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UNIVALENT FUNCTIONS WITH UNIVALENT GELFOND-LEONTEV DERIVATIVES

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Let $\{d_n\}_1^\infty$ be a nondecreasing sequence of positive numbers. We consider Gelfond-Leontev derivative $Df(z)$, of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, defined by $Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}$, for univalence and growth properties, and extend some results of Shah and Trimble. Set $e_n = (d_1 d_2 \ldots d_n)^{-1}$, $n \geq 1$, $e_0 = 1$, $p(z) = \sum_{n=0}^{\infty} e_n z^n$. Let r be the radius of convergence of $p(z)$. We state parts of Theorem 1 and Corollaries. Let f and all $D^k f$, $k = 1, 2, \ldots$, be analytic and univalent in the unit disk U. Then

(i) $|f(z)| \leq |a_0| + (|a_1| d_1/2d_2)\{p(2d_2|z|)-1\}$, $|z| < r/2d_2$,

(ii) $|D^k f(z)| \equiv \left|\sum_{n=k}^{\infty} (e_{n-k}/e_n) a_n z^{n-k}\right| \leq |a_1| d_1 (2d_2)^{k-1} p(2d_2|z|)$, $k \geq 1$,

(iii) if p is entire and of growth (ρ, T) then f must be entire and of growth not exceeding $(\rho, 2d_2 T)$,

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(iv) if D corresponds to the shift operator $(d_n \equiv 1)$, then

$$|f(z)| = O(1-2|z|)^{-1} \text{ as } z \to \tfrac{1}{2} .$$

Another class of functions is defined by a condition of the form $|a_{n+1}/a_n| \le b_{n+1}/d_{n+1}$, where $\{b_n\}_1^\infty$ is a sequence of positive numbers satisfying an inequality, and it is shown that all functions in this class along with all their Gelfond-Leontev successive derivatives are regular and univalent in U . An extension of the definition of a linear invariant family is given and results analogous to (i) and (ii) are stated.

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the disc $|z| < R$. Let $\{d_n\}_{n=1}^{\infty}$ denote a non-decreasing sequence of positive numbers and D the operator which transforms the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

into

$$Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1} . \tag{1.2}$$

For $k = 1, 2, \ldots$, the kth iterate of D is given by

$$D^k f(z) = \sum_{n=k}^{\infty} d_n \ldots d_{n-k+1} a_n z^{n-k} = \sum_{n=k}^{\infty} \frac{e_{n-k}}{e_n} a_n z^{n-k} \tag{1.3}$$

where $e_0 = 1$ and $e_n = (d_1 d_2 \ldots d_n)^{-1}$, $n = 1, 2, \ldots$. If $d_n \equiv n$, D corresponds to the ordinary derivative whereas if $d_n \equiv 1$, D corresponds to the shift operator S^* which transforms

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ into } S^* f(z) = \sum_{n=1}^{\infty} a_n z^{n-1} . \tag{1.4}$$

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The operator D is called the Gelfond-Leontev derivative [7] of f. The operators D have been investigated extensively by Kazmin [9], Buckholtz and Frank [3, 4] and others.

Set

$$p(z) = \sum_{n=0}^{\infty} e_n z^n . \tag{1.5}$$

It is clear that $p(0) = 1$ and $Dp(z) = p(z)$. Thus $p(z)$ bears the same relationship to the operator D which the exponential function bears to the ordinary differentiation. If r be the radius of convergence of $p(z)$ then we have

$$r = \lim_{n\to\infty} d_n = \sup_{1\le n<\infty} d_n . \tag{1.6}$$

Define the p-type of the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ to be the number

$$\tau_p(f) = \limsup_{n\to\infty} |a_n/e_n|^{1/n} . \tag{1.7}$$

If $r < \infty$, it is easy to check that

$$\tau_p(f) = r/R . \tag{1.8}$$

If $p(z)$ is entire, p-type is a growth measure introduced by Nachbin [2, p. 6] and [11] which can be related to the maximum modulus of f. Further, for $p(z)$ entire, $\tau_p(f) < \infty$ implies that f is entire.

Shah and Trimble have, in a series of papers (see, for example, [17] to [22]), studied properties of functions f such that f and its successive ordinary derivatives are univalent in the unit disc U. They showed that such an f must be an entire function of exponential type. In the present paper we consider functions f such that f along with its Gelfond-Leontev derivatives is univalent in U and show that such an f must be of finite p-type. We shall suppose throughout that the operator D is defined by (1.2) and that $p(z)$, given by (1.5), has radius of convergence r $(0 < r \le \infty)$.

2. The class $E(D)$

Let S denote, as usual, the family of functions h of the form

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2.1}$$

which are analytic and univalent in the unit disc U. It is well known (see, for example, [13, p. 20]) that

$$|b_2| \le 2 \tag{2.2}$$

and

$$\left|b_3 - b_2^2\right| \le 1 \tag{2.3}$$

both the inequalities being sharp.

It is further known [13, p. 44] that if h, defined by (2.1), is analytic in U and if

$$\sum_{n=2}^{\infty} n|a_n| \le 1 \tag{2.4}$$

then h is univalent in U, that is, $h \in S$. This condition (2.4), on the moduli of the coefficients alone, is best possible in the sense that if (2.4) does not hold then the arguments of the coefficients can be so altered that the new function defined by

$$z + \sum_{2}^{\infty} a_n z^n$$

is no longer univalent in U (see [8], [10]).

Let $E(D)$ denote the family of functions of the form (2.1) such that f and all its Gelfond-Leontev derivatives $D^k f$ are analytic and univalent in U. Note that when $d_n \equiv n$, D corresponds to the ordinary derivative and $E(D)$ is then the class E considered by Shah and Trimble [17].

THEOREM 1. *Let f represented by* (1.1) *be such that f and all $D^k f$ are analytic and univalent in U; then*

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(2.5) f *is of finite* p*-type not exceeding* $2d_2$,

(2.6) $|f(z)| \le |a_0| + \{|a_1|d_1/2d_2\}|p(2d_2|z|)-1|$, $|z| < r/2d_2$,

(2.7) $|D^k f(z)| \le |a_1|d_1(2d_2)^{k-1}p(2d_2|z|)$, $k \ge 1$,

(2.8) $E(D)$ *is a normal family in* $|z| < t < r/2d_2$ *for all* t *satisfying* $0 < t < r/2d_2$.

Proof. Since, for $k = 1, 2, \ldots$,

$$D^k f(z) = \sum_{n=k}^{\infty} d_n \ldots d_{n-k+1} a_n z^{n-k}$$

is univalent in U , it follows that $a_{k+1} \ne 0$. Define H_k in U by

$$H_k(z) = \frac{D^k f(z) - D^k f(0)}{d_2 \ldots d_{k+1} a_{k+1}} = z + \frac{d_{k+2}}{d_2} \frac{a_{k+2}}{a_{k+1}} z^2 + \ldots .$$

It is clear that $H_k \in S$. By (2.2), we have, therefore,

$$\left| \frac{d_{k+2}}{d_2} \cdot \frac{a_{k+2}}{a_{k+1}} \right| \le 2 , \quad k = 1, 2, \ldots . \tag{2.9}$$

An induction process gives

$$|a_k| \le e_k (2d_2)^{k-1} d_1 |a_1| , \quad k = 1, 2, \ldots . \tag{2.10}$$

Thus

$$\limsup_{k \to \infty} \left| \frac{a_k}{e_k} \right|^{1/k} \le 2d_2$$

showing that f is of finite p-type not exceeding $2d_2$.

Using the estimates (2.10) in the relations

$$|f(z)| \le \sum_{n=0}^{\infty} |a_n| |z|^n$$

and

$$|D^k f(z)| \le \sum_{n=k}^{\infty} \frac{e_{n-k}}{e_n} |a_n| |z|^{n-k}$$

easily leads to (2.6) and (2.7). The assertion (2.8) is a consequence of local boundedness of $E(D)$ obtained from (2.6). Hence the theorem.

COROLLARY 1. *If* p *is entire, then* $f \in E(D)$ *must be entire and relations* (2.6) *and* (2.7) *hold for all* $z \in \mathbb{C}$. *Assertion* (2.8) *is valid on every compact subset of* $\mathbb{C}$.

COROLLARY 2. *If* p *is an entire function of growth* (ρ, T) *(cf.* [*1*, p. 8]*) then* $f \in E(D)$ *must be an entire function of growth not exceeding* $(\rho, 2d_2T)$.

It is known that if f is analytic in $|z| < R$ $(< \infty)$, its order ρ_0 is defined by (see, for example, [*23*])

$$\rho_0 = \limsup_{r \to R} \frac{\log^+ \log^+ M(r)}{\log\left(Rr/(R-r)\right)} \tag{2.11}$$

where $M(r) = \max_{|z|=r} |f(z)|$. Taking this into consideration, we have

COROLLARY 3. *If* D *corresponds to the shift operator* S^* *defined by* (1.4), *then, by* (2.6), $|f(z)| = O(1-2|z|)^{-1}$ *as* $|z| \to \frac{1}{2}$ *and* $f \in E(D)$ *must be of zero order in* $|z| < \frac{1}{2}$.

REMARK. Theorem 1 could also be modelled in terms of "admissible property" as done in Theorem 1 of [*19*], thereby generalizing that theorem.

3. Functions in the class S

A function $f \in S$ may have f' univalent in U but its Gelfond-Leontev derivative Df may not be univalent in U . To see this, let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be such that a_n is negative for $n \ge 3$, $a_2 = 4/(5+8 \log 2)$, $|a_n| = a_2/\left(2^{n-3} \cdot n(n-1)\right)$. Using (2.4), it is easy to check that f' is univalent in U . However, if we take d_n to be a non-decreasing sequence in which $d_3 = 3d_2$, then Df is seen not to be univalent in U . In fact, it is possible to have such an f with p-type as large as we please. This is demonstrated by the following

THEOREM 2. *There exists an* $f \in S$ *such that* f' *is univalent in* U *; its Gelfond-Leontev derivative* Df *is not univalent in* U *and* p*-type of* f *is as large as we please.*

Proof. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

where we choose $a_n = a_2/n(n-1)2^{n-3}$ for $n = 3, 4, \ldots$ and $a_2 > 0$ such that $\sum_{n=2}^{\infty} na_n \le 1$. Then $f \in S$. Further

$$f'(z) = 1 + \sum_{n=2}^{\infty} na_n z^{n-1}$$

is easily seen to satisfy a condition of the form (2.4) and so f' is univalent in U . Now choose $d_1 = 1$, $\{d_n\}\uparrow$, and for some $N \ge 2$, choose nondecreasing $d_{N+1} = d_2(N+2) \cdot 2^{N-2}$, $d_{N+k} = d_{N+1}$ for all $k > 1$. Then

$$\frac{d_{N+1}a_{N+1}}{d_2a_2} = \frac{N+2}{N(N+1)} > \frac{1}{N} . \tag{3.1}$$

Hence, by a theorem of Qin Yuan-Xun [14] there exists a real number φ such that

$$z + e^{i\varphi} \sum_{n=3}^{\infty} \frac{d_n a_n}{d_2 a_2} z^{n-1} \tag{3.2}$$

is not univalent in U . Now let

$$F(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n e^{i\varphi} z^n . \tag{3.3}$$

Since

$$\sum_{n=2}^{\infty} n\left|a_n e^{i\varphi}\right| = \sum_{n=2}^{\infty} na_n \le 1 ,$$

by (2.4), $F \in S$. Further, F' is also seen to be univalent in U . Now, by (1.2),

$$DF(z) = d_1 + d_2 a_2 z + d_3 a_3 e^{i\varphi} z^2 + \ldots + d_{N+1} a_{N+1} e^{i\varphi} z^N + \ldots .$$

In view of (3.2), $DF(z)$ is not univalent in U . Further

$$\limsup_{n\to\infty} \left|\frac{a_n}{e_n}\right|^{1/n} = \frac{d_{N+1}}{2} .$$

Thus by choosing d_{N+1} sufficiently large, we can have p-type of $F(z)$ as large as we please.

4. Radius of univalence of $D^n f$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$ and let ρ_n be the largest number with the property that $D^n f$ is analytic and univalent in an open disc about the origin of radius ρ_n . We now investigate the relation between the growth of $\{\rho_n\}$ and the radius of convergence of f about the origin. We thus have

THEOREM 3. *Let f be defined by* (1.1) *with radius of convergence R and let ρ_n be the radius of univalence of $D^n f$. Then*

$$\liminf_{n\to\infty} d_n \rho_n \le \liminf_{n\to\infty} \left[\prod_{i=N}^{n} \rho_i d_i\right]^{1/n} \le 2d_2 R , \tag{4.1}$$

where N denotes the smallest non-negative integer such that for $n \ge N$, $\rho_n > 0$. Further, if $|a_n/a_{n+1}|$ is eventually a positive and non-decreasing sequence, then

$$\limsup_{n\to\infty} d_n \rho_n \le 2d_2 R ; \tag{4.2}$$

in case f is of finite p-type, then we also have

$$\limsup_{n\to\infty} d_n \rho_n \le d_2 R \sqrt{d_3/(d_3-d_2)} . \tag{4.2$'$}$$

Proof. If $\rho_n = 0$ for an infinity of n, (4.1) is trivially true. If $\rho_n > 0$ for $n \geq N$, then it is obvious that $a_{n+1} \neq 0$ for $n \geq N$. Let

$$F_n(z) := D^n f(\rho_n z) = \frac{a_n}{e_n} + \frac{e_1}{e_{n+1}} a_{n+1}\rho_n z + \frac{e_2}{e_{n+2}} a_{n+2}\rho_n^2 z^n + \ldots , \quad z \in U .$$

Then the function K_n defined by

$$(4.3) \quad K_n(z) = \frac{F_n(z)-F_n(0)}{\left(e_1/e_{n+1}\right)a_{n+1}\rho_n} = z + \frac{e_2}{e_1}\frac{e_{n+1}}{e_{n+2}}\frac{a_{n+2}}{a_{n+1}}\rho_n z^2 + \frac{e_3}{e_1}\frac{e_{n+1}}{e_{n+3}}\frac{a_{n+3}}{a_{n+1}}\rho_n^2 z^3 + \ldots$$

is in S.

Applying (2.2) we have

$$e_2 e_{n+1}|a_{n+2}|\rho_n \leq 2e_1 e_{n+2}|a_{n+1}| ,$$

$$(4.4) \qquad \rho_n d_{n+2} \leq 2d_2|a_{n+1}/a_{n+2}| , \quad n \geq N ;$$

that is

$$(4.5) \qquad \liminf_{n\to\infty} \rho_n d_n \leq 2d_2 R .$$

Inductive process applied to (4.4) gives

$$\prod_{k=N}^{n} \rho_k d_k \leq \left(2d_2\right)^{n-N+1}|a_{N+1}/a_{n+2}| .$$

Hence

$$(4.6) \qquad \liminf_{n\to\infty} \left[\prod_{k=N}^{n} \rho_k d_k\right]^{1/n} \leq 2d_2 R .$$

Since left-hand inequality of (4.1) is readily seen to be true, (4.6) gives (4.1).

If f is such that $|a_n/a_{n+1}|$ is a positive and nondecreasing sequence then $R = \lim_{n\to\infty} |a_n/a_{n+1}|$ and (4.4) gives

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$$\limsup_{n\to\infty} \rho_n d_n \le 2d_2 R \,. \tag{4.7}$$

If $R = \infty$, (4.2)' obviously holds. In case $R < \infty$ and f is of finite p-type then, by (1.7), $r < \infty$ and so $d_n \sim d_{n+1}$ as $n \to \infty$.

We now apply (2.3) to the function K_n defined by (4.3) and obtain

$$\left| \left[\frac{d_{n+2}}{d_2} \frac{a_{n+2}}{a_{n+1}} \rho_n \right]^2 - \frac{d_{n+3}}{d_3} \frac{d_{n+2}}{d_2} \frac{a_{n+3}}{a_{n+1}} \rho_n^2 \right| \le 1 \;;$$

that is,

$$\left| \frac{d_{n+2}}{d_2} \right|^2 \left| \frac{a_{n+2}}{a_{n+1}} \right|^2 \rho_n^2 - \left| \frac{d_{n+3}}{d_3} \frac{d_{n+2}}{d_2} \right| \left| \frac{a_{n+3}}{a_{n+2}} \right| \left| \frac{a_{n+2}}{a_{n+1}} \right| \rho_n^2 \le 1$$

or

$$\limsup_{n\to\infty} (d_n \rho_n)^2 \left| \frac{1}{d_2^2 R^2} - \frac{1}{d_1 d_3} \frac{1}{R^2} \right| \le 1 \,,$$

$$\limsup_{n\to\infty} d_n \rho_n \le d_2 \sqrt{d_3/(d_3 - d_2)}\, R \,. \tag{4.8}$$

(4.8) gives (4.2)'. Hence the theorem.

COROLLARY. *If* $\lim_{n\to\infty} d_n \rho_n = \infty$, *then* f *is a transcendental entire function.*

If we take $f(z) = z/(1-z)$, then $R = 1$ and taking $d_n \equiv n$, we have $\rho_n = \sin(\pi/(n+1))$. Thus (4.1) gives

$$\pi \le \liminf_{n\to\infty} \left[\prod_{i=1}^{n} i \sin \frac{\pi}{i+1} \right]^{1/n} \le 4 \,.$$

5. Entire functions of finite order

We now obtain relations between the radii of univalence of the Gelfond-Leontev derivatives of an entire function and its growth constants namely, order, lower order, and so on. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order α and the lower order β. It is known [15] that

$$(5.1)\qquad \liminf_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{\log n} \le \frac{1}{\alpha} \le \frac{1}{\beta} \le \limsup_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{\log n}$$

and that equality holds at both ends of (5.1) if $|a_n/a_{n+1}|$ forms a non-decreasing function of n for $n > N$. For $0 < \alpha < \infty$, let $f(z)$ be of type T and lower type t; then ([1, p. 11], [16]),

$$(5.2)\qquad e\alpha T = \limsup_{n\to\infty} n|a_n|^{\alpha/n} ,$$

$$(5.3)\qquad e\alpha t \ge \liminf_{n\to\infty} n|a_n|^{\alpha/n} ,$$

where equality holds in (5.3) if $|a_n/a_{n+1}|$ forms a nondecreasing function of n for $n > N$. We now have

THEOREM 4. *Let f, defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, be a transcendental entire function of order α $(0 < \alpha < \infty)$, lower order β, type T and lower type t. Let D denote the Gelfond-Leontev operator defined by* (1.2) *and ρ_n the radius of univalence of $D^n f$. If $\theta = \liminf_{n\to\infty} \log d_n/\log n$, one has*

$$(5.4)\qquad \liminf_{n\to\infty} \frac{\log\rho_n}{\log n} \le \frac{1}{\alpha} - \theta ,$$

$$(5.5)\qquad \liminf_{n\to\infty} \frac{\rho_{n-2}d_n}{n^{1/\alpha}} \le \liminf_{n\to\infty} \left[\prod_{k=N}^{n} \frac{\rho_{k-2}d_k}{k^{1/\alpha}}\right]^{\alpha/n} \le \frac{[2d_2]^{\alpha}}{\alpha T} ,$$

where N denotes the non-negative integer such that for $n \ge N$, $\rho_n > 0$. Further if $|a_n/a_{n+1}|$ is eventually a positive and nondecreasing sequence, then

$$(5.6)\qquad \limsup_{n\to\infty} \frac{\log\rho_n}{\log n} \le \frac{1}{\beta} - \theta ,$$

(5.7) $$\limsup_{n\to\infty}\left[\prod_{k=N}^{n}\frac{\rho_{k-2}d_k}{k^{1/\alpha}}\right]^{\alpha/n}\le\frac{|2d_2|^{\alpha}}{\alpha t}\ .$$

Proof. By (4.4) we have

$$\log|a_{n+1}/a_{n+2}|+\log(2d_2)>\log(\rho_n d_n)\ .$$

This, coupled with extreme left inequality of (5.1), gives

$$\frac{1}{\alpha}\ge\liminf_{n\to\infty}\frac{\log(\rho_n d_n)}{\log n}\ge\liminf_{n\to\infty}\frac{\log\rho_n}{\log n}+\theta$$

which results in (5.4).

Now, by (5.2),

$$\begin{aligned}e\alpha T&=\limsup_{n\to\infty}n|a_n|^{\alpha/n}\\&=\limsup_{n\to\infty}n\left[\prod_{k=N+2}^{n}|a_k/a_{k-1}|\right]^{\alpha/n}\\&=\limsup_{n\to\infty}n\left[\prod_{k=N+2}^{n}\frac{2d_2}{\rho_{k-2}d_k}\right]^{\alpha/n}\\&=(2d_2)^{\alpha}\limsup_{n\to\infty}n\left[\prod_{k=N+2}^{n}\rho_{k-2}d_k\right]^{-\alpha/n}\end{aligned}$$

or

$$\liminf_{n\to\infty}\left[\prod_{k=N}^{n}\frac{\rho_{k-2}d_k}{k^{1/\alpha}}\right]^{\alpha/n}\le\frac{(2d_2)^{\alpha}}{\alpha T}\ .$$

The left-hand inequality of (5.5) being obvious, the proof of (5.5) is complete. If $|a_n/a_{n+1}|$ is eventually a nondecreasing sequence, then, by (5.1),

$$\frac{1}{\beta}=\limsup_{n\to\infty}\frac{\log|a_n/a_{n+1}|}{\log n}$$

so that, for $\varepsilon>0$,

$$\frac{1}{\beta}+\varepsilon>\frac{\log|a_n/a_{n+1}|}{\log n}\ ,\quad n>n_0=n_0(\varepsilon)\ .$$

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(4.4) now gives

$$\frac{1}{\beta} \geq \limsup_{n\to\infty} \frac{\log\left(\rho_n d_n\right)}{\log n} \geq \limsup_{n\to\infty} \frac{\log\rho_n}{\log n} + \theta .$$

This easily leads to (5.6). The proof of (5.7) is similar to that of (5.5) except that one has to use the relation

$$e\alpha t = \liminf_{n\to\infty} n|a_n|^{\alpha/n} .$$

COROLLARY. *(i) If* $|a_n/a_{n+1}|$ *is eventually a positive and non-decreasing sequence and* $\beta\theta > 1$ *then* $\lim_{n\to\infty} \rho_n = 0$.

(ii) If $\liminf_{n\to\infty} d_n n^{-1/\alpha} > 0$ *and* $\lim_{n\to\infty} \rho_n = \infty$ *, then* $T = 0$.

We take $f(z) = \left(e^{\pi z}-1\right)/\pi$ and $d_n \equiv n$, then $\alpha = \beta = 1$, $\theta = 1$, $\rho_n = 1$. Thus equality holds in (5.4) and (5.6).

6. The class $E(D)$

In the present section, we obtain a set of conditions on the coefficients of f and on $\{d_n\}$ such that $f \in E(D)$.

Let $\{b_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that

$$(6.1) \quad b_1 = 1 , \quad \sum_{M=1}^{\infty} \frac{M+1}{d_1 d_2 \cdots d_{M+1}} \prod_{j=k+2}^{M+k+1} b_j \leq 1 \quad \text{for } k = 0, 1, 2, \ldots .$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that

$$(6.2) \quad a_0 = 0 , \quad a_1 = 1 \quad \text{and} \quad \left|\frac{a_{n+1}}{a_n}\right| \leq \frac{b_{n+1}}{d_{n+1}} \quad \text{for } n = 1, 2, \ldots .$$

Let $E(D)$ denote the class of functions f such that

$$(6.3) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

satisfies condition (6.2). We now show that $E(D) \subset E(D)$. In fact we have the following

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THEOREM 5. *If* $f \in \mathrm{E}(D)$, *then* f *is starlike univalent in* U *and* $D^k f$ *for* $k = 1, 2, \ldots$ *is univalent in* U . *Further* $\mathrm{E}(D)$ *is properly contained in* $E(D)$.

Proof. From (1.3) we have

$$D^k f(z) = \sum_{n=k}^{\infty} \frac{e_{n-k}}{e_n} a_n z^{n-k} , \quad k = 0, 1, 2, \ldots .$$

By (2.4), $D^k f$ is univalent in U if

$$\sum_{n=2}^{\infty} \frac{d_1 d_2 \cdots d_{n+k}}{d_1 d_2 \cdots d_n} |a_{n+k}| n \le \frac{d_1 d_2 \cdots d_{1+k}}{d_1} |a_{k+1}| . \tag{6.4}$$

However (6.2) easily gives

$$|a_{n+k}| \le \frac{b_{k+n} \cdots b_{k+2}}{d_{k+n} \cdots d_{k+2}} |a_{k+1}| .$$

Thus (6.4) will follow if

$$\sum_{N=2}^{\infty} \frac{b_{k+N} \cdots b_{k+2}}{d_N \cdots d_2} |a_{k+1}| N \le |a_{k+1}| ,$$

that is,

$$\sum_{M=1}^{\infty} \frac{(M+1)}{d_1 d_2 \cdots d_{M+1}} \prod_{j=k+2}^{M+k+1} b_j \le 1 \quad \text{for} \quad k = 0, 1, 2, \ldots ,$$

which is condition (6.1). Thus $f \in E(D)$.

To show that $\mathrm{E}(D)$ is properly contained in $E(D)$, we take $d_n \equiv n$. Then $E(D) = E$ and $\mathrm{E}(D) = \mathrm{E}$. The function $\left(e^{\pi z}-1\right)/\pi$ belongs to E but is not in E (see [5]). Hence the theorem.

The class $\mathrm{E}(D)$ may contain, in general, functions that are not entire. To see this, let

$$d_n = 1 \quad \text{for} \quad n = 1, 2, \ldots , \quad b_1 = 1 , \quad b_2 = b_3 = \cdots = 1 - \frac{1}{\sqrt{2}} = x , \text{ say.}$$

Then (6.1) becomes

$$\sum_{M=1}^{\infty} (M+1) x^M = \frac{1}{(1-x)^2} - 1 = 1 ;$$

that is, (6.1) is satisfied for $k = 0, 1, 2, \ldots$.

Let

$$B(z) = z + \sum_{n=2}^{\infty} (b_1 b_2 \ldots b_n) z^n = z + \sum_{n=2}^{\infty} b_n^* z^n .$$

Since (6.1) and (6.2) are satisfied, $B(z)$ is in $E(D)$. However, if R be the radius of convergence of $B(z)$, then

$$R = \lim_{n\to\infty} \left|\frac{b_n^*}{b_{n+1}^*}\right| = \lim_{n\to\infty} \frac{1}{b_{n+1}} = \frac{\sqrt{2}}{\sqrt{2}-1}$$

so that $B(z)$ is not entire.

Our next theorem gives conditions under which every $f \in E(D)$ is entire.

THEOREM 6. *If* $b_n/d_n = o(1)$ *and* $f \in E(D)$ *, then* f *is entire.*

Proof. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $E(D)$. Then, by (6.2),

$$|a_{n+1}/a_n| \le b_{n+1}/d_{n+1} = o(1) .$$

Therefore

$$|a_n/a_{n+1}| \to \infty \quad \text{as} \quad n \to \infty .$$

Thus f is entire.

REMARK 1. The condition $b_n/d_n = o(1)$ is sharp. We can construct $f \in E(D)$ such that if $b_n/d_n \ne o(1)$ then f is not entire.

Suppose $\limsup_{n\to\infty} b_n/d_n = l \ne 0$. We may suppose $0 < l \le 1 - 1/\sqrt{2}$.

Let $d_n \equiv 1$, $a_n = b_1 \ldots b_n$ where we choose $\{b_n\}$ as follows:

$$b_1 = 1 ,$$

$$b_n = l \text{ if } n \text{ is not a prime, } n > 1 ,$$

$$= l/2 \text{ if } n \text{ is a prime, } n > 1 .$$

If $\pi(n)$ equals the number of primes less than or equal to n , then

$$a_n^{1/n} \sim \exp\left\{\frac{1}{n}\left(\pi(n)\log\frac{l}{2} + \left(n-\pi(n)\right)\log l\right)\right\}$$
$$\to l \quad \text{as} \quad n \to \infty \quad \text{since} \quad \frac{\pi(n)}{n} \to 0 \ .$$

Further

$$\sum_{M=1}^{\infty} (M+1)b_{k+2} \cdots b_{M+1+k} \le \sum_{M=1}^{\infty} (M+1)l^M = \frac{1}{(1-l)^2} - 1 \le \frac{1}{[1-1+(1/\sqrt{2})]^2} - 1 = 1$$

so that (6.1) is satisfied. Thus $f(z) = z + \sum_{n=2}^{\infty} \left(b_1 b_2 \cdots b_n\right) z^n$ is in $E(D)$ and f is not entire.

REMARK 2. There exist functions $f \in E(D)$ such that f is entire and $b_n/d_n \ne o(1)$.

Let $d_n = 1$, $a_n = b_1 b_2 \cdots b_n$ for $n \ge 1$, where we choose $\{b_n\}$ as follows. Let $b_1 = 1$. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers such that $kn_{k-1} = o\left(n_k\right)$ and $n_1 = 10^{10}$, and

$$b(n) = 1/\left(n+n_1\right)! \ , \text{ if } \ n \ne n_k \ , \quad n = 2, 3, \ldots ,$$
$$b\left(n_k\right) = L \equiv 1/n_1 \ , \text{ for } \ k = 1, 2, \ldots \ .$$

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $f \in E(D)$, f is entire and $\limsup_{n\to\infty} b_n/d_n = L > 0$. We omit the details of the proof here and also in the next

REMARK 3. Even if $b_n/d_n = o(1)$ and conditions (6.1) and (6.2) are satisfied, the entire function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ can be made to have any order ρ $(0 \le \rho \le \infty)$ by appropriate choice of $\{b_n\}$ and $\{d_n\}$. Thus $f(z)$ will be of infinite order if one takes $b_n \equiv 1$, $d_n = \log(n+c)$, $n > 1$, where c is a positive constant such that $\log(2+c) > \eta \equiv 10^{10}$. Let $a_1 = d_1 = 1$, $a_{n+1}/a_n = b_{n+1}/d_{n+1}$.

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7. Final remarks

Some of the work of Shah and Trimble [*17*, *18*, *19*, *22*] has been extended to linear invariant families by Campbell [*6*]. The concept of linear invariant family was introduced by Pommerenke [*12*] who defined a linear invariant family to be a family of functions of the form $f(z) = z + \ldots$ which are analytic and locally univalent $\left(f'(z) \neq 0\right)$ in U such that the function

$$\Lambda_\varphi f(z) = \frac{f\left(\varphi(z)\right)-f\left(\varphi(0)\right)}{\varphi'(0)f'\left(\varphi(0)\right)} = z + \ldots$$

is again a member of the family for every Mobius transformation φ of U onto U. If M is a linear invariant family, then the order of M is defined as $\alpha = \sup\{|f''(0)/2| : f \in M\}$.

Let u_α denote the union of all linear invariant families of order at most α. The family u_α is itself linear invariant. It is known [*12*] that if $\alpha < 1$, then u_α is empty; u_1 is precisely the set of all normalized convex univalent functions. The class S of normalized univalent functions is contained in u_2; in fact u_2 is much larger than S since it contains functions of infinite valence also.

The results of the present paper can be extended for the linear invariant families u_α if instead of assuming f and its (normalized) Gelfond-Leontev derivatives to be in S, one assumes their normalized forms to be in u_α. Thus, we say that $f(z)$ has the property of $u(\alpha)$ if and only if $\left(f(z)-f(0)\right)/f'(0)$ is in u_α. Given the Gelfond-Leontev operator D, let $T_\alpha(D)$ be the set of all $f(z) = z + \ldots$ which are analytic in U and for which $D^n f(z)$ has the property $u(\alpha)$ for all $n \geq 0$. Whatever has been obtained for the class $E(D)$ in this paper can be easily extended to the wider class $T_\alpha(D)$; for example Theorem 1 in this case would read as follows:

THEOREM 7. *Let* $f \in T_\alpha(D)$. *Then*

(i) *f is of finite p-type not exceeding* αd_2;

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(ii) $|f(z)| \leq (d_1/\alpha d_2)|p(\alpha d_2|z|) - 1|$, $|z| < r/\alpha d_2$;

(iii) $|D^k f(z)| \leq d_1(\alpha d_2)^{k-1} p(\alpha d_2|z|)$, $k \geq 1$;

(iv) $T_\alpha(D)$ *is a normal family in* $|z| < t < r/\alpha d_2$ *for all* t *satisfying* $0 < t < r/\alpha d_2$.

We omit the proof.

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