A certain relation between coaxial circles and conics.

By WILLIAM FINLAYSON.

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Theorem. If a point be taken on the radical axis of a coaxial system of circles, and from it tangents be drawn to any circle of the system, these tangents are cut in points on a conic, by the radical axis of the circle and a given fixed point. The two points are the foci of the conic. (Fig. 1.) Let W_1W_2 be the line of

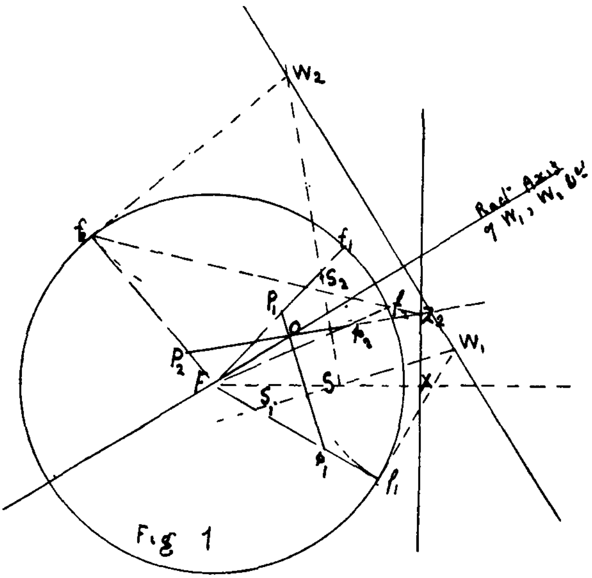
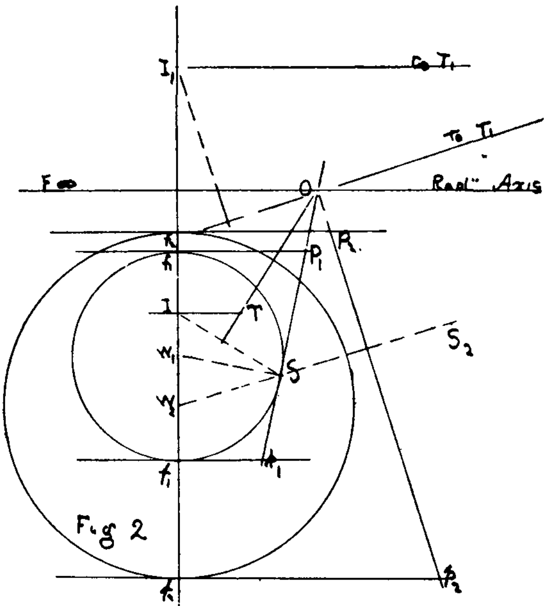


Fig 1

centres, and W_1, W_2, two circles, and RF the radical axis of the system, and S any point, internal or external, to the orthogonal circle whose centre is F. Then if we take S_1 inverse to S with regard to W_1 and bisect the segment SS_1 by a line at right angles to SS_1 we obtain the radical axis of W_1 and S. Let the tangents Ff_1 and Ff_1 from F to W_1 cut this radical axis in P_1, p_1. Then, since P_1 is on the radical axis of W_1 and S and P_1f_1 is a tangent to W_1, we have $P_1f_1 = P_1S$, and therefore $P_1F + P_1S = P_1f_1 + P_1F =$ radius of F, which, being a constant for all circles of the system, gives an ellipse as the locus of P_1P_2, &c., when, as in Fig, 1, S is internal to F. If S were an external point, we should have $P_1F - P_1S = P_1F - P_1f_1 =$ radius of $F =$ constant, and the locus of P_1, P_2, &c., would be a hyperbola. When F is at infinity on the radical axis, $P_1S = P_1f_1$, and P_1f_1 being at right angles to W_1W_2, the conic is a parabola, and the line of centres the directrix.

In all cases, S and F are evidently the foci of the conic. The point O, in which the chords of the conic intersect, is evidently the radical centre of the given system and S, and therefore is always on the radical axis of the system. It will therefore be internal to the conic when the system is intersecting, and external to the conic when the system is of the common inverse point type.

When O is external to the conic, the tangents to the conic from O will be the radical axes of the points, S and I, S and I_1, where I and I_1 are the limiting points of the given system, and the point of contact T is the point in which this radical axis is cut by the tangent from F to the point circle I (see Fig. 2.) Any line through O will cut, touch, or will not meet the conic, according to the position of the centre of the circle, to which and S it is the radical axis. The first two positions are those given for chords and tangents, but if the centre be taken between the limiting points, then the radical axis will not meet the conic.



Let C be the centre and CA the semi-transverse axis of the conic. Since $CA : CS = Ff_1 : FS$, $CA = \frac{1}{2}Ff_1$, $CS = \frac{1}{2}FS$ and the directrix is the polar of S to C, it

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is therefore also the radical axis of S to F, and since P_2p_2 is the radical axis of W_2 and S and f_2z_2 the radical axis of W_2 and F, these three being concurrent, we have the following:—The chord of the conic and its corresponding chord of the circles meet on the directrix.

When the directrix is taken as the line of centres, the circles all pass through S, and the tangents to the circles at S are therefore the radical axes, and determine focal chords in the conic.

When the polar of S is taken as the line of centres, the radical centre of the system and S coincide with C, and all the chords are diameters of the conic.

When the line of centres passes through S, the radical axes are all at right angles to the line of centres, and therefore are all parallel to each other. But in this case, if F be at infinity, the circles become a concentric system having their centre at the point where the line through S cuts the directrix.

When the line of centres is a tangent to F, the radical centre is on the conic, and all the chords therefore pass through P_1, while p_1 traces out the conic, as the centre W_1 moves along the line of centres.

Note.—It will be noticed that it is quite immaterial where the line of centres of the coaxial system be situated, as the conic is determined by the radius of the circle F, the distance FS and the orthogonal relation *between F* and the system.