Ergod. Th. & Dynam. Sys. (1984), **4**, 117–134
Printed in Great Britain

Bowen's equidistribution theory and the Dirichlet density theorem

WILLIAM PARRY
Mathematics Institute, University of Warwick, Coventry, CV4 7AL, England

(*Received* 5 *July* 1983)

Abstract. Let ϕ be an Axiom A flow restricted to a basic set, let g be a C^∞ function and let $\pi_g(x)=\sum_{e^{\lambda(\tau)h}\le x}(\lambda_g(\tau)/\lambda(\tau))$, where $\lambda_g(\tau)$ is the g length of the closest orbit τ, $\lambda(\tau)$ is the period of τ and h is the topological entropy of ϕ. We obtain an asymptotic formula for π_g which includes the 'prime number' theorem for closed orbits. This result generalizes Bowen's theorem on the equidistribution of closed orbits. After establishing an analytic extension result for certain zeta functions the proofs proceed by orthodox number theoretical techniques.

0. *Introduction*

For some time now, in fact probably since Selberg's paper [**16**], there has been a growing awareness of affinities between the distribution problems of number theory and those of dynamical systems. Hejhal, for example, showed how the closed orbits of certain geodesic flows could be counted using number theoretical techniques based on Selberg's trace formula ([**6**]). Sarnak pursued related problems in his thesis and described, in [**15**], the distributional behaviour of closed orbits of horocycle flows on non-compact finite volume surfaces of constant negative curvature.

Without doubt much of the recent work in this area was stimulated by Margulis's thesis and by his announcement in [**8**] of an asymptotic formula for the number of closed geodesics with length less than x when the compact manifold has negative curvature. Margulis's result can be formulated so that it closely resembles the prime number theorem. If one similarly formulates the main result of Bowen [**3**], for Axiom A flows, one can view Bowen's estimate as an analogue of Chebychev's theorem.

These asymptotic formulae for the number of closed orbits are not merely formally analogous to number theoretical results; for the present author has, together with Pollicott [**10**], established a precise formula for the Axiom A flow case by using a certain zeta function introduced by Ruelle [**14**] and by following the Wiener-Ikehara recipe for the proof of the prime numbers theorem [**18**]. Our results subsume those of Margulis and Bowen. However, they rely on the remarkable groundwork which had been prepared by Bowen in [**4**] and Ruelle in [**14**]. Using Bowen's combinatorial formula (an extension of Manning's [**7**]) for counting the number of periodic orbits of a fixed period in terms of those of certain related suspensions of shifts of finite type, we were able to express the zeta function of an Axiom A flow in terms of the zeta functions of shifts. (Ruelle had used the same technique in [13] and [14].) We could then establish a variety of analytic properties of the zeta function by invoking Ruelle's theory. The final step was to follow the Wiener-Ikehara proof of the prime number theorem.

In this paper we take the next natural step and ask is there an analogue of the Dirichlet density theorem for Axiom A flows? The density version of Dirichlet's theorem says that the number of primes in each residue class mod m ($m \geq 2$) is asymptotic to $x/\phi(m)\log x$ (where ϕ is Euler's function) if the class can contain more than one prime. There are two aspects to this theorem which coincide in the number theoretical case but which separate in the dynamical case. On the one hand the theorem gives an asymptotic formula for the number of primes in an arithmetical progression. On the other hand it says that the primes are equally distributed in residue classes (if we neglect uninteresting classes.)

Let ϕ_t be an Axiom A flow (restricted to a basic set Ω) and let m be the unique ϕ invariant probability of maximal entropy. If B is a Borel set with boundary of zero m measure we ask, in analogy with the two aspects of the density theorem, for:

(i) an asymptotic formula for $\sum_{\lambda(\tau)\leq x} \lambda_B(\tau)/\lambda(\tau)$, where the sum is over closed orbits τ, $\lambda(\tau)$ is the least period of τ and $\lambda_B(\tau)/\lambda(\tau)$ is the fraction of the orbit within B;

(ii) an asymptotic formula for the average time spent in B by closed orbits whose periods lie between $x-\varepsilon$ and $x+\varepsilon$.

The answers to these questions appear in theorems 4, 5 and 7.

Theorems 5 and 7 are, of course, due to Bowen ([1], [3], [4]). Our proofs are significantly different, inspired as they are by number theoretical techniques, yet grounded, to be sure, in Bowen's and Ruelle's theories.

Many of the ideas for this paper (and also for [10]) appear in embryonic form in [9]. Whilst working on this present paper, the author was introduced to a preprint of Sunada's [18] in which a number of closely related problems and results are discussed.

This work has benefited from numerous conversations with Mark Pollicott who has pursued other related questions in [12]. His contribution to extending the zeta function was indispensible.

1. *Preliminaries*

Let A be an aperiodic irreducible $k\times k$ zero-one matrix and let

$$X_A=\left\{x\in \prod_{n=-\infty}^{\infty}\{1,\ldots,k\}: A(x_n, x_{n+1})=1 \quad \text{for all } n\in\mathbb{Z}\right\}$$

$$X_A^+=\left\{x\in \prod_{n=0}^{\infty}\{1,\ldots,k\}: A(x_n, x_{n+1})=1 \quad \text{for all } n\in\mathbb{N}\right\}.$$

The *shift of finite type* σ_A (*one-sided shift of finite type* $\sigma_+=\sigma_{A,+}$) is defined by $(\sigma_A x)_n = x_{n+1}$ on X_A ($(\sigma_+ x)_n = x_{n+1}$ on X_A^+).

If $f \in C(X_A)$ $(f \in C(X_A^+))$ is complex-valued we define

$$\mathrm{var}_n f = \sup \{|f(x) - f(y)|: x_i = y_i, |i| \leq n\}$$

$(\mathrm{var}_n f = \sup \{|f(x) - f(y)|: x_i = y_i, i \leq n\})$, and for $0 < \theta < 1$,

$$\|f\|_\theta = \sup_{n \in \mathbb{N}} \mathrm{var}_n f / \theta^{2n+1},$$

$(\|f\|_\theta = \sup_{n \in \mathbb{N}} \mathrm{var}_n f / \theta^n)$. The space

$$\mathcal{F}_\theta = \{f \in C(X_A): \|f\|_\theta < \infty\},$$

$(\mathcal{F}_\theta^+ = \{f \in C(X_A^+): \|f\|_\theta < \infty\})$ is a Banach space with respect to the norm

$$|||f|||_\theta = \max (\|f\|_\infty, \|f\|_\theta)$$

$(|||f|||_\theta = \max (\|f\|_\infty, \|f\|_\theta))$.

If $f \in \mathcal{F}_\theta$ is *real valued* the *pressure* of f is defined by

$$P(f) = \sup_\mu \int f \, d\mu + h_\mu(\sigma_A),$$

where the supremum is taken over all σ_A-invariant probabilities μ and where $h_\mu(\sigma_A)$ denotes measure-theoretic entropy. There is a unique σ_A-invariant probability m such that

$$P(f) = \int f \, dm + h_m(\sigma_A),$$

called the *equilibrium state* of f.

Two functions $f, g \in \mathcal{F}_\theta$ are said to be *cohomologous* if there exists $u \in C(X_A)$ such that

$$f = g + u \circ \sigma_A - u.$$

If f is cohomologous to zero then it is called a *coboundary*. For cohomologous real functions $f, g \in \mathcal{F}_\theta$ we have $P(f) = P(g)$, and for a real constant a we have $P(f + a) = P(f) + a$.

If $f \in \mathcal{F}_\theta$ is real and strictly positive one defines *the f suspension space* X_A^f as the identification space of $\{(x, y): x \in X_A, 0 \leq y \leq f(x)\}$ where $(x, f(x))$ is identified with $(\sigma_A x, 0)$. X_A^f is provided with the obvious direct product compact topology. The *f suspension* is the flow σ^f generated by $\sigma_t^f(x, y) = (x, y + t)$ when $0 \leq y \leq f(x)$, $0 \leq y + t \leq f(x)$. There is a unique σ_A-invariant probability m on X_A for which the topological entropy $h(\sigma^f)$ of σ^f (or σ_1^f) equals $h_m(\sigma_A)/\int f \, dm$. Moreover $h(\sigma^f)$ is the unique real number c such that $P(-cf) = 0$.

If $f \in \mathcal{F}_\theta$ one defines

$$\zeta(f) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\mathrm{Fix}_n} \exp f^n(x),$$

when this converges. Here, $f^n(x) = \sum_{i=0}^{n-1} f(\sigma_A^i x)$ and $\mathrm{Fix}_n = \{x: \sigma_A^n x = x\}$. Clearly $\zeta(f) = \zeta(g)$ when f and g are cohomologous. In fact ζ is defined and analytic in the open set $\{f: P(\mathcal{R}f) < 0\}$ as is shown in [**14**]. (P is continuous on the real functions of $\mathcal{F}_\theta$ and has an ana.ytic continuation to an open set containing all real functions.)

We shall need to extend ζ meromorphically beyond the boundary $P(\mathscr{R}f)=0$, and to do this we shall generalize certain results of [**14**]. Here we shall use a theorem of Pollicott's. (Pollicott has also pursued certain generalizations in [**12**].)

The above definitions and results have analogues for functions in $\mathscr{F}_\theta^+$. A key tool associated with $\mathscr{F}_\theta^+$ is the Ruelle operator. Let $f\in\mathscr{F}_\theta^+$ then *the Ruelle operator* $\mathscr{L}_f$ (associated with f) is defined by

$$(\mathscr{L}_f g)(x)=\sum_{\sigma_+ y=x}\exp f(y)\cdot g(y).$$

$\mathscr{L}_f$ is a bounded linear operator on $\mathscr{F}_\theta^+$.

If $f'\in\mathscr{F}_\theta$ then there exists $u(f')\in\mathscr{F}_{\theta^{1/2}}$ such that

$$f=f'+u(f')\circ\sigma_A-u(f')$$

is a function depending on future coordinates only (i.e. $f(x)=f(y)$ if $x_i=y_i$, $i\geq 0$) and can therefore be interpreted as a function defined on X_A^+. As such, $f\in\mathscr{F}_\theta^+$. Although u is by no means unique, it can be defined so that $f'\to f$ is a bounded linear operator V from $\mathscr{F}_\theta$ to $\mathscr{F}_\theta^+$. ([**17**]). This fact enables one to employ the Ruelle operator when considering functions in $\mathscr{F}_\theta$. As we have noted before $\zeta(f')=\zeta(f)$ when $P(\mathscr{R}f)=P(\mathscr{R}f')<0$.

2. *Analytic extension of pressure*

The possibility of analytically extending pressure to certain complex functions in $\mathscr{F}_\theta^+$, and thereby to certain complex functions in $\mathscr{F}_\theta$, depends on:

PROPOSITION 1. (Ruelle–Perron–Frobenius.) *If $f\in\mathscr{F}_\theta^+$ is real then $e^{P(f)}$ is a simple eigenvalue of $\mathscr{L}_f$ with strictly positive eigenfunction. The rest of the spectrum of $\mathscr{L}_f$ is contained in a disc with radius less than $e^{P(f)}$.*

We say that any $f\in\mathscr{F}_\theta^+$ (complex) has property P if $\mathscr{L}_f$ has a simple eigenvalue $\beta(f)$ and if the rest of the spectrum is contained in a disc with radius less than $|\beta(f)|$. If f has property P, then by perturbation theory (cf. [**5**]) there exists $\varepsilon>0$ such that each

$$g\in D_\varepsilon(f)=\{h\in\mathscr{F}_\theta^+ : |||h-f|||_\theta<\varepsilon\}$$

also has property P. This depends also on the fact that the map $f\to\mathscr{L}_f$ is analytic ([**14**]). Hence $U^+=\{f\in\mathscr{F}_\theta^+ : f$ has property P$\}$ is an open set containing all real functions of $\mathscr{F}_\theta^+$. We note that if $g=f+r\sigma_+-r+a+2\pi iM$ where $f,\ r,\ M\in\mathscr{F}_\theta^+$ and a is constant, M integer valued then

$$\mathscr{L}_g=e^a\Delta_r\mathscr{L}_f\Delta_r^{-1},$$

where Δ_r is the operator which multiplies by e^r Thus the spectrum and eigenvalues of $\mathscr{L}_g$ are obtained from the spectrum and eigenvalues of $\mathscr{L}_f$ by multiplying by e^a. In particular if $f\in U^+$ then $g\in U^+$. Moreover $\beta(g)=e^a\beta(f)$. We see, therefore, that U^+ is invariant under the operations

$$f\to f+r\sigma_+-r+a+2\pi iM.$$

https://doi.org/10.1017/S0143385700002315 Published online by Cambridge University Press

Moreover we can define locally in U^+ an analytic function (pressure) by

$$P(f) = \log \beta(f),$$

which extends the definition already given for real functions, and

$$P(f + r\sigma_+ - r + a + 2\pi iM) = P(f) + a \bmod 2\pi i.$$

We extend these results to $\mathcal{F}_\theta$.

PROPOSITION 2. *Let V be the linear map from $\mathcal{F}_\theta$ to $\mathcal{F}_\theta^+$ defined in* § 1, *and let $U = V^{-1}U^+$, an open set in $\mathcal{F}_\theta$. Then pressure P can be defined locally in U so that it is analytic and so that*

$$P(f' + r'\sigma_A - r' + a' + 2\pi iM') = P(f') + a' \bmod 2\pi i,$$

when f', r', $M' \in \mathcal{F}_\theta$, $f' \in U$, M' is integer valued and a' is constant.

Proof. In fact we need only define $P(f') = P(Vf')$ where Vf' is considered as a member of $\mathcal{F}_\theta^+$. If $f' \in \mathcal{F}_\theta$ and $r, s \in \mathcal{F}_\theta$ satisfy $f' + r\sigma_A - r \in U^+$ and $f' + s\sigma_A - s \in U^+$ then we require

$$\beta(f' + r\sigma_A - r) = \beta(f' + s\sigma_A - s).$$

Evidently this is the same as requiring

$$\beta(f + t\sigma_A - t) = \beta(f)$$

when $f \in \mathcal{F}_\theta^+$, $t \in C(X_A)$ and $t\sigma_A - t \in \mathcal{F}_\theta^+$. This follows from the fact that $t \in \mathcal{F}_\theta^+$ (i.e. t depends only on future coordinates). For we have

$$t(\sigma_A^n x) - t(\sigma_A^n y) = t(\sigma_A^{n-1} x) - t(\sigma_A^{n-1} y), \qquad n = 1, 2, \ldots \tag{2.1}$$

when $x_i = y_i$, $i \geq 0$. Moreover $\sigma_A^n x$ and $\sigma_A^n y$ are asymptotic as $n \to \infty$ so the equations (2.1) show that $t(x) = t(y)$ when $x_i = y_i$, $i \geq 0$. We can therefore consider t as a member of $C(X_A^+)$. Finally if $\mathcal{L}_f v = \beta(f)v$, $v \in \mathcal{F}_\theta^+$, then

$$\mathcal{L}_{f + t\sigma_+ - t} \Delta_t v = \beta(f) \Delta_t v.$$

Hence $e^t v \in \mathcal{F}_\theta^+$ from which one concludes $e^t \in \mathcal{F}_\theta^+$ and $t \in \mathcal{F}_\theta^+$. □

3. *Extending the zeta function on $\mathcal{F}_\theta^+$, $\mathcal{F}_\theta$*

We have already mentioned the fact that ζ is analytic in the open set $\{f\colon P(\mathcal{R}f) < 0\}$ of $\mathcal{F}_\theta^+$ (or of $\mathcal{F}_\theta$). To see this it suffices to show that if $P(\mathcal{R}f) < 0$ then there exists $\varepsilon > 0$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\mathrm{Fix}_n} \exp g^n(x) \tag{3.1}$$

is uniformly convergent in $D_\varepsilon(f)$.

If $g = h + ik$ (h, k real functions in $\mathcal{F}_\theta^+$) then the absolute value of (3.1) is dominated by

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\mathrm{Fix}_n} \exp h^n(x),$$

and since $h \leq \mathcal{R}(f) + \varepsilon$ we have

$$\left| \sum_{\mathrm{Fix}_n} \exp h^n(x) \right|^{1/n} \leq e^{\varepsilon} \left| \sum_{\mathrm{Fix}_n} \exp \mathcal{R}(f^n(x)) \right|^{1/n}.$$

However (cf. [**14**]), $|\sum_{\mathrm{Fix}_n} \exp \mathscr{R}(f^n(x))|^{1/n}$ converges to $\exp P(\mathscr{R}f) < 1$, so that for some $\varepsilon > 0$ and some N

$$e^{\varepsilon}\left|\sum_{\mathrm{Fix}_n} \exp \mathscr{R}(f^n(x))\right|^{1/n} < \rho < 1,$$

for $n \geq N$. Uniform convergence of (3.1) follows.

For the time being we shall restrict attention to functions in $\mathscr{F}_\theta^+$. The proof of our next result is a modification of a method employed by Ruelle [**14**].

PROPOSITION 3. *If $f \in \mathscr{F}_\theta^+$ is real and $P(f) = 0$ then there exists $\varepsilon > 0$ such that P extends to an analytic function in $D_\varepsilon(f)$ and*

$$\sum_{n=1}^{\infty} \frac{e^{ian}}{n}\left(\sum_{\mathrm{Fix}_n} \exp g^n - e^{nP(g)}\right) \tag{3.2}$$

converges uniformly in $D_\varepsilon(f)$. (a is a real constant.)

Proof. We shall consider $\mathscr{F}_\theta^+ \subset \mathscr{F}_{\theta'}^+$, with $\theta < \theta' < 1$ and we first choose $\varepsilon > 0$ sufficiently small that when $|||g - f|||_{\theta'} < \varepsilon$ then $\mathscr{L}_g$, acting on $\mathscr{F}_{\theta'}^+$ $(g \in \mathscr{F}_{\theta'}^+)$, has a maximum eigenvalue $e^{P(g)}$ with $|P(g)| < \delta$ and for λ in the remainder of the spectrum we have $|\lambda| < 1 - 2\delta$ where $\delta < \frac{1}{2}$. (Note that $|e^{P(g)}| \geq e^{-\delta} > 1 - 2\delta$.) Here we are using perturbation theory for the operator $\mathscr{L}_f$ acting on $\mathscr{F}_{\theta'}^+$ $(f \in \mathscr{F}_\theta^+ \subset \mathscr{F}_{\theta'}^+)$. We shall decrease $\varepsilon > 0$ as necessary.

Let $g \in D_\varepsilon(f)$ then $|||g - f|||_{\theta'} \leq |||g - f|||_\theta < \varepsilon$ and $\mathscr{L}_g v = e^{P(g)} v$, $v = v_g \in \mathscr{F}_\theta^+$. Suppose that $\varepsilon > 0$ is small enough that $|||v - v_f|||_\theta < \eta$, where $0 < \eta < \min v_f$, for all $v = v_g$, $g \in D_\varepsilon(f)$.

Define g_m, v_m (functions of $x_0, \ldots, x_m$) so that

$$\|g - g_m\|_\infty \leq \|g\|_\theta \theta^m, \qquad \|v - v_m\|_\infty \leq \|v\|_\theta \theta^m$$

and define

$$w_m = \mathscr{L}_{g_m} v_m - e^{P(g)} v_m, \qquad u_m = v_m + (w_m / e^{P(g)})$$

then

$$\|w_m\|_\infty \leq B\theta^m e^{B\theta^m}, \qquad \|(v_m/u_m) - 1\|_\infty \leq B\theta^m e^{B\theta^m},$$

for $m \geq N_1$ for constants B, N_1 depending on f, ε only. Hence, writing $v_m/u_m = e^{r_m + i\gamma_m}$ with γ_m suitably chosen mod 2π and r_m, γ_m real functions of $x_0, \ldots, x_m$, we have $\|r_m\|_\infty \leq C\theta^m$, $\|\gamma_m\|_\infty \leq C\theta^m$ if $m \geq N_2 \geq N_1$ where N_2, C depend on f, ε only. Since

$$\mathscr{L}_{g_m} \frac{v_m}{u_m} \cdot u_m = e^{P(g)} u_m$$

we have $\mathscr{L}_{\bar{g}_m} u_m = e^{P(g)} u_m$, where $\bar{g}_m = g_m + r_m + i\gamma_m$. Since $\|\bar{g}_m - g_m\|_\infty \leq 2C\theta^m$ and $\|g_m - g\|_\infty \leq (|||f|||_\theta + \varepsilon)\theta^m$ it follows that

$$\|\bar{g}_m - g\|_{\theta'} \leq D(\theta/\theta')^m \qquad \text{for } m \geq N_2$$

where D depends on f, ε only. Hence

$$|||\bar{g}_m - g|||_{\theta'} \leq \max(\varepsilon/2, D(\theta/\theta')^m) \leq \varepsilon/2,$$

and $|||g - f|||_{\theta'} \leq \varepsilon/2$ if $g \in D_{\varepsilon/2}(f)$ and $m \geq N_3$. Thus $|||\bar{g}_m - f|||_{\theta'} \leq \varepsilon$ if $g \in D_{\varepsilon/2}(f)$ and $m \geq N_3$. We see then that $\mathscr{L}_{\bar{g}_m}$ acting on $\mathscr{F}_{\theta'}^+$ has an eigenvalue $e^{P(g)}$ with $|P(g)| < \delta$

and its other eigenvalues λ satisfy $|\lambda| < 1-2\delta$. This is therefore also true of the finite dimensional operator L_m acting on functions $q(x_0, \ldots, x_m)$ by

$$(L_m q)(x_0 \ldots x_m) = \sum_{A(i,x_0)=1} \exp \bar{g}_m(i, x_0 \ldots x_{m-1}) \cdot q(i, x_0, \ldots, x_{m-1}).$$

Consequently

$$|\text{Trace } L_m^n - e^{nP(g)}| \le k^m (1-2\delta)^n,$$

where A is $k \times k$. But

$$\text{Trace } L_m^n = \sum_{\text{Fix}_n} \exp \bar{g}_m^n(x),$$

so we have

$$\left| \sum_{\text{Fix}_n} \exp \bar{g}_m^n - e^{nP(g)} \right|^{1/n} < k^\alpha (1-2\delta)$$

if $m = [n\alpha] > N_3$ $(0 < \alpha < 1)$, i.e. if $n > N_4 = (N_3+1)/\alpha$. Choose α so that $k^\alpha(1-2\delta) < 1-\delta$, then for $n > N_4$ we have

$$\left| \sum_{\text{Fix}_n} \exp \bar{g}_m^n - e^{nP(g)} \right|^{1/n} < 1-\delta. \tag{3.3}$$

Finally, we have

$$\begin{aligned} \left| \sum_{\text{Fix}_n} (\exp \bar{g}_m^n - \exp g^n) \right| &\le \sum_{\text{Fix}_n} e^{\mathcal{R}(g^n)} |\exp (\bar{g}_m^n - g^n) - 1| \\ &\le \sum_{\text{Fix}_n} e^{\mathcal{R}(g^n)} n \|\bar{g}_m - g\|_\infty \, e^{n\|\bar{g}_m - g\|_\infty}, \\ &\le \sum_{\text{Fix}_n} e^{\mathcal{R}(g^n)} nE\theta^m \exp(En\theta^m) \qquad \text{if } n \ge N_5 \ge N_4. \end{aligned}$$

Thus

$$\left| \sum_{\text{Fix}_n} (\exp \bar{g}_m^n - \exp g^n) \right|^{1/n} \le \left(\sum_{\text{Fix}_n} \exp (f^n + n\varepsilon) \right)^{1/n} n^{1/n} E^{1/n} \theta^\alpha,$$

and the latter converges to $e^{P(f)} e^\varepsilon \theta^\alpha = e^\varepsilon \theta^\alpha$, so that if $\varepsilon > 0$ is small enough

$$\left| \sum_{\text{Fix}_n} (\exp \bar{g}_m^n - \exp g^n) \right|^{1/n} < \rho < 1 \qquad \text{if } n > N_6 \ge N_5. \tag{3.4}$$

The inequalities (3.3) and (3.4) show that (3.2) converges uniformly in $D_{\varepsilon/2}(f)$ and the proof is complete. □

If we define $\phi(g+ia)$ to be the exponential of (3.2) then ϕ is a non-vanishing analytic function in a neighbourhood of $f+ia$ whenever $f \in \mathcal{F}_\theta^+$ is real, $P(f)=0$ and a is real.

Suppose now that $f \in \mathcal{F}_\theta^+$ is complex and $P(\mathcal{R}f)=0$. There are three cases to consider:

(i) $\mathcal{I}(f)$ is cohomologous to $(2\pi M + a)$, where M is integer valued and a is not a multiple of 2π. In this case $\zeta(g)$ can be extended analytically to a neighbourhood of f, since a neighbourhood of $\mathcal{R}(f)+ia$ maps to a neighbourhood of f under

$$Wg = g + 2\pi i M + r\sigma_+ - r,$$

and $\zeta(g)=\zeta(Wg)$. Moreover we have seen that ϕ is analytic in a neighbourhood of $\mathcal{R}(f)+ia$. So we define

$$\zeta(g)=\frac{\phi(g)}{1-e^{P(g)}}.$$

(ii) When $\mathcal{I}(f)$ is cohomologous to $2\pi M$ where M is integer valued, we define ζ as in (i) except for g with $P(g)=0$ (or a multiple of $2\pi i$).

(iii) When $\mathcal{I}(f)$ is *not* cohomologous to $2\pi M + a$ where M is integer valued and a is a real constant, Pollicott [**12**] (see also [**10**]) has shown that $\mathcal{L}_f$ has a spectral radius strictly less than 1 and this gives rise to a non-vanishing analytic extension of ζ to a neighbourhood of f.

We are now in a position to discuss the zeta function and its extension for functions $f\in\mathcal{F}_\theta$ (defined over X_A rather than X_A^+). For such functions we employ the operator $V\colon \mathcal{F}_\theta\to\mathcal{F}_\theta^+$ (of § 1.) Since $\zeta(Vf)=\zeta(f)$, $P(f)=P(Vf)$ we can transfer our results to $\mathcal{F}_\theta$ and obtain:

THEOREM 1. *Let $f\in\mathcal{F}_\theta$. Then*

(a) *If $P(\mathcal{R}f)<0$ or if $P(\mathcal{R}f)=0$ and $\mathcal{I}(f)$ is not cohomologous to $2\pi M + a$ where M is integer valued and a is a real constant then ζ is well defined non-vanishing and analytic in a neighbourhood of f.*

(b) *If $P(\mathcal{R}f)=0$ and if $\mathcal{I}(f)$ is cohomologous to $2\pi M + a$ where M is integer valued and $0\le a<2\pi$ then*

$$\exp \sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{\mathrm{Fix}_n}\exp g^n - e^{nP(g)}\right)$$

converges uniformly to a non-zero analytic function $\phi(g)$ in a neighbourhood $D_\varepsilon(f)$ of f and ζ can be extended analytically in $D_\varepsilon(f)-E$ by defining

$$\zeta(g)=\phi(g)/1-e^{P(g)}, \qquad \textit{where } E=\{g\colon P(g)=0\}.$$

4. *Restricting the zeta function*

In this section we fix real functions $f, g\in\mathcal{F}_\theta$ and suppose that f is strictly positive with $P(-f)=0$. Define $\zeta(s,z)=\zeta(-sf-zg)$ whenever this is defined by theorem 1 for $(s,z)\in\mathbb{C}\times\mathbb{C}$. Clearly if $\mathcal{R}(s_0)>1$ we have $\zeta(s,z)$ well defined and non-vanishing in $|s-s_0|<\varepsilon$, $|z|<\varepsilon$ for $\varepsilon>0$ small enough. Now let $s_0=1+it_0$. If $t_0\ne 0$ and $t_0 f$ is not cohomologous to a function $2\pi M + a$ (M integer valued, a real) then $\zeta(s,z)$ is again well defined, non-zero and analytic in $|s-1-it_0|<\varepsilon$, $|z|<\varepsilon$. If $t_0 f$ *is* cohomologous to $2\pi M + a$ (and this includes the case $t_0=0$) then $\phi(s,z)=\phi(-sf-zg)$ is non-vanishing and analytic in $|s-1-it_0|<\varepsilon$, $|z|<\varepsilon$. Hence

$$\zeta(s,z)=\phi(s,z)/(1-e^{P(-sf-zg)})$$

is also non-vanishing and analytic in this region and extends ζ, except where $P(-sf-zg)=0$. In this case if $\varepsilon>0$ is small enough we have $P(-sf)\ne 0$ when

$0<|s-1-it_0|<\varepsilon$, for otherwise $P(-sf)$ would be identically zero. So for $0<|s-1-it_0|<\varepsilon$ there exists $\varepsilon(s)>0$ such that $P(-sf-zg)\neq 0$ when $|z|<\varepsilon(s)$.

By differentiating logarithmically with respect to the second variable at $z=0$ we obtain

$$\frac{\zeta_2(s,0)}{\zeta(s,0)}-\frac{P_2(-sf-zg)_{z=0}}{1-e^{P(-sf)}}=\frac{\phi_2(s,0)}{\phi(s,0)}$$

where

$$\eta(s)=\frac{-\zeta_2(s,0)}{\zeta(s,0)}=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\mathrm{Fix}_n} g^n \exp(-sf^n)$$

is analytic in a neighbourhood of $\mathscr{R}(s)\geq 1$ minus the points $1+it_0$ where $t_0 f$ is cohomologous to $2\pi M$ for an integer valued M. If such $t_0\neq 0$ exist there is a least positive one such that all others are multiples of t_0. (See § 5.) We note that in any case

$$\lim_{s\to 1}(s-1)\eta(s)=\frac{P_2(-f-zg)_{z=0}}{P'(-sf)_{s=1}}=\int g\,d\mu\Big/\int f\,d\mu,$$

where μ is the unique equilibrium state for $-f$. (cf. [**14**].)

To summarize, we have the following:

PROPOSITION 4. *Suppose $f, g\in\mathscr{F}_\theta$, f strictly positive and $P(-f)=0$. When there is no $t_0\neq 0$ with $t_0 f$ cohomologous to $2\pi M$ (M integer valued) we have*

$$\eta(s)=\frac{\int g\,d\mu\Big/\int f\,d\mu}{s-1}+\psi(s),$$

where $\psi(s)$ is analytic in a neighbourhood of $\mathscr{R}(s)\geq 1$.

When $t_0 f$ is cohomologous to $2\pi M(t_0>0$ least) (M integer valued) we have η is simply periodic with period it_0 and η is analytic in $\mathscr{R}(s)>1-\varepsilon$, for some $\varepsilon>0$, except for simple poles at $1+nit_0$, $n\in\mathbb{Z}$ with residue $\int g\,d\mu/\int f\,d\mu$. (The justification of the word 'least' appears in the next section.)

5. *Suspensions of shifts of finite type*

Let F be a strictly positive function in $\mathscr{F}_\theta$, and consider the flow σ^f on X_A^f. As with any other flow we say that σ^f is (topologically) *weak-mixing* if the equation

$$g(\sigma_t^f)=e^{iat}g, \qquad \text{all } t\in\mathbb{R},\ g\in C(X_A^f) \tag{5.1}$$

has no solution other than $a=0$, g constant. σ^f is not weak-mixing and (5.1) has a non-trivial solution if and only if af is cohomologous to $2\pi M$ for some integer valued function. (cf. [**11**].) (We used this fact in § 4, when we claimed that if there is a number $t\neq 0$ such that $tf/2\pi$ is cohomologous to an integer valued function, then there is a *least* such positive t_0 and all other t are multiples of t_0.)

Assume that $h\in C(X_A^f)$ and defining

$$g(x)=\int_0^{f(x)} h\sigma_t^f(x,0)\,dt\in C(X_A)$$

assume that $g\in\mathscr{F}_\theta$. For each closed orbit τ of σ^f let $\lambda(\tau)$ be the least period of τ

and define

$$\lambda_h(\tau)=\int_0^{\lambda(\tau)} h\sigma_t^f(x,u)\,dt \qquad \text{where } (x,u)\in\tau.$$

Clearly $\lambda(\tau)=f(x)+\cdots+f(\sigma_A^{m-1}x)$ and $\lambda_h(\tau)=g(x)+\cdots+g(\sigma_A^{m-1}x)$ where $(x,u)\in\tau$ (for some u) and where m is the least σ_A period of x.

For η defined as in § 4 with respect to f, g we have

$$\begin{aligned}\eta(s)&=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\mathrm{Fix}_n} g^n\exp(-sf^n)\\ &=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\tau} k\lambda_h(\tau)\exp(-sk\lambda(\tau))\\ &=\sum_{k=1}^{\infty}\sum_{\tau}\lambda_h(\tau)\exp(-s\lambda(\tau)k).\end{aligned}$$

The following theorem is an immediate consequence of proposition 4.

THEOREM 2. *If $f\in\mathcal{F}_\theta$ is strictly positive and if*

$$g(x)=\int_0^{f(x)} h(\sigma_t^f(x,0))\,dt\in\mathcal{F}_\theta$$

where $h\in C(X_A^f)$ then

$$\eta(s)=\sum_{k=1}^{\infty}\sum_{\tau}\lambda_h(\tau)\exp(-s\lambda(\tau)k)$$

is analytic in $\mathcal{R}(s)>h(\sigma^f)$, and

(a) *if σ^f is topologically weak-mixing then $\eta(s)$ has an analytic extension to a neighbourhood of $\mathcal{R}(s)\geq h(\sigma^f)$ except for a simple pole at $h(\sigma^f)$ with residue $\int g\,d\mu/\int f\,d\mu=\int h\,dm$, where $m=\mu\times$ Lebesgue (locally) is the measure of maximal entropy for σ^f;*

(b) *if σ^f is not topologically weak-mixing then $\eta(s)$ is simply periodic with least period ia, (where a is the least positive eigenfrequency, that is, solution of* (5.1)) *and has an analytic extension to $\mathcal{R}(s)>h(\sigma^f)-\varepsilon$, for some $\varepsilon>0$, except for simple poles at $h(\sigma^f)+nia$, $n\in\mathbb{Z}$, with residue $\int h\,dm$.*

6. *Axiom A flows*

Let M be a compact Riemannian manifold and let $\phi_t: M\to M$ ($t\in\mathbb{R}$) be a differentiable flow. A closed invariant set $\Omega\subset M$ without fixed points is *hyperbolic* if the tangent bundle over Ω is a Whitney sum

$$T_\Omega M=E+E^s+E^u$$

of three $T\phi_t$-invariant sub-bundles, where E is the one-dimensional bundle tangent to the flow and E^s, E^u are exponentially contracting and expanding, respectively:

(i) $\|T\phi_t(v)\|\leq K e^{-\lambda t}\|v\|$ for all $v\in E^s$, $t\geq 0$

(ii) $\|T\phi_{-t}(v)\|\leq K e^{-\lambda t}\|v\|$ for all $v\in E^u$, $t\geq 0$

where $\lambda>0$, K are constants.

A *basic set* Ω of an Axiom A flow ϕ_t is a hyperbolic set in which periodic points are dense, $\phi_t|\Omega$ is topologically transitive and $\Omega = \bigcap_{t\in\mathbf{R}} \phi_t U$ for some open neighbourhood of Ω.

We shall be interested in an Axiom A flow restricted to a basic set Ω and will always assume that Ω is not a topological circle.

We shall need the following result of Bowen's (cf. [4]) to help us count closed orbits.

PROPOSITION 5. *If ϕ_t is an Axiom A flow restricted to the basic set Ω then there exist suspensions of shifts of finite type $\sigma^i = \sigma^{f_i}_{A_i}$, $i = 0, 1, \ldots, q$, Lipschitz maps $\pi_i: X^{f_i}_{A_i} \to \Omega$, $\pi_i\sigma^i_t = \sigma_t\pi_i$, where each π_i is at most N to 1, $f_i \in \mathcal{F}_\theta$ for some $0<\theta<1$ and*

(i) π_0 is surjective and a.e. one-one with respect to the measure m_0 of maximal entropy for σ^0;

(ii) π_i is not surjective if $i \neq 0$;

(iii) if $\nu(\,, x)$ denotes the number of closed orbits of smallest period x then

$$\nu(\phi, x) = \nu(\sigma^0, x) + \sum_{i=1}^{q} (-1)^{l_i}\nu(\sigma^i, x)$$

for certain integers $l_1, \ldots, l_q$.

It is important to note certain ingredients in the proof of proposition 5. A small Markov partition of a transverse section is constructed, from which the shift of finite type (X_{A_0}, σ_{A_0}) is defined. Initially π_0 maps X_{A_0} to this section in such a way that $\pi_0^{-1}x$ consists of at most N points and π_0 is one-one on the interiors of elements of the partition. π_0 semi-conjugates σ_{A_0} to the Poincaré map of the section, and is extended to a semi-conjugacy between a suspension of σ_{A_0} and ϕ_t by following the flow ϕ_t. The suspending function f_0 at x is the time of first return of $\pi_0 x$ to the section. As for the other suspensions $(X^{f_i}_{A_i}, \sigma^i)$, $i \neq 0$, these are defined 'combinatorially' and canonically in terms of the data,

$$\pi_0\sigma^0_t = \phi_t\pi_0.$$

In fact each π_i is initially π_0 composed with a canonical map p_i of X_{A_i} into X_{A_0} and $f_i = f_0 \circ p_i$.

If k is a C^∞ function on M which is strictly positive, it is possible to define a new flow ϕ^k on M which has the same orbits as ϕ and which has a velocity at each point obtained by multiplying the velocity of ϕ by k. It can be shown that ϕ^k is also an Axiom A flow. (The proof of this fact was given by Anosov and Sinai in [1] for Anosov flows. Their proof seems to work just as well for Axiom A flows.)

The corresponding velocity changes for the above suspensions σ^i are effected by the functions $k \circ \pi_i = k_i$ and one obtains new flows on the suspension spaces $X^{f_i}_{A_i}$ which we denote by σ^{k_i}. These flows can also be represented as suspension flows (over X_{A_i}) although, in general, the suspending functions have to be modified. Using proposition 5 we therefore obtain:

PROPOSITION 6. *For k a C^∞ function on M which is strictly positive and with π_i, k_i as defined above we have*

(i) $\pi_i\sigma^{k_i}_t = \phi^k_t\pi_i$;

(ii) *π_0 is surjective and a.e. one-one with respect to the measure m^{k_0} obtained from m by the velocity change;*

(iii) *π_i is not surjective if $i \neq 0$;*

(iv) *if $\nu(\ ,x)$ denotes the number of closed orbits of smallest period x then*

$$\nu(\phi^k, x) = \nu(\sigma^{k_0}, x) + \sum_{i=1}^{q} (-1)^{l_i} \nu(\sigma^{k_i}, x).$$

Strictly speaking, this is deduced from the *proof* of proposition 5 (see the remarks following proposition 5) and this proof also shows that π_0 is a.e. one-one with respect to the maximal measure for σ^{k_0}.

Now let g be a C^∞ real function on M and consider variable real numbers σ, x with σ close to the topological entropy $h(\phi)$ of ϕ and x close to zero so that $k = \sigma + xg$ is strictly positive. Define

$$\zeta_g(\sigma, x) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\tau} \exp(-n(\lambda(\tau)\sigma + x\lambda_g(\tau)))$$
$$= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\tau} \exp(-n\lambda_{\sigma+xg}(\tau)),$$

then by proposition 6 we have

$$\zeta_g(\sigma, x) = \zeta_{g\circ\pi_0}(\sigma, x) \prod_{i=1}^{q} \zeta_{g\circ\pi_i}(\sigma, z)^{(-1)^{l_i}}$$

By theorem 1 and the fact that the maps π_i are at most N to 1, π_0 is surjective and π_i $(i \neq 0)$ are not surjective we know that $\zeta_{g\circ\pi_0}(s,, z)$ is non-zero analytic in $\mathcal{R}(s) > h(\phi)$, (z small), and for $i \neq 0$, $\zeta_{g\circ\pi_i}(s, z)$ is non-zero analytic in $\mathcal{R}(s) > h(\phi) - \varepsilon$, ($z$ small), for some $\varepsilon > 0$. Hence

$$\omega(s, z) = \prod_{i=1}^{q} \zeta_{g\circ\pi_i}(s, z)^{(-1)^{l_i}}$$

is non-zero analytic in $\mathcal{R}(s) > h(\phi) - \varepsilon$, ($z$ small), and

$$\zeta_g(s, z) = \zeta_{g\circ\pi_0}(s, z)\omega(s, z) \tag{6.1}$$

in $\mathcal{R}(s) > h(\phi)$ when z is small. (The smallness of z above may depend on the locality of s.)

By logarithmically differentiating (6.1) we obtain, using theorem 2, the following:

THEOREM 3. *If ϕ is an Axiom A flow restricted to a basic set Ω and if g is a C^∞ real function then*

$$\eta(s) = \sum_{n=1}^{\infty} \sum_{\tau} \lambda_g(\tau) \exp(-s\lambda(\tau)n)$$

is analytic in $\mathcal{R}(s) > h(\phi)$. Moreover

(a) *If ϕ is topologically weak-mixing then $\eta(s)$ has an analytic extension to a neighbourhood of $\mathcal{R}(s) \geq h(\phi)$ except for a simple pole at $h(\phi)$ of residue $\int g\, dm$ where m is the measure of maximal entropy for ϕ.*

(b) *If ϕ is not topologically weak-mixing then $\eta(s)$ is simply periodic with period ia, where a is the least positive eigenfrequency, and has an analytic extenstion to $\mathcal{R}(s) > h(\phi) - \varepsilon'$, for some $\varepsilon' > 0$, except for simple poles at $h(\phi) + nia$, $n \in \mathbb{Z}$, with residue $\int g\, dm$.*

Remark. In part (b) one has to correlate eigenfunctions of ϕ_t with eigenfunctions of σ_t^0. For more details see [**10**].

7. *Equidistribution theorems* I

In this section we consider an Axiom A flow ϕ restricted to a basic set $\Omega \subset M$. Let g be a non-negative C^∞ function defined on M and denote the topological entropy of ϕ by h. Let m be the measure of maximal entropy for ϕ.

Modifying the η function slightly we write

$$\eta_g(s) = \sum_{n=1}^{\infty} \sum_{\tau} h\lambda_g(\tau) \exp(-snh\lambda(\tau)) \tag{7.1}$$

for $\mathcal{R}(s) > 1$, and note that

$$\eta_g(s) = \int_1^\infty x^{-s}\, dF_g(x),$$

where

$$F_g(x) = \sum_{e^{\lambda(\tau)nh} \le x} \lambda_g(\tau)h = \sum_{e^{\lambda(\tau)h} \le x} \lambda_g(\tau)h\left[\frac{\log x}{\lambda(\tau)h}\right]. \tag{7.2}$$

We have to consider the two cases, ϕ weak-mixing, and ϕ not weak-mixing separately.

ϕ weak-mixing. In this case, according to theorem 3,

$$\int_1^\infty x^{-s}\, dF_g(x) = \frac{\int g\, dm}{s-1} + \psi(s),$$

where ψ is analytic in an open neighbourhood of $\mathcal{R}(s) \ge 1$. By Ikehara's Tauberian theorem ([**19**]) we conclude that:

PROPOSITION 7

$$\frac{F_g(x)}{x} \to \int g\, dm$$

as $x \to \infty$.

ϕ not weak-mixing. In this case, $\int_1^\infty x^{-s}\, dF_g(x)$ is simply periodic with least period ia/h where a is the least positive eigenfrequency of ϕ. Moreover this function is analytic in $\mathcal{R}(s) > 1 - \varepsilon'$ ($\varepsilon' > 0$) except for simple poles at $1 + nia/h$, $n \in \mathbb{Z}$, with residue $\int g\, dm$. Thus

$$n_g(s) = \frac{(2\pi h/a)\int g\, dm}{1 - e^{(-2\pi h/a)(s-1)}} + \psi(s),$$

where ψ is analytic in $\mathcal{R}(s) > 1 - \varepsilon'$. Hence

$$\eta_g(s) = \frac{2\pi h}{a}\int g\, dm \sum_{n=0}^{\infty} e^{2\pi nh/a}\, e^{-2\pi nhs/a} + \psi(s),$$

and comparing this equation with (7.1) we conclude that

$$\sum_{\substack{\lambda(\tau)=2\pi l/a\\ l|n}} h\lambda_g(\tau)-\frac{2\pi h}{a}\int g\,dm\cdot e^{2\pi nh/a}\to 0$$

exponentially fast. Consequently:

PROPOSITION 8. *If ϕ is not weak mixing then*

$$F_g(x)\sim\int g\,dm\cdot\frac{2\pi h}{a}\sum_{e^{2\pi nh/a}\le x} e^{2\pi nh/a}$$

By (7.2) we have

$$F_g(x)\le \pi_g(x)\log x \tag{7.3}$$

where $\pi_g(x)=\sum_{e^{\lambda(\tau)h}\le x}\lambda_g(\tau)/\lambda(\tau)$. Asymptotically the inequality in the reverse direction is also true as we shall now show. To do this we note that $\pi_g(x)/x^\sigma\to 0$ as $x\to\infty$ when $\sigma>1$. To see this, consider the inequality

$$\eta_g(\sigma)\ge\sum_{e^{\lambda(\tau)h}\le x}\frac{h\lambda_g(\tau)}{(e^{h\lambda(\tau)\sigma}-1)}\ge\sum_{e^{\lambda(\tau)h}\le x}\frac{\lambda_g(\tau)}{\sigma\lambda(\tau)}\frac{1}{x^{\sigma}},$$

which implies that

$$\eta_g(\sigma)\ge\frac{\pi_g(x)}{x^{\sigma'}}\qquad\text{when } \sigma'>\sigma>1.$$

Hence, since $\sigma>1$ is arbitrary, $\pi_g(x)/x^\sigma\to 0$ as $x\to\infty$.

Now let $\sigma>1$ and define $y=(x/\log x)^\sigma$, then

$$\begin{aligned}\pi_g(x)-\pi_g(y)&=\sum_{y<e^{\lambda(\tau)h}\le x}\lambda_g(\tau)/\lambda(\tau)\le\sum_{e^{\lambda(\tau)h}\le x}h\lambda_g(\tau)/\log y\\ &\le F_g(x)/\log y.\end{aligned}$$

Consequently

$$\pi_g(x)\frac{\log x}{x}\le\frac{\pi_g(y)}{y^{1/\sigma}}+(F_g(x)/x)\frac{\log x}{\sigma(\log x-\log\log x)}$$

so that

$$\overline{\lim}\ \pi_g(x)\log x/F_g(x)\le 1/\sigma\qquad\text{as } x\to\infty,\text{ for all }\sigma>1. \tag{7.4}$$

(Here we have used the fact that $F_g(x)/x$ is bounded away from zero (propositions 7 and 8) and $\pi_g(y)/y^{1/\sigma}\to 0$.) Clearly (7.3) and (7.4) show that

PROPOSITION 9. *Whether or not ϕ is weak mixing we have $\pi_g(x)\log x\sim F_g(x)$.*

Combining propositions 7, 8 and 9 we have proved

THEOREM 4. *If ϕ is weak mixing then*

$$\pi_g(x)\sim\frac{x}{\log x}\int g\,dm. \tag{7.5}$$

If ϕ is not weak mixing then

$$\pi_g(x)\sim\int g\,dm\cdot\frac{2\pi h}{a}\frac{1}{\log x}\sum_{e^{2\pi hn/a}\le x}e^{2\pi nh/a}. \tag{7.6}$$

Remark 1. In this theorem g may be replaced by an arbitrary continuous function on Ω, or for that matter by χ_B when B is a Borel set whose boundary has zero m measure. The proof is provided by a simple approximation argument.

Remark 2. If we put $g \equiv 1$ we recover the main theorem of [**10**] mentioned in the introduction.

Suppose ϕ is weak mixing and let $\varepsilon > 0$. Define a measure $\mu_{x,\varepsilon}$ on Ω by

$$\mu_{x,\varepsilon}(g) = \sum_{x-\varepsilon<\lambda(\tau)\leq x+\varepsilon} \left(\int_\tau g(\phi_t p)\, dt/\lambda(\tau) \right)$$

and normalize it so that $\bar{\mu}_{x,\varepsilon}(g) = \mu_{x,\varepsilon}(g)/\mu_{x,\varepsilon}(1)$. From (7.5) we see that

$$\mu_{x,\varepsilon}(g) \sim (e^{\varepsilon h} - e^{-\varepsilon h})(e^{xh}/xh) \int g\, dm,$$

and

$$\mu_{x,\varepsilon}(1) \sim (e^{\varepsilon h} - e^{-\varepsilon h}) e^{xh}/xh,$$

and hence

$$\bar{\mu}_{x,\varepsilon}(g) \to \int g\, dm \tag{7.7}$$

as $x \to \infty$, when g is a non-negative C^∞ function on M.

When ϕ is not weak mixing we insist that $0 < \varepsilon < 2\pi/a$. With this restriction, however,

$$\mu_{2\pi n/a,\varepsilon}(g) = \sum_{\lambda(\tau)=2\pi n/a} \lambda_g(\tau)/\lambda(\tau) \sim \frac{e^{2\pi n h/a}}{n} \int g\, dm$$

and hence

$$\bar{\mu}_{2\pi n/a,\varepsilon}(g) \to \int g\, dm. \tag{7.8}$$

A simple approximation argument will show that (7.7) and (7.8) are valid for any continuous function on Ω, or for that matter for a characteristic function χ_B of a Borel set when the m measure of the boundary of B is zero. These facts justify:

THEOREM 5 ([**4**]) (Equidistribution theorem; first version). *The closed orbits of an Axiom A system are uniformly distributed with respect to the maximal invariant measure* m.

8. *Equidistribution theorems* II.

We return to our η function

$$\eta_g(s) = \sum_{n=1}^{\infty} \sum_\tau \lambda_g(\tau) h\, e^{-shn\lambda(\tau)}$$

where h is the topological entropy of the Axiom A flow on Ω and g is a non-negative C^∞ function on M. Define

$$\eta_g^1(s) = \sum_\tau \lambda_g(\tau) h\, e^{-sh\lambda(\tau)}$$

and

$$\eta_g^2(s) = \sum_{n=2}^{\infty} \sum_{\tau} \lambda_g(\tau) h\, e^{-shn\lambda(\tau)},$$

so that

$$\eta_g = \eta_g^1 + \eta_g^2.$$

We shall show that η_g^1 contains the essential part of η_g. Evidently

$$\eta_g^2(s) = \sum_{\tau} \lambda_g(\tau) h\, e^{-2sh\lambda(\tau)}(1 - e^{-sh\lambda(\tau)})^{-1}$$

$$= \sum_{\tau} \lambda_g(\tau) h\, e^{-sh\lambda(\tau)}(e^{sh\lambda(\tau)} - 1)^{-1}.$$

Moreover,

$$e^{\sigma h\lambda(\tau)} \le K(e^{\sigma h\lambda(\tau)} - 1)$$

for some constant K when σ is real and $\sigma > 1 - \varepsilon'$, $(\varepsilon' > 0)$. Therefore

$$\sum_{\tau} |\lambda_g(\tau) h\, e^{-sh\lambda(\tau)}(e^{sh\lambda(\tau)} - 1)^{-1}| \le \sum_{\tau} Kh\lambda_g(\tau)\, e^{-2\sigma h\lambda(\tau)},$$

with $s = \sigma + it$. Since $\sum_\tau h\lambda_g(\tau)\, e^{-\sigma h\lambda(\tau)}$ converges for $\sigma > 1$ we see that $\sum_\tau h\lambda_g(\tau)\, e^{-2\sigma\lambda(\tau)}$ converges for $\sigma > \frac{1}{2}$. Hence $\eta_g^2(s)$ is well defined and analytic in $\mathscr{R}(s) > 1 - \varepsilon'$, $(0 < \varepsilon' < \frac{1}{2})$. We see then that

$$\eta_g^1(s) = \sum_{\tau} h\lambda_g(\tau)\, e^{-sh\lambda(\tau)}$$

enjoys all the properties listed for η_g in theorem 3. Clearly

$$\eta_g^1(s) = \int_1^{\infty} x^{-s}\, dF_g^1(x), \tag{8.1}$$

where

$$F_g^1(x) = \sum_{e^{h\lambda(\tau)} \le x} h\lambda_g(\tau).$$

Again we have to consider the weak mixing and not weak mixing cases separately.

ϕ weak mixing. Ikehara's Tauberian theorem applied to (8.1) shows that $F_g^1(x) \sim x \int g\, dm$ as $x \to \infty$, or equivalently

$$\sum_{\lambda(\tau) \le x} \lambda_g(\tau) \sim \frac{e^{xh}}{h} \int g\, dm.$$

ϕ not weak mixing. In this case, as before, we have

$$\eta_g^1(s) = \frac{2\pi h}{a} \int g\, dm \sum_{n=0}^{\infty} e^{2\pi nh/a}\, e^{-2\pi nhs/a} + \psi^1(s),$$

where $\psi^1(s)$ is analytic in $\mathscr{R}(s) > 1 - \varepsilon'$. Comparing this equation with the definition of $\eta_g^1(s)$ we have

$$\sum_{\lambda(\tau) = 2\pi n/a} h\lambda_g(\tau) - \frac{2\pi h}{a} \int g\, dm \cdot e^{2\pi nh/a} \to 0$$

exponentially fast. To summarize, we have

THEOREM 6. *If ϕ is weak mixing then*

$$\sum_{\lambda(\tau)\le x} \lambda_g(x) \sim \frac{e^{hx}}{h}\int g\, dm.$$

If ϕ is not weak mixing then

$$\sum_{\lambda(\tau)\le x} \lambda_g(x) \sim \frac{2\pi}{a}\int g\, dm \cdot \sum_{2\pi n/a\le x} e^{2\pi n h/a}.$$

This time we introduce rather different closed orbital measures. Define

$$\rho_{x,\varepsilon}(g) = \sum_{x-\varepsilon<\lambda(\tau)\le x+\varepsilon} \int_\tau g(\phi_t p)\, dt$$

and

$$\bar\rho_{x,\varepsilon}(g) = \rho_{x,\varepsilon}(g)/\rho_{x,\varepsilon}(1),$$

where $\varepsilon > 0$, and in the not weak mixing case $\varepsilon < 2\pi/a$.

Using our asymptotic formula for η_g^1 note that in the weak mixing case $\bar\rho_{x,\varepsilon}(g)\to \int g\, dm$ as $x\to\infty$. And in the not weak-mixing case we obtain

$$\rho_{2\pi n/a,\varepsilon}(g) = \sum_{\lambda(\tau)=2\pi n/a} \lambda_g(\tau) \sim e^{2\pi n h/a}\int g\, dm$$

so that (using the case $g\equiv 1$) we have

$$\bar\rho_{2\pi n/a,\varepsilon}(g) \to \int g\, dm \text{ as } n\to\infty.$$

We have therefore proved, with a second (equivalent) interpretation

THEOREM 7 ([**2**], [**3**]) (Equidistribution theorem; second version). *The closed orbits of an Axiom A flow are uniformly distributed.*

REFERENCES

[**1**] D. V. Anosov & Ja. G. Sinai. Some smooth ergodic systems. *Russian Math. Surveys No.* 5, **22** (1967) 103–167.
[**2**] R. Bowen. Periodic orbits for hyperbolic flows. *Amer. J. Math.* **94** (1972) 1–30.
[**3**] R. Bowen. The equidistribution of closed geodesics. *Amer. J. Math.* **94** (1972) 413–423.
[**4**] R. Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95** (1973) 429–450.
[**5**] N. Dunford & J. T. Schwartz. *Linear Operators I.* John Wiley (Interscience): New York, 1967.
[**6**] D. A. Hejhal. The Selberg trace formula and the Riemann zeta function. *Duke Math. J.* **43** (1976), 441–482.
[**7**] A. Manning. Axiom A diffeomorphisms have rational zeta functions. *Bull. London Math. Soc.* **3** (1971) 215–220.
[**8**] G. A. Margulis. Applications of ergodic theory to the investigation of manifolds of negative curvature. *Func. Anal. i Ego Prilozhen.* **3** No. 4. (1969), 89–90.
[**9**] W. Parry. An analogue of the prime number theorem for shifts of finite type and their suspensions. *Israel J. Math.* **45** (1983), 41–52.
[**10**] W. Parry & M. Pollicott. An analogue of the prime number theorem for closed orbits of Axiom A flows. *Ann. of Math.* **118** (1983), 573–591.
[**11**] W. Parry & S. Tuncel. *Classification Problems in Ergodic Theory.* London Math. Soc. Lecture Notes **67**, Cambridge University Press: Cambridge (1982).
[**12**] M. Pollicott. A complex Ruelle operator theorem and two counter examples. *Ergod. Th. & Dynam. Sys.* **4** (1984), 135–146.

[13] D. Ruelle. Zeta functions for expanding maps and Anosov flows. *Invent. Math.* **34** (1976), 231–242.
[14] D. Ruelle. *Thermodynamic Formalism.* Addison-Wesley: Reading (1978).
[15] P. Sarnak. Asymptotic behaviour of periodic orbits of the horocycle flow and Eisenstein series. *Comm. Pure and Applied Math.* **34** (1981), 719–739.
[16] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc.* **20** (1956) 47–87.
[17] Ja. G. Sinai. Gibbs measures in ergodic theory. *Russian Math. Surveys No.* 4 (166) (1972), 21–64.
[18] T. Sunada. Geodesic flows and geodesic random walks. To appear.
[19] N. Wiener. *The Fourier Integral and Certain of its Applications.* Cambridge University Press: Cambridge, 1933.