Compositio Math. 142 (2006) 701–718
doi:10.1112/S0010437X06001928

The *cd*-index of fans and posets

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Abstract

The number of flags in a complete fan, or more generally in an Eulerian poset, is encoded in the cd-index. We prove the non-negativity of the cd-index for complete fans, regular CW-spheres and Gorenstein* posets.

1. Introduction

It is well known that if Δ is a complete simplicial n-dimensional fan, then giving the f-vector $(f_0, \dots, f_n)$ of Δ (i.e. f_i is the number of i-dimensional cones in Δ) is equivalent to giving the h-vector $(h_0, \dots, h_n)$, where h_i is the $2i$th Betti number of the associated toric variety. When the fan Δ is not necessarily simplicial, one can similarly encode the number of cones in its first barycentric subdivision in the flag h-vector of Δ. Eliminating all relations among the flag h-numbers, one arrives at the cd-index of Δ (see [BB85, BK91, Sta94]). The goal of this article is to prove that the cd-index has non-negative integer coefficients as conjectured in [BK91, Sta94].

Let us start by recalling the definition of the cd-index for complete fans. If Δ is a complete simplicial fan in $\mathbb{R}^n$, define $\mathcal{A}(\Delta)$ to be the space of continuous conewise polynomial functions on Δ (i.e. functions on $\mathbb{R}^n$ that restrict to polynomials on each cone of Δ). Then $\mathcal{A}(\Delta)$ is a graded free module over the ring A of global polynomial functions. We write its Poincaré polynomial

$$p_n(\Delta) = \sum_k h_k t^k,$$

where $h_k = \dim(\mathcal{A}(\Delta) \otimes_A \mathbb{R})^k$. When the fan Δ is rational, it corresponds to a toric variety $X(\Delta)$ and the numbers h_k are the even Betti numbers of the cohomology of $X(\Delta)$. In any case, they satisfy the Poincaré duality $h_k = h_{n-k}$.

When the fan Δ is complete but not necessarily simplicial, one can apply the same construction to the first barycentric subdivision $B(\Delta)$ of Δ. We can label a one-dimensional cone of $B(\Delta)$ according to the dimension of the cone in Δ whose barycenter it is. This gives a multi-grading on $\mathcal{A}(B(\Delta))$ by $\mathbb{N}^n$. It is also possible to adjust the A-module structure of $\mathcal{A}(B(\Delta))$ so that it preserves this multi-grading. The corresponding Poincaré polynomial is

$$P_n(\Delta) = \sum_S h_S t^S,$$

where $S \in \mathbb{N}^n$, $t^S = t_1^{S_1} \cdots t_n^{S_n}$ and h_S is the dimension of the degree S part of $\mathcal{A}(B(\Delta)) \otimes_A \mathbb{R}$. From the Poincaré duality $h_S = h_{\overline{S}}$, where $\overline{S} = (1, \dots, 1) - S$, we see that h_S is zero unless $0 \leqslant S_i \leqslant 1$. Thus, the sum in the Poincaré polynomial runs over subsets $S \subset \{1, \dots, n\}$.

It turns out that the Poincaré polynomial $P_n(\Delta)$ can be written as a polynomial in non-commuting variables c and d as follows. To a monomial in c and d we associate a polynomial

Received 18 May 2005, accepted in final form 16 September 2005.
2000 Mathematics Subject Classification 14M25 (primary), 06A07 (secondary).
Keywords: cd-index, flag h-vector, Gorenstein* posets.
The author was partially supported by an NSERC Research Grant.

in $t_1, \ldots, t_n$ by replacing c with $t_i + 1$ and d with $t_i + t_{i+1}$, for example,

$$ccdcd = (t_1 + 1)(t_2 + 1)(t_3 + t_4)(t_5 + 1)(t_6 + t_7).$$

The replacement is done by starting from the left and using each t_i exactly once in increasing order. Define the degree of c to be 1 and the degree of d to be 2. Then Bayer and Klapper [BK91] show that $P_n(\Delta)$ is a homogeneous cd-polynomial of degree n, called the cd-index of Δ.

Example 1.1. For any complete n-dimensional fan the coefficient of c^n in the cd-index is 1 because it is the dimension of the degree 0 part in $\mathcal{A}(B(\Delta))$. Thus, c is the cd-index of the complete one-dimensional fan. The cd-index of the complete two-dimensional fan with k maximal cones is $c^2 + (k-2)d$. If Δ is the three-dimensional fan over the faces of a pyramid (with square base) then its cd-index is $c^3 + 3cd + 3dc$.

A simple way to construct examples in low dimension is to find a subfan Π of the $(n-1)$-skeleton $\Delta^{\leqslant n-1}$ of Δ such that Π contains the $(n-2)$-skeleton $\Delta^{\leqslant n-2}$ and is combinatorially equivalent to a complete $(n-1)$-dimensional fan. Let $\sigma_1, \ldots, \sigma_m$ be the $(n-1)$-dimensional cones of $\Delta \smallsetminus \Pi$. Then

$$P_n(\Delta) = P_{n-1}(\Pi)c + \sum_i P_{n-2}(\partial\sigma_i)d.$$

If Δ is the fan over the faces of a three-dimensional polytope, then finding such a Π amounts to finding a Hamiltonian cycle in the edge graph of the polytope.

We prove another combinatorial construction of the cd-index. Let $\Delta^{\leqslant m}$ be the m-skeleton of Δ and $I(\Delta^{\leqslant m})$ the set of functions $f : \Delta^{\leqslant m} \to \mathbb{Z}$. We define the following operations on such functions. First let $E : I(\Delta^{\leqslant m}) \to I(\Delta^{\leqslant m})$ be

$$E(f)(\sigma) = \sum_{\tau \in \Delta^{\leqslant m}, \sigma \leqslant \tau} (-1)^{m - \dim \tau} f(\tau).$$

Also, let $C : I(\Delta^{\leqslant m}) \to I(\Delta^{\leqslant m-1})$ be the restriction, and let $D : I(\Delta^{\leqslant m}) \to I(\Delta^{\leqslant m-2})$ be

$$D = C \circ (E - Id) \circ C.$$

Now if $w(c,d)$ is a cd-monomial of degree n, then applying $w(C,D)$ to an element $f \in I(\Delta)$ gives an element in $I(\Delta^0)$, that is, an integer number.

Proposition 1.2. *Let $1_\Delta \in I(\Delta)$ be the constant function* 1. *If $w(c,d)$ is any cd-monomial of degree n, then $w(C,D)(1_\Delta)$ is the coefficient of the monomial $w(c,d)$ in the cd-index of Δ.*

To prove that the cd-index of a complete fan is non-negative, we consider sheaves on Δ. For a sheaf of finite-dimensional vector spaces, taking the dimension of stalks gives an element in $I(\Delta)$ with non-negative integer values. We show that the operations C and D have their counterparts in the category of sheaves, hence applying any cd-monomial to 1_Δ gives a function with non-negative values.

The cd-index can be defined more generally for Eulerian posets. Bayer and Klapper [BK91] conjectured that the cd-index of a regular CW-sphere has non-negative coefficients. Stanley [Sta94] generalized this conjecture to Gorenstein* posets, which are defined as follows. If Λ is a rank n graded poset, let $B(\Lambda)$ be the simplicial complex of chains in $\Lambda \smallsetminus \{\mathbf{0}, \mathbf{1}\}$. Then Λ is called Gorenstein* if $B(\Lambda)$ is a homology manifold with the homology of the sphere S^{n-1} (see § 2.4 for more on Gorenstein* posets). Complete fans and regular CW-spheres are examples of Gorenstein* posets; in these cases $|B(\Lambda)|$ is homeomorphic to S^{n-1}.

The main result we prove is the following.

Theorem 1.3. *Let Λ be a complete fan or a Gorenstein* poset. Then the cd-index of Λ has non-negative integer coefficients.*

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Stanley [Sta94] has shown that there are no other linear inequalities satisfied by the *cd*-indices of all Gorenstein* posets than the non-negativity of its coefficients.

Note that it suffices to prove the theorem for Gorenstein* posets because this case also covers complete fans. Nevertheless we prefer to use the terminology of fans and prove the theorem for complete fans, after which we explain which parts of the proof also apply to Gorenstein* posets.

As the *cd*-index of a complete fan has non-negative integer coefficients, one can ask what do these coefficients count? We do not give an answer to this question. The closest we come to answering it is the construction using the subfan Π above. Suppose in an ideal world every complete n-dimensional fan Δ has a subfan Π of the $n-1$-skeleton as above. One can think of Π as a maximal submanifold of $\Delta^{\leqslant n-1}$. If the *cd*-index of Δ is $fc+gd$, then f accounts for the 'manifold part' of $\Delta^{\leqslant n-1}$ and g for the rest. Thus, the *cd*-index is a universal invariant that describes the extraction of a maximal submanifold of $\Delta^{\leqslant n-1}$. This description is rather vague, but the principle of extracting the manifold part of $\Delta^{\leqslant n-1}$ also holds in § 3 when we describe how to cut $\Delta^{\leqslant n-1}$ into manifolds with boundary and compute the *cd*-index from these pieces.

2. Preliminaries

We use the terminology of fans [Ful93]. If Δ is a fan and $\sigma \in \Delta$ a cone, let $[\sigma]$ be the fan consisting of all faces of σ, $\partial\sigma = [\sigma]\smallsetminus\{\sigma\}$, and $\operatorname{Star}\sigma = \{\gamma \in \Delta \mid \gamma \geqslant \sigma\}$. For a graded set, the superscript refers to the degree. If Δ is a fan or a graded poset, then $\Delta^{\leqslant m}$ is the m-skeleton of Δ. The dimension of a cone, or the degree of an element in a poset, is denoted $\dim(\sigma)$.

2.1 Fan spaces

Let us recall the definition of a topology on a fan and the sheaves on it [BBFK02, BL03].

Given a fan Δ (considered as a finite partially ordered set), the topology on Δ is generated by subsets $[\sigma]$ for $\sigma \in \Delta$. In other words, open sets in the topology are the subfans of Δ. A sheaf F of $\mathbb{R}$-vector spaces is given by specifying the stalk F_σ for each $\sigma \in \Delta$, and the restriction map $\operatorname{res}^\sigma_\tau : F_\sigma \to F_\tau$ for $\tau < \sigma$, satisfying the compatibility condition $\operatorname{res}^\sigma_\tau \circ \operatorname{res}^\gamma_\sigma = \operatorname{res}^\gamma_\tau$ for $\gamma > \sigma > \tau$. More abstractly, a sheaf is a functor from the category of the poset Δ to the category of vector spaces. For $U \subset \Delta$ an open set, we denote by $F(U)$ the space of sections of F on U. Thus, $F(\Delta) = H^0(\Delta, F)$.

The structure sheaf $\mathcal{A}$ on Δ is defined as follows. For $\sigma \in \Delta^k$, let

$$\mathcal{A}_\sigma = A_k := \mathbb{R}[x_1, \ldots, x_k],$$

and for $\tau < \sigma$, $\tau \in \Delta^l$, let the restriction map $\operatorname{res}^\sigma_\tau$ be the standard surjection $A_k \to A_l$ (sending $x_i \mapsto x_i$ for $i \leqslant l$ and $x_j \mapsto 0$ for $j > l$). Note that if Δ has dimension n, then the sheaf $\mathcal{A}$ is multi-graded by $\mathbb{N}^n$.

Let $\mathcal{M}(\mathcal{A})$ be the category of (multi-graded) locally free flabby $\mathcal{A}$-modules: a sheaf $\mathcal{F}$ lies in $\mathcal{M}(\mathcal{A})$ if the stalks $\mathcal{F}_\sigma$ are finitely generated graded free $\mathcal{A}_\sigma$ modules and the restrictions

$$\mathcal{F}_\sigma \to \mathcal{F}(\partial\sigma)$$

are surjective morphisms, compatible with the module structure and grading. There exists an indecomposable sheaf $\mathcal{L} \in \mathcal{M}(\mathcal{A})$ satisfying the following conditions:

(i) $\mathcal{L}_0 = \mathbb{R}$;

(ii) the map induced by restriction

$$\mathcal{L}_\sigma \otimes_{A_k} \mathbb{R} \to \mathcal{L}(\partial\sigma) \otimes_{A_k} \mathbb{R}$$

is an isomorphism for all $\sigma \in \Delta^k$.

The sheaf $\mathcal{L}$ is unique up to an isomorphism and it can be constructed by induction on the dimension of the cones using the two properties. In fact, $\mathcal{L}(\partial\sigma)$ is a free A_{k-1}-module for any $\sigma \in \Delta^k$ and we can put

$$\mathcal{L}_\sigma = \mathcal{L}(\partial\sigma) \otimes_{A_{k-1}} A_k.$$

Now let $B(\Delta)$ be a barycentric subdivision of Δ. It is constructed by choosing, for each $\sigma \in \Delta$, a vector v_σ in the relative interior of σ and star-subdividing the fan at v_σ, starting with cones of maximal dimension, then one dimension smaller, and so on. The fan $B(\Delta)$ is simplicial and its cones are in one-to-one correspondence with chains of cones $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k$ in Δ. This gives $B(\Delta)$ a multi-grading by subsets $S = \{\dim\sigma_1, \ldots, \dim\sigma_k\} \subset \{1, \ldots, n\}$.

Let us define a sheaf of rings $\mathcal{B}$ on $B(\Delta)$. If $\sigma \in B(\Delta)^S$, set

$$\mathcal{B}_\sigma = B_S := \mathbb{R}[x_i]_{i\in S},$$

and let the restriction maps res^σ_τ be the standard surjections $B_S \to B_T$. This sheaf again is multi-graded by $\mathbb{N}^n$. We can define the indecomposable $\mathcal{B}$-module $\mathcal{L}$ as before, but because $\mathcal{B}$ itself is flabby, it follows that $\mathcal{L} \simeq \mathcal{B}$. To avoid confusion, we use the letter $\mathcal{L}$ for the indecomposable $\mathcal{A}$-module on Δ and the letter $\mathcal{B}$ for the similar $\mathcal{B}$-module on $B(\Delta)$.

It should be noted that the definition of topology, structure sheaf and the sheaf $\mathcal{L}$ only depend on the partially ordered set Δ, and not its realization as a fan. We say that two fans are combinatorially equivalent if their posets are isomorphic. Combinatorially equivalent fans give rise to isomorphic theories of sheaves on them.

Define a map of posets $\pi : B(\Delta) \to \Delta$ by sending a cone in $B(\Delta)$ to the smallest cone in Δ containing it. If $\sigma \in B(\Delta)$ corresponds to a chain $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k$, then $\pi(\sigma) = \sigma_k$. This map is continuous in the fan topologies. We can also extend it to a map of ringed spaces, letting

$$\mathcal{A}_{\pi(\sigma)} = \mathcal{A}_{\sigma_k} = A_{\dim\sigma_k} \to B_{\{\dim\sigma_1,\ldots,\dim\sigma_k\}} = \mathcal{B}_\sigma$$

be the standard projection.

Lemma 2.1. *There exists an isomorphism of sheaves of $\mathcal{A}$-modules on Δ*

$$\mathcal{L} \simeq \pi_*\mathcal{B}.$$

Proof. Let us check that the sheaf $\pi_*\mathcal{B}$ satisfies the two conditions in the definition of $\mathcal{L}$. The first condition is trivially true. For the second, note that combinatorially for any $\sigma \in \Delta^k$

$$B([\sigma]) \simeq B(\partial\sigma) \times [\rho],$$

where ρ is a one-dimensional cone. It follows that

$$\pi_*\mathcal{B}([\sigma]) = \mathcal{B}(B([\sigma])) \simeq \mathcal{B}(B(\partial\sigma)) \otimes_{A_{k-1}} A_k.$$

As we will see below, $\mathcal{B}(B(\partial\sigma))$ is a free A_{k-1}-module, so the sheaf $\pi_*\mathcal{B}$ lies in $\mathcal{M}(\mathcal{A})$, hence is isomorphic to $\mathcal{L}$. $\square$

2.2 Duality

Let us summarize some of the duality results proved in [BBFK02, BL03]. The structure sheaves defined in those papers differ from those used here, but the constructions are general enough to also work for sheaves $\mathcal{A}$ and $\mathcal{B}$. We will mostly be using the category-theoretic constructions from [BL03].

Fix an n-dimensional complete fan Δ.

(i) Let F be a sheaf on Δ. The cellular complex of F is

$$C^\bullet_n(\Delta, F) = 0 \to C^0 \to C^1 \to \cdots \to C^n \to 0,$$

https://doi.org/10.1112/S0010437X06001928 Published online by Cambridge University Press

where

$$C^i = \bigoplus_{\sigma\in\Delta^{n-i}} F_\sigma,$$

and the differentials $\delta : C^i \to C^{i+1}$ are defined by choosing an orientation for each cone $\sigma \in \Delta$ and setting $F_\sigma \to F_\tau$ to be $\pm\,\mathrm{res}^\sigma_\tau$ depending on whether the orientations agree or not.

On a complete fan Δ the sheaf cohomology of F can be computed using the cellular complex:

$$H^i(\Delta, F) \simeq H^i(C^\bullet_n(\Delta, F)).$$

(ii) As the sheaf $\mathcal{L}$ is flabby, its non-zero cohomology vanishes. Moreover, $H^0(\Delta,\mathcal{L})$ is a graded free A_n-module of finite rank. A similar result holds for the sheaf $\mathcal{B}$ on $B(\Delta)$, and from Lemma 2.1 we get

$$H^0(\Delta,\mathcal{L}) \simeq H^0(B(\Delta),\mathcal{B}).$$

Denote the Poincaré polynomial

$$P_n(\Delta) = \sum_{S\in\mathbb{N}^n} h_S t^S,$$

where $h_S = \dim(H^0(\Delta,\mathcal{L})\otimes_{A_n}\mathbb{R})^S$ and $t^S = t_1^{S_1}\cdots t_n^{S_n}$.

(iii) Let $\omega_n = A_n[(-1,\dots,-1)]$. That is, ω_n is the free A_n-module of rank 1 placed in degree $(1,1,\dots,1)$. Then

$$\mathrm{Hom}_{A_n}(H^0(\Delta,\mathcal{L}),\omega_n) \simeq H^0(\Delta,\mathcal{L}).$$

In particular, the numbers h_S satisfy Poincaré duality

$$h_S = h_{\overline{S}},$$

where $\overline{S} = (1,1,\dots,1) - S$. It follows that we may index the numbers h_S by subsets $S \subset \{1,\dots,n\}$.

(iv) More generally, consider the m-skeleton $\Delta^{\leqslant m}$ of Δ. The sheaf cohomology of $\mathcal{L}$ on $\Delta^{\leqslant m}$ vanishes in non-zero degrees because $\mathcal{L}$ is flabby, and it is a free A_m-module in degree 0. There is an isomorphism in the derived category $D^b(\mathcal{A})$ of finitely generated $\mathcal{A}$-modules

$$\mathrm{R}\Gamma(\Delta^{\leqslant m},\mathcal{L}) \simeq \mathrm{RHom}(C^\bullet_m(\Delta^{\leqslant m},\mathcal{L}),\omega_m).$$

In particular, the cellular complex of $\mathcal{L}$ on $\Delta^{\leqslant m}$ has no higher cohomology; the degree zero cohomology is a free A_m-module, dual to the module of global sections. When $m = n$ this together with part (i) gives Poincaré duality.

The cellular complex can be used to express the numbers h_S in terms of flag numbers of Δ. First, let us generalize the definition of Poincaré polynomial.

If M is a graded free A_n module, we define its Poincaré polynomial $P_n(M)$ as above. As the module may have elements in negative degree, $P_n(M)$ is a Laurent polynomial in general. For a complex of finite length $M^\bullet$ of free A_n-modules we define the Poincaré polynomial by taking the alternating sum

$$P_n(M^\bullet) = \sum_i (-1)^i P_n(M^i).$$

Finally, for any finitely generated A_n-module (or a bounded complex of such modules) we first choose a free resolution and then compute its Poincaré polynomial. It is easy to see that this is well defined and in fact it gives the Poincaré polynomial of an object in $D^b(A_n)$. Note also that the Poincaré polynomial of the dual module $\mathrm{Hom}_{A_n}(M, A_n)$ is obtained from the Poincaré polynomial of M by replacing t_i with t_i^{-1}.

As the cellular complex $C_n^\bullet(B(\Delta),\mathcal{B})$ is a resolution of $H^0(B(\Delta),\mathcal{B}) \simeq H^0(\Delta,\mathcal{L})$, we have

$$P_n(\Delta) = \sum_{\sigma\in B(\Delta)} (-1)^{n-\dim\sigma} P_n(\mathcal{B}_\sigma).$$

The A_n-module $\mathcal{B}_\sigma = B_S$ has the Koszul resolution

$$\cdots \to \Lambda^2 V \otimes A_n \to V \otimes A_n \to A_n \to B_S,$$

where V is a graded vector space with basis $\{x_i\}_{i\notin S}$. It follows that if σ has multi-degree S, then

$$P_n(\mathcal{B}_\sigma) = \prod_{i\notin S}(1-t_i).$$

Thus,

$$P_n(\Delta) = \sum_S \sum_{\sigma\in B(\Delta)^S} (-1)^{n-|S|} \prod_{i\notin S}(1-t_i).$$

The numbers $f_S = |B(\Delta)^S|$ are called flag numbers of Δ: they count the number of chains (or flags) in Δ of type S. We can express the coefficient of t^T in the formula above as

$$h_T = \sum_{S\subset\overline{T}} (-1)^{n-|S|-|T|} f_S.$$

As $h_T = h_{\overline{T}}$, the sum can also be written as

$$h_T = \sum_{S\subset T} (-1)^{|T\smallsetminus S|} f_S.$$

This last formula is often used to define the numbers h_T. Note that we can also invert this formula to express f_S in terms of h_T.

2.3 Existence of the *cd*-index

Let c and d be non-commuting variables of degree 1 and 2, respectively, and let $\mathbb{R}[c,d]$ be the polynomial ring in c and d. We define an injective $\mathbb{R}$-vector space homomorphism $\phi : \mathbb{R}[c,d] \to \mathbb{R}[t_1,t_2,\dots]$ by induction on degree. If $f(c,d)$ is a homogeneous polynomial of degree k on which ϕ is defined, let

$$\begin{aligned}\phi(f(c,d)c) &= \phi(f(c,d))(t_{k+1}+1),\\ \phi(f(c,d)d) &= \phi(f(c,d))(t_{k+1}+t_{k+2}).\end{aligned}$$

We say that $g(t_1,\dots,t_n)$ is a *cd*-polynomial if it lies in the image of ϕ. When talking about the degree of a *cd*-polynomial we always mean the degree in c and d.

The following proposition was proved by Bayer *et al.* [BK91]. The proof we give is essentially the same as in [BK91, Sta94].

Proposition 2.2. *Let Δ be a complete n-dimensional fan. Then $P_n(\Delta)$ is a homogeneous cd-polynomial of degree n.*

Proof. By Poincaré duality

$$2P_n(\Delta) = P_n(H^0(\Delta,\mathcal{L})) + P_n(\mathrm{Hom}_{A_n}(H^0(\Delta,\mathcal{L}),\omega_n)).$$

As the cellular complex of $\mathcal{L}$ is a resolution of $H^0(\Delta,\mathcal{L})$, we have

$$2P_n(\Delta) = \sum_{\sigma\in\Delta} (-1)^{n-\dim\sigma}(P_n(\mathcal{L}_\sigma) + P_n(\mathrm{RHom}_{D^b(A_n)}(\mathcal{L}_\sigma,\omega_n))), \tag{1}$$

where $\mathrm{RHom}_{D^b(A_n)}(\mathcal{L}_\sigma,\omega_n)$ is the derived Hom_{A_n}, computed as the Hom_{A_n} of a free resolution of $\mathcal{L}_\sigma$. It suffices to prove that each term in the sum above is a *cd*-polynomial.

Let us fix a $\sigma \in \Delta^k$, $k > 0$. By induction on dimension, we may assume that $P_k(\mathcal{L}_\sigma) = P_{k-1}(\partial\sigma)$ is a homogeneous cd-polynomial of degree $k-1$. From the Koszul resolution we get

$$P_n(\mathcal{L}_\sigma) = P_k(\mathcal{L}_\sigma) \prod_{i>k}(1-t_i).$$

Poincaré duality applied to $\partial\sigma$ implies that

$$P_k(\mathrm{Hom}_{A_k}(\mathcal{L}_\sigma, \omega_k)) = P_k(\mathcal{L}_\sigma)(t_1^{-1}, \ldots, t_k^{-1}) t_1 \cdots t_k = P_k(\mathcal{L}_\sigma) t_k.$$

From this we compute

$$\begin{aligned} P_n(\mathrm{RHom}_{D^b(A_n)}(\mathcal{L}_\sigma, \omega_n)) &= P_n(\mathcal{L}_\sigma)(t_1^{-1}, \ldots, t_n^{-1}) t_1 \cdots t_n \\ &= P_k(\mathcal{L}_\sigma)(t_1^{-1}, \ldots, t_k^{-1}) \prod_{i>k}(1-t_i^{-1}) t_1 \cdots t_n \\ &= P_k(\mathcal{L}_\sigma) t_k \prod_{i>k}(t_i - 1) \\ &= (-1)^{n-k} P_k(\mathcal{L}_\sigma) t_k \prod_{i>k}(1-t_i). \end{aligned}$$

Thus, a term corresponding to σ in the sum (1) is

$$(-1)^{n-k} P_k(\mathcal{L}_\sigma)(1 + (-1)^{n-k} t_k) \prod_{i>k}(1-t_i) = P_k(\mathcal{L}_\sigma)((-1)^{n-k} + t_k) \prod_{i>k}(1-t_i).$$

As $(1-t_i)(1-t_{i+1}) = (1+t_i)(1+t_{i+1}) - 2(t_i + t_{i+1}) = c^2 - 2d$, we get

$$P_k(\mathcal{L}_\sigma)((-1)^{n-k} + t_k) \prod_{i>k}(1-t_i) = \begin{cases} P_k(\mathcal{L}_\sigma) c (c^2 - 2d)^{(n-k)/2} & \text{if } n-k \text{ is even,} \\ -P_k(\mathcal{L}_\sigma)(c^2 - 2d)^{(n-k+1)/2} & \text{if } n-k \text{ is odd.} \end{cases}$$

When σ is the zero cone then a similar computation shows that the contribution from σ to the sum (1) is

$$((-1)^n + 1) \prod_{i>1}^{n}(1-t_i) = \begin{cases} 2(c^2 - 2d)^{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

□

Remark 2.3.

(i) The proof above gives a formula of Stanley [Sta94] for recursively computing the cd-index:

$$P_n(\Delta) = \frac{1}{2}\Bigg[\sum_{\substack{\sigma \in \Delta^k \smallsetminus \{0\} \\ n-k \text{ even}}} P_{k-1}(\partial\sigma) c (c^2 - 2d)^{(n-k)/2} - \sum_{\substack{\sigma \in \Delta^k \smallsetminus \{0\} \\ n-k \text{ odd}}} P_{k-1}(\partial\sigma)(c^2 - 2d)^{(n-k+1)/2} + \epsilon \Bigg],$$

where ϵ is the cd-polynomial (2) above.

(ii) The proof also shows that for any fan Δ, not necessarily complete, the sum of the Poincaré polynomials of the cellular complex of $\mathcal{L}$ and its dual is a cd-polynomial.

2.4 Gorenstein* posets

It is clear that the topology and the ringed space structure on a fan Δ only depend on the poset of Δ. Let us discuss the conditions a poset must satisfy in order to apply the results of this section. Gorenstein* posets defined below form the largest class of posets for which everything works as for fans.

We refer to [Sta96] for the terminology and results on posets. By a graded rank n poset we mean a finite graded poset Λ with a unique minimal element $\mathbf{0}$ in degree 0 and a unique maximal element $\mathbf{1}$ in degree $n+1$. If such elements do not exist, we add them. In particular, the poset Λ being graded implies that all maximal chains have the same length.

For any graded poset, the definition of topology, structure sheaf $\mathcal{A}$ and the irreducible sheaf $\mathcal{L}$ goes through as for fans. Note that by a sheaf on a poset Λ we mean a sheaf on $\Lambda \smallsetminus \{\mathbf{1}\}$. What we do not have for an arbitrary poset Λ is the cellular complex of a sheaf on Λ. To define the cellular complex, we need both the restriction maps res^σ_τ and the orientations $\mathrm{or}^\sigma_\tau = \pm 1$.

Let Λ be a rank n graded poset. Define Λ to be *orientable* if for every $x < y < \mathbf{1}$, where y covers x, we can choose an orientation

$$\mathrm{or}^y_x = \pm 1,$$

such that the sequence $C^\bullet_n(\Lambda, \mathbb{R}_\Lambda)$ for the constant sheaf $\mathbb{R}_\Lambda$ defined the same way as for fans above is a complex, $\delta \circ \delta = 0$. Note that we do not specify or^y_x for $y = \mathbf{1}$ because the maximal element $\mathbf{1}$ does not appear in the cellular complex.

We say that an orientable poset Λ is *Gorenstein* if

$$\dim H^i(C^\bullet_n(\Lambda, \mathbb{R}_\Lambda)) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and recursively, every proper sub-interval $[x, y] \subset \Lambda$ is also Gorenstein. (This is not a standard terminology.) Recall that a graded poset Λ is *Eulerian* if every interval $[x, y]$ for $x < y$ has the same number of elements of even and odd degree. A Gorenstein poset is clearly Eulerian: the Euler characteristic of the cellular complex on any half-open interval $[x, y)$ is 1.

To apply the duality theory of Bressler and Lunts [BL03], we need a Gorenstein poset that is actually a lattice. A complete fan is an example of a Gorenstein lattice. Regular CW-spheres [BK91] are Gorenstein posets, but they are not usually lattices.

The proof of duality in [BL03] for fans also involves the following technicality. If U is a subfan of a complete fan Δ, define its complement $U' = \{\gamma \in \Delta : \gamma \cap |U| = \mathbf{0}\}$. Then for the subfan $[\sigma] \subset \Delta$ we have $[\sigma]'' = [\sigma]$. Also, if $V_\sigma = \Delta \smallsetminus [\sigma]'$, then V_σ (or the corresponding space) has a deformation retraction to σ. It follows that the cellular complex $C^\bullet_n$ applied to the constant sheaf $\mathbb{R}$ on V_σ is acyclic in degrees less than $n-1$. Let us call a graded oriented lattice Λ *well-complemented* if it satisfies these two conditions, and recursively, every proper sub-lattice $[\mathbf{0}, x]$ is also well-complemented. Now the duality theory holds for well-complemented Gorenstein lattices. In fact, everything we prove for complete fans also works word-by-word for such lattices.

Examples of well-complemented lattices are lattices coming from polyhedral complexes, in particular simplicial complexes such as $B(\Lambda)$ below. Indeed, for polyhedral complexes, the second condition of well-complemented is satisfied because we have a deformation retract as in the case of fans. The first condition is equivalent to the statement that if the vertices of a polyhedron Q are also vertices of P, then Q is a face of P.

To define the more general notion of a *Gorenstein** poset [Sta96], let Λ be a graded poset of rank n and let $B(\Lambda)$ be the lattice of chains in $\Lambda \smallsetminus \{\mathbf{0}, \mathbf{1}\}$. Then $B(\Lambda) \smallsetminus \{\mathbf{0}, \mathbf{1}\}$ is a simplicial complex with well-defined boundary maps. In particular, $B(\Lambda)$ is orientable. Define Λ to be Gorenstein* if $B(\Lambda)$ is Gorenstein. From the discussion above it follows that the lattice $B(\Lambda)$ is well-complemented, thus if Λ is Gorenstein*, we can apply the duality theory to $B(\Lambda)$.

Our definition of a Gorenstein* poset differs from that in [Sta96], but the two are equivalent. In [Sta96], a rank n graded poset Λ is Gorenstein* if: (i) the simplicial complex $B(\Lambda) \smallsetminus \{\mathbf{0}, \mathbf{1}\}$ is a homology-manifold, i.e., the link of every simplex has the homology of a sphere of appropriate dimension; and (ii) the entire complex has the homology of the sphere S^{n-1}. These two conditions are equivalent to the cohomology conditions (3) for all intervals $[x, \mathbf{1}]$ in $B(\Lambda)$. For $y \neq \mathbf{1}$, the interval $[x, y]$ is the face lattice of a simplex, hence it satisfies condition (3).

It is also not hard to see that if Λ is a Gorenstein* poset, then any interval $[x, y] \subset \Lambda$ is again Gorenstein*.

The results proved in this section also hold if we replace the fan Δ with a Gorenstein* poset Λ. We define the Poincaré polynomial of Λ using the sheaf $\mathcal{B}$ on $B(\Lambda)$ rather than the sheaf $\mathcal{L}$ on Λ. Then we know that the module of global sections of $\mathcal{B}$ satisfies Poincaré duality.

To prove Proposition 2.2, we work with the cellular complex $C_n^\bullet(B(\Lambda), \mathcal{B})$ instead of $C_n^\bullet(\Lambda, \mathcal{L})$. Write

$$\begin{aligned} P_n(\Lambda) = P_n(C_n^\bullet(B(\Lambda), \mathcal{B})) &= \sum_{x \in B(\Lambda)} (-1)^{n-\dim x} P_n(\mathcal{B}_x) \\ &= \sum_{\sigma \in \Lambda^k} (-1)^{n-k} \sum_{x \in B(\Lambda),\ \pi(x)=\sigma} (-1)^{k-\dim x} P_k(\mathcal{B}_x) \prod_{i>k} (1-t_i) \\ &= \sum_{\sigma \in \Lambda^k} (-1)^{n-k} P_{k-1}([0, \sigma]) \prod_{i>k} (1-t_i). \end{aligned}$$

To the term $P_{k-1}([0, \sigma])$ (which is equal to $P_{k-1}(\partial\sigma)$ in case of fans) we apply induction assumption and proceed as in the proof of Proposition 2.2.

3. Shelling

Stanley [Sta94] has shown that for certain shellable Eulerian posets the cd-index is non-negative. We explain this result for fans. As this section is not essential for the general non-negativity proof, we do not generalize it from fans to posets.

3.1 Fans with boundary

Barthel *et al.* studied fans with boundary in [BBFK02]. A *quasi-convex* fan is a full-dimensional fan Δ in $\mathbb{R}^n$ such that the boundary of Δ is a homology manifold. In other words, the lattice of the boundary fan $\partial\Delta$ is Gorenstein. We also call a fan quasi-convex if it is combinatorially equivalent to a quasi-convex fan as defined above. In Stanley's terminology a quasi-convex fan is *near-Eulerian*. It can be completed by adding an element with boundary $\partial\Delta$. Such a completion is a Gorenstein* poset, not necessarily an actual fan or even a lattice.

It is shown in [BBFK02] that if Δ is a quasi-convex n-dimensional fan, then $\mathcal{L}(\Delta)$ is a free A_n-module. Its dual is the submodule $\mathcal{L}(\Delta, \partial\Delta)$ of sections vanishing on the boundary. The cellular complex of $\mathcal{L}$ has no non-zero cohomology and

$$H^0(C_n^\bullet(\Delta, \mathcal{L})) = \mathcal{L}(\Delta, \partial\Delta).$$

The module of global sections $\mathcal{L}(\Delta)$ can also be expressed using the cellular complex:

$$\mathcal{L}(\Delta) = H^0(C_n^\bullet(\Delta \smallsetminus \partial\Delta, \mathcal{L})).$$

We denote by $P_n(\Delta)$ the Poincaré polynomial of $\mathcal{L}(\Delta)$ as for complete fans.

LEMMA 3.1. *Let Δ be a quasi-convex fan of dimension n. Then*

$$P_n(\Delta) = f(c, d) + P_{n-1}(\partial\Delta)$$

for some homogeneous cd-polynomial f of degree n.

Proof. We complete Δ to $\overline{\Delta}$ by adding an element σ such that $\partial\sigma = \partial\Delta$. As $\mathcal{L}$ is flabby, we get an exact sequence of free A_n-modules

$$0 \to \mathcal{L}([\sigma], \partial\sigma) \to \mathcal{L}(\overline{\Delta}) \to \mathcal{L}(\Delta) \to 0.$$

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Now $P_n(\overline{\Delta}) = f_1(c,d)$ is a homogeneous cd-polynomial of degree n and $P_n(\mathcal{L}([\sigma],\partial\sigma)) = P_{n-1}(\partial\sigma)t_n$. Hence,

$$P_n(\Delta) = f_1(c,d) - P_{n-1}(\partial\Delta)t_n = (f_1(c,d) - P_{n-1}(\partial\Delta)c) + P_{n-1}(\partial\Delta). \qquad \square$$

Let $P_n(\Delta,\partial\Delta) = P_n(\mathcal{L}(\Delta,\partial\Delta))$. Then from the lemma and Poincaré duality we get

$$P_n(\Delta,\partial\Delta) = f(c,d) + P_{n-1}(\partial\Delta)t_n.$$

Let Δ be a complete fan and $\Delta^{\leqslant m}$ its m-skeleton. Then $\mathcal{L}(\Delta^{\leqslant m})$ is a free A_m-module and its Poincaré polynomial is

$$P_m(\Delta^{\leqslant m}) = P_n(\Delta)|_{t_{m+1}=\cdots=t_n=0}.$$

It is easy to see that $P_m(\Delta^{\leqslant m}) = f(c,d) + g(c,d)t_m$ for some homogeneous cd-polynomials f and g of degree m and $m-1$, respectively. For example, if $m = n-1$ and $P_n(\Delta) = f(c,d)c + g(c,d)d$, then setting $t_n = 0$, we have

$$P_{n-1}(\Delta^{\leqslant n-1}) = f(c,d) + g(c,d)t_{n-1}.$$

Lemma 3.2. *Let Δ be a quasi-convex n-dimensional fan and let Δ_i be quasi-convex m-dimensional fans, $m < n$, embedded in the m-skeleton $\Delta^{\leqslant m}$ of Δ, such that every $\sigma \in \Delta^{\leqslant m}$ occurs in $\Delta_i \smallsetminus \partial\Delta_i$ for precisely one i. Then*

$$P_m(\Delta^{\leqslant m}) = \sum_i P_m(\Delta_i,\partial\Delta_i).$$

Proof. According to [BL03], the cellular complex $C^\bullet_m(\Delta^{\leqslant m},\mathcal{L})$ computes the dual of $\mathcal{L}(\Delta)$. It suffices to prove that the Poincaré polynomials of the dual modules are equal:

$$P_m(C^\bullet_m(\Delta^{\leqslant m},\mathcal{L})) = \sum_i P_m(\Delta_i).$$

This follows from the decomposition of $\Delta^{\leqslant m}$ into a disjoint union

$$\Delta^{\leqslant m} = \bigcup_i (\Delta_i \smallsetminus \partial\Delta_i). \qquad \square$$

Definition 3.3. An n-dimensional quasi-convex fan Δ is *shellable* if there exists an ordering of the maximal cones $\sigma_1,\sigma_2,\ldots,\sigma_N$, such that $\partial\Delta$ as well as

$$\sigma_i^- := [\sigma_i] \cap ([\sigma_1] \cup \cdots \cup [\sigma_{i-1}]) \subset \partial\sigma_i$$

for all $i = 2,\ldots,N$ are $(n-1)$-dimensional shellable quasi-convex fans.

Example 3.4.

(i) If Δ is a complete shellable fan, then $\Delta_i = \sigma_i^- \subset \Delta^{\leqslant n-1}$ satisfy the conditions of Lemma 3.2 (the $(n-1)$-skeleton of Δ can be constructed by starting with $\sigma_N^- = \partial\sigma_N$ and attaching to it cells $\sigma_{N-1}^-,\sigma_{N-2}^-,\ldots,\sigma_2^-$), hence

$$P_{n-1}(\Delta^{\leqslant n-1}) = \sum_{i=2}^N P_{n-1}(\sigma_i^-,\partial\sigma_i^-).$$

Let $P_{n-1}(\sigma_i^-,\partial\sigma_i^-) = f_i(c,d) + g_i(c,d)t_{n-1}$ for cd-polynomials f_i and g_i of degree $n-1$ and $n-2$, respectively. Then

$$P_n(\Delta) = \sum_{i=2}^N f_i c + g_i d.$$

(ii) As in the introduction, suppose that Δ is a complete n-dimensional fan and we have

$$\Delta^{n-2} \subset \Pi \subset \Delta^{n-1},$$

where the fan Π is combinatorially equivalent to a complete $(n-1)$-dimensional fan. Let $\Delta^{n-1} \smallsetminus \Pi = \{\sigma_1, \ldots, \sigma_k\}$. Then $\Pi, [\sigma_1], \ldots, [\sigma_k]$ provide the Δ_i in Lemma 3.2, hence

$$P_{n-1}(\Delta^{\leqslant n-1}) = P_{n-1}(\Pi) + \sum_i P_{n-2}(\partial\sigma_i)t_{n-1}.$$

From this it follows that

$$P_n(\Delta) = P_{n-1}(\Pi)c + \sum_i P_{n-2}(\partial\sigma_i)d.$$

PROPOSITION 3.5. *Let Δ be a quasi-convex shellable n-dimensional fan. Then $P_n(\Delta)$ is a cd-polynomial with non-negative integer coefficients.*

Proof. Let us fix a shelling $\sigma_1, \ldots, \sigma_N$ of Δ and complete Δ to $\overline{\Delta}$ by adding an element σ_{N+1} such that $\partial\sigma_{N+1} = \partial\Delta$. (The case when $\partial\Delta = \emptyset$ can be handled similarly and is left to the reader.) Then $\sigma_1, \ldots, \sigma_N, \sigma_{N+1}$ is a shelling of $\overline{\Delta}$ and as in the previous example, we get

$$P_n(\overline{\Delta}) = \sum_{i=2}^{N+1} f_i c + g_i d,$$

where $P_{n-1}(\sigma_i^-, \partial\sigma_i^-) = f_i(c,d) + g_i(c,d)$; also $f_{N+1} = P_{n-1}(\partial\Sigma)$ and $g_{N+1} = 0$. From the exact sequence

$$0 \to \mathcal{L}([\sigma_{N+1}], \partial\sigma_{N+1}) \to \mathcal{L}(\overline{\Delta}) \to \mathcal{L}(\Delta) \to 0$$

we get

$$\begin{aligned} P_n(\Delta) &= P_n(\overline{\Delta}) - P_n([\sigma_{N+1}], \partial\sigma_{N+1}) \\ &= P_n(\overline{\Delta}) - f_{N+1}t_n \\ &= \sum_{i=1}^{N} f_i c + g_i d + P_{n-1}(\partial\Delta). \end{aligned}$$

By induction on dimension we know that f_i, g_i and $P_{n-1}(\partial\Delta)$ are homogeneous *cd*-polynomials with integer coefficients, and so is $P_n(\Delta)$. $\square$

4. Non-negativity of the *cd*-index

In this section we prove the main theorem. An essential step in the proof is to compare $H^0(\Delta^{\leqslant m}, \mathcal{L})$ with its dual module $H^0(C_m^\bullet(\Delta^{\leqslant m}, \mathcal{L}))$. To do this we look for a sheaf F on $\Delta^{\leqslant m}$ such that the cellular complex of $F \otimes \mathcal{L}$ computes $H^0(\Delta^{\leqslant m}, \mathcal{L})$.

Throughout this section we fix a complete n-dimensional fan Δ and an orientation on the cones of Δ (in fact, we only need the signs or_τ^σ for $\sigma > \tau$, $\dim\sigma = \dim\tau + 1$). We consider sheaves on the m-skeleton $\Delta^{\leqslant m}$ of Δ.

DEFINITION 4.1. A sheaf F of finite-dimensional vector spaces on $\Delta^{\leqslant m}$ is called *semi-Gorenstein* if for any $\sigma \in \Delta^{\leqslant m}$ and any $i > 0$

$$H^i(C_m^\bullet(\operatorname{Star}\sigma, F)) = 0.$$

If, moreover, for any $\sigma \in \Delta^{\leqslant m}$

$$\dim H^0(C_m^\bullet(\operatorname{Star}\sigma, F)) = \dim F_\sigma,$$

the sheaf F is called *Gorenstein*.

Example 4.2. (i) The constant sheaf $\mathbb{R}_\Delta$ is Gorenstein on Δ.

(ii) If $\Delta' \subset \Delta^{\leqslant m}$ is an m-dimensional Gorenstein subfan and $\mathbb{R}_{\Delta'}$ the constant sheaf on Δ' extended by zero, then $\mathbb{R}_{\Delta'}$ is a Gorenstein sheaf on $\Delta^{\leqslant m}$.

(iii) If F is a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$, then its restriction to $\Delta^{\leqslant k}$ for $k \leqslant m$ is again semi-Gorenstein.

Remark 4.3. From the definition it is clear that a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$ is determined by its restriction to the degree $[m-1, m]$ part $\Delta^{[m-1,m]}$ of Δ. Indeed, if F is defined on $\operatorname{Star}\sigma \smallsetminus \{\sigma\}$, where $\dim\sigma \leqslant m-2$, then

$$F_\sigma \simeq H^{m-\dim\sigma-1}(C^\bullet_m(\operatorname{Star}\sigma \smallsetminus \{\sigma\}, F)),$$

with the obvious boundary maps. However, not every sheaf on $\Delta^{[m-1,m]}$ extends to a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$.

Let F, G be semi-Gorenstein sheaves on $\Delta^{\leqslant m}$. A morphism $F|_{\Delta^{[m-1,m]}} \to G|_{\Delta^{[m-1,m]}}$ extends uniquely to a morphism $F \to G$ (inductively, $F_\sigma \to G_\sigma$ is defined as a morphism of cohomologies above). As for $\tau \in \Delta^{m-1}$ the restriction map

$$\bigoplus_{\sigma > \tau} F_\sigma \to F_\tau$$

is surjective, a morphism $F \to G$ is uniquely determined by its restriction to Δ^m, but not every morphism $F|_{\Delta^m} \to G|_{\Delta^m}$ extends to a morphism $F \to G$.

Given a semi-Gorenstein sheaf F on $\Delta^{\leqslant m}$, we define a new sheaf $F^\vee$ on $\Delta^{\leqslant m}$ as follows. Set

$$F^\vee_\sigma = H^0(C^\bullet_m(\operatorname{Star}\sigma, F))^*$$

(the vector space dual), and let the restriction maps be induced by the projections

$$C^\bullet_m(\operatorname{Star}\tau, F) \to C^\bullet_m(\operatorname{Star}\sigma, F)$$

for $\tau < \sigma$.

A morphism of semi-Gorenstein sheaves $F \to G$ induces a morphism $G^\vee \to F^\vee$, hence we get a contravariant functor from the category of semi-Gorenstein sheaves to the category of sheaves. This functor is exact: given an exact sequence of semi-Gorenstein sheaves, applying the functor $(\cdot)^\vee$ gives an exact sequence (this follows from the vanishing of higher cohomology: we get an exact sequence of degree zero cohomology).

Lemma 4.4. *If F is semi-Gorenstein on $\Delta^{\leqslant m}$, then $F^\vee$ is also semi-Gorenstein.*

Proof. For $\sigma \in \Delta^m$, let G_σ be the constant sheaf F_σ on $\{\sigma\}$, extended by zero. Consider the exact sequence

$$0 \to K \to F \to \bigoplus_{\dim\sigma=m} G_\sigma \to 0.$$

Here $K = F|_{\Delta^{\leqslant m-1}}$, hence K is semi-Gorenstein on $\Delta^{\leqslant m-1}$. The sheaves G_σ are clearly semi-Gorenstein on $\Delta^{\leqslant m}$. For $\tau \in \Delta^{\leqslant m-1}$ we get an exact sequence of complexes

$$0 \to C^\bullet_m(\operatorname{Star}\tau, K) \to C^\bullet_m(\operatorname{Star}\tau, F) \to \bigoplus_{\dim\sigma=m} C^\bullet_m(\operatorname{Star}\tau, G_\sigma) \to 0,$$

hence an exact sequence of cohomology

$$0 \to H^0(C^\bullet_m(\operatorname{Star}\tau, F)) \to \bigoplus_{\dim\sigma=m} H^0(C^\bullet_m(\operatorname{Star}\tau, G_\sigma)) \to H^0(C^\bullet_{m-1}(\operatorname{Star}\tau, K)) \to 0.$$

Thus, we get an exact sequence of sheaves

$$0 \to K^\vee \to \bigoplus_{\dim\sigma=m} G_\sigma^\vee \to F^\vee \to 0,$$

where $K^\vee$ is constructed as a sheaf on $\Delta^{\leqslant m-1}$. By induction on m we may assume that $K^\vee$ is semi-Gorenstein on $\Delta^{\leqslant m-1}$. The sheaves $G_\sigma^\vee$ are constant sheaves on $[\sigma]$ extended by zero, hence semi-Gorenstein on $\Delta^{\leqslant m}$. Now a long-exact cohomology sequence

$$\cdots \to \bigoplus_{\dim\sigma=m} H^i(C_m^\bullet(\operatorname{Star}\tau, G_\sigma^\vee)) \to H^i(C_m^\bullet(\operatorname{Star}\tau, F^\vee)) \to H^i(C_{m-1}^\bullet(\operatorname{Star}\tau, K^\vee)) \to \cdots$$

shows that $F^\vee$ is also semi-Gorenstein. $\square$

LEMMA 4.5. *The functor $(\cdot)^\vee$ is an anti-involution on the category of semi-Gorenstein sheaves on $\Delta^{\leqslant m}$.*

Proof. We construct a canonical isomorphism $F^{\vee\vee} \simeq F$ on $\Delta^{[m-1,m]}$. If $\dim\sigma = m$, then we have canonically $(F^{\vee\vee})_\sigma = F_\sigma^{**} \simeq F_\sigma$. We claim that this isomorphism extends uniquely to an isomorphism $F^{\vee\vee} \simeq F$. For $\dim\tau = m-1$, $F_\tau^\vee$ is defined by the exact sequence

$$0 \to (F_\tau^\vee)^* \to \bigoplus_{\sigma>\tau} F_\sigma \to F_\tau \to 0.$$

Similarly, $(F^{\vee\vee})_\tau$ is defined by

$$0 \to ((F^{\vee\vee})_\tau)^* \to \bigoplus_{\sigma>\tau} F_\sigma^\vee \to F_\tau^\vee \to 0.$$

As the second sequence is the dual of the first, this gives the isomorphism $(F^{\vee\vee})_\tau \simeq F_\tau$ compatible with restrictions $\operatorname{res}_\tau^\sigma$. $\square$

Let us now return to sheaves of $\mathcal{A}$-modules on $\Delta^{\leqslant m}$. We will be considering sheaves of the type $\mathcal{L} \otimes F$, where F is a semi-Gorenstein sheaf and the tensor product is over $\mathbb{R}$.

PROPOSITION 4.6. *Let F be a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$. Then in $D^b(A_m)$ we have an isomorphism*

$$C_m^\bullet(\Delta^{\leqslant m}, \mathcal{L} \otimes F) \simeq \operatorname{RHom}(C_m^\bullet(\Delta^{\leqslant m}, \mathcal{L} \otimes F^\vee), \omega_m).$$

Proof. Bressler and Lunts [BL03] construct a dualizing functor $\mathbb{D}$ on the derived category $D^b(\mathcal{M}(\mathcal{A}))$ of finitely generated $\mathcal{A}$-modules and show that

$$\mathrm{R}\Gamma(\Delta^{\leqslant m}, \mathbb{D}(\mathcal{F})) \simeq \operatorname{RHom}(C_m^\bullet(\Delta^{\leqslant m}, \mathcal{F}), \omega_m)$$

for any $\mathcal{F} \in D^b(\mathcal{M}(\mathcal{A}))$. This result together with $\mathbb{D}(\mathcal{L}) \simeq \mathcal{L}$ gives duality between $\Gamma(\Delta^{\leqslant m}, \mathcal{L})$ and $C_m^\bullet(\Delta^{\leqslant m}, \mathcal{L})$, hence Poincaré duality when $m = n$.

We show that for $\mathcal{F} \in D^b(\mathcal{M}(\mathcal{A}))$ there exists a canonical isomorphism

$$C_m^\bullet(\Delta^{\leqslant m}, \mathbb{D}(\mathcal{F}) \otimes F) \simeq \operatorname{RHom}(C_m^\bullet(\Delta^{\leqslant m}, \mathcal{F} \otimes F^\vee), \omega_m). \tag{4}$$

Then $\mathbb{D}(\mathcal{L}) \simeq \mathcal{L}$ proves the proposition.

The complex of sheaves $\mathbb{D}(\mathcal{F})$ has stalk at σ isomorphic to

$$\operatorname{RHom}_{D^b(A_m)}(C_m^\bullet([\sigma], \mathcal{F}), \omega_m).$$

To make this an actual complex of sheaves, one first resolves $\mathcal{F}$ in projective sheaves of free A_m-modules of the type $\mathcal{P}_\sigma$, which is the constant sheaf A_m on $[\sigma]$, extended by zero to a sheaf on $\Delta^{\leqslant m}$; then $C_m^\bullet([\sigma], \mathcal{P}_\sigma)$ is a complex of free A_m-modules and we can replace $\operatorname{RHom}_{D^b(A_m)}$ with Hom_{A_m}. Now it suffices to prove the isomorphism (4) for $\mathcal{F} = \mathcal{P}_\sigma$.

Let us fix a $\sigma \in \Delta^{\leqslant m}$ and consider the double complex of free A_m-modules

$$(0 \to K^{-m} \to K^{-m+1} \to \cdots \to K^0 \to 0) \otimes \omega_m,$$

where

$$K^{-m+i} = \bigoplus_{\tau \leqslant \sigma,\, \dim \tau = i} C_m^\bullet(\operatorname{Star} \tau, F),$$

and the differentials are induced by the projections with signs or_τ^σ. We claim that this double complex represents both sides of the isomorphism (4) when $\mathcal{F} = \mathcal{P}_\sigma$.

For a cone $\rho \in \Delta^{\leqslant m}$ the stalk of $\mathbb{D}(\mathcal{P}_\sigma) \otimes F$ is the complex

$$\begin{aligned}(\mathbb{D}(\mathcal{P}_\sigma) \otimes F)_\rho &= \operatorname{Hom}_{A_m}(C_m^\bullet([\rho], \mathcal{P}_\sigma), \omega_m) \otimes F_\rho \\ &= \operatorname{Hom}_{A_m}(C_m^\bullet([\rho \cap \sigma], A_m), \omega_m) \otimes F_\rho \\ &= (C_m^\bullet([\rho \cap \sigma], \mathbb{R}))^* \otimes \omega_m \otimes F_\rho.\end{aligned}$$

This is precisely the contribution from ρ to the double complex: $\rho \in \operatorname{Star} \tau$ for $\tau \leqslant \sigma$ if and only if $\tau \in [\rho \cap \sigma]$.

On the other hand, taking cohomology inside each K^i, the double complex becomes

$$\begin{aligned}\Big((F_0^\vee)^* \to \bigoplus_{\tau \leqslant \sigma,\, \dim \tau = 1} (F_\tau^\vee)^* \to \cdots\Big) \otimes \omega_m &= C_m^\bullet([\sigma], F^\vee)^* \otimes \omega_m \\ &= \operatorname{Hom}_{A_m}(C_m^\bullet([\sigma], F^\vee \otimes A_m), \omega_m) \\ &= \operatorname{Hom}_{A_m}(C_m^\bullet(\Delta^{\leqslant m}, F^\vee \otimes \mathcal{P}_\sigma), \omega_m)\end{aligned}$$

which is the right-hand side of (4). □

To simplify notation, let us denote

$$P_m(F) = P_m(C_m^\bullet(\Delta^{\leqslant m}, \mathcal{L} \otimes F)).$$

Then for F a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$, the proposition gives that $P_m(F)$ and $P_m(F^\vee)$ are dual to each other. If F is Gorenstein, then $P_m(F) = P_m(F^\vee)$.

Lemma 4.7. *Let F be a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$. Assume that:*

(i) *we have a surjective morphism $F^\vee \to F$ with kernel G;*

(ii) *$P_m(F)$ is a cd-polynomial*

$$P_m(F) = f(c, d) + g(c, d),$$

where f and g are homogeneous cd*-polynomials of degree m and $m-1$, respectively.*

Then G is a Gorenstein sheaf on $\Delta^{\leqslant m-1}$ and

$$P_{m-1}(G) = g(c, d).$$

Proof. From the exact sequence

$$0 \to G \to F^\vee \to F \to 0$$

we have $\dim G_\sigma = \dim F_\sigma^\vee - \dim F_\sigma$. Computing the cohomology long-exact sequence of

$$0 \to C_m^\bullet(\operatorname{Star} \sigma, G) \to C_m^\bullet(\operatorname{Star} \sigma, F^\vee) \to C_m^\bullet(\operatorname{Star} \sigma, F) \to 0,$$

we get that $H^i(C_m^\bullet(\operatorname{Star} \sigma, G)) = 0$ for $i > 1$, hence G is semi-Gorenstein on $\Delta^{\leqslant m-1}$, and

$$0 \to H^0(C_m^\bullet(\operatorname{Star} \sigma, F^\vee)) \to H^0(C_m^\bullet(\operatorname{Star} \sigma, F)) \to H^1(C_m^\bullet(\operatorname{Star} \sigma, G)) \to 0$$

https://doi.org/10.1112/S0010437X06001928 Published online by Cambridge University Press

shows that

$$\dim G_\sigma^\vee = \dim F_\sigma^\vee - \dim F_\sigma = \dim G_\sigma,$$

thus G is Gorenstein.

Proposition 4.6 gives that $P_m(F^\vee) = f(c,d) + g(c,d)t_m$. Now from the exact sequence

$$0 \to G \otimes \mathcal{L} \to F^\vee \otimes \mathcal{L} \to F \otimes \mathcal{L} \to 0,$$

we get

$$\begin{aligned} P_{m-1}(G)(1-t_n) = P_m(G) &= -P_m(C_m^\bullet(\Delta^{\leqslant m}, \mathcal{L} \otimes G)) \\ &= P_m(F) - P_m(F^\vee) = g(c,d)(1-t_n). \end{aligned}$$

This proves the last statement. □

To apply the previous lemma, we need to see when a surjective morphism $F^\vee \to F$ exists. As $F^\vee$ is semi-Gorenstein on $\Delta^{\leqslant m}$, it suffices to construct such a morphism on $\Delta^{[m-1,m]}$. Moreover, such a morphism is uniquely determined if we know it on Δ^m. It is also easy to see that such a morphism is surjective if and only if its restriction to Δ^m is surjective. Let us see when a morphism defined on Δ^m extend to a morphism on $\Delta^{[m-1,m]}$.

Suppose that G and F are semi-Gorenstein sheaves on $\Delta^{\leqslant m}$, and let $\phi_\sigma : G_\sigma \to F_\sigma$ be given for all $\sigma \in \Delta^m$. This map extends to a morphism of sheaves $G \to F$ if for all $\tau \in \Delta^{m-1}$ the map $\bigoplus \phi_\sigma$ takes

$$\operatorname{Ker}\Bigl(\bigoplus_{\sigma>\tau} G_\sigma \xrightarrow{\operatorname{res}_\tau^\sigma} G_\tau\Bigr) \to \operatorname{Ker}\Bigl(\bigoplus_{\sigma>\tau} F_\sigma \xrightarrow{\operatorname{res}_\tau^\sigma} F_\tau\Bigr).$$

Indeed, then we can lift an element in G_τ to $\bigoplus_{\sigma>\tau} G_\sigma$ and map it to $\bigoplus_{\sigma>\tau} F_\sigma$ followed by restriction to F_τ.

When $G = F^\vee$, then $G_\sigma = F_\sigma^*$ and

$$G_\tau = \operatorname{Ker}\Bigl(\bigoplus_{\sigma>\tau} F_\sigma \xrightarrow{\operatorname{or}_\tau^\sigma \operatorname{res}_\tau^\sigma} F_\tau\Bigr)^* = \operatorname{Coker}\Bigl(F_\tau^* \xrightarrow{\operatorname{or}_\tau^\sigma (\operatorname{res}_\tau^\sigma)^*} \bigoplus_{\sigma>\tau} F_\sigma^*\Bigr).$$

Now given $\phi_\sigma : F_\sigma^* \to F_\sigma$ for all $\sigma \in \Delta^m$, the compatibility condition becomes that the composition

$$F_\tau^* \xrightarrow{\operatorname{or}_\tau^\sigma (\operatorname{res}_\tau^\sigma)^*} \bigoplus_{\sigma>\tau} F_\sigma^* \xrightarrow{\phi_\sigma} \bigoplus_{\sigma>\tau} F_\sigma \xrightarrow{\operatorname{res}_\tau^\sigma} F_\tau \tag{5}$$

must be zero.

As an example, take F to be the constant sheaf $\mathbb{R}$ on $\Delta^{\leqslant m}$. Then $\phi_\sigma : \mathbb{R}^* \to \mathbb{R}$ is given by a constant c_σ and the sequence (5) becomes

$$\mathbb{R}_\tau^* \xrightarrow{\operatorname{or}_\tau^\sigma} \bigoplus_{\sigma>\tau} \mathbb{R}_\sigma^* \xrightarrow{c_\sigma} \bigoplus_{\sigma>\tau} \mathbb{R}_\sigma \xrightarrow{1} \mathbb{R}_\tau,$$

the composition of which is

$$\sum_{\sigma>\tau} \operatorname{or}_\tau^\sigma c_\sigma.$$

This sum is zero for all τ precisely when the collection (c_σ) defines a class in $H^0(C^\bullet(\Delta^{\leqslant m}, \mathbb{R}))$. When $m < n$ then such a class is defined as follows. Choose $d_\pi \in \mathbb{R}$ for each $\pi \in \Delta^{m+1}$, and set $c_\sigma = \sum_{\pi>\sigma} \operatorname{or}_\sigma^\pi d_\pi$. Then for $\tau \in \Delta^{m-1}$ we have

$$\sum_{\sigma>\tau} \operatorname{or}_\tau^\sigma c_\sigma = \sum_{\pi>\sigma>\tau} \operatorname{or}_\tau^\sigma \operatorname{or}_\sigma^\pi d_\pi = \sum_\pi d_\pi \Bigl(\sum_{\pi>\sigma>\tau} \operatorname{or}_\tau^\sigma \operatorname{or}_\sigma^\pi\Bigr) = 0.$$

Choosing d_π generically ensures that all c_σ are non-zero, hence $\phi : F^\vee \to F$ is surjective.

https://doi.org/10.1112/S0010437X06001928 Published online by Cambridge University Press

Proposition 4.8. *Let F be a semi-Gorenstein sheaf on $\Delta^{\leqslant m+1}$. Then there exists a surjective morphism*

$$F|^{\vee}_{\Delta^{\leqslant m}} \to F|_{\Delta^{\leqslant m}}.$$

Proof. We look for $\phi_\sigma : F^*_\sigma \to F_\sigma$ for $\sigma \in \Delta^m$ in the form of a composition

$$\phi_\sigma : F^*_\sigma \xrightarrow{\mathrm{or}^\pi_\sigma(\mathrm{res}^\pi_\sigma)^*} \bigoplus_{\pi>\sigma} F^*_\pi \xrightarrow{\phi_\pi} \bigoplus_{\pi>\sigma} F_\pi \xrightarrow{\mathrm{res}^\pi_\sigma} F_\sigma,$$

where $\phi_\pi : F^*_\pi \to F_\pi$ are defined for all $\pi \in \Delta^{m+1}$ independently of σ. Plugging this sequence into the sequence (5), we get for each τ

$$F^*_\tau \xrightarrow{\mathrm{or}^\sigma_\tau(\mathrm{res}^\sigma_\tau)^*} \bigoplus_{\sigma>\tau} F^*_\sigma \xrightarrow{\mathrm{or}^\pi_\sigma(\mathrm{res}^\pi_\sigma)^*} \bigoplus_{\pi>\sigma>\tau} F^*_\pi \xrightarrow{\phi_\pi} \bigoplus_{\pi>\sigma>\tau} F_\pi \xrightarrow{\mathrm{res}^\pi_\sigma} \bigoplus_{\sigma>\tau} F_\sigma \xrightarrow{\mathrm{res}^\sigma_\tau} F_\tau,$$

whose composition is zero because for a fixed π and τ we have

$$\sum_{\pi>\sigma>\tau} \mathrm{or}^\sigma_\tau \,\mathrm{or}^\pi_\sigma \,\mathrm{res}^\sigma_\tau \,\mathrm{res}^\pi_\sigma = 0.$$

Thus, it suffices to find a linear map $\phi_\pi : F^*_\pi \to F_\pi$ for all $\pi \in \Delta^{m+1}$, such that the composition ϕ_σ is an isomorphism for each $\sigma \in \Delta^m$. We claim that a general collection of ϕ_π satisfies this condition: for each σ there is a proper algebraic subset in $\bigoplus_\pi \mathrm{Hom}_{\mathbb{R}}(F^*_\pi, F_\pi)$ for which ϕ_σ is not an isomorphism.

Let us fix one σ. The condition for ϕ_σ not being an isomorphism is clearly algebraic. Thus, it suffices to find one collection ϕ_π that gives an isomorphism ϕ_σ. We may assume that $\mathrm{or}^\pi_\sigma = 1$, otherwise replace ϕ_π by $-\phi_\pi$. Now an isomorphism ϕ_π is given by a non-degenerate bilinear form B_π on F^*_π and ϕ_σ is given by the restriction of $\bigoplus_\pi B_\pi$ to the subspace $F^*_\sigma \subset \bigoplus_\pi F^*_\pi$, provided that this restriction is non-degenerate. If we take B_π to be positive definite symmetric bilinear forms, then their restriction is also positive definite, hence defines an isomorphism ϕ_σ. □

Lemma 4.9. *Let F be a semi-Gorenstein sheaf on $\Delta^{\leqslant m+1}$ such that $P_{m+1}(F) = f(c,d) + g(c,d)$, where f and g are homogeneous cd-polynomials of degree $m+1$ and m, respectively. If*

$$f(c,d) = f_m(c,d)c + g_m(c,d)d,$$

then

$$P_m(F|_{\Delta^{\leqslant m}}) = f_m(c,d) + g_m(c,d).$$

Proof. Consider the exact sequence

$$0 \to C^\bullet_m(\Delta^{\leqslant m}, F\otimes\mathcal{L})[-1] \to C^\bullet_{m+1}(\Delta^{\leqslant m+1}, F\otimes\mathcal{L}) \to \bigoplus_{\dim\sigma=m+1} F_\sigma\otimes\mathcal{L}_\sigma \to 0.$$

Let $P_{m+1}(\bigoplus_{\dim\sigma=m+1} F_\sigma\otimes\mathcal{L}_\sigma) = h(c,d)$, which is a homogeneous cd-polynomial of degree m. Then

$$-P_m(F|_{\Delta^{\leqslant m}})(1-t_{m+1}) - P_{m+1}(F) + h(c,d) = 0.$$

It follows that $P_m(F|_{\Delta^{\leqslant m}})$ is the coefficient of t_{m+1} in $P_{m+1}(F)$. □

Let us define operations $\mathcal{C}$ and $\mathcal{D}$ on semi-Gorenstein sheaves. If F is a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$, let $\mathcal{C}(F)$ be its restriction to $\Delta^{\leqslant m-1}$, and let $\mathcal{D}(F)$ be the sheaf G on $\Delta^{\leqslant m-2}$ from Lemma 4.7 and Proposition 4.8. Such a sheaf G is not uniquely defined; however, the dimensions of its stalks are. Given a cd-monomial $w(c,d)$ of degree m, we can apply the operation $w(\mathcal{C},\mathcal{D})$ to a semi-Gorenstein sheaf F on $\Delta^{\leqslant m}$. The result is a semi-Gorenstein sheaf on the cone 0, hence a vector space.

Theorem 4.10. *Let F be a semi-Gorenstein sheaf on $\Delta^{\leqslant m}$ such that $P_m(F) = f(c,d) + g(c,d)$, where f and g are homogeneous cd-polynomials of degree m and $m-1$, respectively. If $w(c,d)$ is a cd-monomial of degree m, then*

$$\dim w(\mathcal{C},\mathcal{D})(F)_0$$

is the coefficient of $w(c,d)$ in $f(c,d)$.

Proof. We prove it by induction on m, the case $m=0$ being trivial.

Let $f = f_{m-1}(c,d)c + g_{m-1}(c,d)d$. Then by the previous lemma

$$P_{m-1}(\mathcal{C}(F)) = f_{m-1}(c,d) + g_{m-1}(c,d),$$

and by Lemma 4.7

$$P_{m-2}(\mathcal{D}(F)) = g_{m-1}(c,d).$$

Now we are done by induction: if $w = w_{m-1}c$, then the coefficient of w in $f(c,d)$ is the coefficient of w_{m-1} in $f_{m-1}(c,d)$, which is $\dim w_{m-1}(\mathcal{C},\mathcal{D})(\mathcal{C}(F))_0$. If $w = w_{m-2}d$, then the coefficient of w in $f(c,d)$ is the coefficient of w_{m-2} in $g_{m-1}(c,d)$, which is $\dim w_{m-2}(\mathcal{C},\mathcal{D})(\mathcal{D}(F))_0$. □

Applying this to $F = \mathbb{R}_\Delta$, we get the following.

Corollary 4.11. *If Δ is a complete n-dimensional fan, then $P_n(\Delta)$ is a homogeneous cd-polynomial with non-negative integer coefficients.*

Corollary 4.12. *If Δ is a quasi-convex n-dimensional fan, then $P_n(\Delta)$ is a cd-polynomial with non-negative integer coefficients.*

Proof. Embed Δ in a complete fan $\overline{\Delta}$ of the same dimension. Let F be the constant sheaf $\mathbb{R}$ on $\Delta \smallsetminus \partial\Delta$, extended by zero to a sheaf on $\overline{\Delta}$. Then

$$P_n(F) = P_n(\Delta) = f(c,d) + P_{n-1}(\partial\Delta).$$

Now F is semi-Gorenstein, in fact $F = \mathbb{R}_\Delta^\vee$, where $\mathbb{R}_\Delta$ is the constant sheaf $\mathbb{R}$ on Δ extended by zero. The theorem gives that $f(c,d)$ has non-negative integer coefficients, the previous corollary proves the same for $P_{n-1}(\partial\Delta)$. □

4.1 Gorenstein* posets

Let Λ be a Gorenstein* poset. Then $B(\Lambda)$ is a well-complemented Gorenstein lattice and the theory of semi-Gorenstein sheaves on $B(\Lambda)$ works the same way as for complete fans. When working with $\mathcal{B}$-modules on $B(\Lambda)$, the difference is that restriction to the m-skeleton should be restriction to $B(\Lambda^{\leqslant m})$, rather than restriction to $B(\Lambda)^{\leqslant m}$. Then for Lemma 4.7 to hold, we need the sheaf G to be supported on $B(\Lambda^{\leqslant m})$.

If F is a sheaf of finite-dimensional vector spaces on $B(\Lambda^{\leqslant m})$, let $\dim F : B(\Lambda^{\leqslant m}) \to \mathbb{Z}$ be the element in $I(B(\Lambda^{\leqslant m}))$ such that

$$\dim F(\sigma) = \dim F_\sigma.$$

We consider sheaves F on $B(\Lambda^{\leqslant m})$ satisfying the condition

$$\dim F = \pi^*(f) \tag{6}$$

for some $f \in I(\Lambda^{\leqslant m})$. Recall that $\pi : B(\Lambda) \to \Lambda$ is the projection.

Lemma 4.13. *If F is a semi-Gorenstein sheaf on $B(\Lambda^{\leqslant m})$ satisfying condition (6), then $F^\vee$ satisfies the same condition.*

Proof. Recall the definition of the operation E given in the introduction. We have

$$\dim F^{\vee} = E(\dim F).$$

Now it suffices to prove that E commutes with π^*:

$$E \circ \pi^*(f) = \pi^* \circ E(f).$$

Then $\dim F^{\vee} = \pi^*(E(f))$.

For $\sigma \in B(\Lambda^{\leqslant m})$,

$$\begin{aligned} E \circ \pi^*(f)(x) &= \sum_{y \geqslant x} (-1)^{m-\dim y} f(\pi(y)) \\ &= \sum_{\sigma \in \Lambda,\, \sigma \geqslant \pi(x)} (-1)^{m-\dim \sigma} f(\sigma) \sum_{y \geqslant x,\, \pi(y)=\sigma} (-1)^{\dim \sigma - \dim y}. \end{aligned}$$

The last sum runs over all elements in $[\sigma, \mathbf{1}) \subset B([\mathbf{0}, x])$. As $B([\mathbf{0}, x])$ is Eulerian, this sum is 1. □

The previous lemma implies that if F satisfies condition (6), then the sheaf G constructed in Lemma 4.7 again satisfies the same condition. It follows that the operations $\mathcal{C}$ and $\mathcal{D}$ are well-defined on such sheaves and the results also satisfy the same condition. Now all proofs above work for semi-Gorenstein sheaves F on $B(\Lambda^{\leqslant m})$ satisfying condition (6) with $\mathcal{L}$ replaced by $\mathcal{B}$.

This proves Theorem 1.3. The operations $\mathcal{C}$ and $\mathcal{D}$ on semi-Gorenstein sheaves correspond to operations C and D defined in the introduction. Thus, Theorem 4.10 implies Proposition 1.2 not only for complete fans but also for Gorenstein* posets.

Acknowledgement

I would like to thank Louis Billera for introducing me to the *cd*-index.

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