

PERFECTING GROUP SCHEMES

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Abstract We initiate a systematic study of the perfection of affine group schemes of finite type over fields of positive characteristic. The main result intrinsically characterises and classifies the perfections of reductive groups and obtains a bijection with the set of classifying spaces of compact connected Lie groups topologically localised away from the characteristic. We also study the representations of perfectly reductive groups. We establish a highest weight classification of simple modules, the decomposition into blocks, and relate extension groups to those of the underlying abstract group.

Introduction

For a (group) scheme over a field k of characteristic $p > 0$, its “perfection” is defined as the inverse limit over the Frobenius homomorphism. In this paper, we study the perfection of group schemes and their representation theory. We place particular emphasis on reductive groups. We obtain an intrinsic characterisation (“perfectly reductive groups”) and give a classification in terms of root data “with p inverted”. We also give a highest weight classification of simple modules for perfectly reductive groups, establish the block decomposition, and make a first step towards the study of multiplicity questions. Finally, we prove that perfectly reductive groups and the classifying spaces of compact Lie groups localised away from p are classified by the same data. This result is the “perfect analogue”

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of the fact that reductive groups over algebraically closed fields and compact Lie groups are both classified by root data.

The motivations for the current work were three-fold and came from several directions. Before describing the structure of this paper in more detail, we outline these motivations and possible future directions.

Perfect representability

Sometimes functors on rings in characteristic p are only well-behaved on perfect algebras (that is, algebras for which the Frobenius homomorphism is an isomorphism). A prominent example is the functor of Witt vectors and its relatives. An important observation (see, for instance, [BS, BD, Zh]) is that if a functor on the category of perfect commutative k -algebras can be represented by a scheme, it determines the scheme “up to perfection”. An example of such a setting is the Witt vector affine Grassmannian, which plays a prominent role in recent advances in the Langlands program.

It is important that passage to the perfection gives an isomorphism of étale topoi. In particular, constructions built via étale sheaves (like étale cohomology, or categories of perverse sheaves in the étale topology) are insensitive to passage to the perfection. This fact plays an important role in [Zh], where a mixed characteristic analogue of the geometric Satake equivalence is obtained. Similarly, it plays an important role in [BD], where the “Serre dual” of a unipotent group is shown to be the perfection of a unipotent group, and its character sheaves are studied.

By the above, also the étale homotopy type of a (simplicial) scheme only depends on its perfection. In [Fr], Friedlander used this homotopy type to construct interesting maps between topological localisations of classifying spaces of compact Lie groups, based on (exceptional) isogenies in positive characteristic. This is one of the main results we rely on to establish our bijection between perfectly reductive groups and localised classifying spaces.

Fractal representation theory

For a (reduced) group scheme defined over \mathbb{F}_p , the Frobenius twist realises its category of representations as a full subcategory of itself. This self-similarity induces a fractal-like structure. For example, Figure 1 shows a classic picture of the non-zero weight spaces of simple modules for SL_2 in characteristic 3. (For the reader unfamiliar with this picture, it may be helpful to note that it simply depicts the (non)-vanishing behaviour of Pascal’s triangle modulo p ; see, for instance, [Wi, §1].) This picture is fractal-like, but not genuinely fractal: one can “zoom out” but one cannot “zoom in” indefinitely since the Frobenius homomorphism is not an isomorphism. By passing to the perfection, one gets a genuine fractal. One goal (not realised in the current paper) is to use this fractal structure to say something about important open questions in representation theory like dimensions and characters of simple modules.

Much of the difficulty in the representation theory of reductive groups in characteristic p remains after perfecting. We do observe two interesting simplifications. Firstly, the complexities of the block decomposition disappear after passage to the perfection (see

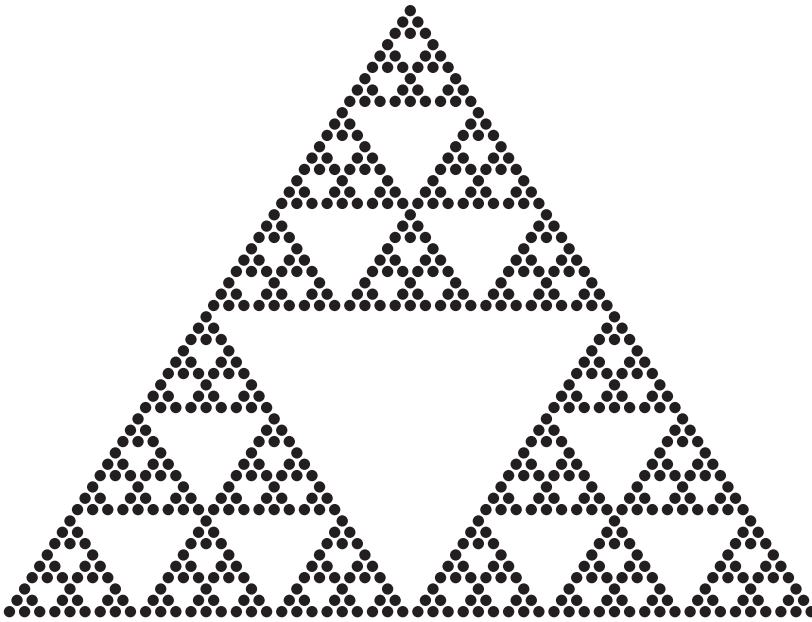


Figure 1. Characters of simple modules for SL_2 in characteristic $p = 3$.

Theorem 5.1.5). Secondly, perfect representation theory appears to provide the correct setting for the generic cohomology results of [CPSV]: for a perfectly reductive group, extensions computed inside algebraic representations agree with extensions computed as abstract groups (see Theorem 5.2.2). (That perfect group schemes provide the correct setting for generic cohomology was suggested by Donkin in the early 1980s and proved by Wang [W].)

Tensor categories in characteristic p

Over fields of characteristic zero, a famous theorem of Deligne classifies those tensor categories which admit a fibre functor to super vector spaces as precisely those of moderate growth [D]. It is a fascinating open problem to find an analogue of this theorem in characteristic p , with many potential applications to modular representation theory. Recently, this problem was solved in [CEO] for tensor categories with exact Frobenius functor. An important technical tool was a limit procedure in [CEO, §6] which, by restriction to representation (tensor) categories of affine group schemes, generalises perfection of group schemes. Remarkably, the “perfection” of a Frobenius exact tensor category of moderate growth essentially returns the representation category of a perfect group scheme. In other words, up to perfection, all Frobenius exact tensor categories of moderate growth arise from (perfect) group schemes. We consider this as further evidence for their importance.

Structure of the paper

Motivated by the above considerations, we initiate a systematic study of perfecting group schemes. The paper is organised as follows:

In Section 1, we investigate the purely combinatorial problem of describing root data over the ring $\mathbb{Z}[1/p]$, for later use in our classification results. In Section 2, we derive some general results on perfect schemes. In Section 3, we start studying the perfections of group schemes. We study perfect subgroups of perfect groups and their quotients. We also obtain criteria for when two group schemes perfect to isomorphic groups and derive some results on the perfections of the additive and multiplicative groups. In Section 4, we classify perfectly reductive groups by their $\mathbb{Z}[1/p]$ -root data. In Section 5, we study the representation theory of perfectly reductive groups. We classify simple modules and realise them as socles of induced modules from Borel subgroups. Then we show that the block decomposition simplifies considerably compared to the non-perfect case; in fact, blocks are governed by the root lattice. We also show that extension groups for the perfected groups are given in terms of generic cohomology in the sense of [CPSV]. This actually implies that extensions in the category of (rational) representations over the perfected reductive group can be computed in the category of representations of the abstract group of $\overline{\mathbb{F}}_p$ -points. In Section 6, we prove that $\mathbb{Z}[1/p]$ -root data also classify the localisations away from p of the classifying spaces of compact connected Lie groups. Finally, in Section 7, we present some explicit computations for extension groups, decomposition multiplicities and line bundle cohomology for perfected SL_2 . We also make explicit the fractal behaviour of perfected representation theory for SL_2 .

1. Root data over rings

For the entire section, we assume that D is a *principal ideal domain of characteristic 0*. By a D -lattice, we understand a finitely generated free D -module. Because D is a PID, we could replace “free” by “projective”, so our definition agrees with standard terminology (e.g. in [CR]). For a lattice V , we have the dual lattice $V^* := \text{Hom}_D(V, D)$.

1.1. Reflection groups and root data

We follow the definition of root data of, for instance, [Gr].

Definition 1.1.1. (0) A **reflection** $\sigma \in \text{Aut}_D(V)$, for a D -lattice V , is a non-trivial automorphism that fixes every element of a submodule $V' \subset V$ for which V/V' is free of rank 1.

- (1) A **D-reflection group** is a pair (W, V) , where V is a D -lattice and $W < \text{Aut}_D(V)$ is a subgroup generated by reflections. We say (W, V) is **finite** if W is a finite group.
- (2) A **D-root datum** is a triple $(W, V, \{P_\sigma\})$, where (W, V) is a finite D -reflection group and $\{P_\sigma\}$ is a collection of rank one submodules of V , indexed by the set $\{\sigma\}$ of reflections in W , satisfying

- (a) $\text{im}(1 - \sigma) \subset P_\sigma$ and
- (b) $w(P_\sigma) = P_{w\sigma w^{-1}}$ for all $w \in W$.

An isomorphism of D -reflection groups $(W, V) \xrightarrow{\sim} (W', V')$ is an isomorphism $\varphi : V \rightarrow V'$ with $W' = \varphi W \varphi^{-1}$. An isomorphism of D -root data $(W, V, \{P_\sigma\}) \xrightarrow{\sim} (W', V', \{P'_\sigma\})$ is such an isomorphism $\varphi : V \rightarrow V'$ satisfying additionally $\varphi(P_\sigma) = P'_{\varphi\sigma\varphi^{-1}}$ for each reflection $\sigma \in W$.

1.1.2. A \mathbb{Z} -reflection group (W, V) yields a D -reflection group $D \otimes (W, V) := (W, D \otimes V)$ via extension of scalars. Similarly, for a root datum $(W, V, \{P_\sigma\})$ over \mathbb{Z} , we have the D -root datum $D \otimes (W, V, \{P_\sigma\}) := (W, D \otimes V, \{DP_\sigma\})$.

Lemma 1.1.3. *If there exists an embedding $D \hookrightarrow \mathbb{Q}$, then every D -root datum is the extension of scalars of a \mathbb{Z} -root datum.*

Proof. Let (W, V) be a D -reflection group. Starting from a \mathbb{Z} -lattice in V and acting on it with W shows there exists a finitely generated $\mathbb{Z}W$ -submodule $V^0 \subset V$ with $D \otimes_{\mathbb{Z}} V^0 \rightarrow V$ an isomorphism; see [CR, Corollary 23.14]. For a D -root datum $(W, V, \{P_\sigma\})$, we can then take the \mathbb{Z} -root datum $(W, V^0, \{P^0_\sigma\})$, with $P^0_\sigma := P_\sigma \cap V^0$. □

Remark 1.1.4.

- (1) Lemma 1.1.3 does not imply that root data over $D \subset \mathbb{Q}$ are “the same” as root data over \mathbb{Z} ; see Example 1.4.2.
- (2) The condition $D \subset \mathbb{Q}$ is necessary in Lemma 1.1.3, as one observes by considering dihedral groups as real reflection groups on \mathbb{R}^2 , or the complex reflection groups generated by a root of unity acting by multiplication on \mathbb{C} .

We will use the following direct computations several times.

Lemma 1.1.5. *Consider a D -lattice V .*

- (1) *For a fixed $\phi \in V^*$ and $\kappa_1, \kappa_2 \in V$ with $\phi(\kappa_1) = 2 = \phi(\kappa_2)$, we have the reflections $s_i : \lambda \mapsto \lambda - \phi(\lambda)\kappa_i$ of order 2 on V . Then*

$$(s_1 s_2)^j(\kappa_1) = \kappa_1 + 2j(\kappa_1 - \kappa_2), \quad \text{for all } j \in \mathbb{N}.$$

- (2) *For a fixed $\kappa \in V$ and $\phi_1, \phi_2 \in V^*$ with $\phi_1(\kappa) = 2 = \phi_2(\kappa)$, we have the reflections $s_i : \lambda \mapsto \lambda - \phi_i(\lambda)\kappa$ of order 2 on V . Then, for all $\lambda \in V$,*

$$(s_1 s_2)^j(\lambda) = \lambda + j(\phi_2(\lambda) - \phi_1(\lambda))\kappa, \quad \text{for all } j \in \mathbb{N}.$$

Lemma 1.1.6. *Consider a prime p and two \mathbb{Z} -root data $(W, V, \{P_\sigma\})$ and $(W', V', \{P'_\sigma\})$. Assume there exists an isomorphism $\varphi : \mathbb{Z}[1/p] \otimes (W, V) \xrightarrow{\sim} \mathbb{Z}[1/p] \otimes (W', V')$ of $\mathbb{Z}[1/p]$ -reflection groups.*

- (1) *If $p = 2$, then φ is actually an isomorphism of $\mathbb{Z}[1/p]$ -root data.*
- (2) *If $p > 2$ and if the further extension along $\mathbb{Z}[1/p] \rightarrow \mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^n$ of φ induces an isomorphism of \mathbb{Z}_2 -root data, then φ is an isomorphism of $\mathbb{Z}[1/p]$ -root data.*

Proof. It is well-known and easy to show that for \mathbb{Z} -root data, we either have

$$P_\sigma = \text{im}(1 - \sigma) \quad \text{or} \quad \text{im}(1 - \sigma) = 2P_\sigma. \tag{1.1}$$

Hence, the additional condition in the definition of an isomorphism of $\mathbb{Z}[1/2]$ -root data is trivially satisfied.

Now assume $p > 2$. In this case, (1.1) shows that in $\mathbb{Z}[1/p] \otimes V'$, we have either $\varphi(P_\sigma) = P'_{\varphi\sigma\varphi^{-1}}$, $\varphi(P_\sigma) = 2P'_{\varphi\sigma\varphi^{-1}}$, or $2\varphi(P_\sigma) = P'_{\varphi\sigma\varphi^{-1}}$. By assumption, after extension to scalars to \mathbb{Z}_2 , only the first option is possible. As 2 is not invertible in \mathbb{Z}_2 , this means that also over $\mathbb{Z}[1/p]$, only the first option was possible. \square

1.2. Real-type root data

The following definition is closer to the classical definition of (reduced) root data.

Definition 1.2.1. A **real-type D -root datum** is a quadruple $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$, where \mathbf{X} and \mathbf{Y} are D -lattices with subsets $\mathbf{R} \subset \mathbf{X}$ and $\mathbf{R}^\vee \subset \mathbf{Y}$, together with

- (a) a perfect bilinear pairing $\langle \cdot, \cdot \rangle : \mathbf{X} \times \mathbf{Y} \rightarrow D$;
- (b) a bijection $\mathbf{R} \rightarrow \mathbf{R}^\vee, \beta \mapsto \beta^\vee$;

such that

- (1) We have $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \mathbf{R}$.
- (2) If $\alpha^\vee \in \mathbf{R}^\vee$ and $a \in D$, then $a\alpha^\vee \in \mathbf{R}^\vee$ if and only if $a \in D^\times$.
- (3) There are only finitely many D^\times -orbits in \mathbf{R}^\vee .
- (4) For each $\alpha \in \mathbf{R}$, the reflection $s_\alpha : \lambda \mapsto \lambda - \langle \alpha, \lambda \rangle \alpha^\vee$ in $\text{Aut}_D(\mathbf{Y})$ preserves \mathbf{R}^\vee .
- (4') For each $\alpha \in \mathbf{R}$, the reflection $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ in $\text{Aut}_D(\mathbf{X})$ preserves \mathbf{R} .

Note that $\langle s_\alpha(\lambda), s_\alpha(\mu) \rangle = \langle \lambda, \mu \rangle$, for $\lambda \in \mathbf{X}$ and $\mu \in \mathbf{Y}$, with s_α as defined in (4) and (4').

To a real-type root datum (X, R, Y, R^\vee) over \mathbb{Z} , we can define a real-type D -root datum $(D \otimes X, D^\times R, D \otimes Y, D^\times R^\vee)$, with obvious bilinear pairing and bijection $D^\times R \rightarrow D^\times R^\vee$ given by $a\lambda \mapsto a^{-1}\lambda^\vee$.

Remark 1.2.2. For $D = \mathbb{Z}$, Definition 1.2.1 is equivalent to the definition of a “donnée radicielle réduite” in [De, §3.6]. We will show below in Lemma 1.2.5 that a real-type D -root datum $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$ also satisfies the following:

- (2') If $\alpha \in \mathbf{R}$ and $a \in D$, then $a\alpha \in \mathbf{R}$ if and only if $a \in D^\times$.
- (3') There are only finitely many D^\times -orbits in \mathbf{R} .

In particular, the definition of real-type root data is closed under duality.

Example 1.2.3. Consider a D -root datum $(W, V, \{P_\sigma\})$. Take a reflection $\sigma \in W$ and a generator $v \in P_\sigma$. By condition 1.1.1(2)(a), there exists (a unique) $\beta \in V^*$ such that

$$\sigma(\lambda) = \lambda - \beta(\lambda)v, \quad \text{for all } \lambda \in V. \tag{1.2}$$

We then define $\mathbf{R}^\vee \subset V$ as the set of generators of the submodules $\{P_\sigma\}$ and $\mathbf{R} \subset V^*$ as the set of elements β constructed by the above procedure. If we denote by $S \subset W$ the set of reflections, then this procedure yields a surjective partially defined function

$$S \times \mathbf{R}^\vee \rightarrow \mathbf{R}, \quad (\sigma, v) \mapsto \beta. \tag{1.3}$$

Theorem 1.2.4. *Assume that for the D -root datum $(W, V, \{P_\sigma\})$, every reflection in W has order 2. Then (1.3) restricts to a bijection $\mathbf{R}^\vee \rightarrow \mathbf{R}$. The quadruple $(V^*, \mathbf{R}, V, \mathbf{R}^\vee)$, equipped with inverse bijection $\mathbf{R} \rightarrow \mathbf{R}^\vee$ and evaluation pairing $V^* \times V \rightarrow D$, is a real-type D -root datum.*

Proof. The assumption $\sigma^2 = 1$ in (1.2) shows that $\beta(v) = 2$, for every pair (v, β) we associate to a reflection in W via the procedure in 1.2.3. First, we observe that this procedure yields a well-defined function $\mathbf{R}^\vee \rightarrow \mathbf{R}$, namely that β only depends on v and not on σ . This is indeed the case since a given $v \in V$ cannot be a generator in both P_{σ_1} and P_{σ_2} for two distinct reflections $\sigma_1, \sigma_2 \in W$, which follows from finiteness of W , Lemma 1.1.5(2) and the fact that D is of characteristic 0. By definition of \mathbf{R} , the function $\mathbf{R}^\vee \rightarrow \mathbf{R}$ is surjective. Next, we prove that this function is injective, and hence a bijection. Assume, therefore, that for two generators $v_1 \in P_{\sigma_1}$ and $v_2 \in P_{\sigma_2}$, we obtain the same $\beta \in V^*$. By finiteness of W and Lemma 1.1.5(1), we find $v_1 = v_2$.

By the above paragraph, 1.2.1(1) is satisfied. To establish 1.2.1(2), it suffices to show that we cannot have non-trivial inclusions $P_{\sigma_1} \subset P_{\sigma_2}$. We can extend scalars along $D \hookrightarrow K$, with K the field of fractions. Now $\sigma_1\sigma_2$ acts as the identity on $KP_{\sigma_1} = KP_{\sigma_2}$, but also as the identity on KV/KP_{σ_1} . Since it has finite order and $\text{char}K = 0$, we find $\sigma_1\sigma_2 = 1$. Property 1.2.1(3) follows immediately from the fact that W is finite. Property 1.2.1(4) is an immediate consequence of 1.1.1(2)(b).

By letting $w \in W$ act on V^* by $w(f) = f \circ w^{-1}$, we also have a reflection group (W, V^*) . We can define $Q_\sigma := D\beta \subset V^*$, for each reflection $\sigma \in W$ with β as in (1.2), since Q_σ does not depend on our choice of generator $v \in P_\sigma$. It follows immediately that $(W, V^*, \{Q_\sigma\})$ is a D -root datum. That 1.2.1(4') is satisfied follows by applying the proof for (4) to $(W, V^*, \{Q_\sigma\})$. \square

We conclude this section with some technical results needed later.

Lemma 1.2.5. *Consider a real-type D -root datum $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$. For $\beta \in \mathbf{R}$ and $b \in D$, we have $b\beta \in \mathbf{R}$ if and only if $b \in D^\times$ and then $(b\beta)^\vee = b^{-1}\beta^\vee$. In particular, $s_{b\beta} = s_\beta$ and conditions (2') and (3') in 1.2.2 hold.*

Proof. Assume first that $b \in D^\times$. We need to prove that $(\sigma, b^{-1}\beta^\vee) \mapsto b\beta$ in (1.3), for $\sigma : \lambda \mapsto \lambda - \beta(\lambda)\beta^\vee$, which is clearly true.

Conversely, assume that $\beta_1 := b\beta \in \mathbf{R}$, for some $b \in D$, and set $\lambda := b\beta_1^\vee - \beta^\vee \in \mathbf{Y}$. Lemma 1.1.5(1) for $\phi = \langle \beta, - \rangle$ and $\kappa_1 = b\beta_1^\vee, \kappa_2 = \beta^\vee$ implies

$$b(s_{\beta_1} s_\beta)^j (\beta_1^\vee) = b\beta_1^\vee + 2j\lambda, \quad \text{for all } j \in \mathbf{N}.$$

Conditions 1.2.1(3) and (4) thus imply $\lambda = 0$. Finally, 1.2.1(2) then implies b is a unit. \square

Lemma 1.2.6. *Consider a real-type D -root datum $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$. If, for $\alpha, \beta, \gamma \in \mathbf{R}$, we have $s_\beta(\alpha^\vee) = \gamma^\vee$, then $s_\beta s_\alpha s_\beta = s_\gamma$ and $s_\beta(\alpha) = \gamma$.*

Proof. A direct calculation shows that, for every $\lambda \in \mathbf{Y}$,

$$s_\beta s_\alpha s_\beta(\lambda) = \lambda - \langle s_\beta(\alpha), \lambda \rangle s_\beta(\alpha^\vee). \tag{1.4}$$

Set $\gamma_1 = s_\beta(\alpha) \in \mathbf{R}$. Then we can apply Lemma 1.1.5(1) to $\phi = \langle \gamma_1, - \rangle$ and $\kappa_1 = \gamma_1^\vee, \kappa_2 = \gamma^\vee$ (so $s_1 = s_{\gamma_1}$ and $s_2 = s_\beta s_\alpha s_\beta$). By 1.2.1(3) and (4), it thus follows that $\gamma_1 = \gamma$. \square

1.3. Real reflection groups

1.3.1. Hypotheses Consider a finite-dimensional real vector space V (without fixed inner product/Euclidean structure), a reflection group $W < \text{Aut}_{\mathbb{R}}(V)$, in the sense of Definition 1.1.1(1), and a fixed *finite* generating set T of reflections in W such that

- The map from T to the set of hyperplanes in V , $s \mapsto H_s := \ker(1 - s)$, is injective.
- We have $wT w^{-1} \subset T$ for all $w \in W$.

Theorem 1.3.2. *Under the assumptions in 1.3.1, W is a finite group.*

Remark 1.3.3. If a real reflection group (W, V) is finite, by Weyl’s unitary trick, we can assume it is Euclidean (meaning there is an inner product on V for which each reflection in W is orthogonal).

The remainder of this section is devoted to the proof.

1.3.4. All topological references consider the Euclidean topology on V . Consider

$$\mathcal{H} = \cup_{t \in T} H_t \subset V$$

and refer to the connected components of the complement of \mathcal{H} in V as chambers. We say that H_t is a **wall** of a chamber A if the intersection of $\overline{A_0}$ and H_t cannot be contained in a codimension 2 hyperplane. Fix one such chamber A_0 . Denote by $S \subset T$ the set of reflections s for which H_s is a wall of A_0 . Our assumptions in 1.3.1 imply that W acts on the (finite) set of chambers.

Lemma 1.3.5. *The set S is a set of generators for W .*

Proof. Denote by $W_S \subset W$ the subgroup generated by $S \subset T$. We need to show that $W_S = W$, or equivalently, $T \subset W_S$.

First, we show that every W_S -orbit in V intersects $\overline{A_0}$. By continuity, it suffices to show that every W_S -orbit in $V \setminus \mathcal{H}$ intersects A_0 . For $v \in V \setminus \mathcal{H}$, there exists a sequence A_0, A_1, \dots, A_l of distinct chambers where $v \in A_l$, and for each $0 \leq i < l$, there is $t_i \in T$ such that H_{t_i} is a wall of $\overline{A_i}$ and of $\overline{A_{i+1}}$.

If $l = 0$, there is nothing to prove, so assume $l > 0$. Then $t_0(A_1) = A_0$, and by assumption, $t_0 \in S$. Now $A_0, A'_1 = t_0(A_2), A'_2 = t_0(A_3), \dots, A'_{l-1} = t_0(A_l)$ forms a chain of distinct

chambers as before and $t_0(v) \in A'_{l-1}$. We can thus perform induction on l to deduce the claim.

Now take an arbitrary $t \in T$ and let A be a chamber which has H_t as a wall. By the above, we know there exists $w \in W_S$ with $A = w(A_0)$. This implies that

$$H_t = w(H_s) = H_{ws w^{-1}}.$$

By the first hypothesis in 1.3.1, this shows that $t = w s w^{-1} \in W_S$. □

Precisely as in [Bo, V.§3.2 Theorem 1], we then find the following consequence.

Corollary 1.3.6. *The pair (W, S) is a Coxeter system.*

Proof of Theorem 1.3.2. The reflections in the Coxeter group (W, S) are by definition the elements of the set $\cup_{w \in W} w S w^{-1}$, which by assumption is included in T and hence finite. By [BB, Corollary 1.4.5], any Coxeter group (W, S) with finitely many reflections is finite. □

1.4. Equivalence of definitions

Theorem 1.4.1. *If there exists an embedding $D \hookrightarrow \mathbb{R}$, then the map in Theorem 1.2.4 is a bijection between the sets of isomorphism classes of D -root data and real-type D -root data.*

Proof. Since we have $D \subset \mathbb{R}$, the only roots of unity in D are ± 1 , and it follows that every reflection of finite order must have order two. Hence, the map in Theorem 1.2.4 is defined on every D -root datum.

To each real-type D -root datum $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$ we will now associate a D -root datum, in a way which is easily seen to be the inverse of the above map. Define the D -reflection group $W < \text{Aut}_D(\mathbf{Y})$ generated by $\{s_\alpha \mid \alpha \in \mathbf{R}\}$. To a reflection s_γ , $\gamma \in \mathbf{R}$, we associate the corresponding rank one submodule $D\gamma^\vee \subset \mathbf{Y}$.

We show that W is finite by considering the corresponding real reflection group acting on $\mathbf{Y} \otimes_D \mathbb{R}$. By Lemma 1.2.5, W is generated by a finite (see 1.2.1(2)) set of reflections $\{s_\alpha \mid \alpha \in \mathbf{R}\}$, such that the reflecting hyperplane $\ker(1 - s_\alpha) = \ker\langle \alpha, - \rangle$ determines s_α . Moreover, we claim that for each $\alpha \in \mathbf{R}$ and $w \in W$, we have $w s_\alpha w^{-1} = s_\gamma$ for some $\gamma \in \mathbf{R}$. Clearly, it suffices to consider the case $w = s_\beta$, which is Lemma 1.2.6. We can now apply Theorem 1.3.2.

Now it follows that the triple $(W, \mathbf{Y}, \{D\gamma^\vee\})$ is a D -root datum. Indeed, by Remark 1.3.3 and [Hu, Proposition 1.14], every reflection in W is equal to s_γ for some $\gamma \in \mathbf{R}$, and property (a) in 1.1.1(2) is automatic, while (b) follows from Lemma 1.2.6. □

Clearly, the bijection in Theorem 1.4.1 exchanges the two notions of extensions of scalars of root data. We conclude this section with some examples of root data which become isomorphic after extension of scalars.

Example 1.4.2.

- (1) The root datum of SO_{2n+1} becomes isomorphic to its dual (the root datum of Sp_{2n}) after extension of scalars to D if and only if 2 is invertible in D .
- (2) The root datum of SL_n becomes isomorphic to its dual (the root datum of PGL_n) after extension of scalars to D if and only if n is invertible in D .

1.5. Isogenies

We fix a prime p and define isogenies of $\mathbb{Z}[1/p]$ -root data. We will work with real-type root data, but we refer to them simply as root data (this is justified by Theorem 1.4.1).

Definition 1.5.1. An isogeny $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee) \rightarrow (\mathbf{X}_1, \mathbf{R}_1, \mathbf{Y}_1, \mathbf{R}_1^\vee)$ of $\mathbb{Z}[1/p]$ -root data is an injective morphism $\varphi : \mathbf{X} \hookrightarrow \mathbf{X}_1$ of $\mathbb{Z}[1/p]$ -modules such that

- (1) the induced $\varphi^\vee : \mathbf{Y}_1 \rightarrow \mathbf{Y}$ is also injective;
- (2) φ restricts to a bijection $\mathbf{R} \rightarrow \mathbf{R}_1$;
- (3) $\varphi^\vee(\varphi(\alpha)^\vee) = \alpha^\vee$, for all $\alpha \in \mathbf{R}$.

Example 1.5.2. An isogeny of \mathbb{Z} -root data, with respect to some prime p , is defined in [St, §1]. It follows immediately that the induction to $\mathbb{Z}[1/p]$ of such an isogeny yields an isogeny of $\mathbb{Z}[1/p]$ -root data. Note that Definition 1.5.1 is simpler than the definition in [St, §1], as the powers of p present in [St, §1] are subsumed by (2), since multiplication by p is invertible on $\mathbb{Z}[1/p]$ -root data.

Conversely, if for \mathbb{Z} -root data RD_1 and RD_2 , there exists an isogeny $\varphi : \mathbb{Z}[1/p] \otimes RD_1 \rightarrow \mathbb{Z}[1/p] \otimes RD_2$, then for some $l \in \mathbb{N}$, the map $p^j \varphi$ restricts to an isogeny $RD_1 \rightarrow RD_2$ in the sense of [St] for all $j \geq l$.

2. Perfection of schemes

Fix a prime p .

2.1. Notation

We recall some basic set-up of algebraic geometry; see, for instance, [DG].

Fix a field k . Denote by Alg_k the category of commutative k -algebras. We consider the categories (where the first two ‘‘inclusions’’ are fully-faithful embeddings)

$$\text{Alg}_k^{\text{op}} \subset \text{Sch}_k \subset \text{Fais}_k \subset \text{Fun}_k. \tag{2.1}$$

Here, Sch_k is the category of k -schemes, and Fun_k is the category of functors $\text{Alg}_k \rightarrow \text{Set}$. The category Fais_k stands for the full subcategory of such functors which are sheaves for the fpqc topology. In other words, a functor F is in Fais_k if and only if

$$F(A) \rightarrow F(B) \rightrightarrows F(B \otimes_A B)$$

is an equaliser for every faithfully flat A -algebra B , and F commutes with finite products. When k is clear, we will usually leave out the subscript in the above categories.

The inclusion $I : \mathbf{Fais} \hookrightarrow \mathbf{Fun}$ has a left adjoint

$$S : \mathbf{Fun} \rightarrow \mathbf{Fais}$$

which commutes with finite limits (as well as all colimits).

By a subgroup of an affine group scheme G , we understand a closed subscheme which inherits a group structure, or in other words, an affine group scheme represented by a quotient of the Hopf algebra representing G .

2.2. Perfection functors

2.2.1. Frobenius morphisms For a commutative \mathbb{F}_p -algebra A , we have the p -th power algebra morphism

$$\mathrm{Fr} = \mathrm{Fr}_A : A \rightarrow A, \quad a \mapsto a^p.$$

For an \mathbb{F}_p -scheme \mathfrak{X} , we have the morphism $\mathrm{Fr} : \mathfrak{X} \rightarrow \mathfrak{X}$, which is the identity on the underlying topological space and given by the p -th power map on the sheaf of algebras. For $F \in \mathbf{Fun}_{\mathbb{F}_p}$, we define the Frobenius morphism $F \rightarrow F$ as the natural transformation given by letting F act on the p -th power morphism. Concretely, the evaluation of the natural transformation at an arbitrary $A \in \mathbf{Alg}_{\mathbb{F}_p}$ is

$$F(A) \xrightarrow{F(\mathrm{Fr}_A)} F(A).$$

These Frobenius morphisms are compatible with the inclusions (2.1).

For an object F of $\mathbf{Fun}_{\mathbb{F}_p}$ or $\mathbf{Alg}_{\mathbb{F}_p}$, the notation $\varinjlim F$ or $\varprojlim F$ will always be used for the direct or inverse limit along the Frobenius morphism. For example, for an \mathbb{F}_p -algebra A , the algebra $A_{\mathrm{perf}} := \varinjlim A$ is the direct limit of the system

$$A \xrightarrow{a \mapsto a^p} A \xrightarrow{a \mapsto a^p} A \xrightarrow{a \mapsto a^p} A \rightarrow \dots$$

For $F \in \mathbf{Fun}_{\mathbb{F}_p}$, it follows directly that

$$(\varinjlim F)(A) = F(\varinjlim A) \quad \text{and} \quad (\varprojlim F)(A) = F(\varprojlim A).$$

An \mathbb{F}_p -scheme (or an algebra or functor) is called **perfect** if the Frobenius map is an isomorphism; see [BS, Definition 3.1].

Lemma 2.2.2. *The endofunctor of $\mathbf{Fun}_{\mathbb{F}_p}$*

$$F \mapsto F_{\mathrm{perf}} := \varinjlim F$$

restricts to endofunctors of \mathbf{Fais} and \mathbf{Sch} . Moreover, for $A \in \mathbf{Alg}_{\mathbb{F}_p}$, we have $(\mathrm{Spec} A)_{\mathrm{perf}} = \mathrm{Spec}(A_{\mathrm{perf}})$.

Proof. It is a standard property that limits exist in a Grothendieck topos and can be computed in the presheaf category, which shows that perfection restricts to \mathbf{Fais} . The remaining properties follow from the explicit realisation in Example 2.2.3 below. \square

Example 2.2.3. For an \mathbb{F}_p -algebra A , set $(X, \mathcal{O}) = \text{Spec}A$. Using the basis of distinguished open subsets, it follows easily that $\text{Spec}(A_{\text{perf}}) = (X, \mathcal{O}')$, with \mathcal{O}' the sheafification of the presheaf $U \mapsto \mathcal{O}(U)_{\text{perf}}$.

It then follows that for an arbitrary \mathbb{F}_p -scheme $\mathfrak{X} = (X, \mathcal{O})$, the scheme $\mathfrak{X}_{\text{perf}}$ can be realised as (X, \mathcal{O}') with \mathcal{O}' the sheafification of the presheaf $U \mapsto \mathcal{O}(U)_{\text{perf}}$.

Remark 2.2.4. In 2.2.3, is essential to take \mathcal{O}' as the direct limit of \mathcal{O} in the category of sheaves (as opposed to presheaves). For example,

- (1) For an infinite family of \mathbb{F}_p -algebras A_i , consider the (non-affine) scheme $(X, \mathcal{O}) = \sqcup_i \text{Spec}A_i$. Then, for general A_i , we have, by the sheaf axioms and Lemma 2.2.2,

$$\Gamma(X, \mathcal{O}') = \prod_i (A_i)_{\text{perf}} \neq \Gamma(X, \mathcal{O})_{\text{perf}} = \left(\prod_i A_i \right)_{\text{perf}}.$$

- (2) Also for non-noetherian affine schemes, this phenomenon occurs. Consider $A = \mathbb{F}_p[x_i \mid i \in \mathbb{N}]/(x_i x_j, i \neq j)$. Let U_i be the distinguished open corresponding to x_i . The (disjoint) union $U = \cup_i U_i$ is the complement of the origin and, as in (1), we find $\mathcal{O}(U)_{\text{perf}} \neq \mathcal{O}'(U)$.

Remark 2.2.5. For \mathbb{F}_p -algebras A, B , we have

$$\text{Alg}(A_{\text{perf}}, B) \cong \varprojlim \text{Alg}(A, B) \cong \text{Alg}(A, \varprojlim B).$$

In particular, if B is perfect, we have $\text{Alg}(A_{\text{perf}}, B) \cong \text{Alg}(A, B)$.

Lemma 2.2.6. *Let \mathfrak{X} be an \mathbb{F}_p -scheme.*

- (1) *We have $\dim \mathfrak{X} = \dim \mathfrak{X}_{\text{perf}}$, and $\mathfrak{X}_{\text{perf}}$ is quasi-compact (resp. connected) if and only if \mathfrak{X} is quasi-compact (resp. connected).*
- (2) *Any radical ideal I in a perfect \mathbb{F}_p -algebra A satisfies $I^2 = I$.*
- (3) *If \mathfrak{X} is perfect, for $x \in \mathfrak{X}$, we have $T_{\mathfrak{X}, x} = 0$.*
- (4) *Perfect schemes are reduced. Moreover, $-_{\text{perf}}$ sends $\mathfrak{X}_{\text{red}} \rightarrow \mathfrak{X}$ to an isomorphism.*

Proof. Part (1) follows immediately from the fact that the underlying topological spaces of \mathfrak{X} and $\mathfrak{X}_{\text{perf}}$ are the same; see Example 2.2.3. Part (2) is obvious. By (2), it is clear that the Zariski cotangent space is zero, which proves (3). Alternatively, for (3), let A be a perfect \mathbb{F}_p -algebra and κ a field. Every algebra morphism $A \rightarrow \kappa[\epsilon]/(\epsilon^2)$ factors through $\kappa \hookrightarrow \kappa[\epsilon]/(\epsilon^2)$. Applying this to $\text{Spec}(\kappa(x)[\epsilon]/(\epsilon^2)) \rightarrow \mathfrak{X}$ shows the claim. Part (4) is immediate. □

2.3. Relative version

Fix a perfect field k of characteristic p for the remainder of the section.

2.3.1. For a fixed \mathbb{F}_p -scheme \mathfrak{T} , perfection naturally yields a functor from the category of \mathfrak{T} -schemes to the category of $\mathfrak{T}_{\text{perf}}$ -schemes. Using the canonical morphism $\mathfrak{T}_{\text{perf}} \rightarrow \mathfrak{T}$, we can also interpret perfection as an endofunctor of the category of \mathfrak{T} -schemes. We will

take the latter point of view for $\mathfrak{T} = \text{Spec}k$ (in which case $\mathfrak{T}_{\text{perf}} \rightarrow \mathfrak{T}$ is an isomorphism), and we will henceforth interpret the perfection functor as an endofunctor of Sch_k .

2.3.2. Sometimes it will be beneficial to consider an alternative realisation of the perfection of k -schemes, in which the morphisms in the chain of which we take the limit are morphisms of k -schemes. For a k -scheme \mathfrak{X} , let $\mathfrak{X}^{(1)}$ denote the extension of scalars of \mathfrak{X} along the Frobenius automorphism $k \rightarrow k$. The morphism $\text{Fr} : \mathfrak{X} \rightarrow \mathfrak{X}$ over \mathbb{F}_p from 2.2.1 then lifts to a morphism $\text{Fr} : \mathfrak{X} \rightarrow \mathfrak{X}^{(1)}$ of k -schemes. For instance, for a k -algebra A , this corresponds to the morphism

$$A^{(1)} \rightarrow A, \quad \lambda \otimes a \mapsto \lambda a^p, \tag{2.2}$$

with $A^{(1)} = k \otimes A$ the extension of scalars along the Frobenius automorphism of k .

By taking iterates of the Frobenius automorphism and its inverse (k is perfect), we define $\mathfrak{X}^{(i)}$ for $i \in \mathbb{Z}$. Then we have (over k)

$$\mathfrak{X}_{\text{perf}} \cong \varprojlim_{i \rightarrow \infty} \mathfrak{X}^{(-i)}.$$

The advantage of the approach in this subsection is that it extends to Fun_k by setting

$$F_{\text{perf}}(A) := \varprojlim_{i \rightarrow \infty} F(A^{(i)}).$$

By construction, perfection commutes with limits – for instance, products – in Fun_k .

Proposition 2.3.3. *Consider a morphism f in Fais_k .*

- (1) *If f is an epimorphism in Fais_k , then so is f_{perf} .*
- (2) *If f is a monomorphism in Fais_k , then so is f_{perf} .*

Proof. For part (1), we can use the criterion from [DG, Corollaire 2.8] to describe that f is an epimorphism, which carries over to f_{perf} by [BS, Lemma 3.4(xii)]. Part (2) is a generality for limits of monomorphisms. □

It is obvious that $\mathfrak{X} \mapsto \mathfrak{X}_{\text{perf}}$ loses a lot of information. For instance,

$$(\mathfrak{X}_{\text{perf}})_{\text{perf}} \cong \mathfrak{X}_{\text{perf}} \cong (\mathfrak{X}_{\text{red}})_{\text{perf}}.$$

A more subtle example is given below.

Example 2.3.4. Assume $p > 2$. Consider the algebra $A = k[x, y]/(y^p - x^2)$ with injective algebra morphism $A \hookrightarrow k[z]$, given by $x \mapsto z^p, y \mapsto z^2$. Then $\mathfrak{X} := \text{Spec}A$ is reduced, but perfection sends $\mathbb{A}_k^1 \rightarrow \mathfrak{X}$ to an isomorphism.

2.4. Perfect finite type

Recall that $\mathfrak{X} \in \text{Sch}_k$ is of finite type (over k) if the underlying topological space is quasi-compact, and for every $x \in \mathfrak{X}$, there exists an affine open neighbourhood isomorphic to the spectrum of a finitely generated k -algebra.

Lemma 2.4.1. *For a perfect commutative k -algebra A , the following conditions are equivalent:*

(a) *There is a finite subset $S \subset A$ such that the set*

$$S' = \{x \in A \mid x^{p^n} \in S \text{ for some } n \in \mathbb{N}\}$$

generates A as a k -algebra;

(b) *There exists a finitely generated k -algebra A_0 with $A \cong (A_0)_{\text{perf}}$.*

If the conditions are satisfied, we say that A is perfectly finitely generated.

Proof. Exercise. □

Proposition 2.4.2. *For a perfect k -scheme \mathfrak{X} , the following are equivalent:*

(a) *\mathfrak{X} is quasi-compact, and for every $x \in \mathfrak{X}$, there exists an affine open neighbourhood corresponding to a perfectly finitely generated k -algebra;*

(b) *There exists a scheme \mathfrak{Y} of finite type over k with $\mathfrak{X} \cong \mathfrak{Y}_{\text{perf}}$.*

If the conditions are satisfied, we say that \mathfrak{X} is of perfect finite type (over k).

Proof. Clearly, (b) implies (a). That (a) implies (b) is proved in [BS, Proposition 3.13]. □

2.5. Perfection of line bundle cohomology and quotients

All schemes and functors are assumed to be over k .

Lemma 2.5.1. *Let \mathfrak{X} be a quasi-compact separated scheme over k and \mathcal{L} a line bundle on \mathfrak{X} . For the pullback $p^*\mathcal{L}$ along $p: \mathfrak{X}_{\text{perf}} \rightarrow \mathfrak{X}$ and $i \in \mathbb{N}$, we have*

$$H^i(\mathfrak{X}_{\text{perf}}, p^*\mathcal{L}) \cong \varinjlim_j H^i(\mathfrak{X}, \mathcal{L}^{\otimes p^j}),$$

where the transition maps are induced from $\mathcal{L}^{\otimes p^j} \rightarrow \mathcal{L}^{\otimes p^{j+1}}$, $f \mapsto f^{\otimes p}$.

Proof. Since \mathfrak{X} is quasi-compact, we can take a finite cover \mathbb{U} by affine opens and, since \mathfrak{X} is separated, intersections of these opens are again affine. Moreover, the cohomology groups $H^i(\mathfrak{X}, -)$ are canonically isomorphic to the Čech cohomology groups $\check{H}^i(\mathbb{U}, -)$. It follows that $H^i(\mathfrak{X}, -)$ commutes with direct limits.

Now, for a line bundle, we have (as sheaves on the underlying topological space of $\mathfrak{X}_{\text{perf}}$ or \mathfrak{X}) isomorphisms

$$p^*\mathcal{L} \cong (\varinjlim_j \mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L} \cong \varinjlim_j \mathcal{L}^{\otimes p^j},$$

which follow easily from Remark 2.5.2 below.

The conclusion follows from the combination of the two paragraphs. □

Remark 2.5.2. In case $\mathfrak{X} = \text{Spec}A$ for a commutative k -algebra A , Lemma 2.5.1 simply states that

$$A_{\text{perf}} \otimes_A \mathcal{L} \cong \varinjlim \mathcal{L}^{\otimes p^j},$$

for every invertible A -module \mathcal{L} . To prove this isomorphism, we only need to observe that the morphism

$$A \otimes_A \mathcal{L} \rightarrow \mathcal{L}^{\otimes p}, \quad a \otimes v \mapsto av^{\otimes p},$$

where the first module is the extension of scalars of \mathcal{L} along the p -th power map, is an isomorphism. Indeed, the morphism commutes with localisation and is obviously an isomorphism when A is local.

2.5.3. For $X \in \text{Fais}$ and G a group object in Fais acting on X (on the right), we consider the corresponding co-equalisers in Fun and Fais of $X \times G \rightrightarrows X$ and denote them by $X/_0G$ and $X/_1G$. In particular, $X/_1G = S(X/_0G)$, for the sheafification functor $S : \text{Fun} \rightarrow \text{Fais}$.

It follows from the definitions that G_{perf} is again a group object and acts on X_{perf} . We create the following commutative diagram in Fun :

$$\begin{array}{ccccc} X_{\text{perf}} & \longrightarrow & X_{\text{perf}}/_0G_{\text{perf}} & \longrightarrow & (X/_0G)_{\text{perf}} \\ & & \downarrow & & \downarrow \\ & & X_{\text{perf}}/_1G_{\text{perf}} & \longrightarrow & (X/_1G)_{\text{perf}}. \end{array} \tag{2.3}$$

The vertical arrows are induced from the adjunction $S \dashv I$ (either directly or via the action of the perfection functor). The left horizontal arrow is the defining one for the co-equaliser. The remaining two arrows are uniquely defined from the co-equaliser properties applied to the perfection of the morphisms $X/_1G \leftarrow X \rightarrow X/_0G$.

Theorem 2.5.4. *Assume that the action of G on X is free. Then the morphism from (2.3)*

$$X_{\text{perf}}/_1G_{\text{perf}} \rightarrow (X/_1G)_{\text{perf}}$$

in Fais_k is an isomorphism.

Proof. By Proposition 2.3.3(1), the composite morphism (from top left to bottom right) in (2.3) is an epimorphism in Fais . In particular, the lower horizontal arrow is an epimorphism. Since isomorphisms in Grothendieck topoi are precisely morphisms which are both monomorphisms and epimorphisms, it now suffices to show this arrow is also a monomorphism.

Since sheafification sends monomorphisms to monomorphisms, it actually suffices to show that $X_{\text{perf}}/_0G_{\text{perf}} \rightarrow (X/_1G)_{\text{perf}}$ is a monomorphism in Fun . Using the assumption that the action is free, we can prove that $X_{\text{perf}}/_0G_{\text{perf}} \rightarrow (X/_0G)_{\text{perf}}$ and $(X/_0G)_{\text{perf}} \rightarrow (X/_1G)_{\text{perf}}$ are monomorphisms.

Indeed, the first case follows from the fact that for an inverse system of sets X_i with free actions of groups G_i ,

$$\varprojlim X_i / \varprojlim G_i \rightarrow \varprojlim (X_i / G_i)$$

is injective. By Proposition 2.3.3(2), for the second morphism, it suffices to demonstrate that $X/{}_0G \rightarrow X/{}_1G$ is a monomorphism. This is equivalent to the claim that the presheaf $X/{}_0G$ is separated for the fpqc topology, meaning that

$$X(A)/G(A) \rightarrow X(B)/G(B)$$

is an injection for every faithfully flat A -algebra B (and the same for finite products of algebras). This is easily verified for free actions; see, for instance, [Ja, §I.5.5]. □

3. Perfection of group schemes

Let k be a perfect field of characteristic $p > 0$. All schemes are assumed to be over k .

3.1. Perfection

For an affine group scheme G over k , clearly G_{perf} is again an affine group scheme over k . Note also that, since k is perfect, G_{red} is a subgroup of G .

3.1.1. The group $p^{\mathbb{Z}}$ We denote by $p^{\mathbb{Z}} < \mathbb{Z}[1/p]^{\times}$ the group of powers of p , an infinite cyclic group. Let G be an affine group scheme over k , which can be defined over \mathbb{F}_p . Then we can choose an isomorphism $\phi : G^{(1)} \xrightarrow{\sim} G$, which yields an automorphism

$$\Phi = \phi_{\text{perf}} \circ \text{Fr} : G_{\text{perf}} \xrightarrow{\sim} G_{\text{perf}}^{(1)} \xrightarrow{\sim} G_{\text{perf}},$$

and a corresponding group homomorphism $p^{\mathbb{Z}} \rightarrow \text{Aut}(G_{\text{perf}})$, $p \mapsto \Phi$. Examples are given in 3.4.5.

Lemma 3.1.2. *For a perfect group scheme G , we have $\text{Lie}G = 0$ and $\text{Dist}G = k$.*

Proof. This is an immediate consequence of Lemma 2.2.6(2) and (3). □

Theorem 3.1.3. *Let \mathbf{G} be a perfect affine group scheme over k . The following are equivalent:*

- (a) *The scheme \mathbf{G} is of perfect finite type (i.e. $k[\mathbf{G}]$ is perfectly finitely generated);*
- (b) *The group scheme \mathbf{G} is a subgroup of $GL(V)_{\text{perf}}$ for a finite-dimensional vector space V ;*
- (c) *There exists an affine group scheme G of finite type with $\mathbf{G} \cong G_{\text{perf}}$;*
- (d) *There exists a reduced affine group scheme G of finite type with $\mathbf{G} \cong G_{\text{perf}}$.*

Proof. Clearly, (d) implies (c). Any affine group scheme G of finite type is a subgroup of some $GL(V)$; see, for example, [DM, Corollary 2.5]. It follows immediately that $G_{\text{perf}} < GL(V)_{\text{perf}}$, so (c) implies (b). That (b) implies (a) follows by the observation that $k[\mathbf{G}]$ is a quotient of the perfectly finitely generated algebra $\varinjlim k[GL(V)]$.

Finally, we prove that (a) implies (d). Let S be a finite set which perfectly generates $k[\mathbf{G}]$ as in Lemma 2.4.1(a). It is possible to replace S by a finite set $S_1 \supset S$ such that the subalgebra that is generated (in the ordinary, non-perfect sense) by S_1 is actually a Hopf subalgebra of $k[\mathbf{G}]$; see [Ab, Lemma 3.4.5]. Let G be the corresponding affine group scheme. By construction $G_{\text{perf}} \cong \mathbf{G}$ and, since $k[G]$ is a subalgebra of $k[\mathbf{G}]$, it follows that G is reduced. \square

Lemma 3.1.4. *Let \mathbf{G} be a perfect affine group scheme and H an affine group scheme. Then $H_{\text{perf}} \rightarrow H$ induces an isomorphism*

$$\text{Hom}(\mathbf{G}, H_{\text{perf}}) \xrightarrow{\sim} \text{Hom}(\mathbf{G}, H),$$

with inverse given by perfection. Moreover, if H is of finite type and G an affine group scheme, then

$$\text{Hom}(G_{\text{perf}}, H_{\text{perf}}) \xrightarrow{\sim} \varinjlim \text{Hom}(G^{(-i)}, H).$$

Proof. This is an immediate application of Remark 2.2.5, or the fact that the perfection functor on \mathbb{F}_p -schemes is right adjoint to the inclusion functor for perfect schemes. \square

3.1.5. For a subgroup H of an affine group scheme G , we denote by G/H , when it exists, the equaliser of $G \times H \rightrightarrows G$ in Sch_k .

Recall from [DG, III, §3 Théorème 5.4] that for G of finite type, the quotient $G/{}_1H$ in Fais_k is a scheme and of finite type over k , and so in particular is equal to G/H .

Theorem 3.1.6. (1) *For every perfect subgroup \mathbf{H} of an affine group scheme \mathbf{G} of perfect finite type, the quotient \mathbf{G}/\mathbf{H} exists, is of perfect finite type and is isomorphic to $\mathbf{G}/{}_1\mathbf{H}$.*

(2) *For an affine group scheme G , every perfect subgroup of G_{perf} is the perfection of a subgroup of G . More precisely, every perfect (normal) subgroup of G_{perf} is the perfection of a reduced (normal) subgroup of $G_{\text{red}} < G$.*

(3) *For an affine group scheme G of finite type with subgroup H , the quotient $G_{\text{perf}}/H_{\text{perf}}$ exists and is isomorphic to $(G/H)_{\text{perf}}$ and $G_{\text{perf}}/{}_1H_{\text{perf}}$.*

Proof. We will freely use the results from [DG] recalled above. Part (1) is then an immediate consequence of parts (2) and (3).

Now we prove part (2). Take a perfect subgroup $\mathbf{H} < G_{\text{perf}}$. We have a commutative square, where \hookrightarrow denotes the inclusion of a subgroup and \twoheadrightarrow denotes a faithfully flat homomorphism

$$\begin{array}{ccc} \mathbf{H} & \hookrightarrow & G_{\text{perf}} \\ \downarrow & & \downarrow \\ L & \hookrightarrow & G_{\text{red}}, \end{array}$$

where L is just defined to be the image of the composite diagonal homomorphism. By Lemma 3.1.4, perfecting the lower path in the square yields homomorphisms $\mathbf{H} \twoheadrightarrow L_{\text{perf}} \hookrightarrow$

G_{perf} which compose to the original inclusion $\mathbf{H} \rightarrow G_{\text{perf}}$. Clearly, $\mathbf{H} \twoheadrightarrow L_{\text{perf}}$ must be an isomorphism. If \mathbf{H} is a normal subgroup, it follows easily that so is $L < G_{\text{red}}$.

Part (3) is an application of Theorem 2.5.4. □

Remark 3.1.7. A perfect group scheme can have non-perfect subgroups; for instance,

$$\mu_{p^\infty} := \varprojlim \mu_{p^i}, \quad \text{i.e. } k[\mu_{p^\infty}] = k[x^{1/p^\infty}]/(x-1)$$

is a subgroup of $(\mathbb{G}_m)_{\text{perf}}$. Here, we use the convention

$$k[x^{1/p^\infty}] = k[x, x^{1/p}, x^{1/p^2}, \dots] = \cup_i k[x^{1/p^i}].$$

Moreover, $(\mathbb{G}_m)_{\text{perf}}/\mu_{p^\infty} \cong \mathbb{G}_m$.

For our purposes, it is most convenient to define short exact sequences of affine group schemes as those sequences $N \rightarrow G \rightarrow Q$ in which $G \rightarrow Q$ is faithfully flat and N is the kernel of the latter morphism.

Lemma 3.1.8. *The perfection functor acting on a short exact sequence of affine group schemes*

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

yields a short exact sequence

$$1 \rightarrow N_{\text{perf}} \rightarrow G_{\text{perf}} \rightarrow Q_{\text{perf}} \rightarrow 1.$$

Proof. Faithful flatness is preserved by perfection; see [BS, Lemma 3.4]. Taking inverse limits of affine group schemes always respects kernels. □

Corollary 3.1.9. *A reduced affine group scheme G is solvable if and only if G_{perf} is solvable.*

Proof. Lemma 3.1.8 shows that for any solvable affine group scheme, its perfection is again solvable. However, assume that G_{perf} is solvable and G reduced. Applying Theorem 3.1.6(2) iteratively allows us to construct a finite chain of reduced normal subgroups such that the perfection of the quotients are abelian. A reduced affine group scheme with abelian perfection is clearly abelian itself. □

3.2. Isomorphic perfections

We gather some examples and results about affine group schemes with isomorphic perfections.

Example 3.2.1. (1) Let G be a finite group scheme (i.e. $k[G]$ is finite dimensional). We have $G_{\text{perf}} \cong G_{\text{red}}$, and so, in particular, G is infinitesimal if and only if G_{perf} is trivial.

(2) Reduced affine group schemes can also become isomorphic after perfection. For instance, if q is a power of p , then $(SL_q)_{\text{perf}} \cong (PGL_q)_{\text{perf}}$. This is an example of Lemma 3.2.2 below, or follows from 4.2.3 below and Example 1.4.2. Moreover,

the latter results also show that, conversely, $(SL_n)_{\text{perf}} \cong (PGL_n)_{\text{perf}}$ implies that n must be a power of p .

- (3) As follows from Remark 2.2.5, a necessary condition for affine group schemes G, H to have isomorphic perfection is that there exists an isomorphism $G(k) \cong H(k)$ as abstract groups.

Recall that an **isogeny** is a faithfully flat homomorphism of affine group schemes with finite kernel N . An isogeny is **infinitesimal** if N is infinitesimal. An isogeny $G \rightarrow Q$ between reduced and connected affine group schemes is **purely inseparable** if the induced morphism $k(Q) \rightarrow k(G)$ between the fields of fractions is a purely inseparable field extension.

Lemma 3.2.2. *For an isogeny $q : G \rightarrow Q$, the following conditions are equivalent:*

- (a) q is infinitesimal;
- (b) The perfection of q is an isomorphism.

Moreover, if G, Q are reduced and connected, the above properties are equivalent to

- (c) q is purely inseparable.

Proof. The equivalence between (a) and (b) is an immediate application of Example 3.2.1(1) and Lemma 3.1.8.

Condition (b) implies that for every $a \in k[G]$, there is $i \in \mathbb{N}$ such that a^{p^i} is in the image of $k[Q] \rightarrow k[G]$, from which (c) follows immediately. Conversely, that (c) implies (b) follows similarly, by exploiting the equality

$$k[Q] = k(Q) \cap k[G], \quad \text{inside } k(G).$$

To prove the displayed equality (of which the inclusion is obvious), we consider one $f \in k(G)$ which belongs to $k(Q) \subset k[G]$. By viewing G as the inverse limit of quotient group schemes of finite type $\varprojlim G_\alpha$ (which induces $Q \cong \varprojlim Q_\alpha$), we can easily reduce to the case where G and Q are of finite type, by taking α for which $f \in k[G_\alpha] \subset k[G]$ as well as $f \in k(Q_\alpha) \subset k(G_\alpha)$.

Assume thus that G is of finite type and consider the span S of $\{g(f) \mid g \in G(k)\}$. This is the $G(k)$ -subrepresentation of the rational left regular representation $k[G]$; in particular, S is finite dimensional. Clearly, the action of $G(k)$ on f factors through the canonical action of $Q(k)$ on $k(Q)$, and by our finite type assumption, we have $G(k) \twoheadrightarrow Q(k)$. Hence, the elements $h \in k[Q]$ for which $hS \subset k[Q]$ form a non-zero (by finite dimensionality of S) $Q(k)$ -invariant ideal $I < k[Q]$, and hence, $I = k[Q]$. So $f \in S \subset k[Q]$, as desired.

Note that an alternative argument considers the purely inseparable isogeny $G \rightarrow G/(\ker q)^0$, which is also étale and therefore an isomorphism. □

Proposition 3.2.3. *The following conditions are equivalent on two reduced affine group schemes G, H of finite type.*

- (a) $G_{\text{perf}} \cong H_{\text{perf}}$;
- (b) There exists $j \in \mathbb{N}$ and an infinitesimal isogeny $G \rightarrow H^{(j)}$;

(c) *There are $i, j \in \mathbb{N}$ and homomorphisms $\phi : G \rightarrow H^{(j)}$, $\psi : H \rightarrow G^{(i)}$ for which the following triangles are commutative:*

$$\begin{array}{ccc}
 G & \xrightarrow{\phi} & H^{(j)} \\
 & \searrow \text{Fr}^{i+j} & \downarrow \psi^{(j)} \\
 & & G^{(i+j)}
 \end{array}
 \qquad
 \begin{array}{ccc}
 H & \xrightarrow{\psi} & G^{(i)} \\
 & \searrow \text{Fr}^{i+j} & \downarrow \phi^{(i)} \\
 & & H^{(i+j)}.
 \end{array}$$

Proof. That (b) implies (a) is a special case of Lemma 3.2.2.

Now we prove that (c) implies (b). The kernel of the Frobenius homomorphism is infinitesimal; hence, the left diagram shows that the kernel of ϕ is infinitesimal. Since we assume that G and H are reduced, the Frobenius homomorphism is faithfully flat. The right diagram thus proves that ϕ is faithfully flat.

Applying Lemma 3.1.4 to the isomorphism in (a) yields morphisms $\phi : G^{(-j)} \rightarrow H$ and $\psi : H^{(-i)} \rightarrow G$. Expressing that these induce mutually inverse homomorphisms on the perfected groups then states that there exists $l \in \mathbb{N}$ such that the composition

$$H^{(-i-j-l)} \xrightarrow{\text{Fr}^l} H^{(-i-j)} \xrightarrow{\psi^{(-j)}} G^{(-j)} \xrightarrow{\phi} H$$

is Fr^{i+j+l} . Since Fr^l is faithfully flat, we arrive precisely at the conditions in (c), so (a) implies (c). □

3.3. Tannakian point of view

For an affine group scheme G , we denote by $\text{Rep}G$ its category of (rational) representations which are finite dimensional over k . Its category of all representations will be denoted by $\text{Rep}^\infty G \cong \text{IndRep}G$.

3.3.1. Let us interpret G_{perf} as $\varprojlim G^{(-i)}$ as in 2.3.2. We refer to [CEO, §6] for an overview of the notion of the direct limit of (tensor) categories. In particular, we have

$$\text{Rep}(G_{\text{perf}}) \cong \varinjlim \text{Rep}G^{(-i)}, \tag{3.1}$$

where the k -linear (exact) tensor functors in the chain are given by the pullback along $G^{(-i-1)} \rightarrow G^{(-i)}$. These functors fit into commutative diagrams:

$$\begin{array}{ccc}
 \text{Rep}G & \longrightarrow & \text{Rep}G^{(-1)} \\
 & \searrow -^{(1)} & \downarrow \sim \\
 & & \text{Rep}G
 \end{array}
 \qquad
 \begin{array}{ccc}
 (V \rightarrow V \otimes k[G]) & \longmapsto & (V \rightarrow V \otimes k[G]^{(-1)}) \\
 & \searrow & \downarrow \\
 & & (V^{(1)} \rightarrow V^{(1)} \otimes k[G]).
 \end{array}
 \tag{3.2}$$

The non-horizontal arrows are only k -linear up to twist. By definition, $V \rightarrow V \otimes k[G]^{(-1)}$ comes from $k[G] \rightarrow k[G]^{(-1)}$ in (2.2). The downwards arrow is given by applying $-^{(1)}$ to both vector space and co-action. Note that we can equivalently realise the G -representation $V^{(1)}$ from the bottom right in (3.2) as the subquotient of $\otimes^p V$ given by the image of $\Gamma^p V \rightarrow S^p V$. This gives a more palatable definition of the Frobenius twist from the Tannakian point of view.

Diagram (3.2) allows us to realise $\text{Rep}G_{\text{perf}}$ alternatively as

$$\text{Rep}(G_{\text{perf}}) \cong \varinjlim \text{Rep}G, \tag{3.3}$$

in the spirit of interpretation 2.3.1. Compared to (3.1), we no longer have to work with twists of G , but we do have the drawback that the defining functors are not k -linear.

3.3.2. Notation For $M \in \text{Rep}G$, we denote by $M[i]$ the object of $\varinjlim \text{Rep}G$ where M is placed in the i -th copy of $\text{Rep}G$ in the chain, and we use the same notation for the corresponding object in $\text{Rep}(G_{\text{perf}})$, via (3.3). Note that every object in $\text{Rep}(G_{\text{perf}})$ is of this form, and furthermore, $M[i] \cong M^{(1)}[i + 1]$.

Lemma 3.3.3. [CEO, Remark 6.5] *Let G be an affine group scheme over k .*

- (1) G is reduced if and only if $-^{(1)} : \text{Rep}G \rightarrow \text{Rep}G$ is full.
- (2) G is perfect if and only if $-^{(1)} : \text{Rep}G \rightarrow \text{Rep}G$ is an equivalence.

Proof. If G is reduced, the p -th power map is injective on $k[G]$, and the fullness in (1) follows. However, if $-^{(1)}$ is not full, then (by applying adjunction) there is a G -representation V with a vector $v \in V$ which is not G -invariant, but for which $1 \otimes v \in V^{(1)}$ is G -invariant. Looking at the $k[G]$ coaction then provides a non-zero $f \in k[G]$ with $f^p = 0$.

Via diagram (3.2), the functor is an equivalence if and only if $G^{(-1)} \rightarrow G$ is an isomorphism, which is equivalent to G being perfect. □

Remark 3.3.4. As for any direct limit of abelian categories, for objects $M[i], N[i] \in \text{Rep}(G_{\text{perf}})$, using notation from 3.3.2, we have

$$\varinjlim_j \text{Ext}_G^l(M^{(j)}, N^{(j)}) \cong \text{Ext}_{G_{\text{perf}}}^l(M[i], N[i]).$$

3.4. Additive, multiplicative and unipotent groups

For convenience, we let k be algebraically closed in this section.

3.4.1. To lighten expressions, we introduce the following notation:

$$\mathbf{G}_a := (\mathbb{G}_a)_{\text{perf}} \quad \text{and} \quad \mathbf{G}_m := (\mathbb{G}_m)_{\text{perf}}$$

for the perfection of the additive and multiplicative group of k .

Proposition 3.4.2. *Assume that k is algebraically closed.*

- (1) *Let \mathbf{G} be a connected affine group scheme of perfect finite type and of dimension 1; then either $\mathbf{G} \cong \mathbf{G}_a$ or $\mathbf{G} \cong \mathbf{G}_m$.*
- (2) *Let G be a reduced affine group scheme of finite type with $G_{\text{perf}} \cong \mathbf{G}_a$ (resp. $G_{\text{perf}} \cong \mathbf{G}_m$); then $G \cong \mathbb{G}_a$ (resp. $G \cong \mathbb{G}_m$).*

Proof. By [Sp, Theorem 3.4.9], for any connected reduced affine group scheme G of finite type and of dimension 1, we must have $G \cong \mathbb{G}_a$ or $G \cong \mathbb{G}_m$. This implies part (2), by Lemma 2.2.6(1). Part (1) follows similarly, using characterisation 3.1.3(d). □

By a torus we mean an affine group scheme isomorphic to $\mathbf{G}_m^{\times n}$ for some $n \in \mathbb{N}$. Similarly, **perfect tori** are the affine group schemes isomorphic to $\mathbf{G}_m^{\times n}$ for some $n \in \mathbb{N}$.

Lemma 3.4.3. *For a connected reduced affine group scheme G of finite type, the following are equivalent:*

- (a) G is a torus;
- (b) G_{perf} is a perfect torus.

Proof. Clearly, (a) implies (b). However, (b) implies that G is connected and, since

$$\text{Rep}G \rightarrow \text{Rep}G_{\text{perf}}$$

is exact and fully faithful, see Lemma 3.3.3(1), it follows that $\text{Rep}G$ is semisimple and pointed (every simple representation has dimension one). That G is a torus then follows from [Sp, 3.2.3 and 3.2.7(ii)]. □

Recall that an affine group scheme G is **unipotent** if and only if every representation has an invariant vector (equivalently every simple object in $\text{Rep}G$ is trivial).

Lemma 3.4.4. *For an affine group scheme G over k , G_{perf} is unipotent if and only if G_{red} is unipotent.*

Proof. By equivalence (3.1), if G (or G_{red}) is unipotent, then so is G_{perf} . However, if G_{perf} is unipotent, then the fully faithful exact functor from $\text{Rep}G_{\text{red}}$ to $\text{Rep}G_{\text{perf}}$ shows that also G_{red} is unipotent. □

3.4.5. We have a ring isomorphism

$$\mathbb{Z}[1/p] \xrightarrow{\sim} \text{End}(\mathbf{G}_m), \quad a \mapsto \{\lambda \mapsto \lambda^a\}, \tag{3.4}$$

where λ stands for an element of $\mathbf{G}_m(A) = \varprojlim A^\times$ for a commutative k -algebra A . The restriction to $p^\mathbb{Z} \hookrightarrow \text{Aut}(\mathbf{G}_m)$ is the homomorphism from 3.1.1.

For \mathbf{G}_a , the latter homomorphism extends to an isomorphism

$$k^\times \rtimes p^\mathbb{Z} \xrightarrow{\sim} \text{Aut}(\mathbf{G}_a), \quad (\kappa, n) \mapsto \{\theta_\kappa^n : \lambda \mapsto \kappa \lambda^n\}. \tag{3.5}$$

3.4.6. For a perfect affine group scheme \mathbf{G} , Lemma 3.1.4 shows that characters of \mathbf{G} correspond to homomorphisms $\mathbf{G} \rightarrow \mathbf{G}_m$. As the latter formulation carries more structure, we define

$$\mathbf{X}(\mathbf{G}) := \text{Hom}(\mathbf{G}, \mathbf{G}_m) \cong \text{Hom}(\mathbf{G}, \mathbf{G}_m) = X(\mathbf{G}).$$

This is a $\mathbb{Z}[1/p]$ -module via (3.4). Consequently, we will define cocharacters of \mathbf{G} to be the $\mathbb{Z}[1/p]$ -module

$$\mathbf{Y}(\mathbf{G}) := \text{Hom}(\mathbf{G}_m, \mathbf{G}).$$

We have the obvious bilinear pairing

$$\mathbf{X}(\mathbf{G}) \times \mathbf{Y}(\mathbf{G}) \rightarrow \mathbb{Z}[1/p]. \tag{3.6}$$

It is non-degenerate if \mathbf{G} is a perfect torus.

3.5. Induction

3.5.1. For a homomorphism $f : H \rightarrow G$ of affine group schemes, we will sometimes abbreviate $f_* := \text{Ind}_H^G$ and $f^* := \text{Res}_H^G$. This gives an adjoint pair $f^* \dashv f_*$ of functors between $\text{Rep}^\infty G$ and $\text{Rep}^\infty H$.

3.5.2. For a commutative square of group homomorphisms

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ \downarrow a & & \downarrow b \\ A & \xrightarrow{g} & B, \end{array}$$

the adjunction morphisms yield a natural transformation

$$\xi : b^* g_* \Rightarrow f_* a^*.$$

In particular, for $M \in \text{Rep}^\infty A$, the morphism ξ_M is zero if and only if the composite

$$g^* g_* M \rightarrow M \rightarrow a_* a^* M$$

is zero.

3.5.3. Now we consider two affine group schemes of the form $A = \varprojlim A_i$ and $B = \varprojlim B_i$ for inverse systems of affine group schemes $(A_i \mid i \in \mathbb{N})$ and $(B_i \mid i \in \mathbb{N})$. We label the homomorphisms $p_i : A \rightarrow A_i$, $q_i : B \rightarrow B_i$ and $a_{[i,j]} : A_i \rightarrow A_j$ for $i > j$. Assume also given homomorphisms $A_i \rightarrow B_i$, leading to $A \rightarrow B$.

Proposition 3.5.4. (1) For $M_i \in \text{Rep}^\infty A_i$ and morphisms $a_{[i+1,i]}^* M_i \rightarrow M_{i+1}$, the evaluations at M_i of the natural transformations in 3.5.2 lead to an isomorphism

$$\varinjlim q_i^* \text{Ind}_{A_i}^{B_i} M_i \xrightarrow{\sim} \text{Ind}_A^B \varinjlim p_i^* M_i.$$

(2) For $M \in \text{Rep}^\infty A_j$ and $n \in \mathbb{N}$, we have a canonical isomorphism

$$\varinjlim q_i^* (R^n \text{Ind}_{A_i}^{B_i} (a_{[i,j]}^* M)) \xrightarrow{\sim} R^n \text{Ind}_A^B (p_j^* M).$$

Proof. The right-hand side in part (1) is given by the A -invariants in the vector space

$$\varinjlim (M_i \otimes k[B_i]).$$

Since direct limits commute with co-equalisers, this is isomorphic to the direct limit of A_i -invariants in the above spaces. This proves part (1).

Note that the case $n = 0$ in part (2) is the special case of part (1) where we set $M_i := a_{[i,j]}^* M$. We can prove the analogue of part (1) for derived functors, which similarly specialises to part (2), as follows. For a chain $\{M_i\}$ as in part (1), we consider injective hulls in $\text{Rep}^\infty A_i$, yielding short exact sequences

$$0 \rightarrow M_i \rightarrow I_i \rightarrow Q_i \rightarrow 0.$$

The defining property of injective modules gives a chain map from the action of $a_{[i+1,i]}^*$ on the above sequence and the corresponding sequence for $i + 1$. In particular, this makes $\{I_i\}$ and $\{Q_i\}$ into chains of representations as in part (1), and we have a short exact sequence

$$0 \rightarrow \varinjlim p_i^* M_i \rightarrow \varinjlim p_i^* I_i \rightarrow \varinjlim p_i^* Q_i \rightarrow 0$$

in $\text{Rep}^\infty A$. The claim now follows by induction on n , using long exact sequences in homology if we observe that $I := \varinjlim p_i^* I_i$ is injective in $\text{Rep}^\infty A$. Exactness of

$$\text{Hom}_A(-, I) : \text{Rep}A \rightarrow \text{Vec}^\infty$$

follows from the observation $\text{Rep}A \cong \varinjlim \text{Rep}A_i$. The latter exactness is sufficient to conclude that I is injective. Indeed, we can consider an injective hull $I \subset I'$ and an intermediate module $I \subset I'' \subset I'$ for which I''/I is finite dimensional. Now we must have $I'' \cong I \oplus I''/I$ (apply $\text{Hom}(-, I)$ to a finite submodule of I'' which still surjects onto I''/I), which violates $\text{soc}I = \text{soc}I'$ unless $I'' = I$. □

Corollary 3.5.5. *Let G be an affine group scheme with subgroup H . Set $\mathbf{G} = H_{\text{perf}}$, set $\mathbf{H} = H_{\text{perf}}$ and take $n \in \mathbb{N}$. For $M \in \text{Rep}H$ such that $R^n \text{Ind}_H^G M^{(i)}$ is finite dimensional for each $i \in \mathbb{N}$. For each $j \in \mathbb{N}$, we have an isomorphism*

$$\varinjlim_{i \geq j} \left(R^n \text{Ind}_H^G (M^{(i-j)})[i] \right) \xrightarrow{\sim} R^n \text{Ind}_{\mathbf{H}}^{\mathbf{G}} (M[j]),$$

with notation as in 3.3.2.

Proof. We start by applying Proposition 3.5.4(2), using the interpretation $G_{\text{perf}} = \varinjlim G^{(-i)}$, applied to $M^{(-j)} \in \text{Rep}H^{(-j)}$. By assumption, all the representations appearing in the direct limit in 3.5.4(2) are finite dimensional. After passing from (3.1) to (3.3), this allows us to use the notation from 3.3.2 to rewrite the isomorphism in the desired form. □

Remark 3.5.6. (1) For group schemes of finite type, we can prove Corollary 3.5.5 alternatively using Theorem 3.1.6(3) and (a generalisation from line bundles to general quasi-coherent sheaves with identical proof of) Lemma 2.5.1.

(2) Assume that M is one-dimensional, and $\text{Ind}_H^G (M^{(j)}) \neq 0$ for all j . It follows from 3.5.2 that the morphisms in the directed system for the left-hand side in Corollary 3.5.5 for $n = 0$ are all non-zero.

4. Perfectly reductive groups

In this section, we assume that k is algebraically closed of characteristic $p > 0$.

4.1. Definition

Recall that a reductive group over k is a connected reduced (smooth) affine group scheme of finite type which has no non-trivial normal reduced unipotent subgroup.

Theorem 4.1.1. (1) *For an affine group scheme \mathbf{G} , the following are equivalent:*

- (a) \mathbf{G} is a connected affine group scheme of perfect finite type and has no non-trivial perfect normal unipotent subgroups;
- (b) $\mathbf{G} \cong G_{\text{perf}}$ for a reductive group G ;
- (c) \mathbf{G} is an affine group scheme of perfect finite type and, whenever $\mathbf{G} \cong H_{\text{perf}}$ for a reduced affine group scheme of finite type H , then H must be reductive.

If these conditions are satisfied, we call \mathbf{G} perfectly reductive.

- (2) *For every connected affine group scheme of perfect finite type \mathbf{H} , there exists a short exact sequence of perfect affine group schemes*

$$1 \rightarrow \mathbf{U} \rightarrow \mathbf{H} \rightarrow \mathbf{Q} \rightarrow 1,$$

where \mathbf{U} is unipotent and \mathbf{Q} is perfectly reductive.

Proof. First we show that 1(b) implies 1(a). Set $\mathbf{G} = G_{\text{perf}}$ for a reductive group G . Let $\mathbf{U} \triangleleft \mathbf{G}$ be a perfect normal unipotent subgroup. Then by Theorem 3.1.6(2), there exists a reduced normal subgroup $U \triangleleft G$ with $U_{\text{perf}} = \mathbf{U}$. By Lemma 3.4.4, U is unipotent, so U is trivial. Consequently, \mathbf{U} is trivial.

That 1(c) implies 1(b) follows from Theorem 3.1.3.

For \mathbf{H} as in (2), we know that $\mathbf{H} = H_{\text{perf}}$ for a connected reduced affine group scheme of finite type H by Theorem 3.1.3. By taking the perfection of the short exact sequence corresponding to the unipotent radical $R_u H \triangleleft H$ (see [Mi, §6.4.6]), we get a short exact sequence as desired in (2) (provided we define, for now, perfectly reductive groups as the perfections of reductive groups), with $\mathbf{U} := (R_u H)_{\text{perf}}$, by Lemma 3.1.8.

Since $R_u H$ is reduced, \mathbf{U} is trivial if and only if $R_u H$ is trivial, which shows that 1(a) implies 1(c). □

Remark 4.1.2. In addition to Theorem 4.1.1(2), we can also observe that every affine group scheme of perfect finite type \mathbf{G} admits a short exact sequence $\mathbf{H} \rightarrow \mathbf{G} \rightarrow Q$ where Q is a finite abstract group and \mathbf{H} is a connected affine group scheme of perfect finite type. This follows from perfecting the classical theory; see [Mi, §2.g].

4.1.3. A **perfect Borel** subgroup \mathbf{B} of a perfectly reductive group is a maximal solvable perfect subgroup. For a reductive group G , every perfect Borel subgroup of G_{perf} is the perfection of a Borel subgroup of G by Theorem 3.1.6(2) and Corollary 3.1.9. In particular, every two perfect Borel subgroups are conjugate.

Similarly, by Theorem 3.1.6(2) and Lemma 3.4.3, every maximal perfect torus \mathbf{T} in G_{perf} is the perfection of a maximal torus $T < G$, and so maximal perfect tori are unique up to conjugation.

We henceforth use freely that every such choice $\mathbf{T} < \mathbf{B} < G_{\text{perf}}$, for a reductive group G , can be obtained as the perfection of a corresponding choice $T < B < G$ of Borel subgroup and maximal torus in G .

4.2. Classification

We freely use the equivalence between D -root data and real-type D -root data for $D = \mathbb{Z}$ and $D = \mathbb{Z}[1/p]$ from Theorem 1.4.1.

Theorem 4.2.1. *There is a canonical bijection between the set of isomorphism classes of perfectly reductive groups over k and the set of isomorphism classes of $\mathbb{Z}[1/p]$ -root data.*

Remark 4.2.2. (1) As in the classical case, this theorem can be extended to cover isogenies. We do this in 4.3 below.

- (2) Theorem 4.2.1 implies in particular that for two reductive groups to have isomorphic perfections, they must have Weyl groups that are isomorphic as Coxeter groups. Slightly more restrictive, they must have the same Dynkin diagram, except that we can have perfected isomorphisms between types B and C when $p = 2$.

4.2.3. Idea of the proof We will prove that, when characterising a reductive group in terms of its root datum, two reductive groups become isomorphic after perfection if and only if their root data become isomorphic after extension of scalars to $\mathbb{Z}[1/p]$.

More explicitly, denote by $D\text{-RD}$ the set of isomorphism classes of D -root data. Furthermore, we let ReGr , resp. PeReGr , denote the set of isomorphism classes of reductive groups, resp. perfectly reductive groups, over k . We can exploit the classical bijection between $\mathbb{Z}\text{-RD}$ and ReGr (see, for instance, [De]) and include it in the following (commutative) diagram:

$$\begin{array}{ccc}
 \text{ReGr} & \xrightarrow{\quad \text{---}^{\text{perf}} \text{---}} & \text{PeReGr} \\
 \uparrow \scriptstyle 1:1 & & \uparrow \ \backslash \\
 \mathbb{Z}\text{-RD} & \xrightarrow{\quad \mathbb{Z}[1/p] \otimes \text{---}} & \mathbb{Z}[1/p]\text{-RD}
 \end{array} \tag{4.1}$$

The upper surjection is given by the definition in 4.1.1(1)(b) of perfectly reductive groups. The lower surjection comes from Lemma 1.1.3. To prove Theorem 4.2.1, it suffices to show the dashed arrows in (4.1) exist. This is established in the following two propositions.

Proposition 4.2.4. *If the root data of two reductive groups G_1, G_2 extend to isomorphic root data over $\mathbb{Z}[1/p]$, then $(G_1)_{\text{perf}}$ and $(G_2)_{\text{perf}}$ are isomorphic. In particular, the upwards dashed arrow in (4.1) exists.*

Proof. Let $(X_i, R_i, Y_i, R_i^\vee)$, with $i \in \{1, 2\}$, denote the root datum of G_i . Consider an isomorphism of root data given by $\psi : \mathbb{Z}[1/p] \otimes X_1 \xrightarrow{\sim} \mathbb{Z}[1/p] \otimes X_2$. Replacing ψ by $p^l \psi$ if necessary (as is allowed by Lemma 1.2.5), we can assume that ψ restricts to an embedding $X_1 \hookrightarrow X_2$. Furthermore, we get a bijection $R_1 \rightarrow R_2$, by associating to $\alpha \in R_1$ the unique element $\alpha' \in R_2$ for which $\psi(\alpha) = p^j \alpha'$ for some $j \in \mathbb{Z}$. Again, after replacing ψ by $p^l \psi$ if necessary, we can assume $j \in \mathbb{N}$. Now it follows quickly that this $X_1 \hookrightarrow X_2$ satisfies the requirements to apply [St, Theorem 1.5], which yields an isogeny $G_1 \rightarrow G_2$.

We can apply the same procedure to ψ^{-1} to obtain an isogeny $G_2 \rightarrow G_1$. As we might need to replace ψ^{-1} again by a composition with multiplication by a power of p , our two isogenies will not necessarily be induced by mutually inverse maps $X_1 \leftrightarrow X_2$, but by maps which compose to p^l times the identity for some $l \in \mathbb{N}$. Uniqueness of isogenies in [St, Theorem 1.5] then states that composition of the isogenies between G_1 and G_2 yields morphisms $\phi^l \circ \text{Fr}^l$, for isomorphisms $\phi : G_i^{(1)} \rightarrow G_i$ as in 3.1.1 (up to possible composition with inner automorphisms). That $(G_1)_{\text{perf}}$ and $(G_2)_{\text{perf}}$ are isomorphic now follows from Proposition 3.2.3. □

Establishing the existence of the downwards dashed arrow will take more work. Note that an alternative proof of this fact will be given in Section 6. As that proof moves via topology, it seems preferable to have this direct algebraic proof too. The following lemma can be proved by looking at the Hopf algebra morphisms.

Lemma 4.2.5. *Consider a reduced affine group scheme H with a homomorphism $\phi : H^{(-i)} \rightarrow \mathbb{G}_m$, such that the diagram*

$$\begin{array}{ccc}
 \mathbb{G}_m \times \mathbb{G}_a & \xrightarrow{(\lambda, \mu) \mapsto \lambda\mu} & \mathbb{G}_a \\
 \phi \times \text{id} \uparrow & & \uparrow \\
 H^{(-i)} \times \mathbb{G}_a & \xrightarrow{\text{Fr}^i \times \text{id}} & H \times \mathbb{G}_a
 \end{array}$$

can be completed with dashed arrow to a commutative square. Then ϕ factors through $\text{Fr}^i : H^{(-i)} \rightarrow H$.

Definition 4.2.6. Let \mathbf{G} be a perfectly reductive group with maximal perfect torus \mathbf{T} . An **rt-pair** is a pair (x, α) of a subgroup inclusion $x : \mathbf{G}_a \rightarrow \mathbf{G}$ and $\alpha \in \mathbf{X}$ (i.e. $\alpha : \mathbf{T} \rightarrow \mathbb{G}_m$), for which the following square is commutative:

$$\begin{array}{ccc}
 \mathbf{G}_m \times \mathbf{G}_a & \xrightarrow{(\lambda, \mu) \mapsto \lambda\mu} & \mathbf{G}_a \\
 \alpha \times \text{id} \uparrow & & \downarrow x \\
 \mathbf{T} \times \mathbf{G}_a & \xrightarrow{(t, \mu) \mapsto tx(\mu)t^{-1}} & \mathbf{G}
 \end{array} \tag{4.2}$$

If we apply this definition to ordinary reductive groups (we replace every perfect group in (4.2) by its finite type analogue), we get precisely the pairs of inclusions of root subgroups and their corresponding root. Since root subgroups are unique, the inclusion of the root subgroup is unique up to scalar in $k^\times \cong \text{Aut}(\mathbb{G}_a)$.

Lemma 4.2.7. *Let \mathbf{G} be a perfectly reductive group with maximal perfect torus \mathbf{T} .*

(1) *The group $k^\times \rtimes p^\mathbb{Z}$ acts on the set of rt-pairs as*

$$(x, \alpha) \xrightarrow{(\kappa, n)} (x \circ (\theta_\kappa^n)^{-1}, n\alpha), \quad \text{for all } (\kappa, n) \in k^\times \rtimes p^\mathbb{Z}.$$

(See 3.4.5 for the definition of θ_κ^n and 3.4.6 for the action of $n \in \mathbb{Z}[1/p]$ on \mathbf{X} .)

(2) *If for one $\alpha : \mathbf{T} \rightarrow \mathbf{G}_m$ we have two rt-pairs (x, α) and (z, α) , then $z = x \circ \theta_\kappa^1$ for some $\kappa \in k^\times$.*

(3) *Consider a reductive group G with $G_{\text{perf}} \cong \mathbf{G}$, with maximal torus T which perfects to \mathbf{T} . Consider an rt-pair (x, α) for which x is the perfection of some $x_0 : \mathbb{G}_a \hookrightarrow G$; then α is the perfection of a root homomorphism $T \rightarrow \mathbb{G}_m$ and x_0 is an inclusion of the corresponding root subgroup.*

(4) *For every rt-pair (x, α) , there exists $n \in p^\mathbb{Z}$ such that $x \circ \theta_1^n$ is the perfection of the inclusion of a root subgroup $\mathbb{G}_a \hookrightarrow G$ and $n^{-1}\alpha$ is the perfection of the corresponding root homomorphism $T \rightarrow \mathbb{G}_m$.*

Proof. Part (1) follows from a direct calculation.

For part (3), the homomorphism $\alpha : \mathbf{T} \rightarrow \mathbf{G}_m$ is induced from $T^{(-i)} \rightarrow \mathbb{G}_m$ for some $i \in \mathbb{N}$, as in Lemma 3.1.4. We find a diagram

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_a & \longrightarrow & \mathbb{G}_a \\ \uparrow & & \downarrow \\ T^{(-i)} \times \mathbb{G}_a & \xrightarrow{\text{Fr}^i \times \text{id}} T \times \mathbb{G}_a \longrightarrow & G \end{array}$$

which “perfects” to diagram (4.2). More precisely, after perfecting the above diagram and removing the automorphism which is the perfection of $\text{Fr}^i \times \text{id}$, we recover (4.2). In particular, we find that the above diagram is commutative. It now follows from Lemma 4.2.5 that α comes from $\alpha_0 : T \rightarrow \mathbb{G}_m$, and it follows immediately that α_0 is a root.

Now we prove part (4). By Theorem 3.1.6(2) and Proposition 3.4.2(2), there exists a group monomorphism $\mathbb{G}_a \hookrightarrow G$ which perfects to $x \circ \phi_\kappa^n$ for some $n \in p^\mathbb{Z}, \kappa \in k^\times$. By identifying k^\times with $\text{Aut}(\mathbb{G}_a)$, we might as well take $\kappa = 1$. The claim about α now follows from parts (1) and (3).

Finally, we prove part (2). Since roots of reductive groups cannot be multiples of one another, part (4) implies that there is $n \in p^\mathbb{Z}$ for which both $x \circ \theta_1^n$ and $z \circ \theta_1^n$ are the perfections of inclusions of the same root subgroup. Those inclusions must be the same, up to a scalar in $k^\times \cong \text{Aut}(\mathbb{G}_a)$, from which the claim follows. \square

Proposition 4.2.8. *Consider a reductive group G corresponding to a root datum RD . One can extract $\mathbb{Z}[1/p] \otimes RD$ from the group G_{perf} . In particular, the downwards dashed arrow in (4.1) exists.*

Proof. We construct a $\mathbb{Z}[1/p]$ -root datum $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$ from $\mathbf{G} := G_{\text{perf}}$. It will follow from the construction that $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$ is isomorphic to $\mathbb{Z}[1/p] \otimes RD$.

First, we let \mathbf{T} be a maximal perfect torus in \mathbf{G} . Recall \mathbf{T} is the perfection of a maximal torus $T < G$ (and hence unique up to conjugation). We choose such a T . We set $\mathbf{X} = \mathbf{X}(\mathbf{T})$

and $\mathbf{Y} = \mathbf{Y}(\mathbf{T})$, with (non-degenerate) pairing (3.6). We denote the root datum of G corresponding to T by $(X, R, Y, R^\vee) \cong RD$.

Following Lemma 3.1.4, we find

$$\mathbb{Z}[1/p] \otimes X = \mathbb{Z}[1/p] \otimes \text{Hom}(T, \mathbb{G}_m) \xrightarrow{\sim} \varinjlim \text{Hom}(T^{(-i)}, \mathbb{G}_m) \xrightarrow{\sim} \text{Hom}(\mathbf{T}, \mathbf{G}_m) = \mathbf{X}. \tag{4.3}$$

We define $\mathbf{R} \subset \mathbf{X}$ as the set of α for which there exists an rt-pair (x, α) . It follows from Lemma 4.2.7(1) and (4) that $\mathbf{R} \subset p^{\mathbb{Z}}R$, under $R \subset X \subset \mathbf{X}$ from (4.3). That $R \subset \mathbf{R}$, so $p^{\mathbb{Z}}R \subset \mathbf{R}$ by Lemma 4.2.7(1), follows from perfecting the root homomorphisms into G . In conclusion, $\mathbf{R} = p^{\mathbb{Z}}R$.

Having defined $(\mathbf{X}, \mathbf{R}, \mathbf{Y})$, we now define the injection $-^\vee : \mathbf{R} \rightarrow \mathbf{Y}$ completing the root datum. Choose $\alpha \in R \subset \mathbf{R}$. Denote by \mathbf{H} the minimal perfect subgroup of \mathbf{G} which contains the images of the two morphisms $\mathbf{G}_a \rightarrow \mathbf{G}$ corresponding to $\alpha, -\alpha$. That the images only depend on $\alpha, -\alpha$ follows from Lemma 4.2.7(2). That there exists such a minimal $\mathbf{H} < \mathbf{G}$ follows from the noetherian property of G and Theorem 3.1.6(2). The latter also shows that \mathbf{H} must be isomorphic to $(SL_2)_{\text{perf}}$ or $(PGL_2)_{\text{perf}}$ and that $\mathbf{T} \cap \mathbf{H}$ is a (maximal) torus in \mathbf{H} . We use this to define $\mathbf{G}_m \rightarrow \mathbf{T}$, determined up to $\text{Aut}(\mathbf{G}_m) = \mathbb{Z}[1/p]^\times$, either as the inclusion of this maximal torus of $(SL_2)_{\text{perf}}$ or by similarly restricting the homomorphism $(SL_2)_{\text{perf}} \rightarrow (PGL_2)_{\text{perf}} \hookrightarrow \mathbf{G}$. Finally, we can then define $\alpha^\vee : \mathbf{G}_m \rightarrow \mathbf{T}$ as the unique such morphism for which composition with $\alpha : \mathbf{T} \rightarrow \mathbf{G}_m$ yields $2 \in \text{End}(\mathbf{G}_m)$. By construction, α^\vee is defined independently of G but clearly corresponds to the direct definition via $\mathbf{G} = G_{\text{perf}}$. For $p^i\alpha$, we set $(p^i\alpha)^\vee = p^{-i}\alpha^\vee$, which extends the definition to $\mathbf{R} = p^{\mathbb{Z}}R$. \square

Remark 4.2.9. An alternative to the proof of Proposition 4.2.8 is given by the proof of $(a) \Rightarrow (b) \Rightarrow (c)$ in Theorem 6.1.1. However, the latter uses deep results about 2-compact groups and étale homotopy types of reductive groups; hence, it is preferable to have this direct proof.

4.3. Isogenies

We establish a connection between isogenies of perfectly reductive groups and our notion of isogenies of $\mathbb{Z}[1/p]$ -root data from Section 1.5.

Theorem 4.3.1. *Consider two perfectly reductive groups \mathbf{G}_1 and \mathbf{G} with perfect maximal tori \mathbf{T}_1, \mathbf{T} and the corresponding $\mathbb{Z}[1/p]$ -root data $(\mathbf{X}_1, \mathbf{R}_1, \mathbf{Y}_1, \mathbf{R}_1^\vee)$ and $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee)$. There is a bijection between the sets of*

- (a) *Isogenies $(\mathbf{X}, \mathbf{R}, \mathbf{Y}, \mathbf{R}^\vee) \rightarrow (\mathbf{X}_1, \mathbf{R}_1, \mathbf{Y}_1, \mathbf{R}_1^\vee)$;*
- (b) *Equivalence classes of isogenies $\mathbf{G}_1 \rightarrow \mathbf{G}$ which send \mathbf{T}_1 to \mathbf{T} , where two isogenies are equivalent if one is obtained from the other by composition with an inner automorphism affected by an element of $\mathbf{T}_1(k) = T_1(k)$.*

Proof. Let G, G_1 denote reductive groups which perfect to \mathbf{G}, \mathbf{G}_1 , with maximal tori T, T_1 which perfect to \mathbf{T}, \mathbf{T}_1 , and denote their root data by (X, R, Y, R^\vee) and $(X_1, R_1, Y_1, R_1^\vee)$.

We will prove both sets (a) and (b) are in bijection with the set of

- (c) Equivalence classes of pairs (θ, i) of an isogeny $\theta : (X, R, Y, R^\vee) \rightarrow (X_1, R_1, Y_1, R_1^\vee)$ (in the sense of [St, §1]) and $i \in \mathbb{Z}$, where the equivalence relation is generated by $(\theta, i) \sim (p\theta, i - 1)$.

That sending a pair (θ, i) to the extension of scalars along $\mathbb{Z} \rightarrow \mathbb{Z}[1/p]$ of $p^i\theta$ yields a bijection between (c) and (a) can be easily derived from Remark 1.5.2.

An isogeny $\mathbf{G}_1 \rightarrow \mathbf{G}$ yields a faithfully flat morphism $G_1^{(-i)} \rightarrow G$ for high enough i , via Lemma 3.1.4. Its kernel is an affine group scheme of finite type which perfects to a finite group scheme. It must therefore be a finite group scheme and $G_1^{(-i)} \rightarrow G$ is an isogeny. This principle allows us to establish a bijection between the set of isogenies $\mathbf{G}_1 \rightarrow \mathbf{G}$ and the set of equivalence classes of isogenies $G_1^{(-i)} \rightarrow G$, with equivalence generated by the condition that $\phi : G_1^{(-i)} \rightarrow G$ be equivalent to $\phi \circ \text{Fr} : G_1^{(-i-1)} \rightarrow G$.

The above connection between isogenies $\mathbf{G}_1 \rightarrow \mathbf{G}$ and equivalence classes of isogenies $G_1 \rightarrow G$ allows us to use the classical Isogeny Theorem [St, 1.5] to establish the bijection between (b) and (c). □

5. Perfected representation theory

Let k be an algebraically closed field of characteristic p .

5.1. Simple and induced modules and block structure

Let \mathbf{G} be a perfectly reductive group, \mathbf{B} a perfect Borel subgroup and $\mathbf{T} < \mathbf{B}$ a maximal perfect torus. We consider the set $\mathbf{R}^+ \subset \mathbf{R}$ of positive roots, which are the ones for which the corresponding $\mathbf{G}_\alpha \rightarrow \mathbf{G}$ does not land in \mathbf{B} (i.e. we let \mathbf{B} be the negative Borel).

We set $\mathbf{X} = \mathbf{X}(\mathbf{T})$ and $\mathbf{X}_+ \subset \mathbf{X}$ the subset of $\lambda \in \mathbf{X}$ which satisfy $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \mathbf{R}^+$. We have a canonical bijection $\{\lambda \mapsto k_\lambda\}$ between \mathbf{X} and the set of isomorphism classes of simple \mathbf{B} -representations (which are all one-dimensional).

Theorem 5.1.1. (1) *The representation*

$$\nabla(\lambda) := \text{Ind}_{\mathbf{B}}^{\mathbf{G}} k_\lambda \in \text{Rep}^\infty \mathbf{G}$$

is zero if $\lambda \notin \mathbf{X}_+$. If $\lambda \in \mathbf{X}_+$, it has simple socle, which we denote by $\mathbf{L}(\lambda)$.

- (2) *The above association $\lambda \mapsto \mathbf{L}(\lambda)$ is a bijection between \mathbf{X}_+ and the set of isomorphism classes of simple representations in $\text{Rep} \mathbf{G}$.*

Proof. We choose a reductive group G with $G_{\text{perf}} \cong \mathbf{G}$ and maximal torus and Borel subgroup $T < B < G$ which perfect to \mathbf{T} and \mathbf{B} . We will use the notation in 3.3.2. By equivalence (3.3), every simple object in $\text{Rep} \mathbf{G}$ is of the form $L(\mu)[i]$ for some $\mu \in X_+$ and $i \in \mathbb{N}$. Since $L(\mu)^{(1)} \cong L(p\mu)$ (see [Ja, II.3]), we can define unambiguously the simple object $\tilde{L}(p^{-i}\mu) := L(\mu)[i]$ for $i \in \mathbb{N}$ and $\mu \in X_+$. This clearly gives a bijection between \mathbf{X}_+ and the set of isomorphism classes of simple objects in $\text{Rep} \mathbf{G}$.

By Corollary 3.5.5, we have

$$\nabla(\lambda) \cong \varinjlim \nabla(p^i \lambda)[i], \tag{5.1}$$

where the chain of which we take the limit starts at i where $p^i\lambda \in X \subset \mathbf{X}$. It follows now from [Ja, II.2] that $\nabla(\lambda) = 0$ whenever $\lambda \notin \mathbf{X}_+$.

Now consider $\lambda \in \mathbf{X}_+$. The morphisms in the direct system for (5.1) are not zero by Remark 3.5.6(2). Hence, the defining morphisms $\nabla(p^i\lambda)^{(1)} \rightarrow \nabla(p^{i+1}\lambda)$ are unique (up to scalar) and injective since $\nabla(p^i\lambda)^{(1)}$ has socle $L(p^{i+1}\lambda)$ by Remark 5.1.4 below and $\nabla(p^{i+1}\lambda)$ is the injective hull of this socle in a Serre subcategory containing $\nabla(p^i\lambda)^{(1)}$. In particular, for $\lambda, \mu \in \mathbf{X}_+$, we have

$$\text{Hom}_{\mathbf{G}}(\tilde{L}(\mu), \nabla(\lambda)) \cong \varinjlim \text{Hom}_G(L(p^i\mu), \nabla(p^i\lambda)) \cong k\delta_{\lambda, \mu}.$$

We can therefore identify $\tilde{L}(\mu)$ with the socle of $\nabla(\mu)$. □

Remark 5.1.2. (1) We can alternatively construct the modules $\nabla(\lambda)$ as the global sections of line bundles on $\mathbf{G}/\mathbf{B} \cong (G/B)_{\text{perf}}$.

(2) It follows, for instance, from the proof of Theorem 5.1.1 that

$$[\nabla(\lambda) : \mathbf{L}(\mu)] = [\nabla(p^j\lambda) : \mathbf{L}(p^j\mu)], \quad \text{for all } \lambda, \mu \in \mathbf{X}_+ \text{ and } j \in \mathbb{Z}.$$

In fact, the proof even shows that

$$[\nabla(\lambda) : \mathbf{L}(\mu)] = \lim_{m \rightarrow \infty} [\nabla(p^m\lambda) : L(p^m\mu)]. \tag{5.2}$$

The sequence on the right-hand side in equation (5.2) is monotone increasing, and one can ask when it is bounded. For G of rank 1, it is clearly bounded, but the following rank 2 example was communicated to us by Stephen Donkin.

Example 5.1.3 (Donkin). Set $p = 2$ and let E be a 4-dimensional space with symplectic form, so that E is the natural representation of $Sp(E) \simeq Sp_4$. Let λ be such that $E \simeq L(\lambda)$. Then

$$[\nabla(2^m\lambda) : L(0)] = 2^{m-1} + 1, \quad \text{so} \quad [\nabla(\lambda) : \mathbf{L}(0)] = \infty.$$

Indeed, more generally, $\nabla(m\lambda)$ is the symmetric power $S^m E$, and by [EK, Lemma 4.6], we have

$$\begin{aligned} [S^{2m} E : L(0)] &= [(S^{m-2} E)^{(1)} : L(0)] + [(S^m E)^{(1)} : L(0)] + [\wedge^2 E \otimes (S^{m-1} E)^{(1)} : L(0)] \\ &= [S^{m-2} E : L(0)] + [S^m E : L(0)] + 2[S^{m-1} E : L(0)], \end{aligned}$$

where we used that $[\wedge^2 E] = 2[L(0)] + [L(\omega)]$, with ω the second fundamental weight, so that $[L(\omega) \otimes M^{(1)} : L(0)] = 0$ for all representations M . It follows by induction on m that

$$[S^{2m} E : L(0)] = m + 1.$$

Remark 5.1.4. Let G be a reductive group. The canonical morphism

$$\text{Ext}_G^*(M, N) \rightarrow \text{Ext}_G^*(M^{(1)}, N^{(1)})$$

is injective. This is proved in [Ja, II.10.14] if $(p-1)\rho \in X$ (e.g. $p \neq 2$).

If $\rho \notin X$, we can extend $X \subset X' \subset \mathbb{Q} \otimes X$ by taking the lattice maximal X' for which $\langle -, \alpha^\vee \rangle$ still takes values in \mathbb{Z} for every $\alpha^\vee \in R^\vee$. In particular, $\rho \in X'$. Taking the

appropriate $Y' \subset Y$ yields a new root datum (X', R, Y', R^\vee) corresponding to a reductive group G' . By [St, Theorem 1.5], there is an isogeny $G' \rightarrow G$. The claim then reduces to the previous case via an obvious commutative square.

As in any category of finite dimensional modules over a coalgebra over a field, the blocks in $\text{Rep} \mathbf{G}$ are determined by the first extensions between simple objects; see [Ja, §ii.7.1].

Theorem 5.1.5. *For $\lambda, \mu \in \mathbf{X}^+$, the simple representations $\mathbf{L}(\lambda)$ and $\mathbf{L}(\mu)$ are in the same block of $\text{Rep} \mathbf{G}$ if and only if $\lambda - \mu \in \mathbb{Z}[1/p]\mathbf{R} = \mathbb{Z}\mathbf{R}$.*

Proof. We resume the notation and conventions from the proof of Theorem 5.1.1. By Remarks 3.3.4 and 5.1.4, $\mathbf{L}(\lambda)$ and $\mathbf{L}(\mu)$ are in the same block if and only if there exists $j \in \mathbb{N}$ for which $p^j \lambda, p^j \mu \in X$ and $L(p^j \lambda)$ and $L(p^j \mu)$ are in the same block of $\text{Rep} G$.

Assume first that $\mathbf{L}(\lambda)$ and $\mathbf{L}(\mu)$ are in the same block. By the above, $p^j \lambda - p^j \mu \in \mathbb{Z}R$ for some j , from which $\lambda - \mu \in \mathbb{Z}[1/p]R = \mathbb{Z}[1/p]\mathbf{R}$ follows.

Now assume that $\lambda - \mu \in \mathbb{Z}[1/p]\mathbf{R}$. Then there exists $j \in \mathbb{N}$ such that

- (i) $p^j \lambda, p^j \mu \in X$,
- (ii) $p^j \lambda - p^j \mu \in p\mathbb{Z}R$,
- (iii) $\langle \alpha^\vee, p^j \lambda + \rho \rangle \notin p\mathbb{Z}$ for some $\alpha \in R$.

Indeed, for (iii), it suffices to take α simple and j such that $p^j \lambda \in pX$.

From (ii), it follows that $p^j \lambda$ and $p^j \mu$ are in the same $(\rho$ -shifted) orbit of $p\mathbb{Z}R \rtimes W$, so by (iii) and [Ja, II.7.2(2)], $L(p^j \lambda)$ and $L(p^j \mu)$ are in the same block of $\text{Rep} G$ from which the conclusion follows. □

Remark 5.1.6. Theorem 5.1.5 can be explained by the observation that the orbits of the affine Weyl group $W \ltimes p\mathbb{Z}R$ on X describe most of the block decomposition in $\text{Rep} G$, and the orbits of $W \ltimes p\mathbb{Z}[1/p]R$ on \mathbf{X} coincide with those of $\mathbb{Z}[1/p]R$.

5.1.7. Let n be the length of the longest element w_0 of the Weyl group (i.e. the dimension of \mathbf{G}/\mathbf{B}). We also set

$$\mathbf{X}_{++} = \{\lambda \in \mathbf{X} \mid \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \mathbf{R}^+\} \subset \mathbf{X}_+.$$

Another class of \mathbf{G} -representations which seems of interest is

$$\mathbf{W}(\lambda) := R^n \text{Ind}_{\mathbf{B}}^{\mathbf{G}} k_{w_0(\lambda)} \cong \varinjlim_i \Delta(p^i \lambda - 2\rho)[i], \quad \lambda \in \mathbf{X}_{++}$$

where the isomorphism is an instance of Corollary 3.5.5, using [Ja, II.5.11, Remark (1)].

Note that we do not use the notation $\Delta(\lambda)$ for the module $\mathbf{W}(\lambda)$, as the former would more logically be reserved for a pro-object dual to the ind-object $\nabla(\lambda)$.

5.2. Generic cohomology

In this section, we show how the result from [CPSV] can be formulated very elegantly in terms of perfected groups. We also refer to [BNP] for a more modern treatment of generic cohomology with sharper bounds.

5.2.1. Let G be an affine group scheme over k , which is the extension of scalars of an affine group scheme over \mathbb{F}_p , which we also denote by G . Set $\mathbb{F} = \overline{\mathbb{F}_p}$. We can consider the forgetful functor from rational G -representations to representations over k of the abstract group $G(\mathbb{F})$ (for instance, using the homomorphism $G(\mathbb{F}) \rightarrow G(k)$), which induces comparison morphisms

$$\mathrm{Ext}_G^i(M, N) \rightarrow \mathrm{Ext}_{kG(\mathbb{F})}^i(M, N).$$

Theorem 5.2.2. *Let \mathbf{G} be a perfectly reductive group; then the morphism*

$$\mathrm{Ext}_{\mathbf{G}}^i(M, N) \rightarrow \mathrm{Ext}_{k\mathbf{G}(\mathbb{F})}^i(M, N)$$

is an isomorphism for all $M, N \in \mathrm{Rep}\mathbf{G}$ and $i \in \mathbb{N}$.

We start the proof by pointing out a technical generality.

Remark 5.2.3. Consider a chain of finite abstract groups $\{H_n \mid n \in \mathbb{N}\}$ and set $H = \varprojlim H_n$. For two finite dimensional H -representations M, N , the canonical morphism

$$\mathrm{Ext}_{kH}^i(M, N) \rightarrow \varprojlim \mathrm{Ext}_{kH_n}^i(M, N)$$

is an isomorphism. Indeed, using group cohomology, $\mathrm{Ext}_{kH}^i(M, N)$ is the cohomology of the inverse limit of chain complexes with cohomology $\mathrm{Ext}_{kH_n}^i(M, N)$. Since all vector spaces involved are finite-dimensional, the Mittag-Leffler property leads to the conclusion.

Proof of Theorem 5.2.2. Let G be a reductive group with perfection \mathbf{G} and recall that $\mathbf{G}(\mathbb{F}) = G(\mathbb{F})$ and $\mathrm{Rep}\mathbf{G} \cong \varprojlim \mathrm{Rep}G$. Without loss of generality, we assume that M, N factor over the natural map $\overline{\mathbf{G}} \rightarrow G$. By Remark 3.3.4, we have

$$\mathrm{Ext}_{\mathbf{G}}^i(M, N) \cong \varprojlim_a \mathrm{Ext}_G^i(M^{(a)}, N^{(a)}).$$

It is proved in [CPSV] that the directed system in the above limit stabilises, and moreover, for fixed M, N , for large enough a and q , all morphisms

$$\mathrm{Ext}_G^i(M^{(a)}, N^{(a)}) \rightarrow \mathrm{Ext}_{kG(\mathbb{F}_q)}^i(M^{(a)}, N^{(a)})$$

are isomorphisms. Note that $G(\mathbb{F}_q)$, being a finite abstract group, is perfect in the group scheme sense. Hence, for large enough q , we find the composite isomorphism

$$\varprojlim_a \mathrm{Ext}_G^i(M^{(a)}, N^{(a)}) \xrightarrow{\sim} \varprojlim_a \mathrm{Ext}_{kG(\mathbb{F}_q)}^i(M^{(a)}, N^{(a)}) \xleftarrow{\sim} \mathrm{Ext}_{kG(\mathbb{F}_q)}^i(M, N).$$

Hence, also the inverse system in

$$\varprojlim \mathrm{Ext}_{kG(\mathbb{F}_{p^n})}^i(M, N) \cong \mathrm{Ext}_{kG(\mathbb{F})}^i(M, N)$$

stabilises, and we find the isomorphism in the theorem.

Strictly speaking, [CPSV] only deals with semisimple groups. However, for a general reductive group G , we have a short exact sequence $N \rightarrow G \rightarrow G/N$ with G/N semisimple and N a torus. Since both N and $N(\mathbb{F}_q) \cong C_{q-1}^{\times r}$ have semisimple representation theory over k , the result extends easily – for instance, via a collapsing Hochschild-Serre spectral sequence. □

Remark 5.2.4. The comparison morphism in Theorem 5.2.2 is not always an isomorphism for arbitrary perfect groups; for instance,

$$\text{Ext}_{\mathbb{G}_a}^1(k, k) \rightarrow \text{Ext}_{k\mathbb{F}^+}^1(k, k)$$

is the canonical inclusion

$$k\mathbb{Z} \hookrightarrow k\hat{\mathbb{Z}} \cong k\text{Gal}(\mathbb{F} : \mathbb{F}_p) \hookrightarrow \varprojlim k\text{Gal}(\mathbb{F}_q : \mathbb{F}_p),$$

where we used Remark 5.2.3 and the linear isomorphism (with $\iota : \mathbb{F}_q \hookrightarrow k$)

$$k\text{Gal}(\mathbb{F}_q : \mathbb{F}_p) \xrightarrow{\theta \mapsto \iota \circ \theta} \text{Hom}(\mathbb{F}_q^+, k^+) \cong H^1(\mathbb{F}_q^+, k).$$

Question 5.2.5. *The formulation of Theorem 5.2.2 suggests the question of whether the monomorphism (by Theorem 5.2.2)*

$$\text{Ext}_{\mathbb{G}}^i(M, N) \rightarrow \text{Ext}_{k\mathbb{G}(k)}^i(M, N)$$

is also an isomorphism. This is equivalent to the question of whether the epimorphism

$$\text{Ext}_{kG(k)}^i(M, N) \rightarrow \text{Ext}_{kG(\mathbb{F})}^i(M, N) \tag{5.3}$$

is an isomorphism for all $M, N \in \text{Rep}G$ and $i \in \mathbb{N}$. Note that (5.3) does not involve any perfection.

Example 5.2.6. The question in 5.2.5 has an affirmative answer for $G = \mathbb{G}_m$. Consider the short exact sequence

$$1 \rightarrow \mathbb{G}_m(\mathbb{F}) \rightarrow \mathbb{G}_m(k) \rightarrow Q \rightarrow 1.$$

By the Lyndon-Hochschild-Serre spectral sequence, showing (5.3) is an isomorphism can be quickly reduced to showing the group cohomology $H^i(Q, k)$ is zero for $i > 0$. Now the group structure on Q extends (uniquely) to a \mathbb{Q} -vector space (since $\mathbb{G}_m(\mathbb{F}) < \mathbb{G}_m(k)$ is the group of roots of unity and k is algebraically closed), so we only need to show $H^i(\mathbb{Q}, k) = 0$ for $i > 0$. The case $i = 1$ is obvious. One can compute directly that $H_i(\mathbb{Q}, -) = 0$ for $i > 1$ (or via $B\mathbb{Q}$; see [Su, (10) on p42]). Hence, $H^i(\mathbb{Q}, -) = 0$ for $i > 2$. Finally,

$$H^2(\mathbb{Q}, k) \cong \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, k)$$

must be an abelian group admitting both the structure of a \mathbb{Q} -vector space as well as a k -vector space; hence, it is zero.

We conclude with an example showing that (5.3) being an isomorphism is also something which should not be expected to hold outside of reductive groups.

Example 5.2.7. If instead of a reductive group, we consider $G = \mathbb{G}_a$, as well as $i = 1$, $M = N = k$ in (5.3), we obtain the morphism between spaces of group homomorphisms

$$\text{End}(k^+) \rightarrow \text{Hom}(\mathbb{F}^+, k^+),$$

induced by restriction along $\mathbb{F} \hookrightarrow k$. This is not a bijection as soon as $\mathbb{F} \neq k$.

6. Localisation of classifying spaces

Fix a prime p .

6.1. Main result

Following [Su], by a **simple space** we mean a connected topological space having the homotopy type of a CW complex and abelian fundamental group which acts trivially on the homotopy and homology of the universal covering space. Let F be a connected topological group of homotopy type of a CW complex (below we will consider more specifically complex Lie groups). Then its classifying space BF is a simple space (note that $\pi_1(BF) = \pi_0(F)$ is trivial). Following [Su, Chapter 2], to each simple space X we can associate a (simple) space $X_{\frac{1}{p}}$, the localisation of X away from p . Note that *loc. cit.* $X_{\frac{1}{p}}$ is denoted by X_ℓ where ℓ is the set of all primes different from p .

Theorem 6.1.1. *Let G, H be split (connected) reductive groups over \mathbb{Z} . The following are equivalent:*

- (a) *The perfection of G_k and H_k are isomorphic for $k = \overline{\mathbb{F}}_p$;*
- (b) *The localisations $BG(\mathbb{C})_{\frac{1}{p}}$ and $BH(\mathbb{C})_{\frac{1}{p}}$ are homotopy equivalent;*
- (c) *The root data of G and H become isomorphic after extension to $\mathbb{Z}[1/p]$.*

Proof of (c) \Leftrightarrow (a) \Rightarrow (b). The equivalence of (a) and (c) is already established in 4.2.3.

Assume that the perfection of G_k and H_k are isomorphic. By Proposition 3.2.3, after replacing H_k with an (isomorphic) Frobenius twisted version, there is an infinitesimal isogeny $G_k \rightarrow H_k$. By Lemma 3.2.2, this isogeny is purely inseparable. It then follows from [Fr, Theorem 1.6] that $BG(\mathbb{C})_{\frac{1}{p}}$ and $BH(\mathbb{C})_{\frac{1}{p}}$ are homotopy equivalent. \square

The rest of this chapter is devoted to the proof of (b) \Rightarrow (c).

6.2. Some useful facts

- (a) For a complex reductive group F and a maximal compact subgroup $K < F$ (the corresponding compact connected Lie group), the homomorphism $K \rightarrow F$ is a homotopy equivalence, and hence, $BK \simeq BF$. We will therefore henceforth replace BF by BK .
- (b) For a simple space X , the defining map $X \rightarrow X_{\frac{1}{p}}$ (see [Su, Chapter 2]) induces an isomorphism

$$H_*(X; \mathbb{Z}) \otimes \mathbb{Z}[1/p] \xrightarrow{\sim} H_*(X_{\frac{1}{p}}; \mathbb{Z}).$$

- (c) For a commutative ring D in which p is invertible, by (b) and the universal coefficient theorem, we have a natural isomorphism $H^*(X_{\frac{1}{p}}; D) \cong H^*(X; D)$. In particular, a map $X_{\frac{1}{p}} \rightarrow Y_{\frac{1}{p}}$ induces a graded algebra morphism $H^*(Y; D) \rightarrow H^*(X; D)$.

- (d) For a flat morphism $A \rightarrow B$ of commutative algebras and a topological space X for which $H_i(X; \mathbb{Z})$ are all finitely generated, $H^*(X; A) \otimes_A B \rightarrow H^*(X; B)$ is an isomorphism.
- (e) For a torus $T \cong (S^1)^{\times r}$, $H^*(BT; \mathbb{Z})$ is a polynomial ring with r generators in degree 2. Consequently, for any commutative ring R , $H^*(BT; R) \cong H^*(BT; \mathbb{Z}) \otimes R$ is a polynomial ring and we have a bijection between R -linear morphisms $\theta : H^2(BT'; R) \rightarrow H^2(BT; R)$ and graded R -algebra morphisms $\tilde{\theta} : H^*(BT'; R) \rightarrow H^*(BT; R)$.
- (f) For a compact connected Lie group G , with maximal torus T , and a commutative ring R , consider canonical isomorphisms

$$R \otimes X(T) \cong R \otimes H^1(T; \mathbb{Z}) \cong R \otimes H^2(BT; \mathbb{Z}) \cong H^2(BT; R).$$

The Weyl group W thus acts R -linearly on $H^2(BT, R)$. Moreover, the image of

$$H^*(BG; R) \rightarrow H^*(BT; R)$$

takes values in the algebra of W -invariants; see [Bor, §27].

- (g) For a prime q and a connected CW complex Y , we denote by $Y_{\hat{q}}$ the profinite completion at q of Y ; see [Su, Chapter 3]. If $q \neq p$, then the universality of $X \rightarrow X_{\frac{1}{p}}$ in the definition in [Su, Chapter 2] shows that the latter map induces a homotopy equivalence $X_{\hat{q}} \simeq (X_{\frac{1}{p}})_{\hat{q}}$.
- (h) For G a compact connected Lie group, BG satisfies the requirement in (c) (i.e. the homology groups $H_i(BG; \mathbb{Z})$ are finitely generated). One can observe this, for instance, via induction on i using the Serre fibration $G \rightarrow EG \rightarrow BG$. Note that EG is contractible, BG is simply connected and G is a finite cell complex. The Leray-Serre spectral sequence thus implies that the trivial group can be obtained, starting from $H_i(BG; \mathbb{Z})$ by a finite iteration of taking kernels of morphisms to finitely generated groups (subquotients of $H_a(BG; H_b(G))$, with $a < i$). Consequently, $H_i(BG; \mathbb{Z})$ must also be finitely generated.

6.3. Some results of Adams and Mahmud

We reformulate some results of Adams and Mahmud in the form we will need.

Theorem 6.3.1 (Adams - Mahmud). *Let G and G' be two compact connected Lie groups, with maximal tori T, T' and Weyl groups W, W' .*

- (1) *For a map $f : (BG)_{\frac{1}{p}} \rightarrow (BG')_{\frac{1}{p}}$, there exists a $\mathbb{Z}[1/p]$ -linear morphism*

$$\theta : H^2(BT'; \mathbb{Z}[1/p]) \rightarrow H^2(BT; \mathbb{Z}[1/p])$$

yielding a commutative diagram of graded algebra homomorphisms

$$\begin{array}{ccc} H^*(BG; \mathbb{Z}[1/p]) & \longleftarrow & H^*(BG'; \mathbb{Z}[1/p]) \\ \downarrow & & \downarrow \\ H^*(BT; \mathbb{Z}[1/p]) & \xleftarrow{\tilde{\theta}} & H^*(BT'; \mathbb{Z}[1/p]), \end{array}$$

where the upper horizontal arrow is induced from f as in (c) and $\tilde{\theta}$ is as in (e). Moreover, for every $w \in W$, there exists $x \in W'$ with $w\theta = \theta x$.

- (2) Let D be an integral domain of characteristic zero and consider two D -linear morphisms

$$\theta_1, \theta_2 : H^2(BT'; D) \rightarrow H^2(BT; D)$$

for which the following composite is equal:

$$\begin{array}{ccc} & & H^*(BG'; D) \\ & & \downarrow \\ H^*(BT; D) & \xrightleftharpoons[\tilde{\theta}_2]{\tilde{\theta}_1} & H^*(BT'; D). \end{array}$$

Then there exists $x \in W'$ such that $\theta_2 = \theta_1 x$.

Proof. For the first statement of part (1), consider the morphism $H^*(BG'; \mathbb{Q}) \rightarrow H^*(BG; \mathbb{Q})$ obtained from f via (c), or equivalently, via (d) from the map displayed in part (1). Then [AM, Theorem 1.5(a)] implies the existence of a morphism $H^2(BT'; \mathbb{Q}) \rightarrow H^2(BT; \mathbb{Q})$ yielding the commutative diagram in part (1) with $\mathbb{Z}[1/p]$ replaced by \mathbb{Q} . That the latter is induced from a morphism $H^2(BT'; \mathbb{Z}[1/p]) \rightarrow H^2(BT; \mathbb{Z}[1/p])$ follows from [AM, Theorem 1.5(b)] and the discussion after [AM, Lemma 1.2]. That the diagram over $\mathbb{Z}[1/p]$ is commutative follows from faithful flatness of $\mathbb{Z}[1/p] \rightarrow \mathbb{Q}$.

The case $D = \mathbb{Q}$ of part (2) is a reformulation of [AM, Theorem 1.7]. The proof *loc. cit.* works for any field of characteristic zero. The case of integral domains follows from extension of scalars to the field of fractions, using (d).

The second statement of part (1) now follows from part (2) and fact (f), by using $\theta_1 = \theta$ and $\theta_2 = w\theta$. □

Corollary 6.3.2. *With notation as in Theorem 6.3.1, assume that f is a homotopy equivalence. Then the morphism*

$$t : \mathbb{Z}[1/p] \otimes X(T') \rightarrow \mathbb{Z}[1/p] \otimes X(T),$$

obtained from θ via the isomorphisms in (f), induces an isomorphism of $\mathbb{Z}[1/p]$ -reflection groups $(W', \mathbb{Z}[1/p] \otimes X(T')) \rightarrow (W, \mathbb{Z}[1/p] \otimes X(T))$.

6.4. Conclusion of the proof of Theorem 6.1.1

Proof of (b) \Rightarrow (c). Assume first that $p = 2$. Then the result follows from Corollary 6.3.2 and Lemma 1.1.6(1). Similarly, for $p > 2$, by Lemma 1.1.6(2) it is sufficient to prove that

the extension of scalars along $\mathbb{Z}[1/p] \rightarrow \mathbb{Z}_2$ of t yields an isomorphism of \mathbb{Z}_2 -root data. This allows us to resort to the established theory of 2-compact groups; see [AG].

Starting from a homotopy equivalence f as in Theorem 6.3.1, we have our θ from 6.3.1(1) which induces the isomorphism of reflection groups in Corollary 6.3.2, and using (g), we have a homotopy equivalence f_2 . We consider the diagram

$$\begin{array}{ccc} (BG)_2 & \xrightarrow{f_2} & (BG')_2 \\ \uparrow & & \uparrow \\ (BT)_2 & \dashrightarrow & (BT')_2. \end{array}$$

Now $(BT)_2 \rightarrow (BG)_2$ is a maximal torus of the 2-compact group $(BG)_2$, in the sense of [Gr, Theorem 2.2]. By uniqueness of such maximal tori, see *loc. cit.*, there exists a homotopy equivalence corresponding to the dashed arrow in the above diagram so that the diagram is commutative up to homotopy.

By [Su, Theorem 3.9], this induces

$$\phi : H^2(BT'; \mathbb{Z}_2) \rightarrow H^2(BT; \mathbb{Z}_2),$$

yielding a commutative diagram

$$\begin{array}{ccc} H^*(BG; \mathbb{Z}_2) & \longleftarrow & H^*(BG'; \mathbb{Z}_2) \\ \downarrow & & \downarrow \\ H^*(BT; \mathbb{Z}_2) & \xleftarrow{\tilde{\phi}} & H^*(BT'; \mathbb{Z}_2). \end{array}$$

By uniqueness in Theorem 6.3.1(2) applied to $D = \mathbb{Z}_2$, we may assume that ϕ is actually induced from θ by extension of scalars $\mathbb{Z}[1/p] \rightarrow \mathbb{Z}_2$. Finally, the \mathbb{Z}_2 -root data of the 2-compact group $(BG')_2$, as defined in [AG], is obtained from the map $BT'_2 \rightarrow BG'_2$ and by construction yields the extension of scalars along \mathbb{Z}_2 of the classical root datum of G . The homotopy equivalence $(BG)_2 \simeq (BG')_2$ with commutative diagram therefore indeed implies that our isomorphism of $\mathbb{Z}[1/p]$ -reflection groups extends to an isomorphism of \mathbb{Z}_2 -root data. □

7. Perfected SL_2

Let k be an algebraically closed field of characteristic p . For λ in \mathbb{N} or $\mathbb{N}[1/p] := \mathbb{Z}[1/p] \cap \mathbb{R}_{\geq 0}$, we consider its p -adic expansion $\lambda = \sum_i \lambda_i p^i$ with $0 \leq \lambda_i < p$.

7.1. Fractal

7.1.1. For $i, j \in \mathbb{N}$, denote by $\binom{i}{j}_0$ the zero coefficient of the p -adic expansion of the binomial coefficient. By convention, $\binom{i}{j}_0 = 0$ if $j > i$. For $i, j \in \mathbb{N}[1/p]$, we set

$$\binom{i}{j}_0 := \binom{p^l i}{p^l j}_0 \in \{0, 1, \dots, p-1\},$$

for some $l \in \mathbb{N}$ for which $p^l i, p^l j \in \mathbb{N}$. By Lucas' theorem, this definition does not depend on the choice of l .

7.1.2. We can describe the weight spaces in simple $(SL_2)_{\text{perf}}$ -modules by

$$\dim_k \mathbf{L}(n)_{n-2j} = \begin{cases} 1 & \text{if } \binom{n}{j}_0 \neq 0 \\ 0 & \text{if } \binom{n}{j}_0 = 0, \end{cases}$$

for $n, j \in \mathbb{N}[1/p]$. In particular, the set

$$F := \{(n, i) \mid \dim_k \mathbf{L}(n)_i \neq 0\} \subset \mathbb{Z}[1/p]^2 \subset \mathbb{R}^2$$

is a fractal. Concretely, $(a, b) \in F$ if and only if $(p^l a, p^l b) \in F$ for all $l \in \mathbb{Z}$.

The integer points of the fractal F for $p = 3$ are displayed in Figure 1.

7.2. First extensions

In this section and the next, we will apply the short exact sequence

$$\nabla(\kappa)^{(1)} \hookrightarrow \nabla(p\kappa) \twoheadrightarrow \nabla(\kappa - 1)^{(1)} \otimes L(p - 2), \quad \kappa \in \mathbb{N} = X_+ \tag{7.1}$$

(see [Pa, equation (3)]), as well as the isomorphism

$$\nabla(p\kappa - 1) \cong \nabla(\kappa - 1)^{(1)} \otimes L(p - 1), \quad \kappa \in \mathbb{Z}_{>0}. \tag{7.2}$$

Proposition 7.2.1. For $\lambda, \mu \in X_+ = \mathbb{N}[1/p]$, we have

$$\dim \text{Ext}^1(\mathbf{L}(\lambda), \nabla(\mu)) = \begin{cases} 1 & \text{if there is } i \in \mathbb{Z} \text{ with } \lambda_i + \mu_i = p - 2 \text{ and} \\ & \lambda - p^i \lambda_i = \mu - p^i \mu_i + p^{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim \text{Ext}^1(\mathbf{L}(\lambda), \mathbf{L}(\mu)) = \begin{cases} 1 & \text{if there is } i \in \mathbb{Z} \text{ with } \lambda_i + \mu_i = p - 2, \\ & |\lambda_{i+1} - \mu_{i+1}| = 1 \text{ and } \lambda_j = \mu_j \text{ for } j \notin \{i, i + 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall from equation (5.1) that $\nabla(\lambda)$ is a direct limit of pullbacks to G_{perf} of costandard G -modules. Since $\text{Ext}^1(\mathbf{L}(\lambda), -)$ commutes with direct limits in the second argument, we can use Remark 3.3.4 to conclude

$$\text{Ext}^1(\mathbf{L}(\lambda), \nabla(\mu)) = \varinjlim \text{Ext}^1(L(p^l \lambda), \nabla(p^l \mu)).$$

The transition maps are given by the composite

$$\text{Ext}^1(L(p^l \lambda), \nabla(p^l \mu)) \rightarrow \text{Ext}^1(L(p^l \lambda)^{(1)}, \nabla(p^l \mu)^{(1)}) \rightarrow \text{Ext}^1(L(p^{l+1} \lambda), \nabla(p^{l+1} \mu)).$$

Here, the first map is given by the action of the Frobenius twist, so it is injective by Remark 5.1.4. The second map comes from the inclusion $\nabla(p^l \mu)^{(1)} \hookrightarrow \nabla(p^{l+1} \mu)$. From the description of the cokernel of the inclusion in (7.1), it follows that the second map is

also injective for $p > 2$. For $p = 2$, the second map need not be injective, but one shows easily that the composite is still injective.

Assume $p > 2$. The case $p = 2$ can be proved similarly. It follows quickly from [Pa, Corollary 6.2] that for $0 \leq i < p$,

$$\text{Ext}^1(L(pa + i), \nabla(pb + i)) \cong \text{Ext}^1(L(a), \nabla(b)),$$

while for $0 \leq i < p - 1$,

$$\dim \text{Ext}^1(L(pa + p - 2 - i), \nabla(pb + i)) = \delta_{b+1, a}.$$

Together with the block decomposition, this allows us, by iteration, to calculate the first set of extensions.

It follows from Remark 3.3.4 that

$$\text{Ext}^1(\mathbf{L}(\boldsymbol{\lambda}), \mathbf{L}(\boldsymbol{\mu})) = \varinjlim \text{Ext}^1(L(p^l \boldsymbol{\lambda}), L(p^l \boldsymbol{\mu})),$$

where the transition maps are injective by Remark 5.1.4.

Assume $p > 2$. The case $p = 2$ can be proved similarly. By [Pa, Theorem 4.3], we have for $0 \leq i < p$,

$$\text{Ext}^1(L(pa + i), L(pb + i)) \cong \text{Ext}^1(L(a), L(b)), \tag{7.3}$$

while for $0 \leq i < p - 1$

$$\text{Ext}^1(L(pa + i), L(pb + p - 2 - i)) \cong \text{Hom}(L(a), L(b) \otimes L(1)).$$

However, we have

$$\dim \text{Hom}(L(a), L(b) \otimes L(1)) = \begin{cases} 1 & \text{if } b_0 = 0 \text{ and } a = b + 1 \\ 1 & \text{if } 0 < b_0 < p - 1 \text{ and } a = b \pm 1 \\ 1 & \text{if } b_0 = p - 1 \text{ and } a = b - 1 \\ 0 & \text{otherwise.} \end{cases}$$

The cases $b_0 < p - 1$ follow immediately from the Steinberg tensor product theorem. If $b_0 = p - 1$, we know by parity that the space is zero unless $a_0 < p - 1$, in which case we can use symmetry between a and b to reduce to the already known cases. \square

Remark 7.2.2. Equation (7.3) shows that $\text{Rep}SL_2 \rightarrow \text{Rep}(SL_2)_{\text{perf}}$ yields isomorphisms on first extensions between simple objects for $p > 2$. This is not true for $p = 2$.

7.3. Costandard modules

We describe the multiplicities of the simple modules in $\nabla(\boldsymbol{\lambda})$. By Remark 5.1.2(2), it is sufficient to consider $\boldsymbol{\lambda} \in \mathbb{N}$ (with the case $\boldsymbol{\lambda} = 0$ trivial).

Proposition 7.3.1. *For $\lambda \in \mathbb{Z}_{>0}$, consider the finite sets*

$$E^0(\lambda) := \{\nu \in \mathbb{N} \mid [\nabla(\lambda) : L(\nu)] \neq 0\} \quad \text{and} \quad E^\infty(\lambda) := \{\nu \in \mathbb{Z}_{>0} \mid [\nabla(\lambda - 1) : L(\nu - 1)] \neq 0\}.$$

Then, for all $\mu \in \mathbb{N}[1/p]$, we have

$$[\nabla(\lambda) : \mathbf{L}(\mu)] = \begin{cases} 1 & \text{if } \mu \in E^0(\lambda) \\ 1 & \text{if } \mu = \nu - \frac{2}{p^i} \text{ with } \nu \in E^\infty(\lambda) \text{ and } i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, $\nabla(\lambda)$ has a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \dots$ with $\cup_i M_i = \nabla(\lambda)$ and

$$M_1 = \nabla(\lambda)[0] \quad \text{and} \quad M_{i+1}/M_i \cong \left(\nabla(\lambda - 1)^{(i)} \otimes L(p^i - 2) \right)[i], \quad \text{for } i > 0.$$

Proof. Recall from the proof of Theorem 5.1.1 that $\nabla(\lambda) = \varinjlim \nabla(p^i \lambda)[i]$, where every morphism in the chain is injective. The corresponding filtration is the desired one. Indeed, the subquotients are given by the cokernel in (7.1) for $\kappa = p^i \lambda$ on which we can apply iteratively (7.2) and the Steinberg tensor product theorem. \square

Example 7.3.2. Combining Propositions 7.2.1 and 7.3.1 shows the following:

- (1) Consider $0 < \lambda < p$. Then the socle filtration of $\nabla(\lambda)$ is given by $\text{soc} \nabla(\lambda) = \mathbf{L}(\lambda)$ and

$$\text{soc}^i \nabla(\lambda) = \mathbf{L}\left(\lambda - \frac{2}{p^i}\right), \quad i > 0.$$

- (2) The socle filtration of $\nabla(2p - 1)$ is given by $\text{soc} \nabla(2p - 1) = \mathbf{L}(2p - 1)$, $\text{soc}^1 \nabla(2p - 1) = \mathbf{L}(2p - 1 - 2/p)$ and

$$\text{soc}^{i+1} \nabla(2p - 1) \cong \mathbf{L}\left(2p - 1 - \frac{2}{p^{i+1}}\right) \oplus \mathbf{L}\left(1 - \frac{2}{p^i}\right), \quad i > 0.$$

Remark 7.3.3. We can explicitly realise $\nabla(\lambda)$ as the space of “degree λ ” elements in $k[x^{1/p^\infty}, y^{1/p^\infty}]$ – that is, the span of $\{x^\mu y^\nu \mid \mu, \nu \in \mathbb{N}[1/p], \mu + \nu = \lambda\}$.

7.4. Line bundle cohomology

We consider the representations $\mathbf{W}(\lambda)$, $\lambda \in \mathbb{N}[1/p] \setminus \{0\} = \mathbf{X}_{++}$, from 5.1.7.

Proposition 7.4.1. Recall the finite sets E^0, E^∞ from Proposition 7.3.1. For $\lambda \in \mathbb{Z}_{>0}$ and $\mu \in \mathbb{N}[1/p]$, we have

$$[\mathbf{W}(\lambda) : \mathbf{L}(\mu)] = \begin{cases} 1 & \text{if } \mu \in E^0(\lambda - 2), \text{ (with } E^0(-1) := \emptyset) \\ 1 & \text{if } \mu = \nu - \frac{2}{p^i} \text{ with } \nu \in E^\infty(\lambda) \text{ and } i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, $\mathbf{W}(\lambda)$ has a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \dots$ with $\cup_i M_i = \mathbf{W}(\lambda)$ and (with convention $\Delta(-1) = 0$)

$$M_1 = \Delta(\lambda - 2)[0] \quad \text{and} \quad M_{i+1}/M_i \cong \left(\Delta(\lambda - 1)^{(i)} \otimes L(p^i - 2) \right)[i], \quad \text{for } i > 0.$$

Proof. Using Čech cohomology (see proof of Lemma 2.5.1), it follows easily that the morphisms in the directed system in 5.1.7 are injective. The result then follows as in the proof of Proposition 7.3.1, by now using [Pa, (3)] for $i = p - 2$. \square

Remark 7.4.2. It follows that in the Grothendieck group of $\text{Rep}(SL_2)_{\text{perf}}$, we have

$$[\nabla(\lambda)] - [\mathbf{W}(\lambda)] = [\Delta(\lambda)[0]] - [\Delta(\lambda - 2)[0]].$$

Example 7.4.3. We have a short exact sequence

$$0 \rightarrow \mathbf{L}(1) \rightarrow \nabla(1) \rightarrow \mathbf{W}(1) \rightarrow 0.$$

For $1 < \lambda < p - 1$, we have

$$\nabla(\lambda)/\mathbf{L}(\lambda) \cong \mathbf{W}(\lambda)/\mathbf{L}(\lambda - 2).$$

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References

- [Ab] E. ABE, *Hopf Algebras* (Cambridge Tracts in Mathematics) vol. 74 (Cambridge University Press, Cambridge-New York, 1980).
- [AM] J. F. ADAMS AND Z. MAHMUD, Maps between classifying spaces, *Invent. Math.* **35** (1976), 1–41.
- [AG] K. S. ANDERSEN AND J. GRODAL, The classification of 2-compact groups, *J. Amer. Math. Soc.* **22**(2) (2009), 387–436.
- [BNP] C. P. BENDEL, D. K. NAKANO AND C. PILLEN, Extensions for finite Chevalley groups III: rational and generic cohomology, *Adv. Math.* **262** (2014), 484–519.
- [BB] A. BJÖRNER AND F. BRENTI, *Combinatorics of Coxeter Groups* (Graduate Texts in Mathematics) vol. 231 (Springer, New York, 2005).
- [BS] B. BHATT AND P. SCHOLZE, Projectivity of the Witt vector affine Grassmannian, *Invent. Math.* **209**(2) (2017), 329–423.
- [Bor] A. BOREL, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. of Math. (2)* **57** (1953), 115–207.
- [Bo] N. BOURBAKI, *Lie Groups and Lie Algebras*, chapters 4–6. (Elements of Mathematics) (Springer-Verlag, Berlin, 2002).
- [BD] M. BOYARCHENKO AND V. DRINFELD, Character sheaves on unipotent groups in positive characteristic: foundations, *Selecta Math (N.S.)* **20**(1) (2014), 125–235.
- [CPSV] E. CLINE, B. PARSHALL, L. SCOTT AND W. VAN DER KALLEN, Rational and generic cohomology, *Invent. Math.* **39**(2) (1977), 143–163.
- [CEO] K. COULEMBIER, P. ETINGOF AND V. OSTRIK, On Frobenius exact symmetric tensor categories, *Ann. of Math. (2)* **197**(3) (2023), 1235–1279. With an appendix by A. Kleshchev.
- [CR] C. W. CURTIS AND I. REINER, *Methods of Representation Theory* (Pure and Applied Mathematics) vol. 1 (John Wiley & Sons, Inc., New York, 1981). With applications to finite groups and orders.
- [D] P. DELIGNE, Catégories tensorielles, *Mosc. Math. J.* **2**(2) (2002), 227–248.
- [De] M. DEMAZURE, Schémas en groupes réductifs, *Bull. Soc. Math. France* **93** (1965), 369–413.

- [DG] M. DEMAZURE AND P. GABRIEL, *Groupes algébriques, Tome I: Géométrie algébrique, généralités, groupes commutatifs* (North-Holland Publishing Co., Amsterdam, 1970).
- [DM] P. DELIGNE AND J. S. MILNE, Tannakian categories, in *Hodge Cycles, Motives, and Shimura Varieties* (Lecture Notes in Mathematics) vol. 900 (Springer-Verlag, Berlin-New York, 1982), 101–228.
- [EK] K. ERDMANN AND L. G. KOVÁCS, Metabelian Lie powers of the natural module for a general linear group, *J. Algebra* **352** (2012), 232–267.
- [Fr] E. M. FRIEDLANDER, Exceptional isogenies and the classifying spaces of simple Lie groups, *Ann. of Math. (2)* **101** (1975), 510–520.
- [Gr] J. GRODAL, The classification of p -compact groups and homotopical group theory, in *Proceedings of the International Congress of Mathematicians*, vol. II (Hindustan Book Agency, New Delhi, 2010), 973–1001.
- [Hu] J. E. HUMPHREYS, *Reflection Groups and Coxeter Groups* (Cambridge Studies in Advanced Mathematics) vol. 29 (Cambridge University Press, Cambridge, 1990).
- [Ja] J. C. JANTZEN, *Representations of Algebraic Groups* (Mathematical Surveys and Monographs) second edn, vol. 107 (American Mathematical Society, Providence, RI, 2003).
- [Mi] J. S. MILNE, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field* (Cambridge Studies in Advanced Mathematics) vol. 170 (Cambridge University Press, Cambridge, 2017).
- [Pa] A. PARKER, Higher extensions between modules for SL_2 , *Adv. Math.* **209**(1) (2007), 381–405.
- [Sp] T. A. SPRINGER, *Linear Algebraic Groups* (Perspectives in Mathematics) (Birkhäuser, Basel, 1984), 455–495.
- [St] R. STEINBERG, The isomorphism and isogeny theorems for reductive algebraic groups, *J. Algebra* **216**(1) (1999), 366–383.
- [Su] D. P. SULLIVAN, *Geometric Topology: Localization, Periodicity and Galois Symmetry: The 1970 MIT Notes* (K-Monographs in Mathematics) vol. 8 (Springer, Dordrecht, 2005).
- [W] J. P. WANG, Quasirational modules and generic cohomology, *Northeast. Math. J.* **1**(1) (1985), 90–100.
- [Wi] G. WILLIAMSON, Modular representations and reflection subgroups, in *Current Developments in Mathematics* (Int. Press. Boston, 2019).
- [Zh] X. ZHU, Affine Grassmannians and the geometric Satake in mixed characteristic, *Ann. of Math. (2)* **185**(2) (2017), 403–492.