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ON SIMPLE, PRIMITIVE AND PRIME RINGS RELATIVE TO A TORSION THEORY

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For a hereditary torsion theory τ, we show by examples that the concepts of τ-Artinian τ-simple, τ-Artinian τ-primitive, τ-Artinian τ-prime, τ-Artinian τ-semiprimitive, and τ-Artinian τ-semiprime rings are different from each other and thus answer a question raised by Bland in his book [Topics in Torsion Theory, (Mathematical Research, **103**, Wiley-VCH, 1998)]. The example of a τ-Artinian τ-primitive ring which is not τ-simple here appears to be a counter-example to a result of Bland in the same publication.

INTRODUCTION

Throughout, R is an associative ring with identity, modules will be unitary right R-modules, and $\tau = (\mathcal{T}_\tau, \mathcal{F}_\tau)$ is a hereditary torsion theory on Mod-R, the category of all right R-modules. The following concepts can be found in [**1**]: A nonzero module M is called *τ-simple* if $M \in \mathcal{F}_\tau$ and $M/N \in \mathcal{T}_\tau$ for any nonzero submodule N of M; $J_\tau(R)$ is defined to be the intersection of all those right ideals I of R such that R/I is τ-simple; the ring R is called *τ-Artinian* if any descending chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of right ideals of R with all $R/I_n \in \mathcal{F}_\tau$ terminates; R is called *τ-primitive* if the right annihilator of a cyclic τ-simple R-module is 0; R is said to be *τ-semiprimitive* if $J_\tau(R) = 0$; a two-sided ideal A of R is said to be *completely τ-pure* if $M/(MA) \in \mathcal{F}_\tau$ for any $M \in \mathcal{F}_\tau$; R is defined to be *τ-prime* if whenever $AB = 0$ for completely τ-pure ideals A and B we have $A = 0$ or $B = 0$; R is called *τ-semiprime* if $A^2 = 0$ always implies $A = 0$ for any completely τ-pure ideal A; finally R is called a *τ-simple ring* if $R \in \mathcal{F}_\tau$ and whenever $R/I \in \mathcal{F}_\tau$ for an ideal I of R we have $I = 0$ or $I = R$. The following ring implications were proved by Bland in [**1**]: τ-simple $\Rightarrow$ τ-primitive $\Rightarrow$ τ-prime $\Rightarrow$ τ-semiprime and τ-primitive $\Rightarrow$ τ-semiprimitive $\Rightarrow$ τ-semiprime. It was claimed in [**1**, Proposition 6.1.17] that any τ-Artinian τ-primitive ring is τ-simple and it was then asked what other implications given above reverse under the assumption that R is τ-Artinian (see [**1**, p.142]).

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In this short paper, we first give a counter-example to the result of Bland that any τ-Artinian τ-primitive ring is τ-simple, and then answer Bland's question by giving examples of the following: A τ-Artinian τ-semiprimitive ring which is not τ-prime; a τ-Artinian τ-prime ring which is not τ-primitive; a τ-Artinian τ-semiprime ring which is neither τ-semiprimitive nor τ-prime.

1. A τ-Artinian τ-primitive ring need not be τ-simple

There exists a ring R and a hereditary torsion theory τ such that R is a τ-Artinian τ-primitive ring, but R is not a τ-simple ring. For a module M_R, $M^{\perp}$ is the annihilator of M in R.

Example 1.1. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field, $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then $R \supset I \supset J \supset 0$ is a composition series of right ideals of R, so R is right Artinian. Let $M_R = I$ and $N_R = J$. Then M_R is cyclic faithful and N_R is simple. Let $\mathcal{K} = \{X \in \text{Mod-}R : \forall 0 \neq Y \subseteq X, N \hookrightarrow Y\}$. Then $\mathcal{K}$ is a natural class, that is, $\mathcal{K}$ is closed under submodules, direct sums, injective hulls and isomorphic copies. Since R is right Artinian, R has DCC on $\{L \subseteq R_R : R/L \in \mathcal{K}\}$. By [**2**, Proposition 21], $\mathcal{K}$ is a hereditary torsionfree class. Let τ be the hereditary torsion theory such that $\mathcal{F}_\tau = \mathcal{K}$. Then R is τ-Artinian. Clearly, $M \in \mathcal{K}$. Note that N is the only non-trivial submodule of M and $M/N \not\cong N$. So, $M/N \notin \mathcal{K}$. This shows that M is a τ-simple module. So, R is a τ-primitive ring.

Note that I is a two-sided ideal of R and $(R/I)_R \cong N \in \mathcal{K}$. If R is a τ-simple ring, then it must be that $I = 0$ or $I = R$, a contradiction.

In the above example, M is a faithful cyclic τ-simple R-module and N is a cyclic τ-simple R-module with $N^{\perp} = I \neq 0$. So, for a τ-Artinian τ-primitive ring R, the annihilator of some cyclic τ-simple R-module may be non-zero. It seems that the incorrect statement that "the annihilator of each cyclic τ-simple R-module over the τ-Artinian τ-primitive ring R is zero" has been used in the proof of [**1**, Proposition 6.1.17] (see [**1**, line -6, p.142]).

2. Response to Bland's question

As a response to Bland's question above, we give the following examples.

Example 2.1. A τ-Artinian τ-semiprimitive ring which is not τ-prime: Let R be an Artinian semisimple ring, but one that is not simple. Let $\tau = (\mathcal{T}_\tau, \mathcal{F}_\tau)$ be the hereditary torsion theory in which every R-module is τ-torsionfree, that is, $\mathcal{T}_\tau = \{(0)\}$ and $\mathcal{F}_\tau = \text{Mod-}R$. Then every ideal of R is completely τ-pure. Since R is semisimple but not simple, there exist nonzero ideals A and B such that $AB = 0$. Thus, R is not τ-prime. But, clearly, R is τ-Artinian τ-semiprime. We can further prove R is τ-semiprimitive. By the definition of τ, an R-module is τ-simple if and only if it is simple. Thus, for a right ideal I of R, R/I is τ-simple if and only if R/I is simple if and only if I is a maximal right ideal. It follows that $J_\tau(R)$ is equal to the Jacobson radical $J(R)$ of R. But, clearly, $J(R) = 0$.

The next two examples give τ-Artinian τ-semiprime rings which are neither τ-prime nor τ-semiprimitive.

EXAMPLE 2.2. Let R be an Artinian semisimple ring with two simple R-modules, say M and N, up to isomorphism. Let $\tau = (\mathcal{T}_\tau, \mathcal{F}_\tau)$ be the hereditary torsion theory cogenerated by M, that is, $X \in \mathcal{F}_\tau$ if and only if $X \hookrightarrow E(M)^I$ for some index set I, where $E(M)$ is the injective hull of M. Note that R is Artinian semisimple and every R-module is injective. So, $X \in \mathcal{F}_\tau$ if and only if $X \cong M^{(J)}$ for some index set J. Then an R-module X is τ-simple if and only if $X \cong M$, and thus $X^\perp = M^\perp$. By [**1**, Proposition 2.2.8], $J_\tau(R)$ is the intersection of the annihilators of all cyclic τ-simple R-modules. It follows that $J_\tau(R) = M^\perp$. By the assumption on R, $M^\perp$ is a nonzero proper ideal of R. So, R is not τ -semiprimitive. But, it is easy to see that R is τ-Artinian τ-semiprime. To see that R is not τ-prime, note that $X \in \mathcal{F}_\tau$ if and only if $X \cong M^{(I)}$ for some index set I, and in this case, for any ideal A of R, $X/(XA) \cong M^{(J)} \in \mathcal{F}_\tau$ for some index set J. This means that every ideal of R is completely τ-pure. By the assumptions of R, R is not prime and so R is not τ-prime.

EXAMPLE 2.3. Let $Q = \prod_{i=1}^{\infty} F_i$, where $F_i = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, be the direct product of rings F_i, R the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Let $M = R\Big/\Big(\bigoplus_{i=1}^{\infty} F_i\Big)$. Note that M is an injective R-module. Let τ be the hereditary torsion theory cogenerated by M. Then $X \in \mathcal{F}_\tau$ if and only if $X \hookrightarrow M^K$ for some index set K. Thus, for a right ideal I of R, R/I is τ-torsionfree if and only if $I = \bigoplus_{i=1}^{\infty} F_i$ or $I = R$. So, R is τ-Artinian. And a cyclic module N is τ-simple if and only if $N \cong M$, so, in this case, $N^\perp = M^\perp = \bigoplus_{i=1}^{\infty} F_i \neq 0$. Thus,

$$J_\tau(R) = \text{the intersection of the annihilators of all cyclic } \tau\text{-simple } R\text{-modules} = \bigoplus_{i=1}^{\infty} F_i \neq 0.$$

Thus, R is not τ-semiprimitive. The above discussion shows that $\bigoplus_{i=1}^{\infty} F_i \subseteq X^\perp$ for every τ-torsion free R-module X. It follows that every ideal of R contained in $\bigoplus_{i=1}^{\infty} F_i$ is a completely τ-pure ideal. This implies that R is not τ-prime. Since R is semiprime, R is τ-semiprime.

EXAMPLE 2.4. A τ-Artinian τ-prime ring which is not τ-primitive: Let R be a prime ring but not a right primitive ring. Let $\tau = (\mathcal{T}_\tau, \mathcal{F}_\tau)$ be the hereditary torsion theory such that every R-module is a τ-torsion module, that is, $\mathcal{T}_\tau = \text{Mod-}R$ and $\mathcal{F}_\tau = \{(0)\}$. Then for any right ideal I of R, R/I is τ-torsion free if and only if $I = R$. So, R is τ-Artinian and τ-prime. Since the zero module (0) is the only τ-torsion free module, there does not exist any τ-simple module. So, there does not exist a τ-simple module whose annihilator is zero. Therefore, R is not τ-primitive.

Note that Example 2.4 is not desirable for Bland's question because it was assumed in [**1**, Section 6.1,p.135] that τ is a torsion theory on Mod-R such that τ-simple R-modules exist. So, the remaining question is the following: Does there exist a ring R and a hereditary torsion theory τ on Mod-R such that τ-simple R-modules exist, R is τ-Artinian τ-prime, but R is not τ-primitive ? The answer is "Yes", as shown by the next example.

EXAMPLE 2.5. Let R be any ring with nonzero right ideals I and J, $I \cap J = 0$, satisfying the following properties:

(1) $I^2 \neq I$ and $ab = ba$ for all $a, b \in I$;

(2) for any $r \in R \setminus J$, $0 \neq rs \in I$ for some $s \in R$, and dually for any $r \in R \setminus I$, $0 \neq rt \in J$ for some $t \in R$;

(3) for any $0 \neq L \subseteq I$, $L^{\perp} = J$, and dually for any $0 \neq P \subseteq J$, $P^{\perp} = I$;

(4) lastly, $R/(I \oplus J)$ satisfies the DCC on right submodules.

Let $M_R = I$ and $\mathcal{K} = \{X \in \text{Mod-}R : \forall 0 \neq Y \subseteq X, \exists 0 \neq Z \subseteq Y \text{ such that } Z \hookrightarrow M\}$. Then $\mathcal{K}$ is a natural class, that is, $\mathcal{K}$ is closed under submodules, direct sums, injective hulls and isomorphic copies. Observe that

$$(*) \qquad \text{for any } \ 0 \neq X \in \mathcal{K}, \ \text{ and any } \ 0 \neq m \in I, \ Xm \neq 0 \ \text{ by (3).}$$

Let $R/K \in \mathcal{K}$. Then $JI = 0$ by (3) with $P = J$. Hence $[(J + K)/K]I = \bar{0}$. By $(*)$, $J \subseteq K$. Thus, $K = J$ or $K \supset J$. Suppose that $K \supset J$. Let $r \in K \setminus J$. Then by (2), $0 \neq m = rs \in I$. But then $Im = mI \subseteq I \cap K$ by (1). Thus $[I/(I \cap K)]m = \bar{0}$ and $I/(I \cap K) \cong (I + K)/K \in \mathcal{K}$, and by $(*)$, $I \subseteq K$. Therefore $I \oplus J \subseteq K$. So, we have proved that if $R/K \in \mathcal{K}$ then $K = J$ or $K \supseteq I \oplus J$. In view of (4), R has DCC on $\{L \subseteq R_R : R/L \in \mathcal{K}\}$. By [**2**, Proposition 21], $\mathcal{K}$ is a hereditary torsionfree class. Let $\tau = (\mathcal{T}_\tau, \mathcal{F}_\tau)$ be the hereditary torsion theory such that $\mathcal{F}_\tau = \mathcal{K}$. Then R is τ-Artinian. Clearly, $I \in \mathcal{F}_\tau$. If $0 \neq L \subset I$, then $(I/L)m = \bar{0}$ for $0 \neq m \in L$ (by (1)), so $I/L \notin \mathcal{K}$. So, I is a τ-simple R-module. Therefore, τ-simple R-modules exist. Note that $J_R \notin \mathcal{F}_\tau$ by $(*)$ and (3), and thus $R_R \notin \mathcal{F}_\tau$. It follows from [**1**, Corollary 6.1.4, p.136] that R is not a τ-primitive ring. To show that R is τ-prime, let $A \neq 0$ and $B \neq 0$ be two completely τ-pure ideals of R such that $AB = 0$. Suppose first that $A \not\subseteq J$. Then for $a \in A \setminus J$, by (2), $0 \neq ar \in I$. Thus, $(I/IA)(ar) = \bar{0}$ and $I/(IA) \in \mathcal{K}$, and by $(*)$, $I = IA \subseteq A$. But then $IB \subseteq AB = 0$, and $B \subseteq J$ by (3). If $A = I$, then $I = IA \subseteq I^2 \subset I$ is a contradiction. So, $I \subset A$. By (2), for $b \in A \setminus I$, $0 \neq bt \in J$. Then $(btR)B \subseteq AB = 0$ implies that $B \subseteq I$ by (3). It follows that $B \subseteq I \cap J = 0$. This contradiction shows that $A \subseteq J$. Consequently $AB = 0$ implies that $B \subseteq I$ by (3). For $0 \neq m \in B \subseteq I$, $(I/IB)m = \bar{0}$ and $I/IB \in \mathcal{K}$. By $(*)$, $I = IB$, and this implies that $I = I^2$, a contradiction. Thus, R is τ-prime.

Below are two examples where the ring R with the right ideals I and J satisfies the conditions (1)–(4) in Example 2.5.

1. Let $T = \mathbb{Z} \oplus \mathbb{Z}$ be the ring direct sum, and $R = (2\mathbb{Z} \oplus 2\mathbb{Z}) + \mathbb{Z}1_T$ be the subring of T generated by $2\mathbb{Z} \oplus 2\mathbb{Z}$ and 1_T. Let $I = 2\mathbb{Z} \oplus (0)$ and $J = (0) \oplus 2\mathbb{Z}$. Then I and J are ideals of R and $R/(I \oplus J) = R/(2\mathbb{Z} \oplus 2\mathbb{Z}) = \{\overline{(0,0)}, \overline{(1,1)}\} \cong \mathbb{Z}_2$.
2. Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_2[t]$ be the polynomial ring. Let $T = \mathbb{Z}_2[t] \oplus \mathbb{Z}_2[t]$ be the ring direct sum, and R the subring of T generated by $t\mathbb{Z}_2[t] \oplus t\mathbb{Z}_2[t]$ and 1_T. Let $I = t\mathbb{Z}_2[t] \oplus (0)$ and $J = (0) \oplus t\mathbb{Z}_2[t]$. Then I, J are ideals of R and $R/(I \oplus J) \cong \mathbb{Z}_2$.

References

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