$. Let $\Phi: L^\infty(\mathbb{G}) \otimes L^1(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ be a completely contractive left inverse to Γ which is a right $L^1(\mathbb{G})$ -module map. As a unital complete contraction, Φ is completely positive and $\Phi|_{C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})} \neq 0$ since $C(\mathbb{G})$ is unital. By [1, Theorem 3.3] we also know that $C(\mathbb{G})$ is nuclear, so let $(\Psi_a)_{a \in A}$ be a net of finite-rank, unital completely positive maps converging to $\text{id}_{C(\mathbb{G})}$ in the point-norm topology. For $a \in A$, consider the unital completely positive map $\Phi_a: C(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ given by

$$\Phi_a = \Phi \circ (\text{id} \otimes \Psi_a) \circ \Gamma|_{C(\mathbb{G})}.$$

Since Ψ_a is finite rank, there exist $x_1^a, \dots, x_{n_a}^a \in C(\mathbb{G})$ and $\mu_1^a, \dots, \mu_{n_a}^a \in M(\mathbb{G})$ such that $\Psi_a(x) = \sum_{n=1}^{n_a} \langle \mu_n^a, x \rangle x_n^a$, for $x \in C(\mathbb{G}), a \in A$. For each $a \in A$, and $1 \leq n \leq n_a$, let $\Phi_{(a,n)}: C(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ be defined by $\Phi_{(a,n)}(x) = \Phi(x \otimes x_n^a)$, for $x \in C(\mathbb{G})$. Then $\Phi_{(a,n)}$ is completely bounded with $\|\Phi_{(a,n)}\|_{cb} \leq \|x_n^a\|_{C(\mathbb{G})}$, and is a right $L^1(\mathbb{G})$ -module map. Hence, $\Phi_{(a,n)} \in \mathcal{CB}_{L^1(\mathbb{G})}(C(\mathbb{G}), L^\infty(\mathbb{G})) = \Theta^r(M_{cb}^r(L^1(\mathbb{G})))$. Since $M_{cb}^r(L^1(\mathbb{G})) = \tilde{\rho}(C_u(\mathbb{G})^*)$ by Proposition 5.10, there exist $v_n^a \in C_u(\mathbb{G})^*$ such that $\Phi_{(a,n)} = \Theta^r(v_n^a)$, where $\Theta^r(v_n^a)$ denotes the map $\Theta^r(\tilde{\rho}(v_n^a))$ for simplicity.

Let $x \in C(\mathbb{G})$. Then

$$\begin{aligned} \Phi_a(x) &= \Phi((\text{id} \otimes \Psi_a)(\Gamma(x))) = \sum_{n=1}^{n_a} \Phi(((\text{id} \otimes \mu_n^a)\Gamma(x)) \otimes x_n^a) \\ &= \sum_{n=1}^{n_a} \Phi_{(a,n)}(((\text{id} \otimes \mu_n^a)\Gamma(x))) = \sum_{n=1}^{n_a} \Theta^r(v_n^a)(\Theta^r(\mu_n^a)(x)) \\ &= \sum_{n=1}^{n_a} \Theta^r(v_n^a * \mu_n^a)(x) = \Theta^r\left(\sum_{n=1}^{n_a} v_n^a * \mu_n^a\right)(x). \end{aligned}$$

Letting $\mu_a = \sum_{n=1}^{n_a} v_n^a * \mu_n^a$, we obtain $\Phi_a = \Theta^r(\mu_a)$. Then $\mu_a \in M(\mathbb{G})$ as $M(\mathbb{G})$ is a two-sided ideal in $C_u(\mathbb{G})^*$, and since $\Theta^r|_{C_u(\mathbb{G})_+^*}: C_u(\mathbb{G})_+^* \rightarrow \mathcal{CP}_{L^1(\mathbb{G})}^\sigma(L^\infty(\mathbb{G}))$ is an isometric order bijection [11, Theorem 5.2], we have $\mu_a \in M(\mathbb{G})_+$ and

$$\|\mu_a\|_{M(\mathbb{G})} = \|\Theta^r(\mu_a)\|_{cb} = \|\Phi_a\|_{cb} = 1, \quad a \in A.$$

Convergence of Φ_a to $\text{id}_{C(\mathbb{G})}$ in the point-norm topology entails

$$\mu_a \star x = \Theta^r(\mu_a)(x) \rightarrow x, \quad x \in C(\mathbb{G}).$$

Let μ be a weak* cluster point of $(\mu_a)_{a \in A}$ in the unit ball of $M(\mathbb{G}) = C(\mathbb{G})^*$. Then μ is a right identity of $M(\mathbb{G})$. The restricted unitary antipode R maps $C(\mathbb{G})$ into $C(\mathbb{G})$ and satisfies $R^*(\mu \star \nu) = R^*(\nu) \star R^*(\mu)$ for all $\nu \in M(\mathbb{G})$. Hence, $R^*(\mu)$ is a left identity of $M(\mathbb{G})$. It follows that $\varepsilon := \mu + R^*(\mu) - \mu \star R^*(\mu)$ is an identity for $M(\mathbb{G})$. Hence, \mathbb{G} is co-amenable by [1, Theorem 3.1]. ■

Given a completely contractive Banach algebra \mathcal{A} with a contractive approximate identity, any essential module $X \in \mathbf{mod}\text{-}\mathcal{A}$ is induced by [10, Proposition 6.4]. Since a locally compact quantum group \mathbb{G} is co-amenable if and only if $L^1(\mathbb{G})$ has a contractive approximate identity [20, Theorem 2], the next proposition supports the idea that our methods may be applicable to the general duality problem of amenability and co-amenableity.

Proposition 5.13 *Let \mathbb{G} be a locally compact quantum group for which the dual $\widehat{\mathbb{G}}$ is amenable. Then for any closed right ideal $I \trianglelefteq L^1(\mathbb{G})$, the multiplication map yields a completely isometric isomorphism $\widetilde{m}_I: I \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \cong \langle I \star L^1(\mathbb{G}) \rangle$. In particular, if I is essential, then $I \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \cong I$, that is, I is an induced $L^1(\mathbb{G})$ -module.*

Proof First observe that for any self-induced completely contractive Banach algebra \mathcal{A} and any closed right ideal $J \trianglelefteq \mathcal{A}$, the map $m_{\mathcal{A}/J}: (\mathcal{A}/J) \widehat{\otimes} \mathcal{A} \rightarrow \mathcal{A}/\langle J \cdot \mathcal{A} \rangle$ induces a completely isometric isomorphism $\widetilde{m}_{\mathcal{A}/J}: (\mathcal{A}/J) \widehat{\otimes}_{\mathcal{A}} \mathcal{A} \cong \mathcal{A}/\langle J \cdot \mathcal{A} \rangle$. Indeed, letting $q: \mathcal{A} \twoheadrightarrow \mathcal{A}/J$ be the complete quotient map and identifying

$$(\mathcal{A}/\langle J \cdot \mathcal{A} \rangle)^* = \langle J \cdot \mathcal{A} \rangle^\perp \subseteq \mathcal{A}^*,$$

it follows that $(q \otimes \text{id})^* \circ (\widetilde{m}_{\mathcal{A}/J})^*: \langle J \cdot \mathcal{A} \rangle^\perp \rightarrow (\mathcal{A} \widehat{\otimes}_{\mathcal{A}} \mathcal{A})^*$ is equal to $(\widetilde{m}_{\mathcal{A}})^*|_{\langle J \cdot \mathcal{A} \rangle^\perp}$. In particular, $(\widetilde{m}_{\mathcal{A}/J})^*$ is a complete isometry. If $X \in (N_{\mathcal{A}/J})^\perp$, then $(q \otimes \text{id})^*(X) \in (N_{\mathcal{A}})^\perp$. So there exists $F \in \mathcal{A}^*$ such that $(q \otimes \text{id})^*(X) = (\widetilde{m}_{\mathcal{A}})^*(F)$ as \mathcal{A} is self-induced. Clearly, $F \in \langle J \cdot \mathcal{A} \rangle^\perp$, so $(q \otimes \text{id})^*(X) = (q \otimes \text{id})^* \circ (\widetilde{m}_{\mathcal{A}/J})^*(F)$, implying $X = (\widetilde{m}_{\mathcal{A}/J})^*(F)$, whence $(\widetilde{m}_{\mathcal{A}/J})^*$ is surjective.

Since $\widehat{\mathbb{G}}$ is amenable, by Theorem 5.1 $L^\infty(\mathbb{G})$ is 1-injective in $\mathbf{mod}\text{-}L^1(\mathbb{G})$. Then for every 1-exact sequence of right \mathcal{A} -modules $0 \rightarrow Y \hookrightarrow Z \twoheadrightarrow Z/Y \rightarrow 0$, the induced sequence

$$0 \rightarrow \mathcal{CB}_{L^1(\mathbb{G})}(Z/Y, L^\infty(\mathbb{G})) \hookrightarrow \mathcal{CB}_{L^1(\mathbb{G})}(Z, L^\infty(\mathbb{G})) \twoheadrightarrow \mathcal{CB}_{L^1(\mathbb{G})}(Y, L^\infty(\mathbb{G})) \rightarrow 0$$

is 1-exact, where 1-exactness refers to an exact sequence of morphisms such that the injection (\hookrightarrow) is a complete isometry and the surjection (\twoheadrightarrow) is a complete quotient map. Taking the pre-adjoint of the above sequence, we obtain the 1-exact sequence

$$0 \rightarrow Y \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \hookrightarrow Z \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \twoheadrightarrow Z/Y \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \rightarrow 0.$$

In particular, take $Y = I$ and $Z = L^1(\mathbb{G})$, and consider the commutative diagram:

$$\begin{array}{ccccc}
 I \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) & \hookrightarrow & L^1(\mathbb{G}) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) & \twoheadrightarrow & (L^1(\mathbb{G})/I) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \\
 \downarrow \widetilde{m}_I & & \downarrow \widetilde{m}_{L^1(\mathbb{G})} & & \downarrow \widetilde{m}_{(L^1(\mathbb{G})/I)} \\
 \langle I \star L^1(\mathbb{G}) \rangle & \hookrightarrow & L^1(\mathbb{G}) & \twoheadrightarrow & L^1(\mathbb{G})/\langle I \star L^1(\mathbb{G}) \rangle
 \end{array}$$

As $L^1(\mathbb{G})$ is self-induced, the last two columns are completely isometric isomorphisms, and since both rows are 1-exact, it follows that $\widetilde{m}_I: I \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \cong \langle I \star L^1(\mathbb{G}) \rangle$ completely isometrically. ■

A locally compact quantum group \mathbb{G} is said to be *regular* (see [22, §3], for instance) if $\mathcal{K}(L^2(\mathbb{G})) = \langle (\text{id} \otimes \omega)(\sigma V) \mid \omega \in \mathcal{T}(L^2(\mathbb{G})) \rangle$, where $\mathcal{K}(L^2(\mathbb{G}))$ denotes the ideal of compact operators on $L^2(\mathbb{G})$, σ denotes the flip map on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$, and, as usual, $\langle \cdot \rangle$ denotes the closed linear span. Examples of regular quantum groups include Kac algebras, as well as discrete and compact quantum groups (see [22]). Under the assumption of regularity, we now obtain a version of Theorem 5.1 at the predual level.

Theorem 5.14 *Let \mathbb{G} be a locally compact quantum group. Consider the following conditions:*

- (i) $\widehat{\mathbb{G}}$ is compact (equivalently, \mathbb{G} is discrete);
- (ii) $\mathcal{T}(L^2(\mathbb{G}))$ is relatively 1-projective in $(\mathcal{T}(L^2(\mathbb{G})), \triangleright)$ -**mod**;
- (iii) $L^1(\mathbb{G})$ is 1-projective in $L^1(\mathbb{G})$ -**mod**.

Then (i) \Leftrightarrow (ii) \Rightarrow (iii), and when \mathbb{G} is regular, the conditions are equivalent.

Proof The implication (i) \Rightarrow (ii) follows by an argument similar to the proof of [7, Proposition 5.8] using a normal two-sided invariant mean \widehat{m}' on $L^\infty(\widehat{\mathbb{G}}')$, which exists by compactness.

(ii) \Rightarrow (i). Condition (ii) yields a *normal* completely contractive morphism

$$\Phi: \mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$$

that is a right inverse to Γ^r . As in the proof of Theorem 5.1, it follows that

$$\Phi(L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}))) \subseteq L^\infty(\mathbb{G}).$$

Hence, $(\Phi|_{L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})})_*: L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G})$ is a completely contractive left $L^1(\mathbb{G})$ -module right inverse to the multiplication $m_{L^1(\mathbb{G})}: L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G})$, and $L^1(\mathbb{G})$ is relatively 1-projective in $L^1(\mathbb{G})$ -**mod**. Moreover,

$$\Phi \circ \Gamma^l: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\mathbb{G})$$

is a *normal* conditional expectation, so the quotient map $\pi: \mathcal{T}(L^2(\mathbb{G})) \twoheadrightarrow L^1(\mathbb{G})$ is an admissible surjection. Viewing $\mathcal{B}(L^2(\mathbb{G}))$ as a right $L^1(\mathbb{G})$ -module under the \triangleleft -action, and considering the associated left $L^1(\mathbb{G})$ -module structure on $\mathcal{T}(L^2(\mathbb{G}))$, the relative 1-projectivity of $L^\infty(\mathbb{G})$ yields a completely contractive morphism

$$\Psi: L^1(\mathbb{G}) \rightarrow \mathcal{T}(L^2(\mathbb{G}))$$

satisfying $\pi \circ \Psi = \text{id}_{L^1(\mathbb{G})}$. Then $\Psi^*: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\mathbb{G})$ is a normal condition expectation that is a right $L^1(\mathbb{G})$ -module map, and $\widehat{\mathbb{G}}$ is compact by Proposition 4.8.

(ii) \Rightarrow (iii). By the above we know that $\widehat{\mathbb{G}}$ is compact and $L^1(\mathbb{G})$ is relatively 1-projective in $L^1(\mathbb{G})$ -**mod**. By discreteness of \mathbb{G} we have

$$L^1(\mathbb{G}) \cong \bigoplus_1 \{T_{n_\alpha}(\mathbb{C}) \mid \alpha \in \text{Irr}(\widehat{\mathbb{G}})\},$$

where $T_{n_\alpha}(\mathbb{C})$ is the space of $n_\alpha \times n_\alpha$ trace-class operators. Hence, $L^1(\mathbb{G})$ is 1-projective in \mathbb{C} -**mod** by [2, Proposition 3.6, Proposition 3.7]. The left version of Proposition 2.2 then entails the 1-projectivity of $L^1(\mathbb{G})$ in $L^1(\mathbb{G})$ -**mod**.

Now suppose that \mathbb{G} is regular. Considering again the right $L^1(\mathbb{G})$ -module structure on $\mathcal{B}(L^2(\mathbb{G}))$ given by the \triangleleft -action (which is precisely the degenerated right $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$ -action), it follows from [22, Corollary 3.6] that $\mathcal{K}(L^2(\mathbb{G}))$ is an essential $L^1(\mathbb{G})$ -submodule of $\mathcal{B}(L^2(\mathbb{G}))$, that is, $\mathcal{K}(L^2(\mathbb{G})) = \langle \mathcal{K}(L^2(\mathbb{G})) \triangleleft L^1(\mathbb{G}) \rangle$. We show (iii) \Rightarrow (i).

Since the multiplication $m_{L^1(\mathbb{G})}: L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G})$ is a complete quotient morphism and $L^1(\mathbb{G})$ is 1-projective in $L^1(\mathbb{G})$ -**mod**, for any $\varepsilon > 0$, there exists a morphism $\Phi_\varepsilon: L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G})$ satisfying $m_{L^1(\mathbb{G})} \circ \Phi_\varepsilon = \text{id}_{L^1(\mathbb{G})}$ and $\|\Phi_\varepsilon\|_{cb} < 1 + \varepsilon$. Moreover, we know that $L^\infty(\mathbb{G})$ is 1-injective in **mod**- $L^1(\mathbb{G})$ as the dual of a 1-projective module. Thus, $M'_{cb}(L^1(\mathbb{G})) = \widetilde{\rho}(C_u(\mathbb{G})^*)$ by Proposition 5.10, and $L^\infty(\mathbb{G})$ is a 1-injective operator space. Hence, $L^\infty(\mathbb{G})$ is semi-discrete [4, 5, 14], so there exists a net $(\Psi_i)_{i \in I}$ of normal, unital, completely positive finite-rank maps $\Psi_i: L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ converging to $\text{id}_{L^\infty(\mathbb{G})}$ in the point weak* topology. Using the normal completely bounded morphism $\Phi_\varepsilon^*: L^\infty(\mathbb{G}) \widehat{\otimes} L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$ which is a left inverse of Γ , one can argue in a similar manner to Theorem 5.12 by averaging the normal finite-rank maps Ψ_i into completely positive multipliers and use the fact that $L^1(\mathbb{G})$ is a two-sided ideal in $C_u(\mathbb{G})^*$ to obtain a bounded net $(f_i)_{i \in I}$ in $L^1(\mathbb{G})$ satisfying $f \star f_i - f \rightarrow 0$ weakly for all $f \in L^1(\mathbb{G})$. The standard convexity argument then yields a bounded right approximate identity for $L^1(\mathbb{G})$, and \mathbb{G} is necessarily co-amenable.

Since $\pi: \mathcal{T}(L^2(\mathbb{G})) \rightarrow L^1(\mathbb{G})$ is a complete quotient morphism in $(L^1(\mathbb{G}), \triangleleft)$ -**mod**, for any $\varepsilon > 0$, it also has a right inverse morphism $\Psi_\varepsilon: L^1(\mathbb{G}) \rightarrow \mathcal{T}(L^2(\mathbb{G}))$ with $\|\Psi_\varepsilon\|_{cb} < 1 + \varepsilon$. Then $\Psi_\varepsilon^*: \mathcal{B}(L^2(\mathbb{G})) \rightarrow L^\infty(\mathbb{G})$ is a normal completely bounded right $(L^1(\mathbb{G}), \triangleleft)$ -module projection onto $L^\infty(\mathbb{G})$. Since $L^1(\mathbb{G})$ has a contractive approximate identity and $\mathcal{K}(L^2(\mathbb{G}))$ is an essential $L^1(\mathbb{G})$ -module, we know that $\mathcal{K}(L^2(\mathbb{G}))$ is induced [10, Proposition 6.4], that is,

$$\widetilde{m}_{\mathcal{K}(L^2(\mathbb{G}))}: \mathcal{K}(L^2(\mathbb{G})) \widehat{\otimes}_{L^1(\mathbb{G})} L^1(\mathbb{G}) \rightarrow \mathcal{K}(L^2(\mathbb{G}))$$

is a completely isometric isomorphism. Hence, so too is its dual

$$(\widetilde{m}_{\mathcal{K}(L^2(\mathbb{G}))})^*: \mathcal{T}(L^2(\mathbb{G})) \cong \mathcal{CB}_{L^1(\mathbb{G})}(\mathcal{K}(L^2(\mathbb{G})), L^\infty(\mathbb{G})).$$

Then $\Psi_\varepsilon^*|_{\mathcal{K}(L^2(\mathbb{G}))} \in \mathcal{CB}_{L^1(\mathbb{G})}(\mathcal{K}(L^2(\mathbb{G})), L^\infty(\mathbb{G})) = (\widetilde{m}_{\mathcal{K}(L^2(\mathbb{G}))})^*(\mathcal{T}(L^2(\mathbb{G})))$, so there exists $\rho \in \mathcal{T}(L^2(\mathbb{G}))$ satisfying $(\widetilde{m}_{\mathcal{K}(L^2(\mathbb{G}))})^*(\rho) = \Psi_\varepsilon^*|_{\mathcal{K}(L^2(\mathbb{G}))}$. Then for all $y \in \mathcal{K}(L^2(\mathbb{G}))$ and $f \in L^1(\mathbb{G})$ we have

$$\langle \Psi_\varepsilon^*|_{\mathcal{K}(L^2(\mathbb{G}))}(y), f \rangle = \langle (\widetilde{m}_{\mathcal{K}(L^2(\mathbb{G}))})^*(\rho)(y), f \rangle = \langle \rho, y \triangleleft f \rangle = \langle \rho \triangleleft y, f \rangle.$$

By weak* density of $\mathcal{K}(L^2(\mathbb{G}))$ in $\mathcal{B}(L^2(\mathbb{G}))$, we obtain $\Psi_\varepsilon^*(T) = \rho \triangleleft T$ for all $T \in \mathcal{B}(L^2(\mathbb{G}))$. In particular, $\pi(\rho) \star x = \rho \triangleleft x = \Psi_\varepsilon^*(x) = x$ for all $x \in L^\infty(\mathbb{G})$ as Ψ_ε^* is a projection. Then $\pi(\rho)$ is a right identity for $L^1(\mathbb{G})$, and using the unitary antipode R as in Theorem 5.12, we may construct a two-sided identity for $L^1(\mathbb{G})$, that is, \mathbb{G} is discrete, whence $\widehat{\mathbb{G}}$ is compact. ■

Analogously, there is a right module version of Theorem 5.14.

Theorem 5.15 *Let \mathbb{G} be a locally compact quantum group. Consider the following conditions:*

- (i) $\widehat{\mathbb{G}}$ is compact (equivalently, \mathbb{G} is discrete);
- (ii) $\mathcal{T}(L^2(\mathbb{G}))$ is relatively 1-projective in $\mathbf{mod}\text{-}(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$;
- (iii) $L^1(\mathbb{G})$ is 1-projective in $\mathbf{mod}\text{-}L^1(\mathbb{G})$.

Then (1) \Leftrightarrow (2) \Rightarrow (3), and when \mathbb{G} is regular, the conditions are equivalent.

Remark 5.16 It is not clear at this time whether we can replace relative 1-projectivity of $\mathcal{T}(L^2(\mathbb{G}))$ with 1-projectivity of $\mathcal{T}(L^2(\mathbb{G}))$ in condition (ii) of Theorems 5.14 and 5.15. However, one *cannot* replace 1-projectivity of $L^1(\mathbb{G})$ with relative 1-projectivity of $L^1(\mathbb{G})$ in condition (3) of Theorems 5.14 and 5.15, as, for example, $L^1(G)$ is always relatively 1-projective for any locally compact group G (see [9, Theorem 2.4]).

Remark 5.17 Combining [2, Theorem 3.12] with [24, Corollary 7], it follows that a regular quantum group \mathbb{G} is discrete if and only if $L^1(\mathbb{G})$ is a 1-projective operator space. Theorem 5.14 therefore yields the following equivalence for regular quantum groups: $L^1(\mathbb{G})$ is 1-projective in $\mathbb{C}\text{-mod}$ if and only if $L^1(\mathbb{G})$ is 1-projective in $L^1(\mathbb{G})\text{-mod}$.

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