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THE CANARY TREE

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ABSTRACT. A *canary tree* is a tree of cardinality the continuum which has no uncountable branch, but gains a branch whenever a stationary set is destroyed (without adding reals). Canary trees are important in infinitary model theory. The existence of a canary tree is independent of ZFC + GCH.

A canary tree is a tree of cardinality $2^{\aleph_0}$ which detects the destruction of stationary sets. (A stationary set is *destroyed* in an extension if it is non-stationary in the extension.) More exactly, T is a *canary tree* if $|T| = 2^{\aleph_0}$, T has no uncountable branch, and in any extension of the universe in which no new reals are added and in which some stationary subset of ω_1 is destroyed, T has an uncountable branch. (We will give an equivalent characterization below which does not mention extensions of the universe.) The existence of a canary tree is most interesting under the assumption of CH (if $2^{\aleph_0} = 2^{\aleph_1}$ it is easy to see, as we will point out, that there is a canary tree.) The existence or non-existence of a canary tree has implications for the model theory of structures of cardinality $\aleph_1$ and for the descriptive set theory of ${}^{\omega_1}\omega_1$ ([4]). The canary tree is named after the miner's canary.

In this paper, we will explain the significance of the existence of a canary tree in model theory and prove that the existence of a canary tree is independent from ZFC + CH.

As is well known the standard way to destroy a stationary costationary subset of ω_1 is to force a club through its complement using as conditions closed subsets of the complement ([1]). More precisely if S is a stationary subset of ω_1 we can define $T_S = \{C : C \text{ a closed countable subset of } S\}$, where the order is end-extension. If S is costationary then T_S has no uncountable branch but when we force with T_S we add no reals but do add a branch through T_S. Such a branch is a club subset of ω_1 which is contained in S. (In [1], the forcing to destroy a stationary costationary set E is exactly $T_{\omega_1 \setminus E}$.) Notice that T_S detects the destruction of $\omega_1 \setminus S$ in the sense that in any extension of the universe with no new reals and in which $\omega_1 \setminus S$ is non-stationary, T_S has a branch.

These elementary observations imply that if $2^{\aleph_0} = 2^{\aleph_1}$ then there is a canary tree. The tree can be constructed by having disjoint copies of T_S sitting above a common root where S ranges through the stationary costationary subsets of ω_1. In fact any canary tree must almost contain the union of all the T_S, in the following weak sense.

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THEOREM 1. *Suppose T is a tree of $2^{\aleph_0}$ with no uncountable branch. Then T is a canary tree if and only if for any stationary costationary set S there exists a sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$ of maximal antichains of T_S and there is an order-preserving function $f: \bigcup_{\alpha<\omega_1} X_\alpha \longrightarrow T$. Furthermore $\langle X_\alpha : \alpha < \omega_1 \rangle$ and f are such that: if $\alpha < \beta$ and $s \in X_\alpha$, $t \in X_\beta$ then either s and t are incomparable or $s < t$; if δ is a limit ordinal and $t \in X_\delta$ then $t = \sup\{s < t : s \in X_\beta, \beta < \delta\}$; and f is continuous. (Note that these conditions imply that for all $u \in T_S$ there is δ and $t \in X_\delta$ such that $u < t$.)*

PROOF. First assume that for every stationary costationary set there is such a sequence of antichains and such a function. Suppose that E is a stationary set which is destroyed in an extension of the universe with no new reals. Let $S = \omega_1 \setminus E$ and let f and $\langle X_\alpha : \alpha < \omega_1 \rangle$ be as guaranteed. Let C be a club in the extension which is contained in S. Choose an increasing sequence $\langle s_\alpha : \alpha < \omega_1 \rangle$ of elements of T_S so that for all α, $s_{\alpha+1}$ is greater than some member of X_α and $\max s_\alpha \in C$. The choice of such a sequence is by induction. There is no problem at successor steps. At a limit ordinal δ, we can continue since $\sup \bigcup_{\alpha<\delta} s_\alpha \in C$ and hence in S. Also since no new reals are added $\bigcup_{\alpha<\delta} s_\alpha \cup \sup \bigcup_{\alpha<\delta} s_\alpha \in T_S$. Let b be the uncountable branch through T_S determined by $\langle s_\alpha : \alpha < \omega_1 \rangle$. So in the extension f"$(b \cap \bigcup_{\alpha<\omega_1} X_\alpha)$ is an increasing uncountable subset of T.

Now suppose that T is a canary tree. Let S be a stationary costationary set. Since forcing with T_S destroys a stationary set there is $\tilde{b}$ a T_S-name for a branch of T. We will inductively define the sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$ of maximal antichains of T_S. Let 0 denote be the root of T. Define $X_0 = \{0\}$. In general, let $Y_\alpha = T \setminus \bigcup_{\beta<\alpha} X_\beta$ and let $D_\alpha = \{t \in Y_\alpha : t \text{ decides } \tilde{b} \restriction \alpha\}$. Let X_α be the set of minimal elements of D_α. Since D_α is dense, X_α is a maximal antichain. For $t \in X_\alpha$, choose s so that $t \Vdash s = \tilde{b} \restriction \alpha$ and let $f(t) = s$. ■

It is possible to improve the theorem above to show that T is a canary tree if and only if for every stationary costationary set S there is an order preserving function from T_S to T ([4]). In fact when we show that it is consistent with GCH that there is a canary tree T, we will construct for every stationary costationary set S an order preserving function from T_S to T. It is also worth noting that we get an equivalent definition if we only demand that a canary tree have cardinality at most $2^{\aleph_0}$, since if T is a tree of cardinality less than $2^{\aleph_0}$, then forcing with T_S adds no new branch to T.

1. **The canary tree and Ehrenfeucht-Fraïssé games.** A central idea in the Helsinki school's approach to finding an analogy at ω_1 of the theory of $L_{\infty\omega}$ is the notion of an Ehrenfeucht-Fraïssé game of length ω_1 (see [3] for more details and further references). Given two models, $\mathfrak{A}$ and $\mathfrak{B}$, two players, an isomorphism player and a non-isomorphism player, alternately choose elements from $\mathfrak{A}$ and $\mathfrak{B}$. In its primal form the game lasts ω_1 moves and the isomorphism player wins if an isomorphism between the chosen substructures has been constructed. The analogue of Scott's theorem is the trivial result that two structures of cardinality $\aleph_1$ are isomorphic if and only if the isomorphism player has a winning strategy. In the search for an analogue of Scott height, trees with

no uncountable branches play the role of ordinals. More exactly suppose that T is a tree and $\mathfrak{A}$ and $\mathfrak{B}$ are structures. The game $\mathcal{G}_T(\mathfrak{A}, \mathfrak{B})$ is defined as follows. At any stage the non-isomorphism player chooses an element from either $\mathfrak{A}$ or $\mathfrak{B}$ and a node of T which lies above the nodes this player has already chosen. The isomorphism player replies with an element of $\mathfrak{B}$ if the non-isomorphism player has played an element of $\mathfrak{A}$ and an element of $\mathfrak{A}$ if the non-isomorphism player has played an element of $\mathfrak{B}$. In either case the move must be such that the resulting sequence of moves from $\mathfrak{A}$ and $\mathfrak{B}$ form a partial isomorphism. The first player who is unable to move loses. In analogy with Scott height if $\mathfrak{A}$ and $\mathfrak{B}$ are non-isomorphic structures of cardinality $\aleph_1$ then there is a tree of cardinality at most $2^{\aleph_0}$ with no uncountable branches such that the non-isomorphism player has a winning strategy in $\mathcal{G}_T(\mathfrak{A}, \mathfrak{B})$. (The tree T can be chosen to be minimal.) A defect in the analogy with Scott height is that the choice of the tree depends on the pair $\mathfrak{A}, \mathfrak{B}$ and cannot in general be chosen for $\mathfrak{A}$ to work for all $\mathfrak{B}$ ([2]).

DEFINITION. Suppose $\mathfrak{A}$ is a structure of cardinality $\aleph_1$. A tree T is called a *universal non-equivalence tree* for $\mathfrak{A}$ if T has no uncountable branch and for every non-isomorphic $\mathfrak{B}$ of cardinality $\aleph_1$ the non-isomorphism player has a winning strategy in $\mathcal{G}_T(\mathfrak{A}, \mathfrak{B})$.

As we have mentioned there are structures for which there is no universal non-equivalence tree of cardinality $\aleph_1$. However for some natural structures such as free groups (or free abelian groups) or ω_1-like dense linear orders the existence of a universal non-equivalence tree of cardinality $2^{\aleph_0}$ is equivalent to the existence of a canary tree. We will only explain the case of ω_1-like dense linear orders, the case of groups is similar.

Recall the classification of ω_1-like dense linear orders with a left endpoint. Let η represent the rational order type and for $S \subseteq \omega_1$ let $\Phi(S) = 1 + \eta + \sum_{\alpha<\omega_1} \tau_\alpha$, where $\tau_\alpha = 1 + \eta$ if $\alpha \in S$ and $\tau_\alpha = \eta$ otherwise. It is known that any ω_1-like dense linear order is isomorphic to some $\Phi(S)$ and that for $E, S \subseteq \omega_1$, $\Phi(S) \cong \Phi(E)$ if and only if the symmetric difference of E and S is nonstationary.

THEOREM 2. *There is a universal non-equivalence tree of cardinality* $2^{\aleph_0}$ *for* $\Phi(\emptyset)$ *if and only if there is a canary tree.*

PROOF. Assume that T is a universal non-equivalence tree of cardinality $2^{\aleph_0}$ for $\Phi(\emptyset)$. Consider E, a stationary costationary set. Work now in an extension of the universe in which E is non-stationary and there are no new reals. In that universe, $\Phi(E) \cong \Phi(\emptyset)$. In that universe the isomorphism player can play the isomorphism against the winning strategy of the non-isomorphism player in $\mathcal{G}_T\big(\Phi(\emptyset), \Phi(E)\big)$. At each stage, both players will have a move. So the game will last ω_1 moves and the non-isomorphism player will have chosen an uncountable branch through T. Hence T is a canary tree.

Now suppose that T is a canary tree. Let $T' = T+2$ (*i.e.*, a chain of length 2 is added to the end of every maximal branch of T). We claim that T' is a universal non-equivalence tree for $\Phi(\emptyset)$. Suppose E is a stationary set. The case where E is in the club filter is an easier version of the following argument. Assume that S is stationary where $S = \omega_1 \setminus E$. To fix notation let $\Phi(\emptyset) = 1 + \eta + \sum \tau_\alpha$ and $\Phi(E) = 1 + \eta + \sum \mu_\alpha$. Let $\langle X_\alpha : \alpha < \omega_1 \rangle$ and $f\colon T_S \longrightarrow T$ be as in Theorem 1. Let $X = \bigcup_{\alpha<\omega_1} X_\alpha$. The winning strategy for the non-isomorphism player consists of choosing an increasing sequence $s_\alpha \in X$, playing $f(s_\alpha)$ as the move in the tree T and guaranteeing that at every limit ordinal δ if A is the subset of $\Phi(\emptyset)$ which has been played (by either player) and B is the subset of $\Phi(E)$ which has been played then $\sup \bigcup_{\alpha<\delta} s_\alpha = \sup\{\beta : a \in \tau_\beta, a \in A\} = \sup\{\beta : b \in \mu_\beta, b \in B\}$. The non-isomorphism player continues this way as long as possible. When there are no more moves following this recipe $\sup \bigcup_{\alpha<\delta} s_\alpha$ is an ordinal in E. In that case B has a least upper bound but A doesn't. So the non-isomorphism player only needs two more moves to win the game. ■

The above argument also shows that if there is a canary tree then the ω_1-like dense linear orders share a universal non-equivalence tree of cardinality $2^{\aleph_0}$.

2. Independence results.

THEOREM 3. *It is consistent with GCH that there is no canary tree.*

PROOF. Begin with a model of GCH and add $\aleph_2$ Cohen subsets to ω_1. In the extension GCH continues to hold. Suppose T is a tree of cardinality $\aleph_1$ which has no uncountable branch. Since the forcing to add $\aleph_2$ Cohen subsets of ω_1 satisfies the $\aleph_2$-c.c., T belongs to the extension of the universe by $\aleph_1$ of the subsets. By first adding all but one of the subsets we can work in $V[X]$ where X is a Cohen subset of ω_1 and T is in V. Note that X is a stationary costationary subset of ω_1. Let P be the forcing for adding a Cohen generic subset of ω_1 and let Q be the P-name for T_X. It is easy to see that $P * Q$ is essentially ω_1-closed. Hence forcing with $P * Q$ doesn't add a branch through T. So neither does forcing with T_X over $V[X]$. But forcing with T_X destroys a stationary set, namely, $\omega_1 \setminus X$. ■

It remains to prove the consistency of GCH together with the existence of a canary tree. The proof has two main steps, we first force a very large subtree of ${}^{<\omega_1}\omega_1$. At limit ordinals we will forbid at most one branch from extending. Having created the tree we will then iteratively force order preserving maps of T_S into the tree as S varies over all stationary costationary sets.

THEOREM 4. *It is consistent with GCH that there is a canary tree.*

PROOF. Assume that GCH holds in the ground model. To begin define Q_0 to be

$$\{f : \lim(\omega_1) \longrightarrow {}^{<\omega_1}\omega_1 : \text{dom} f \text{ is countable and for all } \delta \in \text{dom}(f), f(\delta) \in {}^{\delta}\delta\}.$$

If G_0 is Q_0-generic, we can identify G_0 with $\bigcup_{f \in G_0} \text{rge} f$. Let $\mathfrak{C} = \{s \in {}^{<\omega_1}\omega_1 : \text{for all } \delta \leq \ell(s), s \restriction \delta \notin G_0\}$. It is easy to see that in $V[G_0]$, $\mathfrak{C}$ has no uncountable branch and that $V[G_0]$ has no new reals. (In fact forcing with Q_0 is the same as adding a Cohen subset of ω_1, so the claims above follow.)

To complete the proof we need to force embeddings of T_S into $\mathfrak{C}$ as S ranges over stationary sets. Suppose that we are in an extension of the universe which includes a generic set for Q_0 and has no new reals. Fix a stationary set S. An element t of $\mathfrak{C}$ is called an *S-node* if for every limit ordinal $\alpha \notin S$, if $\alpha \leq \ell(t)$ then $t \restriction \alpha \notin {}^{\alpha}\alpha$. Notice that any

S-node has successors of arbitrary height which are S-nodes, since if s is an S-node of height α and δ is a limit ordinal greater than α, then any extension of $s^\frown\langle\delta\rangle$ of length at most δ is an S-node. The poset $P(S)$ will consist of pairs (g, X) where X is a countable subset of ${}^{<\omega_1}\omega_1$ such that each element of X is of successor length and g is a partial order preserving map from T_S to the S-nodes of $\mathfrak{C}$ whose domain is a countable subtree of T_S. Further (g, X) has the following properties.

1. if $c \in \operatorname{dom}(g)$ and $t \in X$ then $t \not\subseteq g(c)$
2. if $c_0 < c_1 < \cdots$ is an increasing sequence of elements of $\operatorname{dom}(g)$ then $\bigcup_{n<\omega} g(c_n) \in \mathfrak{C}$.

If (g, X) is a condition let $o(g, X)$ be the $\sup\{\ell(t) : t \in X \text{ or } t \in \operatorname{rge}(g)\}$. Let $\operatorname{dom}(g, X) = \operatorname{dom} g$. A condition (h, Y) extends (g, X) if

1. $g \subseteq h$,
2. if $c \in \operatorname{dom}(h) \setminus \operatorname{dom}(g)$, then $\ell\big(h(c)\big) > o(g, X)$,
3. $X \subseteq Y$.

CLAIM 4.1. *The poset $P(S)$ is proper.*

Suppose κ is some suitably large cardinal, $N \prec \big(\mathrm{H}(\kappa), \in, <^*\big)$, where $<^*$ is a well-ordering of the model, N is countable, and $P(S) \in N$. We need to show that for every $p \in N \cap P(S)$ there is an N-generic extension. Let $\delta = N \cap \omega_1$. Let f be the Q_0-generic function and $t = f(\delta)$. There are two cases to consider. Either there is a successor ordinal $\alpha < \delta$ so that $\alpha > o(p)$ and $t\restriction\alpha \in N$ or not. Let $p = (g, X)$. If such an ordinal α exists let $p_{-1} = (g, X \cup \{t\restriction\alpha\})$, otherwise let $p_{-1} = p$. Now define a sequence $p_{-1}, p_0, \ldots, p_n, \ldots$ of increasingly stronger conditions so that (for $n \geq 0$) p_n is in the n^{th} dense subset of $P(S)$ which is an element of N. Let $p_n = (g_n, X_n)$ and $q = (h, Y)$ where $h = \bigcup_{n<\omega} g_n$ and $Y = \bigcup_{n<\omega} X_n$. To finish the proof it suffices to see that $q \in P(S)$. The only point that needs to be checked is to verify that if $c_0 < c_1 < \cdots \in \operatorname{dom}(h)$ then $\bigcup_{n<\omega} h(c_n) \in \mathfrak{C}$. If there is m such that $c_n \in \operatorname{dom}(g_m)$ for all n, then we are done. Otherwise, by the second property of being an extension, $\sup\{\ell(h(c_n) : n < \omega\} \geq \sup\{o(g_m, X_m) : m < \omega\}$. However for all $\alpha < \delta$ there is a dense set D such that $(g, X) \in D$ implies $o(g, X) > \alpha$. As D is definable using parameters from N, $D \in N$. Furthermore since the sequence of conditions meets every dense set in N, $\sup\{o(g_m, X_m) : m < \omega\} \geq \delta$. Finally each $h(c_n) \in N$, so $\ell\big(h(c_n)\big) < \delta$ for all n. These facts give the equation, $\sup\{\ell(h(c_n) : n < \omega\} = \delta$. (In the remainder of the paper we will try to point out where a density argument is needed but we will not give it in such detail.) By the choice of p_{-1} and the property 1 of the definition of $P(S)$, $t \neq \bigcup_{n<\omega} h(c_n)$. ∎

Our forcing will be an iteration with countable support of length ω_2. As usual we will let P_i be the forcing up to stage i and will force with Q_i, a P_i-name for a poset. We have already defined Q_0. For i greater than 0, we take $\tilde{S}_i$ a P_i-name for a stationary costationary set and let Q_i be the P_i-name for $P(\tilde{S}_i)$. By Claim 4.2, forcing with P_{ω_2} adds no reals. Also since each Q_i is forced to have cardinality ω_1, if we enumerate the $\tilde{S}_i$ properly every stationary costationary set in the final forcing extension will occur as the interpretation of some $\tilde{S}_i$.

CLAIM 4.2. *For all $i \leq \omega_2$, forcing with P_i adds no new reals.*

The proof is by induction on i. The case $i = 1$ is easy. For successor ordinals the proof can be done along the same lines as Claim 4.1, or by a modification of the limit ordinal case which we do below. Suppose now that i is a limit ordinal and $\tilde{r}$ is a P_i-name for a real. Consider any condition p. We must show that p has an extension which determines all the values of $\tilde{r}$. Choose a countable N so that $N \prec \big(\mathrm{H}(\kappa), \in, <^*\big)$ and $p, P_i, \tilde{r} \in N$. Let $p = p_{-1}, p_0, p_1, \ldots$ be a sequence of increasingly stronger conditions in N so that p_n is in the n^{th} dense subset of P_i which is an element of N. Let $\delta = N \cap \omega_1$. There is an obvious upper bound q for the sequence. Of course q is not a condition. We would like to extend q to a condition q' by choosing some $t \in {}^{\delta}\delta$, letting $q'(0) = q(0)^\frown\langle\delta, t\rangle$ and letting $q'(i) = q(i)$ for $i > 0$. Choose $t \in {}^{\delta}\delta$ so that $t{\restriction}\omega \notin N$. By a density argument we can show that for all i and n, if $p_n{\restriction} i \Vdash c \in \operatorname{dom} p$, then there are m, g, X and $s \in N$ so that $p_m{\restriction} i \Vdash p_n(i) = (g, X)$ and $g(c) = s$. It is straightforward to see that t is as desired. (See the proof of Claim 4.3 for a similar but more detailed argument.) ■

Let G_{ω_2} be P_{ω_2}-generic. We have shown that in $V[G_{\omega_2}]$, for every stationary set S there is an order preserving map from T_S to $\mathfrak{C}$. To finish the proof we must establish the following claim.

CLAIM 4.3. *In $V[G_{\omega_2}]$, $\mathfrak{C}$ has no uncountable branch.*

Suppose that $\tilde{b}$ is forced (for simplicity) by the empty condition to be an uncountable branch of ${}^{<\omega_1}\omega_1$. We will show that there is a dense set of conditions which forces that $\tilde{b}$ is not a branch of $\mathfrak{C}$. Hence $\mathfrak{C}$ has no uncountable branch. Fix a condition $p \in P$. Choose a countable N so that $N \prec \big(\mathrm{H}(\kappa), \in, <^*\big)$ and $p, P_{\omega_2}, \tilde{b} \in N$. Let $p = p_{-1}, p_0, p_1, \ldots$ be a sequence of increasingly stronger conditions in N so that p_n is in the n^{th} dense subset of P_{ω_2} which is an element of N. Let $\delta = N \cap \omega_1$. The sequence $\langle p_n : n < \omega\rangle$ determines a value for $\tilde{b}{\restriction}\delta$. Let this value be t. There is an obvious upper bound q for the sequence. Of course q is not a condition. We would like to extend q to a condition q' by letting $q'(0) = q(0)^\frown\langle\delta, t\rangle$ and letting $q'(i) = q(i)$ for $i > 0$. We will show by induction on i that $q'{\restriction} i$ is a condition in P_i.

The case $i = 1$ and limit cases are easy. So we can assume that $i \in N$ and $q'{\restriction} i \in P_i$. Since forcing with P_i adds no new reals and N is an elementary submodel of $\big(\mathrm{H}(\kappa), \in, <^*\big)$, for all n there is m and (g_n, X_n) so that $p_m{\restriction} i \Vdash p_n(i) = (g_n, X_n)$. Hence $q'{\restriction} i \Vdash q'(i) = (h, Y)$, where $h = \bigcup_{n<\omega} g_n$ and $Y = \bigcup_{n<\omega} X_n$. Suppose now that $c_0 < c_1 < \cdots \in \operatorname{dom}(h)$. We need to show that $q'{\restriction} i \Vdash \bigcup_{n<\omega} h(c_n) \in \mathfrak{C}$. If there is some m so that $c_n \in \operatorname{dom}(g_m)$ for all n, then we are done as in Claim 4.1. Otherwise $\ell\big(\bigcup_{n<\omega} h(c_n)\big) = \delta$ and we only need to show that $\bigcup_{n<\omega} h(c_n) \neq t$.

Notice that for all $\alpha < \delta$, $q'{\restriction} i \Vdash \big(\bigcup_{n<\omega} h(c_n)\big){\restriction}\alpha$ is an $\tilde{S}_i$-node. We will show that there is $\alpha < \delta$ so that then $q'{\restriction} i \Vdash t{\restriction}\alpha$ is not an $\tilde{S}_i$-node. This will complete the proof.

Let $G = \{p \in N \cap P_{\omega_2} :$ there is n so that p_n extends $p\}$. By the choice of the sequence, G is N-generic. Note that by Claim 4.1 and the iteration lemma for proper forcing (or

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by a direct argument similar to Claim 4.1), $\Vdash_{P_{\omega_2}} \tilde{S}_i$ is costationary. Hence for all $i \in N$, $N[G] \models \tilde{S}_i^G$ is costationary and $N[G] \models \{\alpha : \tilde{b}^G \restriction \alpha \in {}^\alpha\alpha\}$ is a club. Hence

$$N[G] \models \text{ there is a limit ordinal } \alpha \text{ so that } \tilde{b}^G \restriction \alpha \in {}^\alpha\alpha \text{ and } \alpha \notin \tilde{S}_i^G.$$

By the forcing theorem there is some n so that $p_n \restriction i \Vdash t \restriction \alpha \in {}^\alpha\alpha$ and $\alpha \notin \tilde{S}_i$. So we have shown $q' \restriction i \Vdash t \restriction \alpha$ is not an $\tilde{S}_i$-node, which was our goal. ∎

Note in the proof above it was necessary to force the embeddings. The forcing Q_0 is the same as adding a Cohen subset of ω_1. So if we add two Cohen subsets of ω_1 and use one to construct the tree, then, by the proof of Theorem 3 the other one gives a stationary set which can be destroyed without adding an uncountable branch.

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