

## Hydrodynamics and thermalization

Since the systems described by quantum fields are by definition extended, it is natural to think that in some limit they may reasonably well be approximated as fluids. This means that the state of the system is parameterized by a few locally well-defined fields such as temperature or energy density, obeying a set of hyperbolic equations of motion. A concrete example is the extensive use of fluid models to describe high-energy collisions [BelLan56, CarDuo73, CoFrSc74, Bjo83, CarZac83].

Our earlier derivation of quantum kinetic theory suggests a way to put this insight on a formal basis. Within its range of validity, the Boltzmann equation will drive the one-particle distribution function towards local thermal equilibrium. On scales much larger than the local thermalization scale, we expect to see hydrodynamical behavior [BeCoPa02]. This is, after all, the usual way of deriving hydrodynamics from kinetic theory [Hua87]. Beware, notwithstanding, that even at the level of classical kinetic theory there are still open questions regarding the cross-over from the kinetic to the hydro regime [KarGor02, KarGor03].

If we understand hydrodynamics as stated in the first paragraph of this chapter, then a system defined in terms of a quantum field may not have a hydrodynamic limit. This has been shown in [Elz02] for the case of a free Fermi field. However, since the hydrodynamic description seems justifiable when applied to the physics of quark–gluon plasmas (see the discussion in Chapter 14) and early universe cosmology [Hu82, Hu83, CalGra02], we shall accept as a working hypothesis that for “interesting” systems whose fundamental description involves quantum fields there is a local thermal equilibrium limit where the system may be described as a fluid. The specifics of quantum fields are manifested through the gap equation and constitutive relations, whose derivation will be our main goal in this chapter.

Let us begin, however, with a brief review of basic thermodynamics, and then its relativistic generalization. The subject of hydrodynamics is one where the generally covariant formulation is actually simpler than the flat spacetime one, and much simpler than the nonrelativistic version. Therefore, it is worth investing some initial effort to familiarize ourselves with the generally covariant approach from scratch. Our presentation follows the review articles by Israel [Isr72, Isr88]; see also [Cal98].

## 12.1 Classical relativistic hydrodynamics

### 12.1.1 A primer on thermodynamics

The basic tenets of thermodynamics we need to keep in mind are the following: we have a (simple) system described by some intensive parameters (temperature  $T$ , chemical potential  $\mu$ , pressure  $p$ , etc.) whose meaning we take for granted (e.g. we already know everything about the zeroth law) and extensive parameters, such as energy  $U$ , entropy  $S$ , volume  $V$ , and particle number  $N$  (we use particle number for concreteness, but any – or several – conserved charge(s) would serve just as well). In equilibrium, all these quantities are position independent. Their first deviations from equilibrium are related by the first law

$$TdS = dU + pdV - \mu dN \quad (12.1)$$

Extensive quantities are homogeneous functions of each other, so we must have

$$TS = U + pV - \mu N \quad (12.2)$$

From the differential of this second identity we obtain the Gibbs–Duhem relation

$$dp = sdT + nd\mu \quad (12.3)$$

where  $s = S/V$  and  $n = N/V$  are the entropy and particle number densities, respectively. This means

$$\left. \frac{\partial p}{\partial T} \right|_{\mu} = s = \frac{\rho + p - \mu n}{T}; \quad \left. \frac{\partial p}{\partial \mu} \right|_T = n \quad (12.4)$$

where  $\rho = U/V$  is the energy density.

Actually, it is convenient to adopt as independent variables  $T$  and  $\alpha = \mu/T$ , whereby

$$\left. \frac{\partial p}{\partial T} \right|_{\alpha} = \frac{\rho + p}{T}; \quad \left. \frac{\partial p}{\partial \alpha} \right|_T = Tn \quad (12.5)$$

We may also write

$$S = \Phi + \left( \frac{1}{T} \right) U - \alpha N \quad (12.6)$$

where the *thermodynamic potential*  $\Phi = pV/T$  is the logarithm of the grand-canonical partition function. Finally we have the second law

$$TdS \geq dQ \quad (12.7)$$

This concludes our mini-tutorial on nonrelativistic thermodynamics.

### 12.1.2 Covariant hydrostatics

We now generalize the above framework of thermodynamics to a relativistic fluid evolving in a spacetime with an arbitrary metric  $g_{\mu\nu}$ . After overcoming some

initial threshold, the reader will be rewarded in the long run by the economy and exactitude of this formulation. All derivatives shall be covariant derivatives with respect to the Levi-Civita connection, so that  $g_{\mu\nu;\rho} \equiv 0$ . We use MTW conventions [MiThWh72].

The divergence of a (contravariant) vector  $X^\mu$  is defined by

$$X^\mu_{;\mu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} X^\mu) \tag{12.8}$$

(where  $\partial$  denotes an ordinary derivative), and the flux of a vector through a hypersurface  $\Sigma$  is

$$\int d^3\mathbf{x} \sqrt{{}^{(3)}g} n_\mu X^\mu \tag{12.9}$$

where  ${}^{(3)}g_{ab}$  is the induced metric on the surface and  $n_\mu$  is the outer normal. If the surface element is space like (that is, the normal is a time-like vector) we adopt the convention that  $n_0 < 0$  (so that  $n^0 > 0$ , recall that  $g^{00} < 0$  in any frame). For example, in ordinary Minkowski space we say a  $t = \text{constant}$  surface is space like, and its normal  $n^\mu = (\partial/\partial t)^\mu = (1, 0, 0, 0)$ . Then we obtain Gauss' theorem

$$\int_V d^4x \sqrt{-g} X^\mu_{;\mu} = \int_{\partial V} d^3\mathbf{x} \sqrt{{}^{(3)}g} (\varepsilon n_\mu) X^\mu \tag{12.10}$$

where  $\varepsilon = 1$  if the normal is space like, and  $-1$  if time-like.

To simplify matters we will describe the construction of a covariant theory in terms of a set of rules:

- (a) *Intensive quantities* ( $T, p, \mu$ ) are associated with scalars, which represent the value of the quantity at a given event, as measured by an observer at rest with respect to the fluid.
- (b) *Extensive quantities* ( $S, V, N$ ) are associated with vector currents  $S^\mu, u^\mu, N^\mu$ , such that for any given space like surface element  $d\Sigma_\mu = n_\mu d\Sigma$ , then the amount of quantity  $X$  within the volume  $d\Sigma$  as measured by an observer with velocity  $n^\mu$  is given by  $-X^\mu d\Sigma_\mu$ . Therefore  $x_n = -n_\mu X^\mu$  is the density of the quantity  $X$  measured by such an observer. If the quantity  $X$  is conserved, then  $X^\mu_{;\mu} = 0$ . The quantity  $u^\mu$  associated with volume is the fluid 4-velocity, and obeys the additional constraint  $u^2 = -1$ . We call density *tout court* the density measured by an observer comoving with the fluid, namely  $x = -u_\mu X^\mu$ .
- (c) Energy and momentum are combined into a single extensive quantity described by an energy-momentum tensor  $T^{\mu\nu}$  which is symmetric. The energy current, properly speaking, is  $U^\mu = -T^{\mu\nu} u_\nu$ , and the energy density  $\rho = T^{\mu\nu} u_\mu u_\nu$ .

We wish to describe a fluid in a state of *equilibrium*. However, we do not assume that the metric is stationary; at the very least, we must allow for the possibility

that the metric appears time dependent in our chosen coordinates. An example is a static de Sitter universe whose metric appears as an expanding spatially flat universe if written locally in Robertson–Walker metric form. Because of the (possibly) changing metric, we cannot expect that the relevant quantities are position independent. We shall only assume that the fluid is isotropic in the rest frame. This means that in the equilibrium state all vector currents are collinear with the velocity, the mixed components of  $T^{\mu\nu}$  vanish in the rest frame, and the spatial components (namely, the momentum flux) are isotropic. In other words, in equilibrium we may decompose

$$N^\mu = nu^\mu, \quad T^{\mu\nu} = \rho u^\mu u^\nu + p\Delta^{\mu\nu}, \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \quad (\Delta^{\mu\nu} u_\nu = 0) \quad (12.11)$$

where  $p$  is the equilibrium or hydrostatic pressure of the fluid. The entropy current  $S^\mu$  is given by  $TS^\mu = -T^{\mu\nu}u_\nu + pu^\mu - \mu N^\mu$ , which we rewrite as

$$S^\mu = \Phi^\mu - \beta_\nu T^{\mu\nu} - \alpha N^\mu \quad (12.12)$$

Here, we have introduced the thermodynamic potential current  $\Phi^\mu = p\beta^\mu$ , and the inverse temperature vector  $\beta^\mu = T^{-1}u^\mu$ . Observe that  $T^{-2} = -\beta^\mu\beta_\mu$ . Contracting with  $u_\mu$  we get  $Ts = p + \rho - \mu n$ , so locally we recover the usual thermodynamics. The first law becomes

$$TdS^\mu = -d(T^{\mu\nu}u_\nu) + pdu^\mu - \mu dN^\mu = -u_\nu dT^{\mu\nu} - \mu dN^\mu \quad (12.13)$$

Contracting the Gibbs–Duhem relation  $(dT)S^\mu = (dp)u^\mu - (d\mu)N^\mu$  with the 4-velocity, we recover the derivatives (12.5). If we regard the thermodynamic potential as a function of  $\beta^\mu$  and  $\alpha$ , then

$$\frac{\partial\Phi^\mu}{\partial\beta^\nu} = p\delta_\nu^\mu + (p + \rho)u^\mu u_\nu = T_\nu^\mu; \quad \frac{\partial\Phi^\mu}{\partial\alpha} = nu^\mu = N^\mu \quad (12.14)$$

so

$$S^\mu_{;\mu} = -\beta_\nu T^{\mu\nu}_{;\mu} - \alpha N^\mu_{;\mu} \quad (12.15)$$

This means that entropy production vanishes in equilibrium, provided the conservation laws of energy–momentum and particle number hold. Now, linear deviations from equilibrium are constrained by the first law (12.13). If we consider a state which deviates linearly from equilibrium, but where the conservation laws still hold, then in this state the entropy production must be

$$d(S^\mu_{;\mu}) = (dS^\mu)_{;\mu} = -\beta_{\nu;\mu} dT^{\mu\nu} - \alpha_{;\mu} dN^\mu \quad (12.16)$$

On the other hand, entropy production must be stationary at equilibrium, so the linear variation must vanish, whatever the deviations  $dT^{\mu\nu}$  and  $dN^\mu$  might be. In equilibrium the inverse temperature vector must be a *Killing field* ( $\beta_{(\nu;\mu)} = 0$ ) and  $\alpha$  must be constant. Being a Killing field means that a coordinate transformation of the type  $x^\mu \rightarrow x^\mu + \varepsilon\beta^\mu$  is a symmetry of the underlying spacetime.

Observe that not every spacetime or field theory may support an equilibrium state.

If we consider the variation of the entropy content within some spatial region  $\Sigma$  as a function of time, the second law demands that the increase in entropy should be higher than the entropy flow through the boundary  $\delta\Sigma$ . Thus the covariant statement of the second law is that entropy production must be positive, i.e.  $S^\mu_{;\mu} \geq 0$ .

### 12.1.3 Ideal and real fluids

In order to generalize the above framework to hydrodynamics (rather than hydrostatics) let us first introduce the concept of an *ideal fluid*, namely a fluid where the decomposition (12.11) is always available, not just under equilibrium conditions. Everything we said about equilibrium states is valid for an ideal fluid even away from equilibrium; this applies in particular to the vanishing of entropy production.

The equations of motion for the perfect fluid are the conservation laws for energy–momentum and particle number. Suppose we know  $\alpha$  to be constant (for example,  $\alpha = 0$ ). Then energy–momentum conservation implies the identities (recall that  $u_\mu u^\mu = 0$ )

$$\rho_{,t} - (\rho + p) u^\nu_{;\nu} = 0; \quad -(\rho + p) u^\mu_{;t} + \Delta^{\mu\nu} p_{,\nu} = 0 \tag{12.17}$$

where  $X_{,t} \equiv -u^\mu X_{,\mu}$ . The second equation is the Navier–Stokes equation for a fluid without viscosity. Since  $\rho$  and  $p$  become space dependent only through their temperature dependence, we may write  $\rho_{,t} = \rho_{,T} T_{,t}$ , and similarly for  $p$ . Using the identity (12.5), equations (12.17) simplify to

$$\frac{1}{T} T_{,t} - c_s^2 u^\nu_{;\nu} = 0; \quad -u^\mu_{;t} + \frac{1}{T} \Delta^{\mu\nu} T_{,\nu} = 0 \tag{12.18}$$

which can be reduced in a standard way to the wave equation with

$$c_s = \sqrt{p_{,T}/\rho_{,T}} \tag{12.19}$$

denoting the speed of sound.

We are interested in weakly nonideal fluids, namely, fluids which are not ideal, but whose properties remain close to a reference ideal fluid. The first obstacle we encounter is an ambiguity in the concept of the velocity of the fluid.

In effect, if the decomposition (12.11) fails, then the motion of mass does not agree with the motion of the conserved charges. In other words, heat transfer implies energy transfer, and therefore mass transfer, even if there is no charge flow. Therefore we must define what we mean by velocity. In practice, two different conventions have proved useful, namely the Eckart and the Landau–Lifshitz prescriptions [Wei72, LanLif59].

In the Eckart prescription, velocity and particle number densities are defined from the particle number current through the equations  $N^\mu = nu^\mu$ ,  $u^2 = -1$ , and

the energy density is read off the energy–momentum tensor  $\rho = T^{\mu\nu}u_\mu u_\nu$ . In the Landau–Lifshitz prescription, the velocity is defined as the (only) normalized time-like eigenvector of the energy–momentum tensor, and the energy density is (minus) the corresponding eigenvalue:  $T^{\mu\nu} = \rho u^\mu u^\nu + T_T^{\mu\nu}$ , with  $T_T^{\mu\nu}u_\nu = 0$ . The number density is read off the number current,  $n = -u_\mu N^\mu$ . In either case, the reference ideal fluid is chosen as having the same velocity, energy and particle number densities as the actual flow. The equation of state prescribes the pressure,  $p_0 = p_0(\rho, n)$  of the reference fluid, and we may parameterize  $T_T^{\mu\nu} = (p_0 + \pi)\Delta^{\mu\nu} + T_{TT}^{\mu\nu}$ , where  $T_{TT}^{\mu\nu}u_\nu = T_{TT\mu}^\mu = 0$ . Here  $\pi$  measures the deviation of the isotropic part of the stress tensor from its local equilibrium value. By definition, it is the product of the *bulk viscosity* of the real fluid times the local rate of expansion.

We must emphasize that, although so far we may think of the Eckart and Landau–Lifshitz prescriptions as simply different conventions, later on, when we impose constitutive relations, say, linking viscous stresses to velocity gradients, these different prescriptions will lead to different physical models of the fluid. For reasons which will become clear in due time, we adopt the Landau–Lifshitz prescription.

Let us write the energy–momentum and particle number currents for a real fluid as

$$T^{\mu\nu} = T_0^{\mu\nu} + \tau^{\mu\nu}; \quad N^\mu = N_0^\mu + j^\mu \quad (12.20)$$

where  $T_0^{\mu\nu} = \rho u^\mu u^\nu + p_0 \Delta^{\mu\nu}$  and  $N_0^\mu = n u^\mu$  are the energy–momentum and particle number current of a reference ideal fluid. We emphasize that, while  $\rho$  and  $n$  have a direct operational meaning,  $p_0$  is a theoretical construct. Concretely,  $p_0$  results from using  $\rho$  and  $n$  as inputs in the equation of state for the ideal fluid. Recall that  $\tau^{\mu\nu}u_\nu = j^\mu u_\mu = 0$ .

In order to complete the specification of the model for the real fluid, we must describe also the entropy current. To do this, let us go back to our discussion of hydrostatics. There we saw that, if we consider an arbitrary state departing by amounts  $dT^{\mu\nu}$  and  $dN^\mu$  from an equilibrium state, then, *to first order in departures from equilibrium*, the entropy production is given by  $(dS^\mu)_{;\mu} = -\beta_{\nu;\mu}dT^{\mu\nu} - \alpha_{,\mu}dN^\mu$ .

We shall apply this formula to two nonequilibrium states of two different fluids (namely, the real and reference ideal ones) rather than to nonequilibrium and equilibrium states of the same fluid, while disregarding the higher order corrections. Suffice it to be forewarned that these assumptions are not rigorous (we shall return to this point later) and will get us in trouble.

We therefore adopt as our model for a real fluid the entropy production formula

$$S^\mu_{;\mu} = -\beta_{\nu;\mu}\tau^{\mu\nu} - \alpha_{,\mu}j^\mu \quad (12.21)$$

which may be easily integrated to a formula for the entropy current, namely

$$S^\mu = \Phi_0^\mu - \beta_\nu T^{\mu\nu} - \alpha N^\mu \quad (12.22)$$

where  $\Phi_0^\mu = p_0\beta^\mu$ . This formula may look like a simple extension of the corresponding one for the equilibrium entropy current, but it is not, since it depends on keeping the ideal form for the thermodynamic potential but replacing the other two terms by their real counterparts.

The second law, in the form of positivity of entropy production, allows us to put further restrictions on the form of  $\tau^{\mu\nu}$  and  $j^\mu$ . For example, if we demand that  $-\alpha_{,\mu}j^\mu$  be nonnegative while  $j^\mu$  is transverse, then we are led to

$$j^\mu = -\kappa\Delta^{\mu\nu}\alpha_{,\nu} \tag{12.23}$$

and the second law implies  $\kappa \geq 0$ . This coefficient is related to heat conductivity.

In the other term, it is convenient to decompose  $\beta_{\nu;\mu}$  in its components along the direction transverse to the velocity, symmetrize, and, in the transverse part, extract the trace part to obtain

$$\beta^{(\mu;\nu)} = \{P_L + P_{LT} + P_T + P_{TT}\}^{\mu\nu\rho\sigma} \beta_{\rho;\sigma} \tag{12.24}$$

where

$$P_L^{\mu\nu\rho\sigma} = u^\mu u^\nu u^\rho u^\sigma \tag{12.25}$$

$$P_{LT}^{\mu\nu\rho\sigma} = \frac{-1}{2} [u^\mu u^\rho \Delta^{\nu\sigma} + u^\nu u^\sigma \Delta^{\mu\rho} + u^\nu u^\rho \Delta^{\mu\sigma} + u^\mu u^\sigma \Delta^{\nu\rho}] \tag{12.26}$$

$$P_T^{\mu\nu\rho\sigma} = \frac{1}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} \tag{12.27}$$

$$P_{TT}^{\mu\nu\rho\sigma} = \frac{1}{2} \left[ \Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} \right] \tag{12.28}$$

Observe that the  $P$ 's are symmetric, mutually orthogonal projectors. Since moreover  $P_L\tau = P_{LT}\tau = 0$ , and  $P_T\tau = \pi\Delta^{\rho\sigma}$ , we obtain

$$-\beta_{\nu;\mu}\tau^{\mu\nu} = -\pi\Delta^{\rho\sigma}\beta_{\rho;\sigma} - T_{TT}^{\mu\nu} [P_{TT}\beta_{\rho;\sigma}]_{\mu\nu} \tag{12.29}$$

which leads us to

$$T_{TT}^{\mu\nu} = -2\eta T [P_{TT}\beta_{\rho;\sigma}]^{\mu\nu}; \quad \pi = -\zeta T \Delta^{\rho\sigma} \beta_{\rho;\sigma} \tag{12.30}$$

where the coefficients  $\eta$  and  $\zeta$  are, respectively, the *shear* (or first) and *bulk* (or second) viscosities, and they must be nonnegative.

The reason why we have introduced factors of temperature explicitly in the above formulae is that they cancel the corresponding powers of  $T^{-1}$  in

$$\beta_{\rho;\sigma} = \frac{1}{T} \left\{ -\frac{T_{,\sigma}}{T} u_\rho + u_{\rho;\sigma} \right\} \tag{12.31}$$

The first term vanishes under the projectors, and the second is purely transverse ( $u^\rho u_{\rho;\sigma} = 0$ ), so

$$T_{TT}^{\mu\nu} = -\eta H^{\mu\nu}; \quad \pi = -\zeta u^\sigma_{;\sigma} \tag{12.32}$$

$$H^{\mu\nu} = 2 [P_{TT}u_{\rho;\sigma}]^{\mu\nu} \tag{12.33}$$

which are the most often quoted forms. It is easy to see that with these constitutive relations, energy–momentum conservation leads to the Navier–Stokes equation in covariant form.

This outcome is very good from the phenomenological point of view, but troublesome from a theoretical point of view, as it can be proven that the covariant Navier–Stokes theory admits no stable solutions [HisLin83, HisLin85]. Eventually we shall learn to live with this contradiction, but let us elaborate on it a little further, to the point where at least we understand what we have settled for.

#### 12.1.4 Stability and the Landau–Lifshitz theory

Rather than attempting a direct study of stability in the Navier–Stokes equations, we shall show that the theory for a real fluid just constructed is a particular case of a class of theories which satisfy the essential condition of causality, whereas the Landau–Lifshitz theory does not. These are the *divergence type theories* of Geroch and Lindblom [GerLin90]. To simplify our discussion, we shall assume  $\alpha \equiv 0$  throughout, since this is the relevant case to compare against the quantum theory of a real scalar field.

In a divergence type theory, the degrees of freedom  $X^A$  of the theory are used to construct currents  $T_A^\mu$ , which are assumed to be *ultralocal* functions of the  $X^A$  (that is, they depend on the  $X^A$  at each point, but not on their derivatives). The equations of motion take the form of conservation laws for the currents  $T_{A;\mu}^\mu = I_A$ . For simplicity, let us assume linear production terms,  $I_A = -V_{AB}X^B$  with a nonnegative matrix  $V_{AB}$  (this will be the relevant case below).

Suppose we consider a linear departure  $\delta X$  from some solution to the equations of motion, say  $X = 0$ . To make it even simpler, suppose that the Cauchy data are homogeneous in space, so that  $\delta X$  depends only on time. Then the equations of motion take the form  $M_{AB}\delta\dot{X}^B = -V_{AB}\delta X^B$ , where  $M_{AB} = \partial T_A^0 / \partial X^B$ . If the solution we are starting from is stable, then  $\delta X$  must regress, and since the matrix  $V$  is nonnegative, for this to be true the matrix  $M$  must be positive definite. In a covariant theory, moreover, this must be true for any choice of time variable, and so we conclude that the matrix  $-t_\mu \partial T_A^\mu / \partial X^B$  must be positive definite for every future-oriented, time-like vector  $t^\mu$ , or, equivalently, the vector

$$\frac{\partial T_A^\mu}{\partial X^B} \delta X^A \delta X^B \quad (12.34)$$

must be time-like and future oriented for any choice of the  $\delta X$  displacements. This is our simple stability criterion.

The Landau–Lifshitz prescription does not directly fit in this scheme, because under the obvious choice  $X^A \rightarrow \beta^\mu$ ,  $T_A^\mu \rightarrow T^{\mu\nu}$ , the energy–momentum tensor is not ultralocal. We must first extend the number of degrees of freedom, so that we can write the theory as first order throughout.

Concretely we introduce a new traceless, symmetric tensor  $\zeta^{\mu\nu}$  ( $\zeta_\mu^\mu = 0$ ) and write (we shall omit indices, for simplicity) the viscous stress tensor as  $\tau = C\zeta$ .



In time, we want this to be equivalent to the constitutive relations (12.30), which we write as  $\tau = -B\nabla\beta$ . To this end, we introduce a new current  $A^{\mu\nu\rho} = (T^2/2)[\Delta^{\mu\nu}\beta^\rho + \Delta^{\mu\rho}\beta^\nu]$ . The divergence of this current may be expressed in terms of first derivatives of the velocity,  $A_{,\rho}^{\mu\nu\rho} = D\nabla\beta$ . Then, by imposing the conservation law  $\nabla A = -(DB^{-1}C)\zeta$ , we see that indeed the new form  $\tau = C\zeta$  of the viscous stress is equivalent to the old one.

So now we have a theory of fields  $\beta^\mu$  and  $\zeta^{\mu\nu}$ , and ultralocal currents  $T^{\mu\nu}$  and  $A^{\mu\nu\rho}$ , and are ready to apply the stability criterion. But then we realize that, while  $T^{\mu\nu}$  depends on all fields,  $A^{\mu\nu\rho}$  depends on  $\beta^\mu$  *only*. The vector in equation (12.34) cannot possibly be time-like for *every* displacement, since a whole diagonal block is missing from  $\partial T_A^\mu/\partial X^B$ . So the Landau–Lifshitz theory fails the stability criterion.

The failure of the Landau–Lifshitz approach to depict real fluids may be attributed to two unwarranted assumptions, namely, that the real fluid could be described with the same set of variables and with the same entropy current as its perfect counterpart. As a matter of fact, all that equilibrium thermodynamics suggests is that, whatever extra variables are brought in to describe the nonequilibrium state, they must vanish in equilibrium, and the entropy current must match its equilibrium value up to first order in the deviations from equilibrium. In other words, when we write the entropy production as  $S_{;\mu}^\mu = -\beta_{\nu;\mu}\tau^{\mu\nu} - \alpha_{,\mu}j^\mu$  we are neglecting second-order deviations from equilibrium. But under the Landau–Lifshitz constitutive relations, the two terms we are retaining are second order themselves. The inconsistency of keeping only *some* second-order terms is the root cause of our problems.

The fact remains that the Navier–Stokes theory is highly successful phenomenologically. The answer to this riddle seems to be that the would-be runaway perturbations of the Landau–Lifshitz theory are in reality high-frequency oscillations around the Navier–Stokes solutions. These oscillations cancel out if evolution is averaged over macroscopic time-scales, and therefore they do not appear in actual observations [NaOrRe94, KrNaOrRe97]. With this understanding, we shall carry on with the Landau–Lifshitz theory [Ger01].

## 12.2 Quantum fields in the hydrodynamic limit

### 12.2.1 Quantum hydrodynamic models

Since thermodynamics alone cannot provide further information on the transport functions, to proceed, we must place the above discussion in the context of a more fundamental description of the field, namely, the quantum kinetic field theory based on the Kadanoff–Baym equations. Let us begin with analyzing the equilibrium states.

Since we shall only discuss the theory of a real scalar field, we may also set  $\alpha = 0$  from scratch. This reflects the fact that a real scalar field is its own antiparticle. Thus our problem is to connect the hydrostatic equilibrium states,

described by an ideal energy–momentum tensor  $T_0^{\mu\nu} = \rho u^\mu u^\nu + p_0 \Delta^{\mu\nu}$ , with the equilibrium kinetic theory states, described by the Bose–Einstein distribution  $f_0 = [\exp |\beta p| - 1]^{-1}$ .

To begin with, let us identify the energy–momentum tensor with the *expectation value* of the corresponding Heisenberg operator. This is derived from the CTPEA  $\Gamma$ . The arguments of  $\Gamma$  are field configurations on a closed time path and in general we will have different metrics  $g_{\mu\nu}^{(1)}$  and  $g_{\mu\nu}^{(2)}$  in the forward and backward branches, respectively. The energy–momentum tensor is defined by the formula (valid for a general state)

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}^{(1)}} \tag{12.35}$$

where only the derivative with respect to the metric in the first time branch is taken. After the derivative is taken we identify  $g_{\mu\nu}^{(1)}$  and  $g_{\mu\nu}^{(2)}$  with the physical metric  $g_{\mu\nu}$ . The effective action itself is given by

$$\Gamma[G] = -\frac{i\hbar}{2} \text{Tr} \ln G + \frac{1}{2} S_{,AB} G^{AB} + \Gamma_2[G], \tag{12.36}$$

where the functional  $\Gamma_2$  is  $-i\hbar$  times the sum of all two-particle-irreducible diagrams with lines given by  $G$  and vertices given by the quartic interaction. The first term  $\text{Tr} \ln G$  does not depend on the metric. Written in full, the second term reads

$$\frac{1}{2} \int d^4x \left\{ \sqrt{-g^{(1)}} (\partial_x^2 - m_b^2) G^{11}(x, x') \Big|_{x'=x} - (1 \rightarrow 2) \right\} \tag{12.37}$$

As usual

$$\frac{\delta \sqrt{-g}}{\delta g_{\mu\nu}} = \frac{1}{2} \sqrt{-g} g^{\mu\nu}; \quad \frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma}} = -g^{\mu\rho} g^{\nu\sigma} \tag{12.38}$$

and so the contribution from this term to  $T^{\mu\nu}$  is

$$\left[ -\partial^\mu \partial^\nu + \frac{1}{2} \eta^{\mu\nu} (\partial_x^2 - m_b^2) \right] G^{11}(x, x') \Big|_{x'=x} \tag{12.39}$$

In the third term, the metric appears through the  $\sqrt{-g}$  factors multiplying the coupling constants. The contribution to  $T^{\mu\nu}$  takes the form

$$-\frac{\lambda_b}{8} \eta^{\mu\nu} [G^{11}(x, x)]^2 - \tilde{\Lambda}_b \eta^{\mu\nu} \tag{12.40}$$

where  $\tilde{\Lambda}_b$  contains all the higher order contributions. To the accuracy desired,  $\tilde{\Lambda}_b$  is position independent, and we shall not analyze it further. Adding the two nontrivial contributions we get

$$T^{\mu\nu} = - \left[ \partial^\mu \partial^\nu - \frac{1}{2} \eta^{\mu\nu} \partial_x^2 \right] G^{11}(x, x') \Big|_{x'=x} - \frac{\eta^{\mu\nu}}{2} \left[ m_b^2 + \frac{\lambda_b}{4} G^{11}(x, x) \right] G^{11}(x, x) - \tilde{\Lambda}_b \eta^{\mu\nu} \tag{12.41}$$

We assume the quasi-particle approximation for  $G^{11}$

$$G^{11} = \frac{(-i\hbar)}{p^2 + M^2 - i\varepsilon} + 2\pi\hbar\delta(p^2 + M^2) f(X, p) \tag{12.42}$$

$f$  is the solution to a kinetic equation of the form

$$p^\mu \frac{\partial f}{\partial X^\mu} - \frac{1}{2} M^2 \frac{\partial f}{\partial p_\mu} = I_{\text{col}}(X, p) \tag{12.43}$$

We only assume energy-momentum conservation

$$\int Dp p^\mu I_{\text{col}}(X, p) = 0; \quad Dp = \frac{d^4 p}{(2\pi)^3} \theta(p^0) \delta(\Omega_0) \tag{12.44}$$

where  $\Omega_0 = p^2 + M^2$ . Observe that equation (12.42) implies that  $f$  must be real and even in  $p$ . In turn, the effective mass  $M^2$  is the solution of the gap equation (11.65) given in Chapter 11.

To write the energy-momentum tensor in terms of the distribution function, observe that  $\partial_x \rightarrow ip + \frac{1}{2}\partial_X$ . We must neglect second derivative terms, and observe that terms involving  $p\partial_X$  eventually vanish because  $G^{11}(X, p)$  is even in  $p$ . So

$$\begin{aligned} T^{\mu\nu}(X) &= \int \frac{d^4 p}{(2\pi)^4} \left[ p^\mu p^\nu - \frac{1}{2} \eta^{\mu\nu} p^2 \right] G^{11}(X, p) \\ &\quad - \frac{1}{2} \eta^{\mu\nu} \left[ m_b^2 + \frac{\lambda_b}{4} G^{11} \right] G^{11} - \tilde{\Lambda}_b \eta^{\mu\nu} \end{aligned} \tag{12.45}$$

Let us isolate

$$T_V^{\mu\nu} = -i\hbar \int \frac{d^d p}{(2\pi)^d} \frac{[p^\mu p^\nu - \frac{1}{2} \eta^{\mu\nu} p^2]}{p^2 + M^2 - i\varepsilon} \equiv -\eta^{\mu\nu} \Lambda_1 \tag{12.46}$$

where

$$\Lambda_1 = \left( \frac{(d-2)}{2d} \right) \eta^{\mu\nu} \mu^\varepsilon \hbar \int \frac{d^d p}{(2\pi)^d} \frac{(-i)p^2}{p^2 + M^2 - i\varepsilon} = \frac{M^4 \hbar}{32\pi^2} \left[ z - \frac{1}{4} - \frac{1}{2} \ln \left( \frac{M^2}{4\pi\mu^2} \right) \right] \tag{12.47}$$

and  $z$  was defined in (11.69) of Chapter 11. Also write

$$\hbar \int \frac{d^4 p}{(2\pi)^4} p^\mu p^\nu 2\pi\delta(\Omega_0) f(X, p) \equiv T_T^{\mu\nu} \tag{12.48}$$

and observe that

$$\int \frac{d^4 p}{(2\pi)^4} \left( -\frac{1}{2} p^2 \right) 2\pi\delta(\Omega_0) f(X, p) = \frac{1}{2} M^2 M_T^2 \tag{12.49}$$

so we get  $T^{\mu\nu} = T_T^{\mu\nu} - \Lambda_b \eta^{\mu\nu}$ , with

$$\Lambda_b = \Lambda_1 + \frac{1}{2} \left[ m_b^2 + \frac{\lambda_b}{4} G^{11} \right] G^{11} + \tilde{\Lambda}_b - \frac{\hbar}{2} M^2 M_T^2 \tag{12.50}$$

If we regard  $G^{11}(x, x)$  and  $M_T^2$  as functions of  $M^2$  defined by the gap equation, then  $\Lambda_b$  is a function of  $M^2$  too, meaning that there is no explicit state dependence other than through  $M^2$ .

Our first concern is to eliminate the formal divergences from these expressions, following the procedure outlined in Chapter 11. With respect to the renormalization of the cosmological constant  $\Lambda_b$  term, we observe that any  $M^2$  independent term may be absorbed in the gravitational action (even if it is formally infinite). So we only need to show that  $d\Lambda_b/dM^2$  is finite. Now, the gap equation yields  $dG^{11}/dM^2 = 2/\lambda_b$ , and

$$\frac{d\Lambda_b}{dM^2} = \frac{-\hbar}{2} M_T^2 \quad (12.51)$$

Consistency requires that we actually neglect the  $O(\lambda^2)$  terms in  $\tilde{\Lambda}_b$ , or at least that we consider them as a true (temperature independent) constant. Equation (12.51) then implies that energy–momentum conservation follows from the transport equation. Henceforth we shall assume that any constant contribution has been subtracted, and drop the  $b$  subscript.

To summarize, what we have done in this section is to introduce a class of theories which, although they receive some support as the long-wavelength limit of an underlying quantum field theory, may be – indeed, should be – studied as *bona fide* models of physical systems. These theories describe fluids with energy–momentum tensor  $T^{\mu\nu} = T_T^{\mu\nu} - \Lambda g^{\mu\nu}$ , where the first term is defined from a one-particle distribution function  $f$  in equation (12.48), and  $\Lambda$  is the solution to equation (12.51), with  $M_T^2$  also defined in terms of  $f$ . The construction of the model is completed by stipulating the collision term in the Boltzmann equation for  $f$  and the gap equation, given by

$$M^2 - \varphi(M^2, \mu^2) = \frac{\hbar\lambda}{2} M_T^2 \quad (12.52)$$

The two functions  $I_{\text{col}}$  and  $\varphi$  allow us to incorporate some higher order effects into the same general scheme.

Charge conservation may be introduced as in classical hydrodynamics through the corresponding currents, provided the collision term has the required symmetry. The entropy current was defined in Chapter 2, equation (2.98). From this, entropy production is given by

$$S_{;\mu}^{\mu} = 2 \int Dp \left[ \ln \frac{(1+f)}{f} \right] I_{\text{col}} \quad (12.53)$$

where  $Dp$  was introduced in (12.44). The positivity of this integral is an expression of the H-theorem.

### 12.2.2 Thermal equilibrium states

Our next task is to investigate the equation of state for an equilibrium state described by a Bose–Einstein distribution function  $f_0$ . The energy–momentum

tensor takes the perfect fluid form. The thermal component  $T_T^{\mu\nu}$  admits a similar decomposition

$$T_{0T}^{\mu\nu} = 2\hbar \int Dp p^\mu p^\nu f_0(X, p) = \rho_T u^\mu u^\nu + p_T \Delta^{\mu\nu} \tag{12.54}$$

where

$$\rho_T = 2\hbar \int Dp (up)^2 f_0(X, p) \tag{12.55}$$

Since  $\rho_T$  and  $M_T^2$  are scalars, we may compute them in the rest frame

$$\rho_T = \frac{\hbar}{2\pi^2} \int_M^\infty d\omega \omega^2 f_0(\omega) \sqrt{\omega^2 - M^2} \tag{12.56}$$

$$M_T^2 = \frac{1}{2\pi^2} \int_M^\infty d\omega f_0(\omega) \sqrt{\omega^2 - M^2} \tag{12.57}$$

For the thermal pressure, we find  $3p_T - \rho_T = -\hbar M^2 M_T^2$ , so

$$p_T = \frac{1}{3} (\rho_T - \hbar M^2 M_T^2) = \frac{\hbar}{6\pi^2} \int_M^\infty d\omega [\omega^2 - M^2]^{3/2} f_0 \tag{12.58}$$

The total energy density and pressure are then  $\rho = \rho_T + \Lambda$  and  $p = p_T - \Lambda$ .

The equilibrium entropy current takes the form

$$S_0^\mu = p\beta^\mu - T_0^{\mu\nu} \beta_\nu = (\rho + p) \beta^\mu = (\rho_T + p_T) \beta^\mu \tag{12.59}$$

On the other hand, equation (2.98) yields  $S_0^\mu = \Phi_{0T}^\mu - T_0^{\mu\nu} \beta_\nu$ , where

$$\Phi_{0T}^\mu = -2 \int Dp p^\mu \ln [1 - e^{-|\hbar\beta_\mu p^\mu|}] \tag{12.60}$$

This form of the thermodynamic potential recalls another equivalent expression for the thermal pressure

$$\frac{p_T}{T} = \frac{-1}{2\pi^2} \int_M^\infty d\omega \omega \sqrt{\omega^2 - M^2} \ln [1 - e^{-\hbar\beta\omega}] \tag{12.61}$$

Together (12.51) and (12.61) imply the thermodynamic relationship  $dp/dT = (p + \rho)/T$ . (Here and hereafter, we shall use  $d/dT$  to denote a total temperature derivative, which accounts for the explicit temperature dependence through  $f_0$  as well as the implicit dependence through  $M^2$ . We shall use  $\partial/\partial T$  when we mean only the former.) Indeed, equation (12.61) implies

$$T \frac{dp_T}{dT} = p_T + \rho_T - \frac{\hbar M_T^2}{2} T \frac{dM^2}{dT} \tag{12.62}$$

but  $p_T + \rho_T = \rho + p$ , and

$$T \frac{dp}{dT} = T \frac{dp_T}{dT} - T \frac{d\Lambda}{dT} = T \frac{dp_T}{dT} + \frac{\hbar M_T^2}{2} T \frac{dM^2}{dT} \tag{12.63}$$

Observe that for  $T^2 \gg M^2$  we recover the Stefan–Boltzmann law and  $p = \rho/3$ , as expected (in this regime, the cosmological constant  $\sim M^2 T^2$  is negligible compared to  $\rho \sim T^4$ ). This concludes our study of the equilibrium states.

**12.2.3 Local equilibria**

We now extend this analysis to local equilibrium states. The idea is to generate a solution to the transport equation as a formal expansion “in derivatives of”  $\beta^\mu$ , replace this solution in the definition of  $T^{\mu\nu}$ , and to compare the result to the Landau–Lifshitz energy–momentum tensor for a real fluid. The first point to realize is that it is not possible to assume arbitrary values for the derivatives of the temperature 4-vector at a given point; they must satisfy constraints derived from the symmetries of the transport equation. These constraints may be used to eliminate the time derivatives of the inverse temperature 4-vector from the equations.

Let us recall the transport equation (12.43). Write  $f = f_0 + f_1$ , where  $f_1$  is “first order,” and observe that, since the collision integral involves no derivatives,  $I_{\text{col}} [f_0] = 0$ . Therefore, to first order, we may write  $I_{\text{col}} [f] = \tilde{K} [f_1]$ , where the operator  $\tilde{K}$  is linear. To analyze the left-hand side, let us assume  $p^0 > 0$ , so that

$$f_0 = \frac{1}{e^{-\hbar\beta_\mu p^\mu} - 1} \tag{12.64}$$

Then

$$\hbar f_0 (1 + f_0) \left\{ p^\mu p^\nu \beta_{\mu;\nu} - \frac{1}{2} \beta^\mu M_{,\mu}^2 \right\} = \tilde{K} [f_1] \tag{12.65}$$

Our goal is to solve for  $f_1$ . However, we must realize there are integrability conditions derivable from (12.44), so a solution exists only when

$$\hbar \int Dp f_0 (1 + f_0) \left\{ p^\kappa p^\mu p^\nu \beta_{\mu;\nu} - \frac{1}{2} p^\kappa \beta^\mu M_{,\mu}^2 \right\} = 0 \tag{12.66}$$

The idea is to use the integrability conditions to eliminate time derivatives from the linearized transport equations, thereby obtaining an equation relating  $f_1$  to spatial derivatives of the inverse temperature tensor only.

Since the integrability conditions are clearly covariant, we may write them down in any frame, in particular, the rest frame. In general, we have  $\beta^\mu = (1/T\sqrt{1-v^2})(1, \mathbf{v})$ . In the rest frame,  $\mathbf{v} = 0$ , the above equations result in

$$\langle \omega^3 \rangle \frac{\dot{T}}{T^2} + \frac{1}{3} \langle \omega (\omega^2 - M^2) \rangle \frac{\nabla \mathbf{v}}{T} - \frac{1}{2} \langle \omega \rangle T M_{,T}^2 \frac{\dot{T}}{T^2} = 0 \quad (\kappa = 0) \tag{12.67}$$

$$\frac{1}{3T} \langle \omega (\omega^2 - M^2) \rangle \left[ (\mathbf{v}) \cdot + \frac{\nabla T}{T} \right] = 0 \quad (\kappa = 1, 2, 3) \tag{12.68}$$

In these expressions, we have introduced the notation

$$\begin{aligned} \langle X \rangle &= \int Dp f_0 (1 + f_0) X \\ &= \frac{1}{2\pi^2} \int_M^\infty d\omega \sqrt{\omega^2 - M^2} f_0 (1 + f_0) X \end{aligned} \tag{12.69}$$

To simplify the integrability conditions, recall that

$$\frac{d\rho}{dT} = \frac{d\rho_T}{dT} + \frac{d\Lambda}{dT} = \frac{\partial\rho_T}{\partial T} + M_{,T}^2 \left[ \frac{\partial\rho_T}{\partial M^2} - \frac{\hbar}{2} M_T^2 \right] \tag{12.70}$$

$$\frac{\partial\rho_T}{\partial T} = \hbar^2 \frac{\langle\omega^3\rangle}{T^2} \tag{12.71}$$

$$\begin{aligned} \frac{\partial\rho_T}{\partial M^2} &= \left(\frac{-\hbar}{2}\right) \frac{1}{2\pi^2} \int_M^\infty d\omega \frac{\omega^2}{\sqrt{\omega^2 - M^2}} f_0 \\ &= \left(\frac{-\hbar}{2}\right) \frac{1}{2\pi^2} \int_M^\infty d\omega \left(\frac{d}{d\omega} \sqrt{\omega^2 - M^2}\right) \omega f_0 = \left(\frac{\hbar}{2}\right) \left[M_T^2 - \frac{\hbar\langle\omega\rangle}{T}\right] \end{aligned} \tag{12.72}$$

so

$$\langle\omega^3\rangle - \frac{1}{2} \langle\omega\rangle T M_{,T}^2 = \frac{T^2}{\hbar^2} \frac{d\rho}{dT} \tag{12.73}$$

On the other hand,

$$\frac{dp_0}{dT} = \frac{dp_{0T}}{dT} - \frac{d\Lambda}{dT} = \frac{\partial p_{0T}}{\partial T} + M_{,T}^2 \left[ \frac{\partial p_{0T}}{\partial M^2} + \frac{\hbar}{2} M_T^2 \right] \tag{12.74}$$

$$\frac{\partial p_{0T}}{\partial M^2} = -\frac{\hbar}{2} M_T^2 \tag{12.75}$$

$$\frac{\partial p_{0T}}{\partial T} = \frac{\hbar^2}{3T^2} \langle\omega(\omega^2 - M^2)\rangle \tag{12.76}$$

Also, recall that

$$\frac{dp_0}{dT} = \frac{p_0 + \rho}{T} \tag{12.77}$$

so finally

$$\langle\omega^3\rangle - M^2 \langle\omega\rangle = \frac{3T}{\hbar^2} (p_0 + \rho) \tag{12.78}$$

The integrability conditions are simply the conservation equations for the ideal energy–momentum tensor built out of  $f_0$ . These equations determine the dynamics of local equilibrium states.

We may regard (12.73) and (12.78) as a system of equations for the two unknowns  $\langle\omega\rangle$  and  $\langle\omega^3\rangle$ , which yields

$$\begin{aligned} \langle\omega^3\rangle &= \frac{T^2}{\hbar^2} \frac{d\rho}{dT} \frac{[M^2 - \frac{3}{2} T M_{,T}^2 c_s^2]}{[M^2 - \frac{1}{2} T M_{,T}^2]} \\ \langle\omega\rangle &= \frac{T^2}{\hbar^2} \frac{d\rho}{dT} \frac{[1 - 3c_s^2]}{[M^2 - \frac{1}{2} T M_{,T}^2]} \end{aligned} \tag{12.79}$$

where  $c_s^2$  is the speed of sound (12.19).

### 12.3 Transport functions in the hydrodynamic limit

While in equilibrium the energy–momentum tensor for the quantum fields takes the ideal fluid form, for mere local equilibrium this will not be so. In general, we may seek a solution of the transport equation as a formal series “in derivatives of the hydrodynamic variables.” The first order in this series is given by the solution to the linearized equation (12.65). When the corrected distribution function is employed to compute the energy–momentum tensor, we get non-ideal terms which are, by construction, linear in gradients. By matching these terms to the Landau–Lifshitz template, we may read off the transport functions, thereby “deriving” the constitutive relations for the quantum real fluid. This is, of course, the traditional way of deriving the transport functions from kinetic theory [ChaCow39, GrLeWe80, Lib98, Hei94]; what is new is the unconventional form of the collision integral. Our treatment here follows [CaHuRa00].

It is amusing to observe that, while in deriving the Kadanoff–Baym equations we had to justify at every step the neglect of higher gradient terms (and were admittedly not always quite convincing), the transport terms are lifted from terms in the energy–momentum tensor which are linear in gradients *by definition*. So many approximations which may be controversial at the quantitative level, are fully legitimate in the context of the derivation of the constitutive relations. We shall not discuss the further issue of whether a first-order theory is a good description of the quantum field in the hydrodynamic limit.

Let us begin by eliminating time derivatives from the left-hand side of the linearized transport equation (12.65). In the rest frame

$$\beta_{\mu,\nu} = \frac{1}{T} \begin{pmatrix} \frac{\dot{T}}{T} & \frac{\nabla T}{T} \\ \dot{\mathbf{v}} & v_{i,j} \end{pmatrix} \quad (12.80)$$

Using the integrability conditions we get

$$\beta_{\mu,\nu} = \frac{1}{T} \begin{pmatrix} -c_s^2 \nabla \mathbf{v} & \frac{\nabla T}{T} \\ -\frac{\nabla T}{T} & v_{i,j} \end{pmatrix} \quad (12.81)$$

Obviously only the symmetric part contributes to the linearized transport equation. Also

$$\frac{1}{2} M_{,\mu}^2 \beta^\mu = \frac{1}{2} M_{,T}^2 \frac{\dot{T}}{T} = \frac{-c_s^2}{2} M_{,T}^2 \nabla \mathbf{v} \quad (12.82)$$

Splitting  $v_{(i,j)}$  into the diagonal and the traceless parts, and reverting to the covariant form, we get the left-hand side of (12.65) as

$$\hbar f_0 (1 + f_0) \left[ \frac{1}{T} p_\mu p_\nu H^{\mu\nu} - \frac{1}{T} \left\{ (p \cdot u)^2 \left[ c_s^2 - \frac{1}{3} \right] + \frac{M^2}{3} - \frac{c_s^2}{2} T M_{,T}^2 \right\} u_\lambda^\lambda \right] \quad (12.83)$$

where  $H^{\mu\nu}$  was defined in (12.33).



12.3.1 The collision term

On the right-hand side of the transport equation the collision integral has the structure of the balance between a gain and a loss term [FiGaJe06]. Let us consider a collision process whereby  $n$  reactant particles are transformed into  $m$  product ones. We get a gain when one of the product particles has the moment  $p_1$ , say, where we wish to evaluate  $I_{\text{col}}$ . Let the other product particles have momenta  $p_2, \dots, p_m$ , and the reactants have momenta  $q_1, \dots, q_n$ . Then the gain term is

$$\hbar \sigma_{n,m}^2 [\mathbf{q}, \mathbf{p}] \delta \left( \sum_{j=1}^m p_j - \sum_{i=1}^n q_i \right) \prod_{j=1}^m (1 + f(p_j)) \prod_{i=1}^n f(q_i) \tag{12.84}$$

where we have made explicit use of the energy-momentum conservation and placed properly the Bose enhancement factor. The corresponding loss term is

$$\hbar \sigma_{n,m}^2 [\mathbf{q}, \mathbf{p}] \delta \left( \sum_{j=1}^m p_j - \sum_{i=1}^n q_i \right) \prod_{i=1}^n (1 + f(q_i)) \prod_{j=1}^m f(p_j) \tag{12.85}$$

We make the micro-reversibility assumption that the cross-section  $\sigma^2$  is the same for both processes. The collision integral is

$$I_{\text{col}} [p_1] = \hbar \sum_{n,m} \int \prod_{i=1}^n Dq_i \prod_{j=2}^m Dp_j \sigma_{n,m}^2 [\mathbf{q}, \mathbf{p}] \delta \left( \sum_{j=1}^m p_j - \sum_{i=1}^n q_i \right) \times \left\{ \prod_{j=1}^m (1 + f(p_j)) \prod_{i=1}^n f(q_i) - \prod_{i=1}^n (1 + f(q_i)) \prod_{j=1}^m f(p_j) \right\} \tag{12.86}$$

In equilibrium, each  $(n, m)$  term vanishes independently. We assume the cross-sections are invariant under permutations of the reactants and products, separately. For reasons made clear below, we are interested in collision integrals which *do not* conserve particle number, meaning

$$\int Dp_1 I_{\text{col}} [p_1] \neq 0 \tag{12.87}$$

Explicitly, this says

$$\sum_{n>m} \int \prod_{i=1}^n Dq_i \prod_{j=1}^m Dp_j [\sigma_{n,m}^2 [\mathbf{q}, \mathbf{p}] - \sigma_{m,n}^2 [\mathbf{p}, \mathbf{q}]] \delta \left( \sum_{j=1}^m p_j - \sum_{i=1}^n q_i \right) \left\{ \prod_{j=1}^m (1 + f(p_j)) \prod_{i=1}^n f(q_i) - \prod_{i=1}^n (1 + f(q_i)) \prod_{j=1}^m f(p_j) \right\} \neq 0 \tag{12.88}$$

so in general we request  $\sigma_{n,m}^2 [\mathbf{q}, \mathbf{p}] \neq \sigma_{m,n}^2 [\mathbf{p}, \mathbf{q}]$  if  $n \neq m$ . In an explicit perturbative calculation, we find that, to order  $\lambda^2$ , only  $\sigma_{2,2}^2$  is not zero, yielding the

usual Boltzmann collision integral. To order  $\lambda^4$ , both  $\sigma_{2,4}^2$  and  $\sigma_{4,2}^2$  are activated, and the inequality may be explicitly verified (in fact,  $\sigma_{2,4}^2 \sim 2\sigma_{4,2}^2$ ).

**12.3.2 The linearized transport equation**

Writing  $f = f_0 + f_1$ ,

$$f_1 = f_0 (1 + f_0) \chi; \quad \tilde{K} [f_1] = \hbar f_{0p} (1 + f_{0p}) K [\chi] \tag{12.89}$$

$$D_\beta p = Dp f_{0p} (1 + f_{0p}) \tag{12.90}$$

we obtain

$$K [\chi] = - \sum_{n,m} \int \prod_{i=1}^n D_\beta q_i \prod_{j=2}^m D_\beta p_j \sigma_{n,m}^2 [\mathbf{q}, \mathbf{p}] \delta \left( \sum_{j=1}^m p_j - \sum_{i=1}^n q_i \right) \times \frac{\{\chi(p_1) + (m-1)\chi(p_2) - n\chi(q_1)\}}{\prod_{i=1}^n (1 + f_0(q_i)) \prod_{j=1}^m f_0(p_j)} \tag{12.91}$$

Thus far we have reduced our problem to that of solving the linear integral equation

$$K [\chi] = \frac{1}{T} p_\mu p_\nu H^{\mu\nu} - \frac{1}{T} \left\{ (p \cdot u)^2 \left[ c_s^2 - \frac{1}{3} \right] + \frac{M^2}{3} - \frac{c_s^2}{2} T M_{,T}^2 \right\} u_{,\lambda}^\lambda \equiv R \tag{12.92}$$

Let us write  $K = K_B + K_1$ , where the former is the lowest order (Boltzmann’s) collision operator.  $K_B$  is a Hermitian operator in the space of functions defined on the positive energy mass shell with inner product

$$\langle \varsigma | \chi \rangle = \int D_\beta p \varsigma^*(p) \chi(p); \quad \langle \chi \rangle \equiv \langle 1 | \chi \rangle \tag{12.93}$$

and  $\langle \chi \rangle$  agrees with the expectation value introduced earlier (12.69).

$K_1$  will not be symmetric, in general. There is a basis of eigenvectors  $|\chi_n\rangle$  of  $K_B$ , with eigenvalues  $a_n$ . Four eigenvectors correspond to the functions  $p^\mu$ , with eigenvalue zero. Because of momentum conservation, these are also eigenvectors of the full collision operator.  $K_B$  admits a fifth null eigenvector, namely a constant: this follows from particle number conservation in Boltzmann’s theory. Let us call  $|\chi_0\rangle$  this (normalized) eigenvector (in conventional notation,  $|\chi_0\rangle = \langle 1 \rangle^{-1/2}$ ). We observe that the inhomogeneous term  $R$  in the linearized transport equation (12.92) is orthogonal to the null eigenvectors  $p^\mu$  (not to  $|\chi_0\rangle$ ). We shall ignore the former, that is, we shall restrict our considerations to the orthogonal space to the  $p^\mu$ ’s.

Writing the unknown  $\chi$  in Dirac’s notation as  $|\chi\rangle = \sum |\chi_n\rangle \langle \chi_n | \chi \rangle$ , we get

$$\langle \chi_0 | K_1 | \chi_0 \rangle \langle \chi_0 | \chi \rangle + \sum_{n \geq 1} \langle \chi_0 | K_1 | \chi_n \rangle \langle \chi_n | \chi \rangle = \langle \chi_0 | R \rangle \tag{12.94}$$

$$\sum_m (a_n \delta_{nm} + \langle \chi_n | K_1 | \chi_m \rangle) \langle \chi_m | \chi \rangle = \langle \chi_n | R \rangle \tag{12.95}$$

if  $n \geq 1$ . From the second equation, we see that  $\langle \chi_m | \chi \rangle \sim O(\lambda^{-2})$  for  $m \neq 0$ ; instead, the first equation suggests that  $\langle \chi_0 | \chi \rangle$  is much larger ( $O(\lambda^{-4})$ ). We are therefore led to the approximation

$$\langle \chi_0 | \chi \rangle = \frac{\langle \chi_0 | R \rangle}{\langle \chi_0 | K_1 | \chi_0 \rangle}; \quad \langle \chi_n | \chi \rangle = \frac{1}{a_n} \langle \chi_n | R \rangle \quad (12.96)$$

As a matter of fact,  $R$  may be split into a term  $R_s$  proportional to the shear tensor  $H^{\mu\nu}$  and a term  $R_b$  proportional to  $u^\lambda$ , and therefore so will the solution. Actually,  $\langle \chi_0 | R_s \rangle = 0$ , so solving the “shear” problem involves only the Boltzmann collision operator. The eigenvalues of this operator are of order  $T/\tau_{\text{rel}}$ , where  $\tau_{\text{rel}}$  is the mean free time, and so the shear linear correction to the distribution function is  $\chi_s \sim (-\tau_{\text{rel}}/T^2) p_\mu p_\nu H^{\mu\nu}$ .

The mean free time may be identified by writing the Boltzmann equation in the collision time approximation, where  $f \sim f_{\text{eq}} + \delta f$  and  $\delta \dot{f} \sim -\delta f/\tau_{\text{rel}}$ . On power counting and dimensional arguments, we find  $\tau_{\text{rel}} \sim 1/\lambda^2 T$ .

On the other hand,  $\langle \chi_0 | R_b \rangle$  is not zero. It follows that the component of the “bulk” solution in the direction of  $|\chi_0\rangle$  is much larger than in any other direction, and we may approximate

$$|\chi_b\rangle = \frac{|\chi_0\rangle \langle \chi_0 | R_b \rangle}{\langle \chi_0 | K_1 | \chi_0 \rangle} = \frac{\langle R_b \rangle}{\langle K_1 [1] \rangle} = \text{constant} \quad (12.97)$$

Expanding in the rest frame

$$\chi_b \equiv c_0 = \frac{-1}{T \langle K_1 [1] \rangle} \left\{ \langle \omega^2 \rangle \left[ c_s^2 - \frac{1}{3} \right] + \langle 1 \rangle \left[ \frac{M^2}{3} - \frac{c_s^2}{2} T M_{,T}^2 \right] \right\} u^\lambda \quad (12.98)$$

### 12.3.3 The temperature shift and the bulk stress

As we have seen, the correction to the distribution function has two components. The one associated with the  $H^{\mu\nu}$  tensor contributes to the shear stress, but it does not induce a change in the energy density. Therefore it is compatible with the Landau–Lifshitz matching conditions. The constant shift of  $\chi$  by  $c_0$ , on the other hand, affects in principle both the energy density and the thermal mass  $M_T$ . So, to enforce the Landau–Lifshitz conditions, it must be partially compensated by a temperature shift. Concretely, if we call  $T$  the temperature of the fiducial equilibrium state, such that  $\rho(T)$  is equal to the energy density in the nonequilibrium state, then the temperature appearing in the local equilibrium distribution function  $f_0$  must be  $T_0 = T + \delta T$ . The effect of this temperature shift is the same as adding another term proportional to  $\omega$  in the first-order correction  $\chi$ .

The distribution function and temperature shifts in turn produce a shift  $\delta M^2$  in the physical mass, which likewise does not affect the transport equation. However, both  $\delta T$  and  $\delta M^2$  enter in the consideration of the bulk stress. Observe that there is no shift in the four velocity  $u^\mu$ .

The three displacements  $c_0$  (12.98),  $\delta T$  and  $\delta M^2$  are related by the constraints that the gap equation must hold, and the total energy density in the

nonequilibrium state must be the same as in the local equilibrium state. Writing the gap equation as in (12.52), the linearized equation then reads

$$\left[ 1 - \varphi' - \frac{\hbar\lambda}{2} \frac{\partial M_T^2}{\partial M^2} \right] \delta M^2 = \frac{\hbar\lambda}{2} \left[ \frac{\partial M_T^2}{\partial T} \delta T + c_0 \langle 1 \rangle \right] \tag{12.99}$$

In fact,  $\partial M_T^2 / \partial T = \hbar \langle \omega \rangle / T^2$ , so  $\delta M^2 = M_{,T}^2 \delta T + M_{,c}^2 c_0$ , where  $M_{,c}^2 = T^2 M_{,T}^2 \langle 1 \rangle / \hbar \langle \omega \rangle$ . Since the gap equation is enforced, we can look at the (cosmological) constant  $\Lambda$  as a function of  $M^2$ , so  $\delta \Lambda_f = -\hbar M_T^2 \delta M^2 / 2$ . Then

$$\delta \rho = \frac{d\rho}{dT} \delta T + \left[ \frac{\partial \rho_T}{\partial M^2} - \frac{\hbar}{2} M_T^2 \right] M_{,c}^2 c_0 + \hbar \langle \omega^2 \rangle c_0 \tag{12.100}$$

Actually

$$\frac{\partial \rho_T}{\partial M^2} = \frac{\hbar}{2} M_T^2 - \hbar^2 \frac{\langle \omega \rangle}{2T} \tag{12.101}$$

so

$$\delta \rho = \frac{d\rho}{dT} \delta T + \left[ \langle \omega^2 \rangle - \frac{\langle 1 \rangle}{2} T M_{,T}^2 \right] \hbar c_0 \tag{12.102}$$

And, since the total energy remains the same,

$$\frac{d\rho}{dT} \delta T = -\hbar c_0 \left[ \langle \omega^2 \rangle - \frac{\langle 1 \rangle}{2} T M_{,T}^2 \right] \tag{12.103}$$

Let us apply the same reasoning to the bulk stress, which results from both the deviation of the pressure from  $p(T)$  and the direct contribution from the new terms in the distribution function

$$\tau = c_s^2 \frac{d\rho}{dT} \delta T + \left[ \frac{\partial p_T}{\partial M^2} + \frac{\hbar}{2} M_T^2 \right] M_{,c}^2 c_0 + \frac{1}{3} [\langle \omega^2 \rangle - M^2 \langle 1 \rangle] \hbar c_0 \tag{12.104}$$

Now  $\partial p_T / \partial M^2 = -\hbar M_T^2 / 2$ , so

$$\tau = -\hbar c_0 \left\{ \left[ c_s^2 - \frac{1}{3} \right] \langle \omega^2 \rangle + \left[ \frac{M^2}{3} - \frac{c_s^2}{2} T M_{,T}^2 \right] \langle 1 \rangle \right\} \tag{12.105}$$

Using (12.98) and the expressions for  $\langle \omega \rangle$  and  $\langle \omega^3 \rangle$  from the last section, we get

$$\tau = -\frac{\hbar^5 u_\lambda^\lambda [M^2 - \frac{1}{2} T M_{,T}^2]^2 \{ \langle \omega^3 \rangle \langle 1 \rangle - \langle \omega^2 \rangle \langle \omega \rangle \}^2}{9T^5 \left( \frac{d\rho}{dT} \right)^2 | \langle K [1] \rangle |} \tag{12.106}$$

where we have used the fact that an explicit calculation shows that  $\langle K [1] \rangle < 0$  to lowest nontrivial order.

### 12.3.4 Shear stress and bulk viscosity

The shear stress can be read off directly from the new terms in  $T_T^{\mu\nu}$ . In the rest frame, we get  $\chi_s = (-\tau_{\text{rel}} / T^2) p_\mu p_\nu H^{\mu\nu}$

$$\tau^{ij} = \frac{-\hbar \tau_{\text{rel}}}{T^2} H^{kl} \langle p^i p^j p_k p_l \rangle \sim \frac{-\hbar \tau_{\text{rel}}}{T^2} H^{ij} \langle p^4 \rangle \tag{12.107}$$

from where we can read out the shear viscosity  $\eta$ . To estimate  $\eta$ , it is enough to keep only the leading (binary scattering) contributions, so  $\eta \sim \lambda^{-2}$ . On dimensional grounds, we recover the usual result,  $\eta \sim T^3/\lambda^2$ .

As expected, things are not so simple with the bulk viscosity. We can read it off from equation (12.106). However, in evaluating it we must consider that  $\langle 1 \rangle$  is logarithmically divergent in the massless limit, so we must correct the sheer dimensional estimate to  $\langle 1 \rangle \sim T^2 \ln(M/T)$ . As for the size of  $|\langle K [1] \rangle|$ , observe that the integral is dominated by the Rayleigh–Jeans tail, where  $f_0 \sim T/\omega \gg 1$ . Thus  $|\langle K [1] \rangle| \sim \hbar^3 \lambda^4 T^6 F(M^2)$ . Since the overall units are  $[\text{mass}]^4$ , it must be that  $|\langle K [1] \rangle| \sim T^6/M^2$ . For the remaining elements we may use the conventional estimates  $\langle \omega^3 \rangle \sim T^5$ ,  $\rho \sim \hbar T^4$ , and thus obtain

$$\zeta \sim \frac{M^2}{\lambda^4 T^3} \left[ M^2 - \frac{1}{2} T M_{,T}^2 \right]^2 \ln^2(M/T) \quad (12.108)$$

which is the folk result. In the limit in which the bare mass vanishes, or equivalently in the  $T \rightarrow \infty$  limit, we may write on dimensional grounds

$$M^2 - \frac{1}{2} T M_{,T}^2 \equiv \frac{1}{2} \mu M_{,\mu}^2 \sim \lambda M^2 \quad (12.109)$$

and since  $M^2 \sim \lambda T^2$  itself, equation (12.108) reduces to  $\zeta \sim \lambda T^3 \ln^2(\lambda)$ .

### 12.3.5 Transport functions for non-abelian plasmas

Although the calculation of transport coefficients in field theories follows the general strategy we have exemplified with a self-interacting scalar field, it is important to keep in mind the particularities of specific theories when aiming for a derivation of those coefficients good enough for a sensible comparison against experimental data.

In this sense, the most important scenario where an estimate of transport coefficients is of crucial relevance is the physics of relativistic heavy ion collisions (RHICs) [Ris98, BaRoWi06a, BaRoWi06b] (which we shall discuss in greater detail in Chapter 14), and correspondingly great effort has been devoted to the derivation of transport functions for hot non-abelian plasmas.

While we shall be content to refer the reader to the comprehensive set of papers by Arnold, Moore and Yaffe on this subject [ArMoYa00, ArMoYa03a, ArMoYa03b, ArDoMo06], we also wish to point out some aspects where the derivation of transport functions for non-abelian plasma differs from the equivalent study in scalar field theory.

First, there is the issue of momentum-dependent interactions and small denominator effects. Because of these, the actual weight of a given diagram may be very different from naive power-counting estimates. We have already encountered this phenomenon in Chapter 10, in our discussion of hard thermal loop resummation for a toy model scalar field.

As in the scalar field case, particle number-changing interactions play a central role in the derivation of the bulk viscosity coefficient [ArDoMo06]. However, the relevant processes are different. In particular, for a hot gluon field the most important contributions to bulk viscosity come from “2 to 1” processes, namely gluon splitting and joining.

In a conformally invariant theory, such as classical pure Yang–Mills theory, the bulk viscosity vanishes. We can see this in two related ways [ArDoMo06]. First, bulk viscosity is related to the departure of the trace part of the stress tensor from its equilibrium value upon isotropic expansion. In a conformally invariant theory, such expansion does not drive the system out of equilibrium – for the Maxwell case, see [Pla59] – so there is no departure. Second, in a conformally invariant theory the energy–momentum tensor must be traceless. This leaves no room for deviations of the trace part of the stress tensor from the value prescribed by the equilibrium equation of state  $p = \rho/3$ .

Therefore, the bulk viscosity in Yang–Mills theory is linked to the trace anomaly of the energy–momentum tensor [Fuj80]. In non-abelian gauge theories the trace anomaly is proportional to the  $\beta$  function which describes the running with scale of the gauge coupling [CoDuJo77]. Arnold, Dogan and Moore [ArDoMo06] observe that, since in principle the  $\beta$  function can have either sign (in a theory with matter fields included), while the bulk viscosity must be positive because of the second law, the bulk viscosity must be related to the *square* of the  $\beta$  function.

We must also mention the Landau–Pomeranchuk–Migdal (LPM) effect [LanPom53a, LanPom53b, Mig56, BaiKAt03]. This effect concerns the suppression of the emission probability for low-frequency photons, and correspondingly the suppression of exchange interactions in the low-frequency sector. It also affects gluon emission at both low and high frequency.

It is crucial that the kinetic equation one takes as take-off point be consistent with the LPM effect. Consistency can be achieved by an explicit calculation of the relevant cross-sections in the collision integral [ArMoYa00, ArMoYa03a, ArMoYa03b]. For an estimate of transport coefficients one does not often require a detailed knowledge of the cross-sections, but rather of certain integrals of them, for which there exist sum rules [AuGeZa02]. In such a case, it is enough to incorporate the LPM effect through the relevant sum rules [BBGM06].

An exciting new development is the possibility of an absolute lower bound for the ratio of shear viscosity to entropy density [KoSoSt05]. A low value for this ratio is usually an indication of a strongly coupled theory (compare with (12.107)). If this “viscosity bound conjecture” is confirmed, it would open up new avenues for the investigation of transport coefficients in a variety of strongly coupled systems, ranging from RHICs to cold atomic gases [Coh07].

More generally, the method of AdS/CFT correspondence [Mal99] is a new tool which is playing an increasing role in the study of strongly coupled gauge theories. A crucial step is the generalization of the correspondence for

the computation of Schwinger–Keldysh (as opposed to Euclidean) propagators [HerSon03]. Similar tools have been used to study the hydrodynamic limit of M theory [Her02, Her03]. Since this field is growing exponentially at the time of writing we cannot even aim to provide a comprehensive list of references. However, see [PoSoSt02a, PoSoSt02b, KoSoSt05, HelJan07] for some key developments.

This concludes our study of the hydrodynamic limit from the kinetic field theory. There are several interesting directions to extend these results, such as including higher order effects [CaDeKo01]. We shall discuss some of these developments in later chapters in the context of applications to concrete problems.

To gain a broader perspective it is instructive to show the derivation of the transport coefficients from a different approach, namely, that of linear response theory, which we now turn to.

### 12.4 Transport functions from linear response theory

Linear response theory aims to provide exact representations for the transport functions as equilibrium expectation values of current correlations. The actual evaluation of these expressions may be technically rather subtle. The reader should consult the literature for details. However, the fact that one has, in principle, a rigorous definition of the transport functions opens up the possibility of implementing nonperturbative techniques, such as extracting the relevant correlations from numerical simulations [AarBer01, AarMar02]. Moreover, the fact that the linear response theory program may be carried through is a beautiful illustration of the deep connection between equilibrium and near-equilibrium dynamics such as embodied in the fluctuation–dissipation theorem, as well as in the stochastic approach to NEqQFT discussed in Chapter 8.

In the literature, there are several equivalent derivations of the linear response expressions for the transport functions. With some over-simplification, they can be traced back to the work of Mori [Mor58, HorSch87], Zubarev [Zub74, HoSata84] and Kadanoff and Martin [KadMar63]. The work of Jeon [Jeo93, Jeo95] and Jeon and Yaffe [JeoYaf96] is also of substantive value. For later developments, see [WaHeZh96, CaDeKo00, WanHei99, WRSG03, Koi07].

Following the presentation of Kadanoff and Martin [KadMar63], we shall first demonstrate this approach with the simpler case of the spin diffusion coefficient for an Ising-like model of a ferromagnetic material, and then derive the linear response theory expressions for the viscosity coefficients  $\eta$  and  $\zeta$ .

#### 12.4.1 The spin diffusion coefficient

We consider a model of some ferromagnetic material where the spin density is described by a continuous scalar quantum field  $\mathbf{m}(t, \mathbf{x})$ . (Here we use bold face to denote quantum fields – not for a vector field – with light face for classical

fields.) The model is nonrelativistic, and for brevity we consider the symmetric phase only. Since the total magnetization is conserved, the Heisenberg equation of motion for the spin density takes the form of a continuity equation

$$\frac{\partial}{\partial t} \mathbf{m}(t, \mathbf{x}) + \nabla \mathbf{J}(t, \mathbf{x}) = 0 \quad (12.110)$$

As we have seen in Chapter 8, it is possible to introduce a classical stochastic field  $m(t, \mathbf{x})$  such that

$$\langle m(t, \mathbf{x}) m(0, \mathbf{y}) \rangle_s = \frac{1}{2} \langle \{ \mathbf{m}(t, \mathbf{x}), \mathbf{m}(0, \mathbf{y}) \} \rangle \quad (12.111)$$

where we have a stochastic average on the left-hand side, and a quantum average on the right.  $m$  satisfies a Langevin equation

$$\frac{\partial}{\partial t} m(t, \mathbf{x}) + \nabla \mathbf{J}(t, \mathbf{x}) = H_s(t, \mathbf{x}) \quad (12.112)$$

To linear order in  $m$ , for slowly varying fields, with consideration of Galilei invariance, we must have

$$\mathbf{J}(t, \mathbf{x}) = -D \nabla m(t, \mathbf{x}) + \dots \quad (12.113)$$

where  $D$  is the spin diffusion coefficient we want to determine.

Since in this approximation the dynamics is linear and space translation invariant, it is convenient to introduce Fourier transforms

$$m(t, \mathbf{x}) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}} m_{\mathbf{k}}(t) \quad (12.114)$$

If the value of the amplitude at  $t = 0$  is  $m_{\mathbf{k}}(0)$ , then for  $t > 0$

$$m_{\mathbf{k}}(t) = m_{\mathbf{k}}(0) e^{-Dk^2 t} + m_{\mathbf{k}}^S(t) \quad (12.115)$$

where  $m_{\mathbf{k}}^S(t)$  depends on the noise between 0 and  $t$ . If the noise and  $m_{\mathbf{k}}(0)$  are uncorrelated, then

$$\langle m_{\mathbf{k}}(t) m_{\mathbf{k}'}(0) \rangle_s = e^{-Dk^2 t} \langle m_{\mathbf{k}}(0) m_{\mathbf{k}'}(0) \rangle_s \quad (t > 0) \quad (12.116)$$

From Onsager's principle of microscopic reversibility [LaLiPi80a], we know that the correlation is even in  $t$ , so this equation determines its value for  $t < 0$  as well, namely,

$$\langle m_{\mathbf{k}}(t) m_{\mathbf{k}'}(0) \rangle_s = e^{-Dk^2 |t|} \langle m_{\mathbf{k}}(0) m_{\mathbf{k}'}(0) \rangle_s \quad (12.117)$$

We now compute the inverse Fourier transform

$$m_{\mathbf{k}}(\omega) = \int dt e^{i\omega t} m_{\mathbf{k}}(t) \quad (12.118)$$

$$\frac{2Dk^2}{\omega^2 + (Dk^2)^2} \langle m_{\mathbf{k}}(0) m_{\mathbf{k}'}(0) \rangle_s = \frac{1}{2} \int dt e^{i\omega t} \langle \{ \mathbf{m}_{\mathbf{k}}(t), \mathbf{m}_{\mathbf{k}'}(0) \} \rangle \quad (12.119)$$



To obtain a prediction for  $D$  from this formula, we take the limits  $k \rightarrow 0$  and  $\omega \rightarrow 0$  in this order [KuToHa91] to get

$$D \langle m_{\mathbf{k}}(0) m_{\mathbf{k}'}(0) \rangle_s = \frac{\omega^2}{4k^2} \int dt e^{i\omega t} \langle \{ \mathbf{m}_{\mathbf{k}}(t), \mathbf{m}_{\mathbf{k}'}(0) \} \rangle \quad (12.120)$$

Since the equilibrium correlation on the right-hand side is time-translation invariant, we may write

$$\begin{aligned} \omega^2 \int dt e^{i\omega t} \langle \{ \mathbf{m}_{\mathbf{k}}(t), \mathbf{m}_{\mathbf{k}'}(0) \} \rangle &= \int dt e^{i\omega t} \left\langle \left\{ \frac{\partial}{\partial t} \mathbf{m}_{\mathbf{k}}(t), \frac{\partial}{\partial t} \mathbf{m}_{\mathbf{k}'}(0) \right\} \right\rangle \\ &= - \int dt e^{i\omega t} k_i k'_j \left\langle \left\{ \mathbf{J}_{\mathbf{k}}^i(t), \mathbf{J}_{\mathbf{k}'}^j(0) \right\} \right\rangle \end{aligned} \quad (12.121)$$

It only remains to compute  $\langle m_{\mathbf{k}}(0) m_{\mathbf{k}'}(0) \rangle_s$ . As  $k \rightarrow 0$ ,

$$m_{\mathbf{k}}(t) \rightarrow M(t) = \int d^3\mathbf{x} m(t, \mathbf{x}) \quad (12.122)$$

where  $M$  is the total magnetization of the sample. Recall that if we turn on an external magnetic field  $H$ , then the Hamiltonian  $\mathbf{H}$  acquires a new term  $-HM$ . Therefore, at constant temperature

$$M = - \left. \frac{\partial F}{\partial H} \right|_T \quad (12.123)$$

where  $F$  is the free energy

$$e^{-\beta F} = \text{Tr} e^{-\beta \mathbf{H}} \quad (12.124)$$

Taking two derivatives we get

$$\langle M^2 \rangle = -k_B T \left. \frac{\partial^2 F}{\partial H^2} \right|_T = k_B T V \chi \quad (12.125)$$

where  $\chi$  is the susceptibility

$$\chi = \frac{1}{V} \left. \frac{\partial M}{\partial H} \right|_T \quad (12.126)$$

(where  $V$  is the volume of the sample). We get

$$D\chi = \frac{-1}{4Vk_B T} \int dt e^{i\omega t} \frac{k_i k'_j}{k^2} \left\langle \left\{ \mathbf{J}_{\mathbf{k}}^i(t), \mathbf{J}_{\mathbf{k}'}^j(0) \right\} \right\rangle \quad (k, k', \omega \rightarrow 0) \quad (12.127)$$

By using the symmetries of the correlator, we may simplify this expression to obtain

$$D\chi = \frac{1}{4k_B T} \int d^3\mathbf{x} \int dt e^{i(\omega t - kx)} \frac{k_i k_j}{k^2} \left\langle \left\{ \mathbf{J}^i(t, \mathbf{x}), \mathbf{J}^j(0, 0) \right\} \right\rangle \quad (k, \omega \rightarrow 0) \quad (12.128)$$

We may also use the KMS theorem to express the anticommutator in terms of a commutator.

We shall now use this calculation as a model for the derivation of the viscosity coefficients.

12.4.2 The bulk and shear viscosity coefficients

We return to the calculation of the viscosity coefficients in scalar quantum field theory. We wish to write them in terms of equilibrium correlations of Heisenberg operators. Observe that in the Landau–Lifshitz prescription there is no heat flux, and for real scalar field theory there is no particle number conservation law. So we have no heat conductivity or particle number diffusion constants. The transport functions to be determined are the shear and bulk viscosities  $\eta$  and  $\zeta$ .

In this subsection we shall not use different types for q or c number quantities. The basic dynamical law, both in the fundamental quantum field theory and in the stochastic field theory formulation, is the conservation of energy–momentum

$$T^{\mu\nu}_{;\nu} = 0 \tag{12.129}$$

Decomposing the energy–momentum tensor as in equation (12.20), we get

$$\dot{\rho} + (\rho + p) u^\lambda_{;\lambda} - \tau^{\mu\nu} u_{\mu;\nu} = 0 \tag{12.130}$$

$$(\rho + p) \dot{u}^\mu + \Delta^{\mu\nu} (p_{,\nu} + \tau^\lambda_{\nu;\lambda}) = 0 \tag{12.131}$$

In the local rest frame of the fluid, when terms of second order in deviations from equilibrium are neglected, they reduce to

$$\frac{\partial \rho}{\partial t} + (\rho + p) u_{i,i} = 0 \tag{12.132}$$

$$(\rho + p) \frac{\partial u_i}{\partial t} + p_{,i} + \tau_{ij,j} = 0 \tag{12.133}$$

They have the form of continuity equations with currents  $J^i_\rho = (\rho + p) u^i$  and  $(J_{u^i})^j = p \delta^j_i + \tau^j_i$ .

Let us now consider the stochastic description. As in the spin diffusion case, the noise terms will not affect the final result, so we will not consider them. We may now parameterize

$$\tau_{ij} = -\eta (u_{i,j} + u_{j,i}) - \left( \zeta - \frac{2}{3} \eta \right) \delta_{ij} u_{s,s} \tag{12.134}$$

The second conservation equation becomes

$$(\rho + p) \frac{\partial u_i}{\partial t} + p_{,i} - \eta u_{i,jj} - \left( \zeta + \frac{1}{3} \eta \right) (u_{j,j})_{,i} = 0 \tag{12.135}$$

We may decompose the velocity field  $u = u^L + u^T$ , where  $\nabla \times u^L = \nabla u^T = 0$ . The transverse part decouples from the energy fluctuations, and obeys the simple heat equation

$$(\rho + p) \frac{\partial u^T_i}{\partial t} - \eta u^T_{i,jj} = 0 \tag{12.136}$$

This is the same as in the spin diffusion case, with  $D$  there replaced by  $D^T_u =$

$\eta/(\rho + p)$  here. We therefore write

$$\frac{\eta}{(\rho + p)} \langle u_{i\mathbf{k}}^T(0) u_{j\mathbf{k}'}^T(0) \rangle_s = \frac{\omega^2}{4k^2} \int dt e^{i\omega t} \langle \{u_{i\mathbf{k}}^T(t), u_{j\mathbf{k}'}^T(0)\} \rangle \quad (k, k', \omega \rightarrow 0) \tag{12.137}$$

For the longitudinal part, observe that  $u_{i,jj}^L = (u_{j,j}^L)_{,i}$ . So we may write

$$\frac{\partial \rho}{\partial t} + (\rho + p) u_{i,i}^L = 0 \tag{12.138}$$

$$(\rho + p) \frac{\partial u_i}{\partial t} + p_{,i} - \left( \zeta + \frac{4}{3} \eta \right) u_{i,jj}^L = 0 \tag{12.139}$$

Introduce the velocity potential  $u^L = -\nabla\phi$ , the sound speed  $p_{,i} = c_s^2 \rho_{,i}$ , and Fourier transform

$$\frac{\partial \rho_{\mathbf{k}}}{\partial t} + (\rho + p) k^2 \phi_{\mathbf{k}} = 0 \tag{12.140}$$

$$(\rho + p) \frac{\partial \phi_{\mathbf{k}}}{\partial t} - c_s^2 \rho_{\mathbf{k}} + \left( \zeta + \frac{4}{3} \eta \right) k^2 \phi_{\mathbf{k}} = 0 \tag{12.141}$$

These are the equations of a damped harmonic oscillator

$$\frac{\partial^2 \phi_{\mathbf{k}}}{\partial t^2} + k^2 c_s^2 \phi_{\mathbf{k}} + 2\Gamma k^2 \frac{\partial \phi_{\mathbf{k}}}{\partial t} = 0 \tag{12.142}$$

where

$$\Gamma = \frac{\left( \zeta + \frac{4}{3} \eta \right)}{2(\rho + p)} \tag{12.143}$$

The secular equation

$$\omega^2 - k^2 c_s^2 - 2i\Gamma k^2 \omega = 0 \tag{12.144}$$

has solutions

$$\omega_{\pm} = i\Gamma k^2 \pm \sqrt{k^2 c_s^2 - \Gamma^2 k^4} \tag{12.145}$$

If  $\Gamma^2 k^2 \ll c_s^2$ , we may expand

$$\omega_{\pm} = \pm k c_s + i\Gamma k^2 + O(k^3) \tag{12.146}$$

so the general solution is

$$\phi_{\mathbf{k}}(t) = e^{-\Gamma k^2 t} [\phi_{\mathbf{k}}(0) \cos(kc_s t) + A_k \sin(kc_s t)] \quad (t > 0) \tag{12.147}$$

At  $t = 0$ , we find

$$\frac{\partial \phi_{\mathbf{k}}}{\partial t} = kc_s A_k - \Gamma k^2 \phi_{\mathbf{k}}(0) \tag{12.148}$$

so

$$(\rho + p) kc_s A_k - c_s^2 \rho_{\mathbf{k}}(0) + \left( \zeta + \frac{4}{3} \eta \right) \frac{k^2}{2} \phi_{\mathbf{k}}(0) = 0 \tag{12.149}$$

$$A_k = \frac{c_s^2 \rho_{\mathbf{k}}(0)}{(\rho + p) kc_s} - \frac{\Gamma k}{c_s} \phi_{\mathbf{k}}(0) \tag{12.150}$$

Assuming that the equal-time potential and energy fluctuations are uncorrelated (see below), we get

$$\langle \phi_{\mathbf{k}}(t) \phi_{\mathbf{k}'}(0) \rangle = e^{-\Gamma k^2 |t|} \left[ \cos(kc_s t) - \frac{\Gamma k}{c_s} \sin(kc_s |t|) \right] \langle \phi_{\mathbf{k}}(0) \phi_{\mathbf{k}'}(0) \rangle \quad (12.151)$$

Upon Fourier transforming

$$\frac{(\zeta + \frac{4}{3}\eta)}{(\rho + p)} \langle u_{i\mathbf{k}}^L(0) u_{j\mathbf{k}'}^L(0) \rangle_s = \frac{\omega^2}{4k^2} \int dt e^{i\omega t} \langle \{ u_{i\mathbf{k}}^L(t), u_{j\mathbf{k}'}^L(0) \} \rangle \quad (12.152)$$

( $k, k', \omega \rightarrow 0$ ). From symmetry considerations, we expect

$$\langle \{ u_{i\mathbf{k}}^L(t), u_{j\mathbf{k}'}^T(0) \} \rangle = 0 \quad (12.153)$$

and so we may combine the longitudinal and transverse correlations into a single expression

$$\begin{aligned} \eta \langle u_{i\mathbf{k}}^T(0) u_{j\mathbf{k}'}^T(0) \rangle_s + \left( \zeta + \frac{4}{3}\eta \right) \langle u_{i\mathbf{k}}^L(0) u_{j\mathbf{k}'}^L(0) \rangle_s \\ = (\rho + p) \frac{\omega^2}{4k^2} \int dt e^{i\omega t} \langle \{ u_{i\mathbf{k}}(t), u_{j\mathbf{k}'}(0) \} \rangle \end{aligned} \quad (12.154)$$

( $k, k', \omega \rightarrow 0$ ). We now have to compute the equal-time averages on the left-hand side. Let us begin by computing the velocity–velocity correlation. Recall that if the center of mass of the system is moving with velocity  $\mathbf{V}$ , then in the statistical operator we must add a new term  $-\mathbf{V}\mathbf{P}$  to the Hamiltonian  $\mathbf{H}$ , where  $\mathbf{P}$  is the total momentum. Therefore

$$\langle \mathbf{P}_i \rangle = -\frac{\partial F}{\partial \mathbf{V}^i} \quad (12.155)$$

and

$$\langle \mathbf{P}_i \mathbf{P}_j \rangle = k_B T \frac{\partial \langle \mathbf{P}_i \rangle}{\partial \mathbf{V}^j} \quad (12.156)$$

To transform this into velocity correlations, we simply observe that from the equilibrium energy–momentum tensor

$$\langle \mathbf{P}_i \rangle = V(\rho + p) \mathbf{V}_i \quad (12.157)$$

so

$$\langle \mathbf{V}_i \mathbf{V}_j \rangle = \frac{k_B T}{V(\rho + p)} \delta_{ij} \quad (12.158)$$

The longitudinal part of the velocity may be obtained from the total velocity by projection

$$u_{i\mathbf{k}}^L(0) = \frac{k_i k^j}{k^2} u_{j\mathbf{k}}(0) \quad (12.159)$$

We do likewise for  $u_{j\mathbf{k}'}^L$ , observe that the correlation must be proportional to  $\delta(\mathbf{k} + \mathbf{k}')$  by translation invariance, and that in the limit  $k \rightarrow 0$ ,  $u_{i\mathbf{k}} \rightarrow V\mathbf{V}_i$ ,

we get

$$\langle u_{i\mathbf{k}}^L(0) u_{j\mathbf{k}'}^L(0) \rangle_s = \frac{k_i k_j}{k^2} \frac{V k_B T}{(\rho + p)} \quad (k, k' \rightarrow 0) \tag{12.160}$$

which also implies

$$\langle u_{i\mathbf{k}}^T(0) u_{j\mathbf{k}'}^T(0) \rangle_s = \left[ \delta_{ij} - \frac{k_i k_j}{k^2} \right] \frac{V k_B T}{(\rho + p)} \quad (k, k' \rightarrow 0) \tag{12.161}$$

Thereby we find

$$\eta \left[ \delta_{ij} + \frac{1}{3} \frac{k_i k_j}{k^2} \right] + \zeta \frac{k_i k_j}{k^2} = \frac{(\rho + p)^2 \omega^2}{V k_B T 4k^2} \int dt e^{i\omega t} \langle \{u_{i\mathbf{k}}(t), u_{j\mathbf{k}'}(0)\} \rangle \tag{12.162}$$

( $k, k', \omega \rightarrow 0$ ). As in the previous case of the spin diffusion coefficient, this may be reduced to an expression involving correlations of the energy–momentum tensor alone. First, use the expressions for  $T^{0i}$  and the conservation laws to write this as

$$\begin{aligned} & \eta \left[ \delta^{ij} + \frac{1}{3} \frac{k^i k^j}{k^2} \right] + \zeta \frac{k^i k^j}{k^2} \\ &= \frac{1}{V k_B T} \frac{k_m k_n}{4k^2} \int dt e^{i\omega t} \langle \{T_{\mathbf{k}}^{im}(t), T_{\mathbf{k}'}^{jn}(0)\} \rangle \quad (k, k', \omega \rightarrow 0) \end{aligned} \tag{12.163}$$

Next, separate  $T_{\mathbf{k}}^{im}$  into scalar and traceless components

$$T_{\mathbf{k}}^{im} = \mathcal{P}_{\mathbf{k}} \delta^{im} + \tau_{\mathbf{k}}^{im}; \quad \tau_{\mathbf{k}i}^i = 0 \tag{12.164}$$

In the limit  $\mathbf{k} \rightarrow 0$ , the tensor structure of the correlations can be expressed in terms of the isotropic tensor  $\delta^{ij}$  alone. By symmetry, we must have

$$\langle \{ \mathcal{P}_0, \tau_0^{im} \} \rangle = 0 \tag{12.165}$$

$$\frac{1}{V} \langle \{ \tau_0^{im}(t), \tau_0^{im}(0) \} \rangle = A(t) \delta^{im} \delta^{jn} + B(t) (\delta^{ij} \delta^{mn} + \delta^{in} \delta^{mj}) \tag{12.166}$$

This last expression must be traceless with respect to  $(im)$ , so  $3A + 2B = 0$ , and

$$\frac{1}{V} \langle \{ \tau_0^{im}(t), \tau_0^{im}(0) \} \rangle = A(t) \left[ \delta^{ij} \delta^{mn} + \delta^{in} \delta^{mj} - \frac{2}{3} \delta^{im} \delta^{jn} \right] \tag{12.167}$$

Contracting  $(ij)$  and  $(mn)$  we get

$$\sigma^2(t) \equiv \frac{1}{V} \langle \{ \tau_0^{im}(t), \tau_0^{im}(0) \} \rangle = 10A(t) \tag{12.168}$$

Substituting this back in equation (12.163) we get

$$\eta = \frac{1}{40k_B T} \int dt e^{i\omega t} \sigma^2(t) \quad (\omega \rightarrow 0) \tag{12.169}$$

$$\zeta = \frac{1}{4k_B T} \int dt e^{i\omega t} \langle \{ \mathcal{P}_0(t), \mathcal{P}_0(0) \} \rangle \quad (\omega \rightarrow 0) \tag{12.170}$$

which are the familiar expressions [Jeo95].

## 12.5 Thermalization

Perhaps the single most important demand on a theory of nonequilibrium quantum fields is that it should describe the means by which equilibrium is reached and sustained by those systems. The process of thermalization plays an important role in all the applications of the theory, such as the behavior of order-parameter fluctuations after a quench (Chapter 9), the dynamics of Bose–Einstein condensates and their associated noncondensed atomic clouds (Chapter 13), the early stages of relativistic heavy ion collisions (Chapter 14) and the physics of reheating after inflation (Chapter 15). In this chapter, we will deploy the knowledge gained so far in the physics of nonequilibrium fields to describe some general features of the thermalization process; then we will discuss some of these applications indicated above.

In spite of this ubiquity, the thermalization process is very hard to access experimentally. Usually all one can actually observe are relics superposed on the equilibrated thermal background, such as topological defects after a nonequilibrium phase transition or the ratios between different particle species after hadronization of the quark–gluon liquid. For this reason, a good deal of our understanding of the thermalization process comes from large-scale numerical simulations. We shall not discuss these simulations *per se*, but will point out below the key entry points to the literature.

Before we proceed, a word is in order about what thermalization is. Quantum field theory is unitary; quantum field theoretic evolution in a closed system cannot create entropy, and so a quantum field starting from a pure state, say, cannot thermalize in the strict thermodynamic sense (unless, e.g. it is coupled to a heat bath, see next section). By thermalization we mean that a restricted set of observables (correlation functions, hydrodynamics variables such as energy density and pressure, equation of state, field configurations over regions of space small compared to the total available volume) evolve in time towards stable, near-stationary values which are robust against changes in the initial conditions and may be approximated by thermal distributions with suitable intensive parameters (temperature, chemical potentials, etc.) [BoDeVe04].

If we talk about thermalization in the context of quantum field theories, the problem becomes slightly academic because nobody has ever solved the full unitary evolution (unless in trivial cases, which do not thermalize). One solves instead the equations of motion for the correlation functions derived, e.g. by some  $n$ -PI effective action functional with a finite  $n$ , which are not time-reversal invariant [IvKnVo99]. However, for classical field theories one can, in principle, actually solve the field equations. Then thermalization in the strict sense is impossible. For example, in a thermal state one should be able to observe arbitrarily high values of the total energy in the field (though large values will be very unlikely); in a numerically correct calculation, one should never see energy values outside the range defined by the initial conditions. Nevertheless, in the

thermodynamic limit the behavior of local observables becomes indistinguishable from equilibrium. This means that thermalization is obtained “for all practical purposes” (FAPP) in the sense defined above. In this chapter, we shall adhere to this use of the term thermalization.

### 12.5.1 A toy model of thermalization

Although our goal is to describe thermalization (FAPP) in an isolated quantum field, it is instructive to consider first the case in which the field is thermalized (*strictu sensu*) by bringing it in contact with a heat bath. This problem was analyzed by Schwinger [Sch61]. The reservoir may be described by one or several quantum fields, and the action will be expanded by adding the action describing these fields, plus the new term describing the system–bath interaction. Probably the most mysterious empirical fact about thermodynamics is that the long-term equilibrium state, if achieved, is totally independent of the details of the bath dynamics and interaction. Therefore we shall leave open the details of the bath, and simply write an interaction term of the form  $g\varphi^A\Psi_A$ , where  $\Psi$  is some (generally composite) bath operator and  $g$  is a coupling constant. We shall assume the usual set-up where system and bath are brought into contact at some initial time  $t = 0$ . We also assume the initial condition is spatially homogeneous, which, neglecting the system’s self-interactions, allows us to decompose it into independent spatial modes. We consider the thermalization of each mode, and in so doing reduce the original theory to a  $1 + 0$  field theory.

As we have seen in Chapter 11, the equations for the (system) Jordan and Hadamard propagators are determined by the dissipation and noise kernels in the 1PI CTPEA for the system field. To lowest order in the system–bath coupling constant  $g$ , they are

$$\begin{aligned} \mathbf{D}(t-t') &= ig^2\hbar^{-1}\theta(t-t')\langle[\Psi(t),\Psi(t')]\rangle \\ \mathbf{N}(t-t') &= \frac{1}{2}g^2\hbar^{-1}\langle\{\Psi(t),\Psi(t')\}\rangle \end{aligned} \quad (12.171)$$

Here, a common or implicit assumption is that the bath is always kept in equilibrium at some temperature  $T$ , and that any back-reaction from the system is negligible. So, writing

$$\langle[\Psi(t),\Psi(t')]\rangle = \hbar \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \text{sign}(\omega) \mathbf{R}(\omega) \quad (12.172)$$

where  $\mathbf{R}(\omega)$  must be even and positive, and following the analysis in Chapter 11, the imaginary part of the retarded propagator becomes

$$\text{Im } G_{\text{ret}}^{-1} = \left(\frac{-g^2}{2}\right) \text{sign}(\omega) \mathbf{R}(\omega) \quad (12.173)$$

The full equation is determined by causality

$$\left[ \omega^2 - m_b^2 - g^2 \int_0^\infty \frac{d\sigma^2}{2\pi} \frac{\mathbf{R}(\sigma)}{(\omega + i\varepsilon)^2 - \sigma^2} \right] G_{\text{ret}} = -1 \quad (12.174)$$

where  $m_b$  is the bare mass of the system field.

If  $\omega^2 = \xi + i\eta$ , the inverse propagator develops an imaginary part

$$\eta \left[ 1 + g^2 \int_0^\infty \frac{d\sigma^2}{2\pi} \frac{\mathbf{R}(\sigma)}{(\xi - \sigma^2)^2 + \eta^2} \right] \quad (12.175)$$

The expression in brackets is positive, so any (first sheet) zero must have  $\eta = 0$ . But on the real axis, the inverse propagator has a cut, with a discontinuity  $g^2 \mathbf{R}(\xi)$  in the imaginary part. So, unless  $\mathbf{R}(\sigma)$  vanishes below some threshold, the inverse propagator cannot be zero. We shall assume this is the case, which means that all excitations of the system are unstable against decay into the bath.

Besides damping, the bath also provides screening. The Debye mass is defined as the closest thing to a zero of the inverse propagator, namely a zero of the real part of the inverse propagator. Therefore

$$M_D^2 - m_b^2 - g^2 PV \left[ \int_0^\infty \frac{d\sigma^2}{2\pi} \frac{\mathbf{R}(\sigma)}{M_D^2 - \sigma^2} \right] = 0 \quad (12.176)$$

We shall assume the physically reasonable condition that  $M_D^2 \geq 0$ . A sufficient condition for this is that the left-hand side of the gap equation changes sign as we go from  $M_D = 0$  to  $\infty$ . If  $\mathbf{R}(\sigma)$  is well behaved (which may require that we perform a subtraction beforehand) the left-hand side is dominated by the first term  $M_D^2$  when this is large, and so it is positive. The sufficient condition for screening (as opposed to anti-screening) boils down to

$$m_b^2 \geq g^2 PV \left[ \int_0^\infty \frac{d\sigma^2}{2\pi} \frac{\mathbf{R}(\sigma)}{\sigma^2} \right] \quad (12.177)$$

We may now write the equation for the retarded propagator as

$$\left\{ (\omega^2 - M_D^2) \left[ 1 + g^2 PV \int_0^\infty \frac{d\sigma^2}{2\pi} \frac{\mathbf{R}(\sigma)}{(\omega^2 - \sigma^2)(M_D^2 - \sigma^2)} \right] + \frac{ig^2 \omega}{2|\omega|} \mathbf{R}(\omega) \right\} G_{\text{ret}} = -1 \quad (12.178)$$

The nice thing about this expression is that it makes it easy to identify the mean life of a field excitation. Indeed, the inverse propagator has a near Ornstein-Zernike structure

$$[\omega^2 - M_D^2 + 2i\gamma\omega] G_{\text{ret}} = -1 \quad (12.179)$$

where  $\gamma \sim g^2 \mathbf{R}(M_D)/4M_D$  is the damping constant, and so we may conclude that the decay of an excitation will be nearly exponential (at very long times it



may turn to power law, depending on the behavior of  $\mathbf{R}(\omega)$  as  $\omega \rightarrow 0$ ). Incidentally, this equation also fixes the field density of states

$$\mathcal{D}(\omega) = \pi^{-1} |\text{Im}G_{\text{ret}}| = \left(\frac{g^2}{2\pi}\right) |G_{\text{ret}}|^2 \mathbf{R}(\omega) \tag{12.180}$$

In particular, the Ornstein-Zernike approximation for the retarded propagator implies a corresponding approximation for the density of states

$$\mathcal{D}(\omega) \sim \frac{1}{\pi} \frac{2\gamma|\omega|}{(\omega^2 - M_D^2)^2 + 4\gamma^2\omega^2} \tag{12.181}$$

So far we have analyzed how the interaction with the heat bath affects the system dynamics, but we have not addressed thermalization *per se*. To do this, it is not efficient to look at the retarded propagator, because this propagator is very robust against thermal corrections. We look instead at the Hadamard propagator, which obeys the equation

$$G_{\text{ret}}^{-1}G_1 = 2\hbar\mathbf{N}G_{\text{adv}} \tag{12.182}$$

We have arrived at the crucial point. The inhomogeneous equation (12.182) admits a particular solution  $G_1 = 2\hbar\mathbf{N}|G_{\text{ret}}|^2$  and also homogeneous solutions which carry the information about the initial conditions. But the homogeneous solutions decay, so after a time long compared to the mean life  $\gamma^{-1}$ , only the particular solution remains. Now, the bath propagators are subject to the KMS theorem (it being insensitive to whether the field  $\Psi$  is fundamental or composite)

$$\mathbf{N} = \frac{g^2}{2} [1 + 2f_0(\omega)] \mathbf{R}(\omega) \tag{12.183}$$

Therefore the asymptotic Hadamard propagator obeys

$$G_1 = 2\pi\hbar [1 + 2f_0(\omega)] \mathcal{D}(\omega) \tag{12.184}$$

This is just the KMS theorem for the *field* (as opposed to the bath) Hadamard propagator.

In conclusion, the essential elements of the thermalization process are that there must be a heat bath, capable of transmitting the KMS condition to the system, and at the same time a damping mechanism so that the field initial conditions may be forgotten in time. Of course, damping does not cease when equilibrium is finally reached, but at late times it is exactly compensated by the inhomogeneous term in the equation for the Hadamard propagator. Thus we arrive at yet another perspective on the KMS theorem, now as a detailed balance condition which enforces the stability of the thermal state.

It is remarkable that when we come to view (12.184) as a relationship between the field Hadamard propagator and density of states, any direct reference to the bath has disappeared. The bath is necessary to validate the KMS theorem on the system, but once this task is accomplished, it can go free. In fact, any bath can perform this function (although the relaxation times will be different) as long as

it is a *good* bath, meaning that it is able to sustain a constant temperature in the face of back-reaction, and that it provides efficient dissipation in all scales. Of course, this is precisely the condition for thermodynamics to prevail.

### 12.5.2 Thermalization of isolated fields

Let us now turn our attention to isolated fields, and examine whether in any sense they fulfill the two conditions above. The answer is yes in both cases. The KMS theorem is built in the Kadanoff–Baym equations, because field configurations consistent with the KMS theorem have slower dynamics, and eventually outlive those that do not. As for dissipation, beware that by restricting ourselves to thermalization in the FAPP sense, we are *de facto* turning the problem into an effectively open system. The interaction between the relevant system and irrelevant sectors (the environment, with its large capacity) brings dissipation and decoherence (or its classical analog, dephasing) to the system, by which the memory of initial conditions is lost. These mechanisms work either for quantum or classical field theories, although we expect them to be more efficient in the quantum case. For example, quantum particle creation may emanate from the vacuum, while classical parametric amplification can only work from a pre-existing seed; thus pumping energy from a heat bath or a classical background into an unpopulated region of the spectrum is easier in quantum theories (for Fermi systems, of course, we have to take Pauli blocking into consideration).

By now, there is a mounting body of (numerical) evidence in support of these statements. Numerical work has focused mostly on scalar field theories with quartic self-interactions, with either one single field or else  $N$  fields with  $O(N)$  symmetry in the large  $N$  limit. Numerical investigations of the equations of motion as derived from the Kadanoff–Baym equations were pioneered by Danielewicz [Dan84a, Dan84b].

As we mentioned in the Introduction, it is not our aim to discuss numerical approaches in detail. However, it is important to know what has been achieved. To this end, it is useful to classify the mounting literature on the subject into the four basic categories of quantum mechanical, classical, semiclassical and quantum field models.

#### *Quantum mechanical models*

The complexity of the field theoretic equations led to the search for simpler systems where at least the basic approximations could be tested. One possible simplification is to consider a field theory in  $0 + 1$  dimensions, namely quantum mechanics. For example, Cooper *et al.* [CDHR98] showed that while the evolution of the coefficients in the quantum mechanical Hartree and leading order (LO) large  $N$  approximations may be described as a chaotic Hamiltonian system, in truth chaos is an artifact of the approximation. Another work using quantum mechanical systems as a testing ground is [BetWet98]. [MACDH00]

matches Hartree, LO and next-to-LO (NLO) large  $N$  against numerical solutions of the Schrödinger equation. [MiDaCo01] analyzes the so-called bare-vertex and dynamic Debye screening approximations. [Hab04] shows that the Gaussian approximation in a closed system leads to the same dynamics for the Wigner and the distribution functions, irrespective of whether the system is quantum or classical.

#### *Classical field models*

Another direction in which the theory may be simplified is by taking the classical limit. Thermalization in classical  $\phi^4$  theory was investigated in [AaBoWe00b, Aar01, BoDeVe04]. Oftentimes a classical field theory arises from a mean field approximation to a quantum problem. In particular, the nonequilibrium dynamics of Bose–Einstein condensates has been thoroughly investigated as described by the Gross–Pitaevskii equation [GaFrTo01, SanShl02, UedSai03, BajaMa04, Adh04].

Both classical field theories and the time-dependent Ginzburg–Landau equation have been investigated as models of defect formation after an instantaneous quench [AntBet97, DzLaZu99, Ste00]. Adding a  $U(1)$  gauge field leads to the Gorkov equations for a type II superconductor [YatZur98, IbaCal99, StBeZu02]. To simulate a quench at a finite rate, it is possible to introduce interaction with a heat bath by adding ohmic dissipation and white noise [Kib80, Kib88, Zur85, Zur96, Riv01, RiKaKa00, LagZur97, LagZur98, YatZur98, AnBeZu99, BeHaLy99, HabLyt00, BeAnZu00, FASA05, AGRS06]. There have also been analyses of classical theories in expanding universes, motivated by the problem of reheating; see [KhlTka96, KoLiSt97, FelTka00, FeKoLi01, FelKof01, FGGKLT01, MicTka04, PFKP06].

Classical field has been extensively used as a test bench for different approximations, for example, the use of a scalar field in 1+1 to compare the Hartree, LO and NLO large  $N$  approximations in [AaBoWe00a].

#### *Semiclassical field models*

One step up in the ladder of increasing complexity we find semiclassical models, often arising from Hartree or leading order  $1/N$  approximations to the full quantum field models [BVHLS95].

This category also includes external field problems beyond the test field approximation (cf. Chapter 4). Among these, the most studied have been electromagnetic and gravitational backgrounds. [KESCM92] compares the semiclassical evolution to a quantum Vlasov equation incorporating Schwinger’s pair creation from the electric field. [CEKMS93] generalizes the above by including the effect of an expanding background geometry. [CHKMPA94] investigates a symmetric scalar  $O(N)$  theory and QED with  $N$  fermion fields. See also [KIMoEi98] and [AarSmi99], which deals with the abelian Higgs model with fermions in 1+1 dimensions.

Fully numerical solution of semiclassical cosmological models presents enormous difficulties, not only because of the intrinsic complexity of general relativity but also because most schemes lead to wildly unstable dynamical equations [ParSim93]. Some questions have been investigated, though, most notably the back-reaction effect of trace anomalies of quantum fields and particle creation leading to avoidance of cosmological singularity and anisotropy damping [FiHaHu79, HarHu79, HarHu80, Har80, Har81].

Another source of semiclassical problems has been the development of spinodal decomposition [CHKM97]. [BBHKP98] considers initial conditions relevant to a relativistic heavy ion collision. See also [BVHS99a, SCHR99]. In particular, the possibility of actually observing disoriented chiral condensates in relativistic heavy ion collisions has focused much attention on the specifics of this problem [CKMP95, LaDaCo96, CoKIMo96, BeRaSt01].

The problem of reheating after inflation combines aspects of both semiclassical theory on curved spacetime backgrounds and spinodal instability. [BoVeHo94] formulates the Hartree and one-loop approximations in an expanding background within the test field approximation. [BVHS96] discusses the effect of anharmonicity on the background field dynamics and the structure of resonances. [RamHu97a, RamHu97b] incorporate fully the back-reaction of quantum fluctuations on the dynamics of the inflaton field as well as the dynamics of the expanding background spacetime. See also [ZiBrSc01].

The problem of condensate collapse within the Hartree–Fock–Bogoliubov approximation has been studied in [WuHoSa05].

#### *Full quantum field models*

At the top of the complexity ladder we find the full quantum field models. Of course, the field theoretic Heisenberg equations being unassailable, some kind of perturbative scheme is necessary. Much of our present understanding of nonequilibrium quantum fields comes from the analysis of  $O(N)$  scalar fields to NLO in the large  $N$  approximation [Ber02, AarBer02, BerSer03a] in  $1+1$  dimensions. This work is reviewed in [Ber04b, BerSer03b, BerSer04, BerBol06].

The  $\lambda\Phi^4$  theory to two loops and beyond leads to a similar phenomenology. It has been investigated in one [AarBer01], two [JuCaGr04] and three space dimensions [ArSmTr05]. Going beyond scalar fields, [BeBoSe03] studies the abelian Higgs model in  $3+1$  dimensions at two loops. Other approximation schemes have been explored [BaaHei03a, BaaHei03b]. A radical new approach to numerical nonequilibrium field theory has been proposed in [BerSta05, BBSS06]. Nonequilibrium Bose–Einstein condensates have been analyzed from a 2PI perspective both in a large  $N$  expansion and to second order in the interaction strength [RHCRC04, GBSS05].

The fields of lattice QCD and hydrodynamical and kinetic models of relativistic collisions targeting the hadronization process are beyond the scope of this book. See [Shu88, Cse94, Wan97, Ris98, BaRoWi06a, BaRoWi06b, TeLaSh01,

HKHRV01, KolRap03, HirTsu02, HirNar04, HeiKol02b]. There is also some numerical work on processes which may speed up thermalization in the early stages of the collision [ArMoYa05, ArnMoo05, Moo05, RomVen06].

*By way of summary*

This brisk enumeration should convince the reader that by now a wide variety of cases has been studied, with a matching diversity of means. The important point is that a coherent picture emerges, since the phenomenology observed in the different cases is consistent. In the remainder of this chapter, we shall tell the prototype thermalization story, by combining the insights gained from these numerical experiments, supporting it whenever possible by analytical arguments. In the remaining chapters of the book we shall contrast this theory with the findings and demands of concrete applications.

### *12.5.3 The stages of thermalization*

Summarizing the results of both numerical and analytical work, we may say that typically the thermalization process in an isolated quantum field goes through three distinctive stages [Son96]:

- (a) early stage;
- (b) intermediate stage;
- (c) late stage.

*The early stage: Preheating and prethermalization*

Description of the earliest stage of thermalization varies a lot from one model to another. It is generally characterized by an explosive pumping of energy into the field, usually because of instabilities. For example, in a quench from a stable to an unstable phase (cf. Chapters 4 and 9), the infrared modes become unstable and begin to grow explosively. A similar phenomenon marks the growth of fluctuations around a collapsing condensate, or the growth of large-scale magnetic fields from an anisotropic distribution of hard gluons after a relativistic heavy ion collision (RHIC). Without involving an actual instability, parametric amplification by a dynamical background is also an efficient way to transfer energy to the field; this occurs in the so-called preheating stage in reheating after inflation.

It is possible to reach an analytic understanding of the early phase if a set of modes may be identified as a linear field on an evolving background. In this case, the early stage may be analyzed within a one-loop or Hartree type approximation. One generic phenomenon is that of decoherence (or dephasing) brought about through quantum diffusion [HKMP96]. As a result, quantities involving contributions from many modes (like the energy density or the pressure) quickly lose memory about the initial conditions, and the equation of state stabilizes to its near-thermal form. This is the phenomenon of pre-thermalization [BeBoWe04].

One basic difficulty in formulating a model of this early stage is accounting for the system–environment interaction which is what drives the system out of equilibrium to begin with. This is usually done by assuming an *ad hoc* time dependence in the field parameters, and/or adding dissipation and noise to the equations of motion. However this procedure is hard to justify on a first principle basis.

There is however an environment which is easy to include into the equations and can bring about the desired effect – a dynamical background spacetime. In an expanding universe only conformally invariant fields may hold on to thermal equilibrium. Once conformal invariance is broken, field modes are relentlessly red-shifted by the expansion, bringing about an effective (in both senses of the word) cooling. We have analyzed this problem in Chapter 4.

The initial stage concludes at the point the infrared peak becomes nonlinear. After a period of parametric amplification, the end result is a nonthermal spectrum with a narrow band of highly populated modes. In the next stage, this far-from-equilibrium spectrum evolves into a Planck distribution through the process of turbulent thermalization.

*Intermediate stage: Turbulent thermalization and kinetic equilibration*

The second stage of the thermalization process is characterized by nonlinear interactions among quantum modes, bringing about an effective thermalization in the energy spectrum, as measured from the Fourier transform of the two-point functions.

As we have seen, the early stage may be described as a theory of linear fields evolving on a classical background. However, as the quantum field amplitude grows, there is a point where a linear model ceases to make sense. We emphasize that the breakdown of linear models beyond a certain point goes over and above formal problems, such as the existence of secular terms [Ber04b, BerSer03b, BerSer04]. If that were the case, it would be enough to resum those terms, for example, by using dynamical renormalization group techniques [BVHS99b, KunTsu06]. The point is that the model of linear fields on a background misses an essential part of the physics, for which there is no formal remedy.

To estimate the point at which the linear approximation is no longer valid, we could for example consider the case of a Bose–Einstein condensate. The Heisenberg operator  $\psi(x)$  which destroys a noncondensate atom at point  $x$  may be regarded as a nonrelativistic scalar field theory with a quartic self-interaction. There is a scattering length  $a = UM/4\pi\hbar^2$  pertaining to the strength  $U$  of the self-interaction and the mass of the atom  $M$ . The cross-section for scattering between noncondensate atoms is  $\sigma \sim a^2$ . If we split the density into its condensate and noncondensate (or anomalous) parts  $n = \Phi^2 + \tilde{n}$ , then the mean free path for a noncondensate atom is  $\lambda \sim 1/\tilde{n}\sigma$ . On the other hand, let  $L$  be a characteristic length of the problem. It could be the distance over which the

condensate varies, or the size of a causal horizon as measured from the start of the nonequilibrium evolution. Once  $\lambda \leq L$ , self-interactions can no longer be ignored, which yields

$$\tilde{n} \geq \frac{(4\pi\hbar^2)^2}{U^2 M^2 L} \quad (12.185)$$

This may come about because  $\tilde{n}$  gets large or  $L$  gets large, for example, if  $L$  grows linearly in time. We observe that for a self-interacting field, there is a nonzero anomalous density even at zero temperature.

$\lambda \leq L$  is also the condition for the validity of the quantum kinetic theory approach, since we expect  $\lambda$  to set the scale for the decay of correlations with respect to the relative variable, and  $L$  to describe the dependence of the correlations on the center-of-mass variable. This affords an enormous simplification of the problem.

Observe, however, that in the same way that it is wrong to apply a linear model in the intermediate stage, it would be wrong to apply a quantum kinetic theory scheme in the early stage [BerBol06, BeBoWe05]. The simplest kinetic theories assume, besides  $\lambda \leq L$ , that all initial non-Gaussian correlations have decayed, and that effectively the initial time may be chosen as in the asymptotic distant past. These restrictions are removed in more complex approaches, but they also set limits to the applicability of quantum kinetic theory at early times.

In a typical problem, the early stage concludes with most of the energy in the field concentrated in a narrow set of modes, and the intermediate stage sees the spread of energy over the full spectrum. There is an initial stage where a cascade is formed between the initial scale  $k_0$  and a moving front  $k_{\max}(t)$ . Within these limits, there is a constant energy current towards higher wavenumbers. This phenomenon closely resembles Kolmogorov's 1941 scenario for fully developed turbulence, and hence the name of turbulent thermalization [Fri95, McC94, LanLif59, Hin75, Bat59, ZaLvFa92].

Turbulent thermalization ends with a self-similar particle number spectrum (as defined by the Fourier transform of the Hadamard propagator, see Chapter 11)  $f(k) \sim k^{-\alpha}$ , with  $\alpha > 1$ . The following stage, or kinetic equilibration, sees the evolution of the spectrum towards the Rayleigh–Jeans form  $f(k) \sim (k_B T) k^{-1}$ . We are assuming of course a high-temperature, weakly coupled scalar field, with high occupation numbers.

The basic features of turbulent thermalization may be understood in terms of a simple quantum kinetic theory model [ZaLvFa92, FelKof01, MicTka04, MuShWo07]. Assume a kinetic equation

$$\omega_k \frac{\partial f}{\partial t}(t, k) = I_{\text{col}} \quad (12.186)$$

For the collision kernel, write

$$I_{\text{col}} = \int \prod_{i=1}^m \frac{d^d p_i}{2\omega_i} \sigma_{m,j}^2 \left\{ \prod_{l=0}^j (1 + f(p_l)) \prod_{l'=j+1}^m f(p_{l'}) \right. \\ \left. - \prod_{l=0}^j f(p_l) \prod_{l'=j+1}^m (1 + f(p_{l'})) \right\} \delta^{(d+1)} \left[ \sum_{l=0}^j p_l - \sum_{l'=j+1}^m p_{l'} \right] \quad (12.187)$$

where  $d$  is the number of spatial dimensions and  $p_0 = k$ . For example, for elastic  $2 \rightarrow 2$  scattering we have  $m = 3, j = 1$ . For large occupation numbers, this simplifies to

$$I_{\text{col}} = \int \prod_{i=1}^m \frac{d^d p_i}{2\omega_i} \sigma_{m,j}^2 \left[ \prod_{l=0}^m f(p_l) \right] \left[ \sum_{l=0}^j f^{-1}(p_l) - \sum_{l'=j+1}^m f^{-1}(p_{l'}) \right] \\ \times \delta \left[ \sum_{l=0}^j p_l - \sum_{l'=j+1}^m p_{l'} \right] \quad (12.188)$$

Observe that  $\omega_k f(t, k)$  is also the energy density in wavenumber space, and so the energy current  $\mathbf{J}_E$  obeys

$$\nabla_k \mathbf{J}_E = -I_{\text{col}} \quad (12.189)$$

Let us assume an isotropic situation  $\mathbf{J}_E = J_E \hat{k}$ . The total energy flux through a shell of radius  $k$  is  $K_E = rk^{d-1} J_E$ , where  $r$  is a constant pure number. Therefore, if  $I_{\text{col}}$  scales as  $k^{-\beta}$ , then  $J_E$  scales as  $k^{1-\beta}$  and  $K_E \sim k^{d-\beta}$ . It follows that turbulent thermalization requires  $\beta = d$ .

On the other hand, if  $f(k) \sim k^{-\alpha}$  then from equation (12.188) we get (assuming  $\sigma_{m,j}^2$  does not scale)

$$\beta = (d + 1) + m(\alpha + 1 - d) \quad (12.190)$$

and finally

$$\alpha = d - 1 - \frac{1}{m} \quad (12.191)$$

The numerical result for  $d = 3$  is  $\alpha = 3/2$ , which corresponds to  $m = 2$  [MicTka04]. This is obtained for a  $g\phi^3$  theory or for a  $\lambda\phi^4$  theory in the presence of a background field. Observe that for  $d = 1$  we get  $\alpha < 0$  whatever the value of  $m$ . Therefore there are no turbulent UV cascades in  $1 + 1$ , also consistent with numerical results. The observed cascade has the Rayleigh–Jeans spectrum  $\alpha = 1$  corresponding to  $K_E = 0$  (cf. equation (12.188)).

The evolution of the wave front  $k_{\text{max}}(t)$  depends upon further details such as whether the total energy (or else the total particle number) contained in the cascade may be considered constant.



As the spectrum spreads it also loses amplitude, and at some point the typical occupation numbers are no longer large. At this point turbulent thermalization ceases. The subsequent relaxation to Rayleigh–Jeans equilibrium may be described by nonequilibrium renormalization group methods. The time-scale for kinetic equilibration may be estimated from a simple Boltzmann equation approach as  $\tau_{\text{rel}} \sim 1/\lambda^2 T$ .

*Late stage: chemical equilibration*

The latest stage concerns chemical, rather than kinetic, equilibration. This means that the energy and particle number spectra already have (local) equilibrium forms, and now the issue is the equilibration among different species. For a real scalar field theory, the two species involved (particles and antiparticles) are identical, and chemical equilibration means the vanishing of the chemical potential.

As in the earlier stages, the basic problem is to find the right tool for the right job. For example, the simplest Boltzmann equation with  $2 \rightarrow 2$  scattering may be successfully used to describe kinetic equilibration. Nevertheless, it fails to describe chemical equilibration, because it has a spurious particle number conservation law built in. To describe chemical equilibration we must go beyond this lowest order kinetic equation, either by considering a more general density of states (as opposed to a sharp mass-shell) or/and by considering higher order terms in the loop or  $1/N$  expansions [CaHuRa00, Wei05b, FiGaJe06]. The relevant terms have been analyzed earlier in this chapter.

Concrete applications may demand other departures from the simple Boltzmann approach. For example, in dealing with a quark–gluon plasma, the relevant kinetic equation is not Boltzmann’s, but rather a Landau-type kinetic equation incorporating the effects of grazing collisions [ChaCow39, Lib98, LifPit81, Mue00a, Mue00b, BjoVen01]. Also we must take into account the color degree of freedom, for example, by analyzing the Wong equations [LitMan02].

Another possibility is to go over directly to a hydrodynamic description. Since the underlying quantum field theory is obviously causal, one expects that the correct hydrodynamic theory would not be a first-order theory (in the classification of Hiscock and Lindblom) but rather a Israel–Stewart or a divergence type theory [CalThi00, CalThi03].

As a concrete example, let us analyze the regression towards zero of the chemical potential in a self-interacting scalar field theory. We assume we are close enough to equilibrium that the chemical potential may be regarded as a linear perturbation, and use the kinetic equation with higher order terms already discussed in this chapter. Assuming the chemical potential is a function of time only, for simplicity, we may write the equation in the spatially translation-invariant case, namely

$$\left[ \omega \frac{\partial}{\partial t} - \frac{1}{2} \left( \frac{dM^2}{dt} \right) \frac{\partial}{\partial \omega} \right] f = I_{\text{col}} [f, M^2] \quad (12.192)$$

The mass is given by the gap equation

$$M^2 - \varphi(M^2, \mu^2) = \frac{\lambda}{2} M_T^2 \tag{12.193}$$

where

$$M_T^2 = \int \frac{d^4p}{(2\pi)^3} \delta(p^2 + M^2) f(X, p) \tag{12.194}$$

Now write

$$f = f_0 + f_0(1 + f_0)\chi \tag{12.195}$$

$$M^2 = M_0^2 + \delta M^2 \tag{12.196}$$

$$I_{\text{col}}[f, M^2] = f_0(1 + f_0)K[\chi] \tag{12.197}$$

$$\omega \frac{\partial \chi}{\partial t} + \frac{\beta}{2} \frac{d\delta M^2}{dt} = K[\chi] \tag{12.198}$$

$$\left[ 1 - \varphi' - \frac{\lambda}{2} \frac{\partial M_T^2}{\partial M^2} \right] \delta M^2 = \lambda \langle \chi \rangle \tag{12.199}$$

The perturbation  $\chi$  may be expanded in eigenfunctions of the linearized Boltzmann collision operator

$$\chi = \beta \left[ \mu + \frac{\delta T}{T} \omega + \bar{\chi} \right] \tag{12.200}$$

$$\bar{\chi} = \sum_{n=1} c_n \chi_n \tag{12.201}$$

so

$$\delta M^2 = 2\omega_0 \left[ \mu + \frac{\langle \omega \rangle}{\langle 1 \rangle} \frac{\delta T}{T} \right] \tag{12.202}$$

$$[\omega + \omega_0] \frac{d\mu}{dt} + \frac{1}{T} \left[ \omega^2 + \frac{\langle \omega \rangle}{\langle 1 \rangle} \omega_0 \right] \frac{d\delta T}{dt} + \omega \frac{\partial \bar{\chi}}{\partial t} = \mu K[1] + K[\bar{\chi}] \tag{12.203}$$

$$\omega_0 = \frac{\lambda \beta \langle 1 \rangle}{2 \left[ 1 - \varphi' - \frac{\lambda}{2} \frac{\partial M_T^2}{\partial M^2} \right]} \tag{12.204}$$

Since we know that  $\mu = \text{constant}$  is a solution if we keep only the lowest order Boltzmann collision term, we expect  $\mu$  to decay on time-scales of the order of  $\lambda^{-4}$  at least.  $\bar{\chi}$  will have a slow part, that will track  $\mu$ , and a fast part, that will decay on time-scales of the order of  $\omega^{-1}$ . Clearly only the slow part is relevant to our discussion, and so we may neglect the  $\partial \bar{\chi} / \partial t$  term.

Observe that now we have an equation of the form

$$K[\bar{\chi}] = A\omega^2 + B\omega + C \tag{12.205}$$

If we look at the right-hand side as a function of  $\omega$ , then the dominant term is the first. The solution is  $\bar{\chi} \sim -(\tau_{\text{rel}}/T) A\omega^2 + \nu\omega + \nu'$ , where the linear and

constant terms enforce the constraints  $\langle \bar{\chi} \rangle = \langle \omega \bar{\chi} \rangle = 0$ . In other words, the effect of the  $K[\bar{\chi}]$  term in equation (12.203) is to compensate the  $d\delta T/dt$  one. Canceling those two, we get the regression equation for  $\mu$  as the average of equation (12.203) (recall that  $\langle K[1] \rangle < 0$ )

$$[\langle \omega \rangle + \omega_0 \langle 1 \rangle] \frac{d\mu}{dt} = -\mu |\langle K[1] \rangle| \quad (12.206)$$

From the estimates in this chapter, we conclude that the characteristic time-scale for chemical equilibration is  $\tau_{\text{chem}} \sim M^2/\lambda^4 T^3$ , parametrically larger than  $\tau_{\text{rel}}$ .

#### 12.5.4 Coda

In this section we have painted a broad outline of the thermalization process, going as far as possible without invoking the specifics of modeling or features of concrete applications. We have seen that a general picture indeed emerges, and that it reveals the communion between quantum field theory and other parts of physics, represented in this chapter by fluid and wave turbulence theory. As all portraits of its kind, it emphasizes more the generalities than the specifics, and so case by case considerations are still useful. In the remainder of this book we shall do just that, at least for the most conspicuous and developed applications of nonequilibrium quantum field theory.

