Glasgow Math. J. **54** (2012) 309–324.
doi:10.1017/S0017089511000619.

GLOBAL EXISTENCE AND BLOW-UP FOR A DOUBLY DEGENERATE PARABOLIC EQUATION SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

YONG-SHENG MI

College of Mathematics and Statistics, Chongqing University, Chongqing 400044, PR China; College of Mathematics and Computer Sciences, Yangtze Normal University, Fuling 408100, Chongqing, PR China
e-mail: miyongshen@163.com

CHUN-LAI MU

College of Mathematics and Statistics, Chongqing University, Chongqing 400044, PR China
e-mail: chunlaimu@yahoo.com.cn

and DENG-MING LIU

College of Mathematics and Statistics, Chongqing University, Chongqing 400044, PR China
e-mail: liudengming08@163.com

(Received 3 May 2011; accepted 24 September 2011; first published online 12 December 2011)

Abstract. In this paper, we deal with the global existence and blow-up of solutions to a doubly degenerative parabolic system with nonlinear boundary conditions. By constructing various kinds of sub- and super-solutions and using the basic properties of M-matrix, we give the necessary and sufficient conditions for global existence of non-negative solutions, which extend the recent results of Zheng, Song and Jiang (S. N. Zheng, X. F. Song and Z. X. Jiang, Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux, *J. Math. Anal. Appl.* **298** (2004), 308–324), Xiang, Chen and Mu (Z. Y. Xiang, Q. Chen, C. L. Mu, Critical curves for degenerate parabolic equations coupled via nonlinear boundary flux, *Appl. Math. Comput*. **189** (2007), 549–559) and Zhou and Mu (J. Zhou and C. L Mu, On critical Fujita exponents for degenerate parabolic system coupled via nonlinear boundary flux, *Pro. Edinb. Math. Soc.* **51** (2008), 785–805) to more general equations.

2000 *Mathematics Subject Classification.* 35K55, 35K65, 35B40.

1. Introduction. In this paper, we investigate the existence and non-existence of global weak solutions to the following doubly degenerate parabolic equation

$$u_{it} = \left(|u_{ix}|^{p_i}\left(u_i^{m_i}\right)_x\right)_x \quad (i = 1, 2, \dots, k), \quad x > 0, 0 < t < T, \tag{1.1}$$

coupled via nonlinear boundary flux,

$$-|u_{ix}|^{p_i}\left(u_i^{m_i}\right)_x(0, t) = \prod_{j=1}^{k} u_j^{q_{ij}}(0, t) \quad (i = 1, 2, \dots, k), \quad 0 < t < T, \tag{1.2}$$

with initial data

$$u_i(x,0) = u_{i0}(x) \quad (i = 1, 2, \ldots, k), \quad x > 0, \tag{1.3}$$

where parameters $k \geq 1$, $m_i \geq 1$, $p_i > 0$, $q_{ij} > 0$ $(i = 1, 2, \ldots, k)$ and u_{i0} $(i = 1, 2, \ldots, k)$ are non-negative continuous functions with compact support in R_+. Let the initial data be appropriately smooth functions and satisfy the compatibility condition.

Nonlinear parabolic equation (1.1) comes from the theory of turbulent diffusion (see [**5**, **11**] and references therein) and appears in population dynamics, chemical reactions, heat transfer and so on. Equation (1.1) includes both the porous medium operator (with $p = 0$) and the gradient-diffusivity p-Laplacian operator (m = 1) as special cases, which have been the subject of intensive study (see [**5**, **7**, **10**, **11**, **13**, **16**, **23–25**, **29**, **36**, **38**] and references therein).

As it is well known that degenerate equations do not possess classical solutions; however, the local in time existence of the weak solution $(u_1, u_2, \ldots, u_k)$ to the problems (1.1)–(1.3), defined in the usual integral way, as well as a comparison principle, can be easily established by using the standard theory of parabolic equations (see [**6**, **15**, **22**, **36**). Let T be the maximal existence time of a solution $(u_1, u_2, \ldots, u_k)$, which may be finite or infinite. If $T < \infty$, then $\| u_1 \|_\infty + \| u_2 \|_\infty + \cdots + \| u_k \|_\infty$ becomes unbounded in finite time and we say that the solution blows up. If $T = \infty$ we say that the solution is global.

The problems on blow-up and global existence conditions and blow-up rates to nonlinear parabolic equations have been intensively studied (see [**1**, **3–5**, **7**, **9**, **10**, **13–15**, **18–20**, **27**, **30**, **32**, **33–41**] and references therein). In particular, the critical Fujita exponents are very interesting for various nonlinear parabolic equations of mathematical physics (see [**5**, **16**, **29**, **30**, **32–36**, **38–41**] and references therein). The concept of the critical Fujita exponents was proposed by Fujita in the 1960s (see [**9**]) during discussion of the heat conduction equation with a nonlinear source.

Galaktionov and Levine in [**10**] studied the single equation case,

$$\begin{array}{ll} u_t = (u^m)_{xx}, & x > 0, 0 < t < T, \\ -(u^m)_x(0,t) = u^p(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0, \end{array} \tag{1.4}$$

and the heat conduction equation with gradient diffusion

$$\begin{array}{ll} u_t = \left(|u_x|^{m-1}u_x\right)_x, & x > 0, 0 < t < T, \\ -|u_x|^{m-1}u_x(0,t) = u^p(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0, \end{array} \tag{1.5}$$

with $m \geq 1, p > 0$ and u_0 has compact support. They proved that for problem (1.4) the critical global exponent is $p_0 = \frac{1}{2}(m+1)$ and the critical Fujita exponent is $p_c = m + 1$, while for problem (1.5) the critical global exponent is $p_0 = \frac{2m}{m+1}$ and the critical Fujita exponent is $p_c = 2m$. The critical global existence exponent and the critical Fujita exponent of (1.5) were also considered in [**10**] for a special case $m = 1$.

Quiros and Rossi in [**27**] considered degenerate equations

$$\begin{array}{lll} u_t = (u^m)_{xx}, & v_t = (v^n)_{xx} & x > 0, 0 < t < T, \\ -(u^m)_x(0,t) = v^p(0,t), & -(v^n)_x(0,t) = u^q(0,t) & 0 < t < T, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x) & x > 0 \end{array} \tag{1.6}$$

with notation

$$\alpha_1 = \frac{2p+n+1}{(m+1)(n+1)-4pq}, \qquad \alpha_2 = \frac{2q+m+1}{(m+1)(n+1)-4pq},$$
$$\beta_1 = \frac{p(m-1-2q)+(n+1)m}{(m+1)(n+1)-4pq}, \ \beta_2 = \frac{q(n-1-2p)+(m+1)n}{(m+1)(n+1)-4pq}.$$

They proved that solutions of (1.6) are global if $pq \leq \frac{1}{4}(m+1)(n+1)$, and may blow up in finite time if $pq > \frac{1}{4}(m+1)(n+1)$. In the case of $pq > \frac{1}{4}(m+1)(n+1)$ if $\alpha_1 + \beta_1 \leq 0$, or $\alpha_2 + \beta_2 \leq 0$, then every non-negative and non-trivial solutions of (1.6) blow up in finite time: If $\alpha_1 + \beta_1 > 0$ and $\alpha_2 + \beta_2 > 0$, then there exist blow-up solutions for large initial and global solutions for small initial data. The critical Fujita exponents to (1.6) are described by $\alpha_i + \beta_i = 0$, $i = 1, 2$, while the blow-up rate of the positive solution is $O((T-t)^{-\alpha_1})$ for component u and $O((T-t)^{-\alpha_2})$ for v as $t \to T$.

Zheng, Song and Jiang [**38**] considered the degenerate equations coupled via nonlinear boundary flux

$$\begin{array}{lll} u_t = (u^m)_{xx}, & v_t = (v^n)_{xx} & x > 0, 0 < t < T, \\ -(u^m)_x(0,t) = u^\alpha(0,t)v^p(0.t), & -(v^n)_x(0,t) = u^q(0,t)v^\beta(0.t) & 0 < t < T, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x) & x > 0 \end{array} \tag{1.7}$$

with notations

$$k_1 = \frac{2p+n+1-2\beta}{4pq-(n+1-2\alpha)(m+1-2\beta)}, \quad l_1 = \frac{1-k_1(m-1)}{2}, \tag{1.8}$$
$$k_2 = \frac{2p+m+1-2\beta}{4pq-(n+1-2\alpha)(m+1-2\beta)}, \quad l_2 = \frac{1-k_2(n-1)}{2}. \tag{1.9}$$

They proved that solutions of (1.7) are global if $\alpha < \frac{1}{2}(m+1)$, $\beta < \frac{1}{2}(n+1)$ and $pq \leq (\frac{1}{2}(m+1)-\alpha)(\frac{1}{2}(n+1)-\beta)$, and may blow up in finite time if $\alpha > \frac{1}{2}(m+1)$ or $\beta > \frac{1}{2}(n+1)$. In the case of $\alpha > \frac{1}{2}(m+1)$, $\beta > \frac{1}{2}(n+1)$ and $pq > (\frac{1}{2}(m+1)-\alpha)(\frac{1}{2}(n+1)-\beta)$ if $l_1 < k_1$, or $l_2 < k_2$ or $l_1 = k_1$ and $l_2 = k_2$, then every non-negative and non-trivial solutions of (1.7) blow up in finite time: If $l_1 < k_1$ and $l_1 > k_1$, then there exist blow-up solutions for large initial and global solutions for small initial data. The critical Fujita exponents to (1.7) are described by $k_i = l_i$ $(i = 1, 2)$, while the blow-up rate of the positive solution is $O((T-t)^{-k_1})$ for component u and $O((T-t)^{-k_2})$ for v as $t \to T$.

Zhou and Mu in [**27**] considered the following problem:

$$\begin{array}{ll} u_t = \left(|u_x|^{m-1}u_x\right)_x, & v_t = \left(|v_x|^{n-1}v_x\right)_x, \quad x > 0, 0 < t < T, \\ -|u_x|^{m-1}u_x(0,t) = u^\alpha(0,t)v^p(0.t), & -|v_x|^{n-1}v_x(0,t) = u^q(0,t)v^\beta(0.t), \quad 0 < t < T, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), \quad x > 0, \end{array} \tag{1.10}$$

where $m > 1, n > 1, p > 0, q > 0, \alpha \geq 0, \beta \geq 0$ and $u_0(x)$, $v_0(x)$ are continuous, non-negative and compactly supported in R_+. They obtained the critical global existence curve and the critical Fujita curve; the blow-up rates of the non-global solution were also obtained.

https://doi.org/10.1017/S0017089511000619 Published online by Cambridge University Press

Xiang, Chen and Mu [37] considered the following degenerate equations:

$$\begin{cases} u_{it} = \left(|u_{ix}|^{m_i-1}u_{ix}\right)_x & (i = 1, 2, \ldots, k), \quad x > 0, \qquad 0 < t < T, \\ -|u_{ix}|^{m_i-1}u_{ix}(0, t) = u_{i+1}^{p_i}(0, t) & (i = 1, 2, \ldots, k), \quad u_{k+1} := u_1, \quad 0 < t < T, \\ u_i(x, 0) = u_{i0}(x) & (i = 1, 2, \ldots, k), \quad x > 0, \end{cases} \tag{1.11}$$

where parameters $k \geq 2, m_i > 1, p_i > 0, u_{i0},\ v_{i0}\ (i = 1, 2, \ldots, k)$ are continuous, non-negative functions. They obtained the critical global existence curve and the critical Fujita type curve.

In [11], Galaktionov and Levine studied the following single equation:

$$\begin{aligned} &u_t = \nabla(|\nabla u|^\sigma \nabla u^m) + u^p, && x \in R^N, \quad t > 0, \\ &u(x, 0) = u_0(x), && x \in R^N, \end{aligned}$$

where $\sigma > 0, \quad m > 1, \quad p > 1$ and $u_0(x)$ is a bounded positive continuous function. They showed that the critical exponent is $p_c = m + \sigma + \frac{\sigma+2}{N}$.

Recently, Jiang and Zheng [13] studied the following single equation:

$$\begin{cases} u_t = \left(|u_x|^\beta (u^m)_x\right)_x, & x > 0, 0 < t < T, \\ -|u_x|^\beta (u^m)_x(0, t) = u^p(0, t), & 0 < t < T, \\ u(x, 0) = u_0(x), & x > 0, \end{cases} \tag{1.12}$$

where $m \geq 1,\ p > 0,\ \ \beta > 0$. They obtained the critical global existence exponent $p_0 = \frac{2\beta+m+1}{\beta+2}$ and the critical Fujita exponent $p_c = 2\beta + m + 1$. These results are the extensions of those of Galaktionov and Levine [10].

In [3], Chen, Mi and Mu studied the following problem:

$$\begin{cases} u_t = \left(|u_x|^{p_1}(u^{m_1})_x\right)_x, & v_t = \left(|v_x|^{p_2}(v^{m_2})_x\right)_x, \quad x > 0,\ 0 < t < T, \\ -|u_x|^{p_1}(u^{m_1})_x(0, t) = u^{\alpha_1}(0, t)v^{\beta_2}(0.t), & \qquad 0 < t < T, \\ -|v_x|^{p_2}(v^{m_2})_x(0, t) = u^{\alpha_2}(0, t)v^{\beta_1}(0.t), & \qquad 0 < t < T, \\ u(x, 0) = u_0(x), & v(x, 0) = v_0(x), \qquad x > 0, \end{cases} \tag{1.13}$$

where parameters $m_i \geq 1,\ p_i > 0,\ \alpha_i > 0,\ \beta_i > 0\ (i = 1, 2)$ and u_0, v_0 are non-negative continuous functions with compact support in R_+. They obtained the critical global existence curve and the critical Fujita type curve, but classification of global existence and non-existence of solutions to system (1.13) is very complicated.

Motivated by the references cited above, the aim of this paper is to give a simple criteria of the classification of global existence and non-existence of solutions to systems (1.1)–(1.3) by using a combination of various kinds of self-similar sub- or super-solutions and the basic properties of the so-called M-matrix for general powers m_i, indices p_{ij} and number $k \geq 1$, which complicate the interaction among various components u_i. Paradoxically, our proof is more simple than that of [3, 38, 41] in the sense that we do not need some specific computations of parameters in the construction of self-similar sub- or super-solutions, even though we are dealing with an abstract system without specific number k.

To proceed further, we introduce some useful symbols from the matrix theory. Following [2, 16], $A \geq 0$ if each element of the vector or matrix A is non-negative, and $A > O$ if at least one element is positive, while $A \gg 0$ if each element is positive. Symbols $\leq, <$ and $\ll$ can be similarly understood. We also need the following important definitions.

DEFINITION 1.1. A $k \times l$ matrix C is said to be reducible if there exists a permutation matrix Q such that

$$QCQ^T = \begin{pmatrix} C_1 & 0 \\ C_2 & C_3 \end{pmatrix},$$

where C_1 and C_2 are square matrices and Q^T is the transpose of Q. Otherwise, C is said to be irreducible.

Throughout this paper, we let

$$P = \left(\frac{(p_i + 2)q_{ij}}{m_i + 2p_i + 1} \right) \tag{1.14}$$

be a matrix of order k. Without loss of generality, we assume that matrix P is irreducible, since if not the case, systems (1.1)–(1.3) can be reduced to two subsystems with one being not coupled with the order. When $\det(I - P) \neq 0$, we denote by $k := (k_1, k_2, \ldots, k_k)^T$ the unique solution of the following linear algebraic system:

$$(I - P)k = \left(-\frac{p_1 + 1}{m_1 + 2p_1 + 1}, -\frac{p_2 + 1}{m_2 + 2p_2 + 1}, \ldots, -\frac{p_k + 1}{m_k + 2p_k + 1} \right)^T, \tag{1.15}$$

where I is an identity matrix of order k, and then define

$$l_i = \frac{1 - p_i - m_i}{p_i + 2} k_i + \frac{1}{p_i + 2} \quad (i = 1, 2, \ldots, k). \tag{1.16}$$

To state our results, we also need some concepts from the theory of M-matrices, which have important applications, for instance, in the study of the Markov chains, in iterative methods in numerical computations and in the blow-up analysis of parabolic systems in bounded domain and source terms (see [**2**, **17**, **21**]). In this paper, we show that M-matrices play a key role on the global existence and non-existence of systems (1.1)–(1.3).

DEFINITION 1.2. A matrix C is called an M-matrix if C can be expressed in the form

$$C = sI - B, \quad s > 0, \quad B \geq 0 \tag{1.17}$$

with $s \geq \rho(B)$, the spectral radius of matrix B.

REMARK 1.1. A matrix C is an M-matrix if and only if all of the principal minors of C are non-negative (see [**2**]). In [**26**, **31**], the authors use the signs of principal minors to describe the global existence and non-existence for a different problem.

Our main results are stated as follows.

THEOREM 1.1. (1) *If $I - P$ is an M-matrix, then every non-negative solution of systems (1.1)–(1.3) is global in time.* (2) *If $I - P$ is not an M-matrix with $k_i > 0$ for some i or there exists i such that $q_{ii} > \frac{p_i+2}{m_i+2p_i+1}$, then systems (1.1)–(1.3) have a solution that blows up.*

REMARK 1.2. Theorem 1.1 suggests that the global existence or non-existence is completely characterised by whether the matrix $I - P$ is M-matrix or not, in case that the algebraic system (1.13) has a solution k with $k_i > 0$ for some i. The assumption on k, which holds naturally if one investigates the systems studied in [**3**, **24**, **37**, **38**, **41**], is rather technical. On the other hand, if there exists i such that $q_{ii} > \frac{p_i+2}{m_i+2p_i+1}$, then $I - P$ is not an M-matrix by Remark 1.1. Therefore, we believe that the critical characterisation of global existence or non-existence of systems (1.1)–(1.3) should be given by $I - P$ being M-matrix or not.

Since we are studying parabolic equations posed on an unbounded interval, in the case that there exist non-global solutions, there should exist another important critical characterisation, the so-called Fujita type critical curve, which describes when all solutions are non-global and there exist global solutions. Our next theorem is related to this question. Note that there are no such results for the problem posed on a bounded domain (see [**8**]).

THEOREM 1.2. *Assume that $I - P$ is not an M-matrix and that system (1.13) has unique solution k with $k_i > 0$ for some i.* (1) *If $max_i\{l_i - k_i\} < 0$, then every non-negative, non-trivial solution of systems (1.1)–(1.3) blows up in finite time.* (2) *If $min_i\{l_i - k_i\} > 0$, then there exists a global non-negative solution to systems (1.1)–(1.3).*

REMARK 1.3. Theorem 1.2 is a partial result of the Fujita type. We believe that the critical Fujita results should be characterised by $min_i\{l_i - k_i\} = 0$. The restriction $max\{l_i - k_i\} < 0$ in Theorem 1.2(2) is rather technical, it comes from the construction of the so-called Zel'dovich–Kompaneetz–Barenblatt profile [**10**, **15**, **28**].

REMARK 1.4. Unfortunately, we cannot obtain the blow-up rates of the non-global solution. We expect to answer this question in near future.

The rest of this paper is organised as follows. In Section 2, we give preliminary properties of M-matrix and the proof of Theorem 1.1. The proof of Theorem 1.2 is shown in Section 3.

2. Proof of Theorem 1.1. In this section, we characterise when solutions to problems (1.1)–(1.3) are global in time for any initial data or may blow up for large initial values. Our methods of establishing the global existence or non-existence are based on M-matrix, the construction of self-similar solutions and the comparison principle. Thus, we begin with presenting the basic properties of M-matrix, whose proof can be found in [**2**, **16**].

LEMMA 2.1. *(1) If C is an irreducible M-matrix of order k, then there exists a vector $x \gg 0$ such that $Cx \geq 0$; (2) if an irreducible matrix C of the form (1.15) is not an M-matrix, then there exists a vector $x \gg 0$ such that $Cx \ll 0$.*

We now prove that all solutions are global if $I - P$ is an M-matrix.

Proof of Theorem 1.1. (1) In order to prove that the solution $(u_1, u_2, \ldots, u_k)$ of (1.1)–(1.3) is global, we look for a globally defined in time super-solution of the self-similar form

$$\overline{u}_i(x,t) = e^{\kappa_{2i-1}t}(M + e^{-L_i x e^{-\kappa_{2i}t}})^{\frac{1}{m_i}}, \quad (i = 1, 2, \ldots, k), \quad x \geq 0, t \geq 0,$$

where parameters L_i and κ_{2i} will be chosen later, $M = max_i\{\| u_{0i} \|_\infty^{m_i} + 1\}$. Obviously, we have $\overline{u}_i(x, 0) \geq u_{0_i}(x)$, $(i = 1, 2, \ldots, k)$ for $x \geq 0$. After a direct computation, we obtain

$$\overline{u}_{it} \geq \kappa_{2i-1} e^{\kappa_{2i-1} t} (M + e^{-L_i x e^{-\kappa_{2i} t}})^{\frac{1}{m_i}} \geq \kappa_{2i-1} e^{\kappa_{2i-1} t} M^{\frac{1}{m_i}},$$

$$|\overline{u}_{ix}|^{p_i} (\overline{u}_i^{m_i})_x = -\frac{L_i^{p_i+1}}{m_i^{p_i}} e^{p_i(\kappa_{2i-1}-\kappa_{2i})t + (m_i \kappa_{2i-1} - \kappa_{2i})t} e^{-(L_i x + p_i L_i x) e^{-\kappa_{2i} t}} (M + e^{-L_i x e^{-\kappa_{2i} t}})^{p_i(\frac{1}{m_i}-1)},$$

$$(|\overline{u}_{ix}|^{p_i} (\overline{u}_i^{m_i})_x)_x \leq (p_i+1) \frac{L_i^{p_i+2}}{m_i^{p_i}} e^{p_i(\kappa_{2i-1}-\kappa_{2i})t + (m_i \kappa_{2i-1} - 2\kappa_{2i})t} M^{p_i(\frac{1}{m_i}-1)}$$

in $R_+ \times R_+$, $i = 1, 2, \ldots, k$. On the other hand, on the boundary we have

$$-|\overline{u}_{ix}|^{p_i} \left(\overline{u}_{ix}^{m_i}\right)_x (0, t) = \frac{L_i^{p_i+1}}{m_i^{p_i}} e^{p_i(\kappa_{2i-1}-\kappa_{2i})t + (m_i \kappa_{2i-1} - \kappa_{2i})t} (M+1)^{p_i(\frac{1}{m_i}-1)},$$

$$\prod_{j=1}^{k} \overline{u}_j^{q_{ij}} (0, t) = (M+1)^{\sum_{j=1}^{k} \frac{q_{ij}}{m_j}} e^{t \sum_{j=1}^{k} q_{ij} \kappa_{2j-1}}.$$

Therefore, we can see that $(\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k)$ is a super-solution of problems (1.1)–(1.3) provided that

$$\kappa_{2i-1} e^{\kappa_{2i-1} t} M^{\frac{1}{m_i}} \geq (p_i+1) \frac{L_i^{p_i+2}}{m_i^{p_i}} e^{p_i(\kappa_{2i-1}-\kappa_{2i})t + (m_i \kappa_{2i-1} - 2\kappa_{2i})t} M^{p_i(\frac{1}{m_i}-1)} \tag{2.1}$$

and

$$\frac{L_i^{p_i+1}}{m_i^{p_i}} e^{p_i(\kappa_{2i-1}-\kappa_{2i})t + (m_i \kappa_{2i-1} - \kappa_{2i})t} (M+1)^{p_i(\frac{1}{m_i}-1)} \geq (M+1)^{\sum_{j=1}^{k} \frac{q_{ij}}{m_j}} e^{t \sum_{j=1}^{k} q_{ij} \kappa_{2j-1}}. \tag{2.2}$$

In order to verify inequalities (2.1) and (2.2), we only need to impose

$$\kappa_{2i-1} \geq (p_i + m_i)\kappa_{2i-1} - (p_i+2)\kappa_{2i} \quad (i = 1, 2, \ldots, k), \tag{2.3}$$

$$p_i(\kappa_{2i-1} - \kappa_{2i}) + m_i \kappa_{2i-1} - \kappa_{2i} \geq \sum_{j=1}^{k} q_{ij} \kappa_{2j-1} \quad (i = 1, 2, \ldots, k) \tag{2.4}$$

and

$$\kappa_{2i-1} M^{\frac{1}{m_i}} \geq (p_i+1) \frac{L_i^{p_i+2}}{m_i^{p_i}} M^{p_i(\frac{1}{m_i}-1)} \quad (i = 1, 2, \ldots, k), \tag{2.5}$$

$$\frac{L_i^{p_i+1}}{m_i^{p_i}} (M+1)^{p_i(\frac{1}{m_i}-1)} \geq (M+1)^{\sum_{j=1}^{k} \frac{q_{ij}}{m_j}} \quad (i = 1, 2, \ldots, k). \tag{2.6}$$

Now we show that such choice in (2.3)–(2.6) is valid. Firstly, by taking

$$L_i = m_i^{\frac{p_i}{p_i+1}} (M+1)^{\frac{1}{p_i+1} \sum_{j=1}^{k} \frac{q_{ij}}{m_j} - \frac{p_i - m_i p_i}{m_i(p_i+1)}},$$

we see that (2.6) holds. Secondly, to obtain (2.3) we take $\kappa_{2i-1} = (p_i + m_i)\kappa_{2i-1} - (p_i + 2)\kappa_{2i}$, $(i = 1, 2, \ldots, k)$, that is

$$\kappa_{2i} = \frac{p_i + m_i - 1}{p_i + 2} \kappa_{2i-1} \quad (i = 1, 2, \ldots, k). \tag{2.7}$$

Meanwhile, we must ensure that such choices are suitable for inequalities (2.4). To this end, we substitute (2.7) into (2.4) and then (2.4) becomes

$$\frac{m_i + 2p_i + 1}{p_i + 2}\kappa_{2i-1} \geq \sum_{j=1}^{k} q_{ij}\kappa_{2j-1} \quad (i = 1, 2, \ldots, k), \tag{2.8}$$

or equivalently

$$\sum_{j=1}^{k}\left(\delta_{ij} - \frac{(p_i + 2)q_{ij}}{m_i + 2p_i + 1}\right)\kappa_{2j-1} \geq 0 \quad (i = 1, 2, \ldots, k). \tag{2.9}$$

As a result, we are left with showing the existence of $(\kappa_1, \kappa_3, \ldots, \kappa_{2k-1})$ satisfying (2.5) and (2.9). To do this, we recall Definition (1.12) of matrix P, and see that (2.9) is equivalent to the existence of non-negative solutions to the algebraic system

$$(I - P)(\kappa_1, \kappa_3, \ldots, \kappa_{2k-1})^T \geq (0, 0, \ldots 0)^T. \tag{2.10}$$

It follows from Lemma 2.1(1) that there exists $(\kappa_1, \kappa_3, \ldots, \kappa_{2k-1})^T \gg (0, 0, \ldots 0)^T$ solving (2.10) under the assumption that $I - P$ is an M-matrix. Since (2.10) is a homogeneous linear system, we can further choose each $\kappa_{2i-1} > 0$ large enough such that (2.5) holds.

Therefore, we have proved that $(\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k)$ is a global super-solution of systems (1.1)–(1.3). Hence, the comparison principle gives $(\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k) \geq (u_1, u_2, \ldots, u_k)$ and we conclude that $(u_1, u_2, \ldots, u_k)$ is global.

(2) For the case $k_i > 0$ for some i, we show that (1.1)–(1.3) has non-global sub-solution of the self-similar form

$$\underline{u}_i(x, t) = (T - t)^{-k_i} f_i(\xi_i), \qquad \xi_i = x(T - t)^{-l_i} \quad (i = 1, 2, \ldots, k), \tag{2.11}$$

where $k_i, l_i (i = 1, 2, \ldots, k)$ were defined as before, T is a positive constant and $f_i \geq 0$ $(i = 1, 2, \ldots, k)$ are the compactly supported functions to be determined.

After some computations, we have

$$\begin{aligned}
&\underline{u}_{it} = (T - t)^{-(k_i+1)}\big(k_i f_i(\xi_i) + l_i \xi_i f_i'(\xi_i)\big),\\
&|\underline{u}_{ix}|^{p_i}\big(\underline{u}_i^{m_i}\big)_x = (T - t)^{-p_i k_i - p_i l_i - m_i k_i - l_i}|f_i'|^{p_i}\big(f_i^{m_i}\big)'(\xi_i),\\
&\big(|\underline{u}_{ix}|^{p_i}\big(\underline{u}_i^{m_i}\big)_x\big)_x = (T - t)^{-p_i k_i - p_i l_i - m_i k_i - 2l_i}\big(|f_i'|^{p_i}\big(f_i^{m_i}\big)'(\xi_i)\big)'
\end{aligned}$$

and

$$\begin{aligned}
&|\underline{u}_{ix}|^{p_i}\big(\underline{u}_i^{m_i}\big)_x(0, t) = (T - t)^{-p_i k_i - p_i l_i - m_i k_i - l_i}|f_i'|^{p_i}\big(f_i^{m_i}\big)'(0),\\
&\prod_{j=1}^{k}\underline{u}_j^{q_{ij}}(0, t) = (T - t)^{-\sum_{j=1}^{k} q_{ij}k_j}\prod_{j=1}^{k} f_j^{q_{ij}}(0).
\end{aligned}$$

By using (1.13) and (1.14), we have

$$k_i + 1 = p_i k_i + p_i l_i + m_i k_i + 2l_i, \quad p_i k_i + p_i l_i + m_i k_i + l_i = \sum_{j=1}^{k} q_{ij}k_j \quad (i = 1, 2, \ldots, k).$$

Thus, $(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k)$ is a sub-solution of (1.1)–(1.3) provided that

$$\big(|f_i'|^{p_i}\big(f_i^{m_i}\big)'(\xi_i)\big)' \geq k_i f_i(\xi_i) + l_i f_i'(\xi_i)\xi_i, \tag{2.12}$$

$$-|f_i'|^{p_i}\big(f_i^{m_i}\big)'(0) \leq \prod_{j=1}^{k} f_j^{q_{ij}}(0). \tag{2.13}$$

Set

$$f_i(\xi_i) = A_i(a_i - \xi_i)_+^{\frac{p_i+1}{p_i+m_i-1}} \quad (i = 1, 2, \ldots, k), \tag{2.14}$$

where $A_i > 0$ and $a_i > 0$ $(i = 1, 2, \ldots, k)$ are constants to be determined. It is easy to see that

$$f_i'(\xi_i) = -A_i \frac{p_i+1}{p_i+m_i-1}(a_i - \xi_i)_+^{\frac{p_i+1}{p_i+m_i-1}-1}, \tag{2.15}$$

$$|f_i'|^{p_i}\big(f_i^{m_i}\big)' = -m_i A_i^{m_i+p_i}\left(\frac{p_i+1}{p_i+m_i-1}\right)^{p_i+1}(a_i - \xi_i)_+^{\frac{p_i+1}{p_i+m_i-1}}, \tag{2.16}$$

$$\big(|f_i'|^{p_i}\big(f_i^{m_i}\big)'\big)' = m_i A_i^{m_i+p_i}\left(\frac{p_i+1}{p_i+m_i-1}\right)^{p_i+2}(a_i - \xi_i)_+^{\frac{p_i+1}{p_i+m_i-1}-1}. \tag{2.17}$$

Substituting (2.14)–(2.17) into (2.12), inequality (2.12) is valid provided that

$$\begin{aligned} &k_i A_i(a_i - \xi_i)_+^{\frac{p_i+1}{p_i+m_i-1}} - l_i\xi_i A_i \frac{p_i+1}{p_i+m_i-1}(a_i - \xi_i)_+^{\frac{p_i+1}{p_i+m_i-1}-1} \\ &\quad - m_i A_i^{m_i+p_i}\left(\frac{p_i+1}{p_i+m_i-1}\right)^{p_i+2}(a_i - \xi_i)_+^{\frac{p_i+1}{p_i+m_i-1}-1} \leq 0 \quad (i = 1, 2, \ldots, k). \end{aligned}$$

To show that the above inequalities hold, we choose a_i with

$$a_i = c_i A_i^{m_i+p_i-1} \quad (i = 1, 2, \ldots, k), \tag{2.18}$$

where

$$c_i = \frac{m_i(p_i+m_i-1)}{k_i(p_i+m_i-1) + |l_i|(p_i+1)}\left(\frac{p_i+1}{p_i+m_i-1}\right)^{p_i+2} \quad (i = 1, 2, \ldots, k),$$

by the assumption $k_i > 0$ for some i, we know that (2.11) is true.

On the other hand, the boundary conditions in (2.12) are satisfied if we have

$$A_i^{m_i+p_i}\rho_i a_i^{\frac{p_i+1}{p_i+m_i-1}} \leq \prod_{j=1}^{k} A_j^{q_{ij}} a_j^{\frac{q_{ij}(p_j+1)}{p_j+m_j-1}}, \tag{2.19}$$

where $\rho_i = m_i(\frac{p_i+1}{p_i+m_i-1})^{p_i+1} > 0$. According to (2.17), we see that (2.18) holds provided we choose A_i $(i = 1, 2, \ldots, k)$ to satisfy

$$A_i^{m_i+2p_i+1}\rho_i c_i^{\frac{p_i+1}{p_i+m_i-1}} \leq \prod_{j=1}^{k} A_j^{q_{ij}(p_j+2)} c_j^{\frac{q_{ij}(p_j+1)}{p_j+m_j-1}} \quad (i = 1, 2, \ldots, k), \tag{2.20}$$

https://doi.org/10.1017/S0017089511000619 Published online by Cambridge University Press

which is equivalent to

$$\prod_{j=1}^{k} \lambda_j^{\delta_{ij}-\frac{(p_i+2)q_{ij}}{m_i+2p_i+1}} \triangleq \prod_{j=1}^{k} \left(A_j^{p_j+2}\right)^{\delta_{ij}-\frac{(p_i+2)q_{ij}}{m_i+2p_i+1}} \geq d_i, \tag{2.21}$$

where

$$d_i = \left(\rho_i^{-1} c_i^{-\frac{p_i+1}{p_i+m_i-1}} \prod_{j=1}^{k} c_j^{\frac{q_{ij}(p_j+1)}{p_j+m_j-1}}\right)^{\frac{p_i+2}{m_i+2p_i+1}} \qquad (i = 1, 2, \ldots, k),$$

then inequality (2.20) can also be written as

$$\sum_{j=1}^{k} \left(\delta_{ij} - \frac{q_{ij}(p_i+2)}{m_i+2p_i+1}\right) \log \lambda_j \leq d_i, \quad (i = 1, 2, \ldots, k). \tag{2.22}$$

We show that this inequality is valid for some sufficiently large λ_j. Indeed, since p is irreducible, it is equivalent to $I - P$, which is also irreducible, and since we have assumed that $I - P$ is not an M-matrix, it follows from Lemma 2.1(2) that we can choose $\lambda_i > 3, (i = 1, 2, \ldots, k)$ such that $(I - P)(\log \lambda_1, \log \lambda_2, \ldots, \log \lambda_k)^T \ll (0, 0, \ldots 0)^T$. Then we can amplify $\log \lambda_i$ such that (2.22) holds.

Therefore, we have shown that $(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k)$ given by (2.11) and (2.14) is a sub-solution of systems (1.1)–(1.3) if we further choose the initial data $(u_{01}, u_{02}, \ldots, u_{0k})$ large enough such that

$$u_{0i} \geq u_i(x, 0) = T^{-k_i} f_i\left(\frac{x}{T^{l_i}}\right) (i = 1, 2, \ldots, k). \tag{2.23}$$

Noticing the construction of $f_i(\xi_i)$ and the assumption $k_i > 0$ for some i, we see that $\lim_{t\to T^-} \underline{u}_i(0, t) = +\infty$ for such i. Then it follows from the comparison principle that there exist non-global solution to systems (1.1)–(1.3).

Finally, we investigate the case that there exists i such that $q_{ii} > \frac{2p_i+m_i+1}{p_i+2}$. Without loss of generality, we assume $q_{11} > \frac{2p_1+m_1+1}{p_1+2}$. Consider the initial data satisfying $(|(u_{0i})'|^{p_i}(u_{0i}^{m_i})')' \geq 0 (i = 1, 2, \ldots, k)$, which imply that $u_{it} > 0$. The existence of such initial data is primary, since u_{0i} are independent of each other. It follows from the results of **[13]** that the following scalar equation

$$\begin{cases} u_{1t} = \left(|u_{1x}|^{p_1}\left(u_1^{m_1}\right)_x\right)_x, & x > 0, 0 < t < T, \\ -|u_{1x}|^{p_1}\left(u_1^{m_1}\right)_x(0, t) = u_1^{q_{11}}(0, t)\prod_{j=2}^{k} u_{0j}^{q_{1j}}(0), & 0 < t < T, \\ u_1(x, 0) = u_{01}(x), & x > 0 \end{cases} \tag{2.24}$$

has non-global solution. On the other hand, it is clear that $(u_{01}(x, t), u_{02}(x, t), \ldots, u_{0k}(x, t))$ would be a sub-solution of systems (1.1)–(1.3). Then the desired result follows from the comparison principle. □

3. Proof of Theorem 1.2. In this section, we consider a more subtle description when there exist non-global solutions to systems (1.1)–(1.3). We shall still prove Theorem 1.2 by constructing self-similar solutions and self-similar super-solutions

https://doi.org/10.1017/S0017089511000619 Published online by Cambridge University Press

and using the comparison principle; however, the fact that we are dealing with a system instead of a single equation forces us to develop some new techniques.

Proof of Theorem 1.2. (1) We construct the following well-known self-similar solution (the so-called Zel'dovich–Kompaneetz–Barenblatt profile [**10**, **15**, **28**]) to (1.1)–(1.3) in the form

$$u_{iB}(x,t) = (\tau+t)^{-\frac{1}{m_i+2p_i+1}} h_i(\xi_i), \qquad \xi_i = x(\tau+t)^{-\frac{1}{m_i+2p_i+1}} \quad (i=1,2,\ldots,k), \tag{3.1}$$

where $\tau > 0$ and

$$h_i(\xi_i) = C_i\left(c_i^{\frac{p_i+2}{p_i+1}} - \xi_i^{\frac{p_i+2}{p_i+1}}\right)_+^{\frac{p_i+1}{p_i+m_i-1}} \quad (i=1,2,\ldots,k), \tag{3.2}$$

with $c_i > 0$ $(i=1,2,\ldots,k)$, and

$$C_i = \left(\frac{1}{m_i(m_i+2p_i+1)}\left(\frac{p_i+m_i-1}{p_i+2}\right)^{p_i+1}\right)^{\frac{i}{p_i+m_i-1}}. \tag{3.3}$$

It is not difficult to check that

$$\begin{aligned}&\left(|h_i'|^{p_i}\left(h_i^{m_i}\right)'\right)'(\xi_i) + \frac{1}{m_i+2p_i+1}\xi_i h_i'(\xi_i) + \frac{1}{m_i+2p_i+1}h_i(\xi_i) = 0,\\ &h_1'(0) = 0 \quad (i=1,2,\ldots,k),\end{aligned}$$

Combining with $h_i'(0) = 0$ $(i=1,2,\ldots,k)$, implies $(u_{iB})_x(0,t) = 0$ $(i=1,2,\ldots,k)$. Since $u_i(x,t)$ $(i=1,2,\ldots,k)$ are non-trivial and non-negative, we see that $u_i(0,t_0) > 0$ $(i=1,2,\ldots,k)$ for some $t_0 > 0$ (compare with the Barenblatt solution of corresponding equations). Noticing that $u_i(x,t_0) > 0$ $(i=1,2,\ldots,k)$ are continuous (see [**12**, **36**]), we can choose τ large enough and c_i small enough so that

$$u_i(x,t_0) > u_{iB}(x,t_0)\ (i=1,2,\ldots,k) \quad for \quad x > 0.$$

A direct calculation shows that $(u_{1B}, u_{2B}, \ldots, u_{kB})$ is a weak sub-solution of (1.1)–(1.3) in $(0,+\infty)\times(t_0,+\infty)$. By the comparison principle, we obtain that

$$u_i(x,t) > u_{iB}(x,t) \quad (i=1,2,\ldots,k) \quad for \quad x>0, t>t_0.$$

Since $\max x_i\,\{l_i - k_i\} < 0$, we get $T^{l_i} \ll T^{k_i}$ for large T. So there exists $t* \geq t_0$ satisfying

$$T^{l_i} \ll (\tau+t^*)^{\frac{1}{m_i+2p_i+1}} \ll T^{k_i} \quad (i=1,2,\ldots,k). \tag{3.4}$$

Let $\underline{u}_i$ $(i=1,2,\ldots,k)$ be the function given by (2.11) and (2.14). Then for any $x > 0$,

$$\underline{u}_i(x,0) \leq u_{iB}(x,t^*) \leq u_i(x,t^*) \quad (i=1,2,\ldots,k).$$

It follows from the comparison principle that

$$\underline{u}_i(x,t) \leq u_i(x,t+t^*) \quad (i=1,2,\ldots,k), \quad for \quad x>0, \quad t>0.$$

As a proof of Theorem 1.1(2), we see that $(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k)$ blows up in a finite time T. Therefore, $(u_1, u_2, \ldots, u_k)$ blows up in a finite time, which is not larger than $T + t^*$.

Observing that (3.4) holds for general non-trivial $(u_{10}, u_{20}, \dots, u_{k0})$, we know that every non-negative and non-trivial solution of (1.1)–(1.3) blows up in finite time.

(2) In order to prove the conclusion, we only need to show that solutions of (1.1)–(1.3), which are small initial data, have global existence, which will be proved by constructing self-similar global super-solution,

$$\overline{u}_i(x,t) = (\tau + t)^{-k_i} F_i(\xi_i), \qquad \xi_i = x(\tau + t)^{-l_i}, \tag{3.5}$$

where k_i, l_i $(i = 1, 2, \dots, k)$ were defined as before, T is a positive constant and $F_i \geq 0$ $(i = 1, 2, \dots, k)$ are compactly supported functions to be determined.

After some computations, we have

$$\begin{aligned}
&\overline{u}_{it} = (\tau + t)^{-(k_i+1)}(-k_i F_i(\xi_i) - l_i \xi_i F_i{}'(\xi_i)),\\
&|\overline{u}_{ix}|^{p_i}\left(\overline{u}_i^{m_i}\right)_x = (\tau + t)^{-p_ik_i - p_il_i - m_ik_i - l_i} |F_i{}'|^{p_i}\left(F_i^{m_i}\right)'(\xi_i),\\
&\left(|\overline{u}_{ix}|^{p_i}\left(\overline{u}_i^{m_i}\right)_x\right)_x = (\tau + t)^{-p_ik_i - p_il_i - m_ik_i - 2l_i}\left(|F_i{}'|^{p_i}\left(F_i^{m_i}\right)'(\xi_i)\right)'
\end{aligned}$$

and

$$\begin{aligned}
&|\overline{u}_{ix}|^{p_i}\left(\overline{u}_i^{m_i}\right)_x(0,t) = (\tau + t)^{-p_ik_i - p_il_i - m_ik_i - l_i}|F_i{}'|^{p_i}\left(F_i^{m_i}\right)'(0),\\
&\prod_{j=1}^{k} \overline{u}_j^{q_{ij}}(0,t) = (\tau + t)^{-\sum_{j=1}^{k} q_{ij}k_j} \prod_{j=1}^{k} F_j^{q_{ij}}(0).
\end{aligned}$$

By using (1.13) and (1.14), we have

$$k_i + 1 = p_ik_i + p_il_i + m_ik_i + 2l_i, \quad p_ik_i + p_il_i + m_ik_i + l_i = \sum_{j=1}^{k} q_{ij}k_j \quad (i = 1, 2, \dots, k).$$

Thus, $(\overline{u}_1, \overline{u}_2, \dots, \overline{u}_k)$ is a super-solution of (1.1)–(1.3) provided that

$$\left(|F_i{}'|^{p_i}\left(F_i^{m_i}\right)'(\xi_i)\right)' + k_iF_i(\xi_i) + l_iF_i'(\xi_i)\xi_i \leq 0, \tag{3.6}$$

$$-|F_i{}'|^{p_i}\left(F_i^{m_i}\right)'(0) \geq \prod_{j=1}^{k} F_j{}^{q_{ij}}(0). \tag{3.7}$$

We choose

$$F_i(\xi_i) = A_iC_i\left((a_ib_i)^{\frac{p_i+2}{p_i+1}} - (\xi_i + a_i)^{\frac{p_i+2}{p_i+1}}\right)_+^{\frac{p_i+1}{p_i+m_i-1}} = A_ih_i(\xi_i + a_i) \quad (i = 1, 2, \dots, k), \tag{3.8}$$

where C_i $(i = 1, 2, \dots, k)$ were defined by (3.3), h_i $(i = 1, 2, \dots, k)$ were defined by (3.2), $a_i > 0$, $b_i > 1$ and $A_i > 0$ $(i = 1, 2, \dots, k)$. We claim that A_i, b_i, a_i $(i = 1, 2, \dots, k)$ exist such that inequality (3.6) is valid for F_i $(i = 1, 2, \dots, k)$ defined by (3.8), then $h_i(\xi_i + a_i)$ $(i = 1, 2, \dots, k)$ satisfy the following equations,

$$\left(|h_i'|^{p_i}\left(h_i^{m_i}\right)'\right)' = -\frac{1}{m_i + p_i + 1}(\xi_i + a_i)h_i' - \frac{1}{m_i + p_i + 1}h_i \quad (i = 1, 2, \dots, k), \tag{3.9}$$

and

$$h_i'(\xi_i + a_i) = -C_i \frac{p_i + 2}{p_i + m_i - 1} \left((a_i b_i)^{\frac{p_i+2}{p_i+1}} - (\xi_i + a_i)^{\frac{p_i+2}{p_i+1}} \right)_+^{\frac{p_i+1}{p_i+m_i-1} - 1} (\xi_i + a_i)^{\frac{1}{p_i+1}}. \quad (3.10)$$

In fact, when $a_i \leq \xi_i + a_i \leq b_i a_i$ $(i = 1, 2, \ldots, k)$ substituting (3.8)–(3.10) into (3.6), denoted by $y_i = \xi_i + a_i$ $(i = 1, 2, \ldots, k)$, then (3.6) can be transformed into the following inequality with respect y_i

$$G_i(y_i) = -e_{i1} y_i^{\frac{p_i+2}{p_i+1}} + e_{i2} a_i y_i^{\frac{1}{p_i+1}} - e_{i3} (a_i b_i)^{\frac{p_i+2}{p_i+1}} \leq 0 \ \ (i = 1, 2, \ldots, k), \quad (3.11)$$

where

$$e_{i1} = \left(k_i - \frac{A_i^{m_i+p_i-1}}{m_i + p_i - 1} \right) + \frac{p_i + 2}{p_i + m_i - 1} \left(l_i - \frac{A_i^{m_i+p_i-1}}{m_i + p_i - 1} \right) \quad (i = 1, 2, \ldots, k),$$
$$e_{i2} = \frac{l_i (p_i + 2)}{p_i + m_i - 1} \quad (i = 1, 2, \ldots, k),$$
$$e_{i3} = \frac{A_i^{m_i+p_i-1}}{m_i + p_i - 1} - k_i \ (i = 1, 2, \ldots, k).$$

Since $\min_i\{l_i - k_i\} > 0$, we can choose a suitable constant $A_1 > 0$ such that

$$l_1 > \frac{A_1^{m_1+p_1-1}}{m_1 + p_1 - 1} > k_1$$

and

$$\left(k_1 - \frac{A_1^{m_1+p_1-1}}{m_1 + p_1 - 1} \right) + \frac{p_1 + 2}{p_1 + m_1 - 1} \left(l_1 - \frac{A_1^{m_1+p_1-1}}{m_1 + p_1 - 1} \right) > 0;$$

for such A_1, it is easy to verify that $e_{1j} > 0$ $(j = 1, 2, 3)$ and $G_1(y_1)$ is a concave function with respect to $y_1^{\frac{1}{p_1+1}}$, then $G_1(y_1)$ attains its maximum at $z_{1*} = \frac{e_{12} a_1}{(p_1+2) e_{11}}$. Therefore, (3.11) is valid provided that

$$G_1(z_{1*}) = a_1^{\frac{p_1+2}{p_1+1}} \left\{ \frac{p_1 + 1}{p_1 + 2} \left(\frac{1}{e_{11}(p_1 + 2)} \right)^{\frac{1}{p_1+1}} e_{12}^{\frac{p_1+2}{p_1+1}} - e_{13} b_1^{\frac{p_1+2}{p_1+1}} \right\} \leq 0. \quad (3.12)$$

So, we only need to choose b_1 sufficiently large such that $b_1 \geq \max\{ (\frac{(p_1+1)e_{12}}{(p+2)e_{13}})^{\frac{p_1+1}{p_1+2}} (\frac{e_{12}}{(p+2)e_{11}})^{\frac{1}{p_1+2}}, 1\}$. Similarly, there exist $A_i > 0,\ b_i > 0$ $(i = 2, 3, \ldots, k)$ such that inequality (3.11) holds. Consequently, we have proved that inequality (3.6) is true.

Now we consider the boundary condition (3.7), we only need to show that

$$(A_i C_i)^{m_i+p_i} m_i \left(\frac{p_i + 2}{p_i + m_i - 1} \right)^{p_i+1} \left(b_i^{\frac{p_i+2}{p_i+1}} - 1 \right)^{\frac{p_i+1}{p_i+m_i-1}} a_i^{\frac{2p_i+m_i+1}{p_i+m_i-1}}$$
$$\geq \prod_{j=1}^{k} (A_j C_j)^{q_{ij}} \left(b_j^{\frac{p_j+2}{p_j+1}} - 1 \right)^{\frac{(p_j+1)q_{ij}}{p_j+m_j-1}} a_j^{\frac{(p_j+2)q_{ij}}{p_j+m_j-1}}$$

https://doi.org/10.1017/S0017089511000619 Published online by Cambridge University Press

with $C_i (i = 1, 2, \ldots, k)$ defined by (3.3). We are left with showing that for $A_i > 0$, d_i $(i = 1, 2, \ldots, k)$ fixed as above, we may take a_i small enough so that above inequality holds. To do this, we rewrite it as

$$\prod_{j=1}^{k} h_j^{\delta_{ij} - \frac{(p_i+2)q_{ij}}{m_i+2p_i+1}} \triangleq \prod_{j=1}^{k} \left(a_j^{\frac{p_j+2}{p_j+m_j-1}} \right)^{\delta_{ij} - \frac{(p_i+2)q_{ij}}{m_i+2p_i+1}} \geq m_i, \tag{3.13}$$

where

$$m_i = ((A_i C_i)^{-m_i-p_i} m_i \left(\frac{p_i + m_i - 1}{p_i + 2} \right)^{p_i+1} \left(b_i^{\frac{p_i+2}{p_i+1}} - 1 \right)^{\frac{-p_i-1}{p_i+m_i-1}}$$
$$\times \prod_{j=1}^{k} \left(A_j C_j)^{q_{ij}} \left(b_j^{\frac{p_j+2}{p_j+1}} - 1 \right)^{\frac{(p_j+1)q_{ij}}{p_j+m_j-1}} \right)^{\frac{p_i+2}{m_i+2p_i+1}}.$$

Without loss of generality, we assume $h_i < 1$ $(i = 1, 2, \ldots, k)$. Then (3.13) is equivalent to

$$\sum_{j=1}^{k} \left(\delta_{ij} - \frac{(p_i + 2)q_{ij}}{m_i + 2p_i + 1} \right) (-\log h_j) \leq -\log m_i. \tag{3.14}$$

Since $I - P$ is irreducible and is not an M-matrix, it follows from Lemma (2.1)(1) that we can choose $h_i \in (0, 1)$ small enough such that (3.14) holds, which completes the proof of (3.13). Thus, for the case $\min_i\{l_i - k_i\} > 0$ we have constructed a class of global self-similar super-solutions defined by (3.5) and (3.8). Owing to the comparison principle, the solution of problems (1.1)–(1.3) is global if the initial datum $(u_{10}, u_{20}, \ldots, u_{k0})$ is small enough. The proof of Theorem 1.2 is complete. □

Acknowledgements. This work is supported in part by NSF of China (11071266), in part by NSF project of CQ CSTC (2010BB9218) and partially by the Foundation Project of China Yangtze Normal University.

REFERENCES

1. G. Astrita and G. Marrucci, *Principles of non-Newtonian fluid mechanics* (McGraw-Hill, New York, 1974).

2. A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics (SIAM, Philadelphia, PA, 1994).

3. B. T. Chen, Y. S. Mi and C. L. Mu, Critical exponents for a doubly degenerate parabolic system coupled via nonlinear boundary flux, *Acta Mathematica Scientia* **31B** (2011), 681–693.

4. Z. J. Cui, Critical curves of the non-Newtonian polytropic filtration equations coupled with nonlinear boundary conditions, *Nonlinear Anal.* **68** (2008), 3201–3208.

5. K. Deng and H. A. Levine, The role of critical exponents in blow-up theorems: The sequel, *J. Math. Anal. Appl.* **243** (2000), 85–126.

6. E. Dibenedetto, *Degenerate parabolic equations* (Springer-Verlag, Berlin, 1993).

7. R. Ferreira, A. de Pablo, F. Quiros and J. D. Rossi, The blow-up profile for a fast diffusion equation with a nonlinear boundary condition, *Rocky Mt. J. Math.* **33** (2003), 123–146.

8. J. Filo, Diffusivity versus absorption through the boundary, *J. Differ. Equ.* **99** (1992), 281–305.

9. H. Fujita, On the blowing up of solutions of the Cauchy problem for$u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect.* I **13** (1966), 109–124.

10. V. A. Galaktionov and H. A. Levine, On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary, *Israel J. Math.* **94** (1996), 125–146.

11. V. A. Galaktionov and H. A. Levine, A general approach to critical Fujita exponents and systems, *Nonlinear Anal.* **34** (1998), 1005–1027.

12. A. V. Ivanov, Hoder estimates for quasilinear doubly degenerate parabolic equations, *J. Soviet Math.* **56** (1991), 2320–2347.

13. Z. X. Jiang, Doubly degenerate parabolic equation with nonlinear inner sources or boundary flux, Doctorate Thesis (Dalian University of Tcchnology, China, 2009).

14. C. H. Jin and J. X. Yin, Critical exponents and non-extinction for a fast diffusive polytropic filtration equation with nonlinear boundary sources, *Nonlinear Anal.* **67** (2007), 2217–2223.

15. A. S. Kalashnikov, Some problems of the qualitative theory of nonlinear degenerate parabolic equations of second order, *Uspekhi Mat. Nauk.* **42** (1987), 135–176 (English translation: *Russian Math. Surveys* **42** (1987), 169–222).

16. H. A. Levine, The role of critical exponents in blow up theorems, *SIAM Rev.* **32** (1990), 262–288.

17. Y. X. Li, Q. L. Liu and C. H. Xie, Semilinear reaction-diffusion systems of several components, *J. Diff. Equ.* **187** (2003), 510–519.

18. Z. P. Li and C. L. Mu, Critical curves for fast diffusive polytropic filtration equation coupled via nonlinear boundary flux, *J. Math. Anal. Appl.* **346** (2008), 55–64.

19. Z. P. Li and C. L. Mu, Critical curves for fast diffusive non-Newtonian equation coupled via nonlinear boundary flux, *J. Math. Anal. Appl.* **340** (2008), 876–883.

20. Z. P. Li, C. L. Mu and Z. J. Cui, Critical curves for a fast diffusive polytropic filtration system coupled via nonlinear boundary flux, *Z. Angew. Math. Phys.* **60** (2009), 284–296.

21. Y. X. Li and C. H. Xie, Quasi-linear parabolic systems of several components, *Math. Ann.* **327** (2003), 395–407.

22. G. M. Lieberman, Second order parabolic differential equations (World Scientific Publishing, River Edge, 1996).

23. Y. S. Mi, C. L. Mu and B. T. Chen, Blow-up analysis for a fast diffusive parabolic equation with nonlinear boundary flux (preprint).

24. Y. S. Mi, C. L. Mu and B. T. Chen, Critical exponents for a nonlinear degenerate parabolic system coupled via nonlinear boundary flux, *J. Korean Math. Soc.* **48** (2011), 513–527.

25. Y. S. Mi, C. L. Mu and B. T. Chen, Critical exponents for a fast diffusive parabolic system coupled via nonlinear boundary flux (preprint).

26. Y. W. Qi, M. X. Wang and Z. J. Wang, Existence and non-existence of global solutions of diffusion systems with nonlinear boundary conditions, *Proc. R. Soc. Edinb.* A **134** (2004), 1199–1217.

27. F. Quiros and J. D. Rossi, Blow-up set and Fujita-type curves for a degenerate parabolic system with nonlinear conditions, *Indiana Univ. Math. J.* **50** (2001), 629–654.

28. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov, *Blow-up in quasilinear parabolic equations* (Walter de Gruyter, Berlin, 1995).

29. J. L. Vazquez, *The porous medium equations: Mathematical theory* (Oxford University Press, Oxford, UK, 2007).

30. M. X. Wang, The blow-up rates for systems of heat equations with nonlinear boundary conditions, *Sci. China Ser.* A **46** (2003), 169–175.

31. M. X. Wang and C. H. Xie, Quasilinear parabolic systems with nonlinear boundary conditions, *J. Differ. Equ.* **166** (2000), 251–265.

32. S. Wang, C. H. Xie and M. X. Wang, Note on critical exponents for a system of heat equations coupled in the boundary conditions, *J. Math. Analysis Applic.* **218** (1998), 313–324.

33. S. Wang, C. H. Xie and M. X. Wang, The blow-up rate for a system of heat equations completely coupled in the boundary conditions, *Nonlinear Anal.* **35** (1999), 389–398.

34. Z. J. Wang, J. X. Yin and C. P. Wang, Critical exponents of the non-Newtonian polytropic filtration equation with nonlinear boundary condition, *Appl. Math. Lett.* **20** (2007), 142–147.

35. C. P. Wang, S. N. Zheng and Z. J. Wang, Critical Fujita exponents for a class of quasi-linear equations with homogeneous Neumann boundary data, *Nonlinearity* **20** (2007), 1343–1359.

36. Z. Q. Wu, J. N. Zhao, J. X. Yin and H. L. Li, *Nonlinear diffusion equations* (World Scientific Publishing, River Edge, NJ, 2001).

37. Z. Y. Xiang, Q. Chen, C. L. Mu, Critical curves for degenerate parabolic equations coupled via non-linear boundary flux, *Appl. Math. Comput.* **189** (2007) 549–559.

38. S. N. Zheng, X. F. Song and Z. X. Jiang, Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux, *J. Math. Anal. Appl.* **298** (2004), 308–324.

39. S. N. Zheng and C. P. Wang, Large time behaviour of solutions to a class of quasi-linear parabolic equations with convection terms, *Nonlinearity* **21** (2008), 2179–2200.

40. J. Zhou and C. L Mu, Critical curve for a non-Newtonian polytropic filtration system coupled via nonlinear boundary flux, *Nonlinear Anal.* **68** (2008), 1–11.

41. J. Zhou and C. L. Mu, On critical Fujita exponents for degenerate parabolic system coupled via nonlinear boundary flux, *Pro. Edinb. Math. Soc.* **51** (2008), 785–805.