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ON AN UPPER BOUND FOR THE HEAT KERNEL ON $\mathrm{SU}^*(2n)/\,\mathrm{Sp}(n)$

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ABSTRACT. Jean-Philippe Anker made an interesting conjecture in [2] about the growth of the heat kernel on symmetric spaces of noncompact type. For any symmetric space of noncompact type, we can write

$$P_t(x) = Ce^{-\|\rho\|^2 t} t^{-q/2} e^{-r^2/(4t)} \phi_0(x) V_t(x)$$

where ϕ_0 is the Legendre function and q, "the dimension at infinity", is chosen such that $\lim_{t\to\infty} V_t(x) = 1$ for all x. Anker's conjecture can be stated as follows: there exists a constant $C > 0$ such that

$$V_t(x) \le C \prod_{\eta\in\Sigma_0^+} \Big(1 + \frac{1+\eta}{t}\Big)^{(m_\eta+m_{2\eta})/2-1}$$

where Σ_0^+ is the set of positive indivisible roots. The behaviour of the function ϕ_0 is well known (see [1]).

The main goal of this paper is to establish the conjecture for the spaces $\mathrm{SU}^*(2n)/\,\mathrm{Sp}(n)$.

Introduction. In [2], Jean-Philippe Anker proves his conjecture for the spaces $\mathbf{U}(p,q)/\mathbf{U}(p)\times\mathbf{U}(q)$ and points out that it is also true for all symmetric spaces of rank 1. The conjecture is immediately verified in the complex case since $V_t(x)$ is then identically equal to 1 (see [7]).

$$V_t(x) \le C \prod_{\eta\in\Sigma_0^+} \Big(1 + \frac{1+\eta}{t}\Big)^{(m_\eta+m_{2\eta})/2-1} \tag{1}$$

We proved in [13] that for the space $\mathrm{Pos}(3,\mathbf{R})$, $V_t(x)$ is bounded above and below by constant multiples of the right hand side of (1). The corresponding results for the heat kernels of the real hyperbolic spaces have been obtained by E. B. Davies and N. Mandouvalos (see Theorem 5.7.2 in [6]). These last results, and the fact that this upper bound is the sharpest suggested for such spaces, make the conjecture particularly interesting.

The space $\mathrm{SU}^*(2n)/\,\mathrm{Sp}(n)$ can be realized as the space of positive definite matrices of determinant 1 over the quaternions $\big(\mathrm{Pos}_1(n,\mathbf{H})\big)$. We will instead work with $\mathrm{Pos}(n,\mathbf{H})$, the space of positive definite matrices over the quaternions. It is simple to translate our results from one space to the other. The Riemannian structure will be induced by the

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bilinear form $\langle X, Y\rangle = \Re \operatorname{tr} XY$ (a scalar multiple of the Killing form). Similar remarks apply to the other symmetric spaces of noncompact type that correspond to A_{n-1}.

It is known that $P_t = \mathcal{A}^{-1}(W_t)$ where $\mathcal{A}$ is the Abel transform and $W_t = Ce^{-\|\rho\|^2 t} t^{-n/2} e^{-r^2/(4t)}$. We will devote a good part of the paper in showing that $\mathcal{A}^{-1}$ is a differential operator with appropriate properties. The work of O. A. Chalykh and A. P. Veselov in [14] is particularly relevant here. They arrive at an explicit expression for the inverse of the Abel transform for the space $SU^*(2n)/Sp(n)$. The focus of their paper is to find a shift operator (as explained in [3]). The aim is to reduce the problem of finding $\mathcal{A}^{-1}$ for the root system A_{n-1} with multiplicities $m = 4$ to the case $m = 2$. We, on the other hand, shift the problem to the root system A_{n-1} with $m = -2$ and go on from there (Lemma 1.3 explains what the shift is). Moreover, the inverse is expressed in a different manner. We will point out the similarities and differences in our approaches in the conclusion. We would like to thank Jean-Philippe Anker for drawing our attention to this recent development.

We take the opportunity to thank Carl S. Herz for his suggestion that showing V_t has a finite expansion in t^{-1} would be a good starting point.

1. **The inverse of the Abel transform for the root system A_{n-1}.** In what follows, the root system under study is A_{n-1} and m is any complex number (the "multiplicity" of the roots). The Abel transform of f is $\mathcal{A}(f; e^H) = e^{\rho(H)} \int_N f(e^h n)\, dn$ (also denoted $F_f(e^H)$ in [8]). It is natural, using the usual Hilbert space structure on L^2 spaces, to define the dual or adjoint of the Abel transform (see [9] and [13]) for functions invariant under the Weyl group W:

$$\langle \mathcal{A}(h;\cdot), f\rangle_{L^2(A/W)} = \langle h, \mathcal{A}^*(f;\cdot)\rangle_{L^2(K\backslash G/K)}.$$

Using the definition of $\mathcal{A}$ and the integral formulas corresponding to the Iwasawa and the Cartan decompositions (refer to [8]), we find that $\mathcal{A}^*(f; e^H) = \int_K e^{-\rho(H(e^H k))} f(e^{H(e^H k)})\, dk$. In particular, the spherical functions are $\phi_\lambda = \mathcal{A}^*(e^{i\lambda};\cdot)$. This is valid for any symmetric space of noncompact type G/K.

In [12], we gave a recursive integral equation for the dual of the Abel transform for the spaces of positive definite matrices over the real numbers, the complex numbers and over the quaternion numbers (Theorem 1.1). It allows us to discuss the generalized Abel transform (or rather its dual) on the root system A_{n-1} as long as $\Re m > 0$.

DEFINITION 1.1. Let us fix m ($\Re m > 0$). For $H \in \mathbf{a}^+$, the diagonal matrices with strictly decreasing entries, we can define the dual of the generalized Abel transform $\mathcal{A}^{(m)}$ for the root system on A_{n-1}:

$$(\mathcal{A}^{(m)})^*(1, f; e^H) = f(e^H) \quad \text{and, for } n \geq 2,$$

(2)
$$(\mathcal{A}^{(m)})^*(n, f; e^H) = \frac{\Gamma(mn/2)}{\left(\Gamma(m/2)\right)^n} \left(d(H)\right)^{1-m} \int_{H_n}^{H_{n-1}} \cdots \int_{H_3}^{H_2} \int_{H_2}^{H_1} (\mathcal{A}^{(m)})^*(n-1, f_{\operatorname{tr} H}; e^{\xi}) \cdot \left[\pm \prod_{j=1}^{n} \prod_{i=1}^{n-1} \sinh(\xi_i - H_j)\right]^{m/2-1} d(\xi)\, d\xi$$

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where $f_{\operatorname{tr} H}(\operatorname{diag}[x_1,\ldots,x_{n-1}]) = f(\operatorname{diag}[x_1,\ldots,x_{n-1},\operatorname{tr} H - \sum_{i=1}^{n-1} x_i])$, $d(H) = \prod_{i<j}\sinh(H_i - H_j)$ and $\pm$ is chosen so that $\pm\prod_{j=1}^{n}\prod_{i=1}^{n-1}\sinh(\xi_i - H_j) \geq 0$ whenever $H_{i+1} \leq \xi_i \leq H_i$ for all i.

Note that $\rho^{(m)} = \frac{1}{2}\sum_{i<j} m(H_i - H_j)$ and the radial part of the generalized Laplace-Beltrami operator is defined as

$$L^{(m)} = \sum_{j=1}^{n}\frac{\partial^2}{\partial H_j^2} + m\sum_{i<j}\coth(H_i - H_j)\Big(\frac{\partial}{\partial H_i} - \frac{\partial}{\partial H_j}\Big). \tag{3}$$

We will see that $(\mathcal{A}^{(m)})^*(1, e^{\rho(m)}; e^H) = 1$ for all H.

The following result is a consequence of Theorem 1.1 in [12] if the space in question is $\operatorname{Pos}(n,\mathbf{F})$ where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ and $m = \dim_{\mathbf{R}}\mathbf{F}$.

THEOREM 1.2. *Suppose $\Re m > 0$ and f is a smooth Weyl invariant function. Then, $(\mathcal{A}^{(m)})^*(n,f;\cdot)$ is smooth on $\mathbf{a}^+$ and*

$$L^{(m)}(\mathcal{A}^{(m)})^*(n,f;\cdot) = (\mathcal{A}^{(m)})^*\big(n,\Gamma(L^{(m)})f;\cdot\big)$$

$(\Gamma(L^{(m)}) = L_A - \|\rho^{(m)}\|^2)$.

PROOF. The result can be proven by induction on n. The idea is to take $\Re m$ large enough in order to use integration by parts without adding new terms. By analytic continuation, the result is true for $\Re m > 0$. The proof is similar, but simpler (the integration being here on a bounded set), to that of Theorem 2.5 in [13]. ∎

We extended the definition of $(\mathcal{A}^{(m)})^*$ to complex values of m other than 1, 2, 4 and 8 ($n = 3$) in order to exploit the following fact:

LEMMA 1.3.

$$(L^{(m)} + \|\rho^{(m)}\|^2)\circ d^{1-m} = d^{1-m}(L^{(2-m)} + \|\rho^{(2-m)}\|^2).$$

In the language Opdam uses in [10], this says that multiplication by d^{1-m} shifts from m to $2-m$. In particular, the situation that concerns us mostly, $m = 4$, shifts to $m = -2$. We will see in Corollary 1.7, that $(\mathcal{A}^{(m)})^*$ corresponds to a differential operator when $m \leq 0$ is an even integer.

The expression in (2) reminds one of fractional integrals. We will want to exploit this.

LEMMA 1.4. *Let $n \geq 2$ be an integer and assume that f is a smooth complex valued function on $\mathbf{R}$. Consider*

$$I^p f = \frac{\Gamma(np)}{\Gamma\big((n-1)p\big)\Gamma(p)}\int_0^1 f(t)t^{p-1}(1-t)^{(n-1)p-1}\,dt$$

for $\Re p > 0$.

$I^p f$ is an analytic function of p and possesses an analytic continuation in the region $\mathcal{S} = \mathbf{C} - \{-\frac{r}{n} : r > 0$ is an integer not a multiple of $n\}$. If f depends smoothly or

analytically on parameters, the same is true of $I^p f$. Furthermore, when $p \leq 0$ is an integer, $I^p f$ is a linear combination of the values of f and of its derivatives at 0 *and at* 1.

PROOF.

$$\frac{\Gamma(np)}{\Gamma\big((n-1)p\big)\Gamma(p)}\int_0^1 f(t)t^{p-1}(1-t)^{(n-1)p-1}\,dt$$
$$= \frac{\Gamma(np)}{\Gamma(p)}\left[\frac{1}{\Gamma\big((n-1)p\big)}\int_{1/2}^1 \big(f(t)t^{p-1}\big)(1-t)^{(n-1)p-1}\,dt\right]$$
$$+ \frac{\Gamma(np)}{\Gamma\big((n-1)p\big)}\left[\frac{1}{\Gamma(p)}\int_0^{1/2}\big(f(t)(1-t)^{(n-1)p-1}\big)t^{p-1}\,dt\right].$$

Note that $\Gamma(z) = \frac{\pi}{\Gamma(1-z)\sin\pi z}$ if z is not an integer. The rest follows from the theory of Riemann-Liouville integrals (see [11]). ∎

LEMMA 1.5. *Let $n \geq 2$ be an integer. Assume that f is a smooth function of $\mathbf{R}^n$ and consider*

$$J^p f = \frac{\Gamma(np)}{\big(\Gamma(p)\big)^n}\int_{\sum_{i=1}^n t_i=1,\; t_i\geq 0} f(t)(t_1t_2\cdots t_n)^{p-1}\,dt$$

for $\Re p > 0$.

$J^p f$ is an analytic function of p and possesses an analytic continuation in the region $\mathcal{R} = \mathbf{C} - \{-\frac{r}{s} : s = 2,3,\ldots,n$ and $r > 0$ is an integer not a multiple of $s\}$. If f depends smoothly or analytically on parameters, the same is true of $J^p f$. Furthermore, when $p \leq 0$ is an integer, $J^p f$ is a linear combination of partial derivatives of f at the points $t = (t_1,\ldots,t_n)$ where one of the t_i is 1 *and all the others are* 0. *Moreover, $J^p 1 = 1$ for all $p \in \mathcal{R}$.*

PROOF. The proof relies on induction on $n \geq 2$. The case $n = 2$ is a special case of Lemma 1.4. The inductive step is as follows:

$$\frac{\Gamma(np)}{\big(\Gamma(p)\big)^n}\int_{\sum_{i=1}^n t_i=1,t_i\geq 0} f(t)(t_1t_2\cdots t_n)^{p-1}\,dt$$
$$= \frac{\Gamma(np)}{\Gamma\big((n-1)p\big)\Gamma(p)}\cdot$$
$$\int_0^1\left[\frac{\Gamma\big((n-1)p\big)}{\big(\Gamma(p)\big)^{n-1}}\int_{\sum_{i=2}^n t_i=1-t_1,\; t_i\geq 0} f(t)(t_2\cdots t_n)^{p-1}dt_2\cdots dt_{n-1}\right]t_1^{p-1}\,dt_1$$
$$= \frac{\Gamma(np)}{\Gamma\big((n-1)p\big)\Gamma(p)}\cdot$$
$$\int_0^1\left[\frac{\Gamma\big((n-1)p\big)}{\big(\Gamma(p)\big)^{n-1}}\int_{\sum_{i=1}^n s_i=1,\; s_i\geq 0} f\big(t_1,(1-t_1)s_i\big)(s_2\cdots s_n)^{p-1}ds_2\cdots ds_{n-1}\right]$$
$$\cdot t_1^{p-1}(1-t_1)^{p(n-1)-1}\,dt_1.$$

It suffices to apply Lemma 1.4 another time. ∎

These lemmas are of interest to us since

$$(\mathcal{A}^{(m)})^*(n,f;e^H) = e^{m(n-1)/2\,\mathrm{tr}\,H} J^{m/2}\Big(e^{-mn/2\,\mathrm{tr}\,\xi(t)}(\mathcal{A}^{(m)})^*(n-1,f_{\mathrm{tr}\,H};e^{\xi(t)})\Big)$$

where $t_j = \frac{\prod_{i=1}^{n-1}(e^{2\xi_i}-e^{2H_j})}{\prod_{i\neq j}(e^{2H_i}-e^{2H_j})}$. Moreover, if $t=(t_1,\dots,t_n)$ is such that one of the t_i is 1 and all the others are 0, then $\mathrm{diag}[\xi(t),\mathrm{tr}\,H-\mathrm{tr}\,\xi(t)]$ corresponds to the image of H by an element of the Weyl group.

THEOREM 1.6. *Let f be a smooth function on A. Then, $(\mathcal{A}^{(m)})^*(n,f;e^H)$ is a smooth function on $\mathbf{a}^+$ and, by analytic continuation, is an analytic function of m in the region $\mathcal{T} = \mathbf{C} - \{-\frac{2r}{s} : s = 2,3,\dots,n$ and $r > 0$ is an integer not a multiple of $s\}$. For $m \in \mathcal{T}$, we have $L^{(m)}(\mathcal{A}^{(m)})^*(n,f;e^H) = (\mathcal{A}^{(m)})^*\big(n,\Gamma(L^{(m)})f;e^H\big)$ and $(\mathcal{A}^{(m)})^*(n,e^{\rho^{(m)}};e^H) = 1$ for all $H \in \mathbf{a}^+$. Furthermore, if $m \leq 0$ is an even integer, then $(\mathcal{A}^{(m)})^*(n,f;H) = \sum_{s\in W}(D_sf)(sH)$ where the D_s are differential operators of the form*

$$D_s = \frac{1}{d^N}\sum_{j\in J} Q_{s,j}\partial^j$$

where N is an integer, J a finite set of indices $j=(j_1,\dots,j_{n-1})$, $Q_{s,j}$ is a polynomial in the exponentials of the roots for each j and, if $j=(j_1,\dots,j_{n-1})$, then $\partial^j = \prod_{k=1}^{n-1}(\frac{\partial}{\partial H_k}-\frac{\partial}{\partial H_{k+1}})^{j_k}$.

PROOF. Much of the result has been proven above. That $L^{(m)}(\mathcal{A}^{(m)})^*(n,f;e^H) = (\mathcal{A}^{(m)})^*\big(n,\Gamma(L^{(m)})f;e^H\big)$, follows from Theorem 1.2 and from analytic continuation.

Let us assume now that $m \leq 0$ is an even integer. We verify the form of the operators D_s using induction. For $n=1$, the result is clear since $D_s = 1$ ($|W|=1$). Assume that the result is true for $n-1$, $n \geq 2$. Take any smooth function f. To avoid confusion, we will use the subscript $n-1$ to refer to objects corresponding to the case $n-1$.

We have

$$\begin{aligned}(\mathcal{A}^{(m)})^*(n,f;e^H) &= e^{m(n-1)/2\,\mathrm{tr}\,H} J^{m/2}\Big(e^{-mn/2\,\mathrm{tr}\,\xi(t)}(\mathcal{A}^{(m)})^*(n-1,f_{\mathrm{tr}\,H};e^{\xi(t)})\Big)\\ &= e^{m(n-1)/2\,\mathrm{tr}\,H}\sum_{s\in W_{n-1}} J^{m/2}\Big(e^{-mn/2\,\mathrm{tr}\,\xi(t)}(D_{n-1})_sf_{\mathrm{tr}\,H}\Big).\end{aligned} \tag{4}$$

It is clear from Lemma 1.5 that $J^{m/2}\big(e^{-mn/2\,\mathrm{tr}\,\xi(t)}(D_{n-1})_sf_{\mathrm{tr}\,H}\big)$ depends only on the values of $e^{-mn/2\,\mathrm{tr}\,\xi(t)}(D_{n-1})_sf_{\mathrm{tr}\,H}$ in the neighbourhoods of the points $t=(t_1,\dots,t_n)$ where one of the t_i is 1 and all the others are 0. In these neighbourhoods, we can write $(D_{n-1})_sf_{\mathrm{tr}\,H} = g(p_1,\dots,p_{n-1})$ where $p_k = \sum_{i_1<i_2<\dots<i_k} e^{2\xi_{i_1}+\dots+2\xi_{i_k}}$ (the elementary symmetric polynomials in the variables $e^{2\xi_i}$). $\prod_{i=1}^{n-1}(e^{2\xi_i}-e^{2H_j}) = \prod_{i\neq j}(e^{2H_i}-e^{2H_j})t_j$ for $1 \leq j \leq n$ is equivalent to $\sum_{q=0}^{n-1}[-e^{2H_j}]^{n-1-q}p_q = \prod_{i\neq j}(e^{2H_i}-e^{2H_j})t_j$ for $1 \leq j \leq n$. This in turn implies that $p_q = \sum_{j=1}^n p_q^j t_j$ where $p_q^j = p_q(e^{2H_1},\dots,e^{2H_{j-1}},e^{2H_{j+1}},\dots,e^{2H_n})$. Now, $e^{-mn/2\,\mathrm{tr}\,\xi(t)}g = p_{n-1}^{-mn/4}g = e^{-mn/2\,\mathrm{tr}\,H}[\sum_{j=1}^n e^{-2H_j}t_j]^{-mn/4}g\big((\sum_{j=1}^n p_q^j t_j)_{1\leq q\leq n-1}\big)$.

Recall that $J^{m/2}$ is a linear combination of partial derivatives with respect to the t_i at the points $t=(t_1,\dots,t_n)$ where one of the t_i is 1 and all the others are 0. These points, in terms of the function f, correspond to points on the orbit of e^H under the action of the

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Weyl group. $J^{m/2}\big(e^{-mn/2\operatorname{tr}\xi(t)}(D_{n-1})_s f_{\operatorname{tr}H}\big)$ is then a linear combination of partial derivatives of g with respect to the variables p_q. The coefficients of these derivatives are polynomial functions of $e^{\pm H_i}$. On the other hand, for each q, $\frac{\partial}{\partial p_q} = \sum_{r=1}^{n-1}\big(p_{qr}(e^{2\xi_1},\ldots,e^{2\xi_{n-1}})/[\prod_{i<j}(e^{2\xi_i}-e^{2\xi_j})]\big)\frac{\partial}{\partial \xi_r}$ where the p_{qr} are polynomials.

Using the integral formula given in (2) and analytic continuation, we find that for $m \in \mathcal{T}$, $(\mathcal{A}^{(m)})^*\big(n,(g\circ \operatorname{tr})\cdot f;e^H\big) = g(\operatorname{tr}H)(\mathcal{A}^{(m)})^*(n,f;e^H)$ and, for any $\eta \in \mathbf{R}$, $(\mathcal{A}^{(m)})^*(n,f;e^{T_\eta(H)}) = (\mathcal{A}^{(m)})^*(n,f\circ T_\eta;e^H)$ where $T_\eta(H) = \operatorname{diag}[H_1+\eta,\ldots,H_n+\eta]$. These observations allow us to conclude the proof. ∎

COROLLARY 1.7. *If $m \leq 0$ is an even integer and f is a smooth Weyl invariant function then $(\mathcal{A}^{(m)})^*(n,f;\cdot) = \mathbf{D}^{(m)}f$ where $\mathbf{D}^{(m)}$ is a differential operator. Moreover,*

$$\mathbf{D}^{(m)} = \frac{1}{d^N}\sum_{j\in J} Q_j \partial^j$$

where $N \geq 0$ is an integer, J a finite set of indices $j = (j_1,\ldots,j_{n-1})$, Q_j is polynomial in the exponentials of the roots for each j and, if $j = (j_1,\ldots,j_{n-1})$, then $\partial^j = \prod_{k=1}^{n-1}(\frac{\partial}{\partial H_k}-\frac{\partial}{\partial H_{k+1}})^{j_k}$.

PROOF. It follows directly from the theorem. ∎

THEOREM 1.8. *If f is a smooth function on A^+ and $m \leq 0$ is an even integer then*

$$L^{(m)}\mathbf{D}^{(m)} = \mathbf{D}^{(m)}\Gamma(L^{(m)}).$$

PROOF. The result is true when f is Weyl invariant (Corollary 1.7). Equality of differential operators is a local property, so the result follows. ∎

To identify further the operator $\mathbf{D}^{(m)}$, we need to discuss the eigenfunctions of the operator $L^{(m)}$. We adapt here the terminology of Chapter IV, §5 in Helgason's [8].

THEOREM 1.9. *Let Λ be the set of all linear combinations of the positive roots having non-negative integer coefficients. We define $'\mathbf{a}_c^*$ to be the set of complex-valued linear functionals on $\mathbf{a}$ such that $i(s\lambda - t\lambda) \notin \tilde{\Lambda}$ (the set of all linear combinations of the positive roots having integer coefficients) for $s \neq t$ in W and $\langle\mu,\mu\rangle \neq 2i\langle\mu,\lambda\rangle$ for all $\mu \in \Lambda-\{0\}$ and $s \in W$. $'\mathbf{a}_{\mathbf{C}}^*$ is a dense open connected subset of $\mathbf{a}_{\mathbf{C}}^*$, the set of complex-valued linear functionals on $\mathbf{a}$. For $\lambda \in {}'\mathbf{a}_{\mathbf{C}}^*$, let*

$$\Phi_\lambda^{(m)}(H) = e^{(i\lambda-\rho^{(m)})(H)}\sum_{\mu\in\Lambda}\Gamma_\mu^{(m)}(\lambda)e^{-\mu(H)} \tag{5}$$

where

$$\{\langle\mu,\mu\rangle - 2\langle i\lambda,\mu\rangle\}\Gamma_\mu^{(m)}(\lambda) = 2m\sum_{\alpha>0}\sum_{k\geq 1}\langle\mu-2k\alpha+\rho^{(m)}-i\lambda,\alpha\rangle\Gamma_{\mu-2k\alpha}^{(m)}(\lambda) \tag{6}$$

and $\Gamma_0(\lambda) = 1$. $\{\Phi_{s\lambda}^{(m)} : s \in W\}$ is a linearly independent set of eigenvectors of $L^{(m)}$ for the eigenvalue $-(\langle\lambda,\lambda\rangle + \langle\rho^{(m)},\rho^{(m)}\rangle)$.

PROOF. It suffices to adapt the proof given in [8] for the "usual" functions Φ_λ. ∎

Note that $\Gamma_\mu^{(m)}(\lambda) = 0$ unless $\mu \in 2\Lambda$.

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LEMMA 1.10. *For* $\lambda \in {}'\mathbf{a}^*_{\mathbf{C}}$, *there exists a constant* $p_0^{(m)}(\lambda)$ *such that* $\mathbf{D}^{(m)}e^{i\lambda} = p_0^{(m)}(\lambda)\Phi_\lambda^{(m)}$.

PROOF. According to Corollary 1.7, $\mathbf{D}^{(m)}e^{i\lambda}$ will have an expansion of the form given in (5), except that the sum will be on $\tilde{\Lambda}$. Denote by $\tilde{\Gamma}_\mu^{(m)}(\lambda)$ the coefficient of $e^{-\mu}$ in that expansion. We claim that $\tilde{\Gamma}_\mu^{(m)}(\lambda) = 0$ for all λ unless $\mu \in \Lambda$. Suppose that is not the case. There can be only finitely many μ exhibiting that statement. Let μ_0 be the smallest of them (we use the lexicographic order on $\tilde{\Lambda} = \{\sum_{\alpha>0} n_\alpha \alpha : n_\alpha \text{ integers}\}$). Our choice of μ_0 implies that $\tilde{\Gamma}^{(m)}_{\mu_0-k\alpha}(\lambda) = 0$ for all $\alpha > 0$, all $k \geq 1$ and for all λ. The coefficients $\tilde{\Gamma}_\mu^{(m)}(\lambda)$ must satisfy (6). In particular, $\{\langle\mu_0,\mu_0\rangle\} - 2\langle i\lambda,\mu_0\rangle\}\tilde{\Gamma}^{(m)}_{\mu_0}(\lambda)$ must be 0 for all λ. This means that $\mu_0 = 0$. The rest follows easily. ■

COROLLARY 1.11. *If* $m \leq 0$ *is an even integer, then*

$$\mathbf{D}^{(m)} = e^{-\rho^{(m)}} \frac{e^{2N\rho^{(m)}/m}}{d^N} \sum_{j\in J} Q_j(e^{-(H_1-H_2)}, \ldots, e^{-(H_{n-1}-H_n)})\partial^j$$

where N *is an integer,* J *a finite set of indices* $j = (j_1, \ldots, j_{n-1})$, Q_j *is a polynomial for each* j *and, if* $j = (j_1, \ldots, j_{n-1})$, *then* $\partial^j = \prod_{k=1}^{n-1}(\frac{\partial}{\partial H_k} - \frac{\partial}{\partial H_{k+1}})^{j_k}$.

PROOF. Just compute $\mathbf{D}^{(m)}e^{i\lambda}$ using Corollary 1.7. ■

COROLLARY 1.12. $p_0^{(m)}(\lambda)$ *is a nonzero polynomial in* λ *of degree at most that of* $D^{(m)}$. *Moreover,* $\Phi^{(m)}(\lambda)$ *is a meromorphic function of* $\lambda \in \mathbf{a}^*_{\mathbf{C}}$ *whose poles are zeros of* $p_0^{(m)}(\lambda)$.

PROOF. Since $(\mathcal{A}^{(m)})^*(n, e^{\rho^{(m)}}; \cdot) = 1$, $D^{(m)}$ cannot be 0 and, consequently, $D^{(m)}e^{i\lambda}$ is not identically equal to 0. This, together with the lemma and the previous corollary, implies that $p^{(m)}(\lambda)$ is a nonzero polynomial (actually, if $i\lambda(H) = \sum_{k=1}^n a_K H_k$, then $p_0^{(m)}(\lambda) = \sum_{j\in J} Q_j(0,\ldots,0)\prod_{k=1}^{n-1}(a_k - a_{k+1})^{j_k}$). The rest follows easily from the fact that $\mathbf{D}^{(m)}e^{i\lambda}$ is analytic in λ. ■

DEFINITION 1.13. The c-function for the root system A_{n-1} and complex multiplicity m is

$$c^{(m)}(\lambda) = 2^{(1-m)(n-2)(n-1)n} \prod_{k=2}^n \frac{\Gamma(mk/2)}{\Gamma(m/2)} \prod_{\alpha>0} \frac{\Gamma(\langle i\lambda, \alpha_0\rangle)}{\Gamma(m/2 + \langle i\lambda, \alpha_0\rangle)}$$

$(\alpha_0 = \alpha/\langle\alpha,\alpha\rangle)$.

$c^{(m)}(\lambda)$ corresponds to the usual c-function when $m = 1, 2, 4$ and 8 ($n = 3$).

LEMMA 1.14. *Let* $m \leq 0$ *be an even integer. Then, there exists a constant* $K \neq 0$ *independent of* λ *such that* $p_0^{(m)}(\lambda) = Kc^{(m)}(\lambda)$.

PROOF. Suppose $i\lambda(H) = \sum_{k=1}^n a_k H_k$ is such that $a_1 > a_2 > \cdots > a_n$ and $(a_i - a_j)/2 = k$ for some $i > j$ and $1 \leq k < -m/2$. Let $\mu \in \Lambda$ be defined by $\mu(H) = 2k(H_i - H_j)$. Equation (6) becomes

$$8k\overbrace{[k - (a_i - a_j)/2]}^{0}\Gamma_\mu^{(m)}(\lambda) = 2m \sum_{\alpha>0}\sum_{k\geq 1}\langle\mu - 2k\alpha + \rho^{(m)} - i\lambda, \alpha\rangle\Gamma^{(m)}_{\mu-2k\alpha}(\lambda).$$

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It is not difficult to see (using the conditions on λ) that for $\alpha > 0$ and $k \geq 1$, $\Gamma^{(m)}_{\mu-2k\alpha}(\lambda) \geq 0$ ($\Gamma^{(m)}_{\mu-2k\alpha}(\lambda) > 0$ for at least one of them), and, consequently, that the right hand side is a positive number. By Corollary 1.12, we conclude that $p_0^{(m)}(\lambda) = 0$ for such λ. This implies that $k - (a_i - a_j)$ divides the polynomial $p_0^{(m)}(\lambda)$. Hence, $p_0^{(m)}(\lambda) = q^{(m)}(\lambda)\prod_{i<j}\prod_{k=1}^{-m/2}\big(k-(a_i-a_j)/2\big)$, $\deg p_0^{(m)}(\lambda) \leq \deg \mathbf{D}^{(m)} \leq (-m/2)n(n-1)/2$. From this, we conclude that $q^{(m)}(\lambda)$ is a constant. ∎

LEMMA 1.15. *If m is an even integer then there exists a constant* $A_m \neq 0$ *independent of* λ *such that* $c^{(m)}(\lambda) = A_m[\pi(-\lambda)c^{(2-m)}(-\lambda)]^{-1}$ *where* $\pi(\lambda) = \prod_{\alpha>0}\langle i\lambda, \alpha_0\rangle$.

PROOF. The result follows from applying the formula $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ to the expression given in Definition 1.13. ∎

THEOREM 1.16. *If* $m \geq 0$ *is an even integer then* $(\mathcal{A}^{(m)})^{-1}$ *is a differential operator.*

PROOF. We will show that $(\mathcal{A}^{(m)})^{-1}$ is a multiple of $d^{1-m}\mathbf{D}^{(2-m)}\partial(\pi)$ where $\partial(\pi)$ is the differential operator with constant coefficients such that $\partial(\pi)e^{i\lambda} = \pi(\lambda)e^{i\lambda}$.

Note first that $d^{1-m}\Phi_\lambda^{(2-m)} = \Phi_\lambda^{(m)}$: the left hand side has the right expansion and satisfies the appropriate differential equation (Lemma 1.3).

Using Lemmas 1.14 and 1.15, we have $d^{1-m}\mathbf{D}^{(2-m)}\partial(\pi)e^{i\lambda} = \pi(\lambda)d^{1-m}\mathbf{D}^{(2-m)}e^{i\lambda} = A_m\pi(\lambda)c^{(2-m)}(\lambda)\Phi_\lambda^{(m)} = B_m[c^{(m)}(-\lambda)]^{-1}\Phi_\lambda^{(m)}$. The result follows (see [3]). ∎

LEMMA 1.17. *If* λ *is a complex-valued linear functional on* **a** *and* $m \leq 0$ *is an even integer, then*

$$\mathbf{D}^{(m)}e^{i\lambda} = e^{-\rho^{(m)}}\sum_{\mu\in\Lambda} p_\mu^{(m)}(\lambda)e^{-\mu} \tag{7}$$

where $p_\mu^{(m)}(\lambda) = B_m[\pi(-\lambda)c^{(2-m)}(-\lambda)]^{-1}\Gamma_\mu^{(m)}(\lambda)$ *is a polynomial in* λ *for each* μ *(*B_m *is a constant independent of* μ *and* λ*).*

PROOF. We already know that the sum in (7) is valid for $\lambda \in {}'\mathbf{a}^*_{\mathbf{C}}$. If we use Corollary 1.11 to compute $\mathbf{D}^{(m)}e^{i\lambda}$, we see immediately that for every λ there is an expansion $\mathbf{D}^{(m)}e^{i\lambda} = e^{-\rho^{(M)}(H)}\sum_{\mu\in\Lambda}\tilde{p}_\mu^{(m)}(\lambda)e^{-\mu(H)}$ where $\tilde{p}_\mu^{(m)}(\lambda)$ is a polynomial. The result follows since for each μ we must have $p_\mu^{(m)}(\lambda) = \tilde{p}_\mu^{(m)}(\lambda)$ whenever $\lambda \in {}'\mathbf{a}^*_{\mathbf{C}}$. ∎

2. The heat kernel of the symmetric space SU*(2n)/ Sp(n): Anker's conjecture.

SU*(2n)/ Sp(n) corresponds to the root system A_{n-1} with $m = 4$. We will omit the superscript (m) when it is equal to (4).

THEOREM 2.1. *We can write the heat kernel for the space* Pos(n, **H**) *as*

$$P_t(e^H) = Ce^{-\|\rho\|^2 t}t^{-n^2/2}e^{-r^2/(4t)}\phi_0(e^H)V_t(e^H)$$

where $\phi_0(e^H) = \int_K e^{-\rho(H(e^H k))}\,dk$ *is the Legendre function,* $V_t(e^H) \geq 0$ *for all H and*

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$\lim_{t\to\infty} V_t(e^H) = 1$ *for all H. There exists a constant* $C > 0$ *such that if* $H \in \mathbf{a}^+$ *then,*

(8) $$V_t(H) \le C \prod_{i<j}\Big(1 + \frac{1 + H_i - H_j}{t}\Big).$$

PROOF. Since $P_t = \mathcal{A}^{-1}(Ce^{-\|\rho\|^2 t} t^{-n/2} e^{-r^2/(4t)}) = Ce^{-\|\rho\|^2 t} t^{-n/2} \frac{1}{d^3}\mathbf{D}^{(-2)}\partial(\pi) e^{-r^2/(4t)}$, we conclude that $e^{-r^2/(4t)}\phi_0(e^H)V_t(e^H) = Ct^{n(n-1)/2}\frac{1}{d^3}\mathbf{D}^{(-2)}\partial(\pi)e^{-r^2/(4t)} = e^{-r^2/(4t)}\frac{1}{d^{3+N}}e^{(N+1)\rho/2}\sum_{k=0}^{n(n-1)/2} R_k(e^H)t^{-k}$ (N is as in Corollary 1.11).

The functions R_k are polynomial functions in the roots and in the exponentials of the negative roots. The functions $\frac{1}{d^{3+N}}e^{(N+1)\rho/2}R_k(e^H)$ are Weyl invariant and analytic (the theory tells us that P_t is Weyl invariant and analytic).

According to the estimate given in [1] (valid for any symmetric space of noncompact type), there exists $C > 0$ such that

$$C^{-1}\prod_{i<j}(1 + H_i - H_j)e^{-\rho(H)} \le \phi_0(H) \le C\prod_{i<j}(1 + H_i - H_j)e^{-\rho(H)}.$$

To prove the theorem, it is then sufficient to show that there exists $C > 0$ such that

$$\left|\frac{e^{(N+3)\rho(H)/2}R_k(e^H)}{d^{N+3}(H)}\right| \le C \sum_{\substack{\sum_{i<j} r_{ij}=k,\\ r_{ij}\in\{0,1\}}} \prod_{i<j}(1 + H_i - H_j)^{1+r_{ij}}$$

for each k.

It will actually be enough to show that there exists $C > 0$ such that

(9) $$|R_k(e^H)| \le C \sum_{\substack{\sum_{i<j} r_{ij}=k,\\ r_{ij}\in\{0,1\}}} \prod_{i<j}(1 + H_i - H_j)^{1+r_{ij}}$$

for each k.

To see this, one observes first that $d(H)$ is like $e^{\rho(H)/2}$ when the roots of H are away from 0. The fact that $e^{(N+3)\rho/2}R_k/d^{N+3}$ is smooth implies that some derivatives of R_k are 0 when the roots are 0. The "trick" can be summarized as follows: if $f(x)/x^3$ is smooth, then $f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{x^3}{2}\int_0^1(1-t)^2 f^{(3)}(tx)\,dt = \frac{x^3}{2}\int_0^1(1-t)^2 f^{(3)}(tx)\,dt$. The rest is to make sure that the bounds in (9) also holds for the appropriate derivatives.

The main factor in the size of the functions R_k is its degree in terms of the roots. It is important to estimate the effect of the operators $\prod_{k=1}^{n-1}(\frac{\partial}{\partial H_k} - \frac{\partial}{\partial H_{k+1}})^{j_k}\partial(\pi)$ on $e^{-r^2/(4t)}$. To simplify the picture, we will consider a simpler situation. $\frac{d^k}{dx^k}e^{-x^2/(4t)} = e^{-x^2/(4t)}\sum_{i=1}^k p_i(x)t^{-i}$ where $\deg p_i \le i$ for each i. Clearly, there exists $C > 0$ independent of $x \ge 0$ and i such that $|p_i(x)| \le C(1 + x)^i$.

To apply this reasoning to our situation, using the notation of Lemma 1.17, we note that if $i\lambda(H) = \sum_{k=1}^n a_k H_k$ then $p_0^{(-2)}(\lambda) = C\prod_{i<j}\big(1 - (a_i - a_j)/2\big)$. Furthermore, if we compute the degree of $p_\mu^{(-2)}(\lambda)$ with respect to any of the differences $a_i - a_j$, we find that $\deg p_\mu(\lambda) = \deg p_0^{(-2)}(\lambda) + \deg \Gamma_\mu^{(-2)}(\lambda) \le \deg p_0^{(-2)}(\lambda)$ since from (6), $\deg \Gamma_\mu^{(-2)}(\lambda) \le 0$. This in turn limits the order of the derivatives that occur in the operator $\frac{1}{d^3}\mathbf{D}^{(-2)}\partial(\pi)$. The rest is straightforward. ■

The same method can be applied to prove the corresponding result for the space $E_{6(-26)}/F_4$:

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THEOREM 2.2. *Anker's conjecture is valid for the space $E_{6(-26)}/F_4$.*

PROOF. The proof is not very different from that of Theorem 2.1 except that now $m = 8$ and $n = 3$. For instance, $\mathcal{A}^{-1} = \frac{1}{d^7}\mathbf{D}^{(-6)}\partial(\pi)$. The main difference is that $p_0^{(-6)}(\lambda) = C\prod_{i<j}\big(1-(a_i-a_j)/2\big)\big(2-(a_i-a_j)/2\big)\big(3-(a_i-a_j)/2\big)$. ∎

CONCLUSION. O. A. Chalykh and A. P. Veselov give in [14] an explicit formula of the inverse of the Abel transform in the case $m = 4$ (it still requires intensive computations if n is large). In their formulation and ours, the inverse of the Abel transform for the space $\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$ is the composition of $n-1$ differential operators with the operator $\partial(\pi)$. Another similarity is that the inverse is computed recursively. They do not use the dual of the Abel transformation but rather give an inductive process to compute the eigenfunctions of the Laplace-Beltrami operator as the images of a differential operator. Most of the work leading to these results is found in [5]. Their formulation of the inverse of the Abel transformation would have permitted us to draw the same conclusion about the heat kernel. Note that their approach, as far as irreducible symmetric spaces are concerned, applies only to those of type A_{n-1} with $m = 2, 4$ and 8 ($n = 3$). We have used the dual of the Abel transform, albeit indirectly, to obtain results in the case $m = 1$ in [13]. It might be interesting to try the same idea on other classes of symmetric spaces.

It would have been nice to show that N in the expression for $\mathbf{D}^{(-2)}$ (Corollary 1.11) can be chosen to be 0. That would imply that $\mathbf{D}^{(-2)}$ could be given as a finite sum:

$$\mathbf{D}^{(-2)} = e^{\rho/2}\sum_{\mu\in\Lambda} e^{-\mu}p_\mu^{(-2)}\Big(\frac{\partial}{\partial H_1},\dots,\frac{\partial}{\partial H_n}\Big)$$

with $p_\mu^{(-2)}(\lambda)$ is as in Lemma 1.17 and the corresponding differential operator with constant coefficients is defined by $p_\mu^{(-2)}(\frac{\partial}{\partial H_1},\dots,\frac{\partial}{\partial H_n})\, e^{i\lambda} = p_\mu^{(-2)}(\lambda)e^{i\lambda}$. We have verified this in some cases but have been unable to give a general proof.

We conclude by summarizing what we now know of Anker's conjecture for the symmetric spaces of noncompact type corresponding to the root systems A_{n-1}, that is, the spaces $\mathrm{Pos}_1(n,\mathbf{F})$ where $\mathbf{F}$ is $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$ (the octonions). For the complex case, there is nothing to do since V_t is identically equal to 1. We just have settled the cases where $\mathbf{F}$ is $\mathbf{H}$ or $\mathbf{O}$. The real case is still, as far as we know, an open problem for $n \geq 4$ (an answer to the case $n = 3$ can be found in [13]).

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