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BOUNDEDNESS OF SOLUTIONS OF PARABOLIC EQUATIONS WITH ANISOTROPIC GROWTH CONDITIONS

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ABSTRACT. In this paper, we consider the parabolic equation with anisotropic growth conditions, and obtain some criteria on boundedness of solutions, which generalize the corresponding results for the isotropic case.

1. **Introduction.** The boundedness of solutions of elliptic equations with anisotropic growth conditions has been investigated by many authors (see [1]–[11]). However, according to our knowledge, there is no paper, the purpose of which is to study parabolic equations with anisotropic growth conditions. In this paper, we will consider the boundedness of its solutions, which generalize the corresponding results for the isotropic case.

Let G be a bounded domain in the n-dimensional Euclidean space E^n. Consider the following equation.

$$u_t - \sum_i \frac{\partial}{\partial x^i} A_i(x,t,u,\nabla u) + B(x,t,u,\nabla u) = 0, \tag{1.1}$$

where $(x,t) \in Q = G \times (0,T)$, $0 < T < \infty$, $A_i(x,t,u,\xi)$ and $B(x,t,u,\xi)$ are defined on $Q \times E^1 \times E^n$, measurable in x,t and continuous in u,ξ, and satisfy the following conditions:

$$\sum_i \xi_i A_i(x,t,u,\xi) \geq \sum_i |\xi_i|^{p_i}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n)$$

$$|A_i(x,t,u,\xi)| \leq \kappa_1 \Big(\sum_j |\xi_j|^{p_j}\Big)^{1-\frac{1}{p_i}}, \quad (i = 1,2,\ldots,n) \tag{1.2}$$

$$|B(x,t,u,\xi)| \leq \sum_i c_i(x,t)|\xi_i|^{\gamma_i} + \kappa |u|^{l-1} + f(x,t),$$

where $\kappa_1 \geq 1$, $\kappa \geq 0$, $p_i > 1$, $i = 1,2,\ldots,n$. p, l, r_i and γ_i are constants and satisfy

$$1 < p < n, \quad \frac{1}{p} = \frac{1}{n}\sum_i \frac{1}{p_i} \qquad p_i \leq p\Big(1 + \frac{2}{n}\Big); \tag{1.3}$$

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$$(1.4)\qquad 1-\frac{n+p}{p(n+2)}\le\frac{\gamma_i}{p_i}\le 1-\frac{n}{p(n+2)},$$

$$(1.5)\qquad l=p(1+\frac{2}{n}),$$

$$(1.6)\qquad c_i(x,t)\in L_{r_i}(Q),$$

$$(1.7)\qquad \begin{array}{lll} \frac{1}{r_i} & =1-\frac{1}{l}-\frac{\gamma_i}{p_i}, & \text{if } 1-\frac{n+p}{p(n+2)}\le\frac{\gamma_i}{p_i}\le 1-\frac{1}{l},\\ r_i & =\infty, & \text{if } \frac{\gamma_i}{p_i}=1-\frac{1}{l}, \end{array}$$

$$(1.8)\qquad f(x,t)\in L_s(Q),\quad s>\frac{n+p}{p}.$$

Let $W^1_{(p_i)}(G)$ denote the anisotropic Sobolev space, in which the norm of u is

$$\|u\|_{W^1_{(p_i)}(G)}=\sum_i\|u_{x_i}\|_{L_{p_i(G)}}+\|u\|_{L_1(G)},$$

and $\overset{0}{W}{}^1_{(p_i)}(G)$ denotes the closure of $C^1_c(G)$ in $W^1_{(p_i)}(G)$, $\big(C^1_c(G)$ is the set of C^1-functions with compact support in $G\big)$. And for $u\in\overset{0}{W}{}^1_{(p_i)}(G)$, the following inequality holds.

$$(1.9)\qquad \Big(\int_G|u|^{p^*}dx\Big)^{\frac{1}{p^*}}\le C(n,p_i)\Big(\int_G\sum_i|u_{x_i}|^{p_i}\,dx\Big)^{\frac{1}{p}}.$$

The function u is said to be a *generalized solution of (1.1)*, if $u\in C\big(0,T;L_2(G)\big)\cap L_p\big(0,T;W^1_{(p_i)}(G)\big)\cap L_p(Q)$ and satisfies

$$(1.1')\qquad \int_0^t\int_G\Big(-v_tu+\sum_i v_{x_i}A_i(x,t,u,\nabla u)+vB(x,t,u,\nabla u)\Big)\,dx\,dt + \int_G\Big(v(x,t)u(x,t)-v(x,0)u(x,0)\Big)\,dx=0$$
$$\forall t\in(0,T);\quad v\in\overset{0}{W}{}^1_2\big(0,T;L_2(G)\big)\cap L_p\big(0,T;W^1_{p_i}(G)\big).$$

For the case of $p_i=p$ and $\gamma_i=\gamma$, it is known that when $p>\frac{2n}{n+2}$ and $p-\frac{n+p}{n+2}\le\gamma\le p-\frac{n}{n+2}$, (in which we do not demand $p<n$), the generalized solution $u\in C\big(0,T;L_2(G)\big)\cap L_p\big(0,T;W^1_p(G)\big)$ of (1.1) is locally bounded in Q; But for the case of $1<p\le\frac{2n}{n+2}$, we need the following restriction for the local boundedness of u:

$$(1.10)\qquad u\in L_{\tilde l,\mathrm{loc}}(Q),\quad \tilde l>\frac{n(2-p)}{p}.$$

Moreover if u is bounded on the parabolic boundary of Q, *i. e.*, there is a constant $M>0$, such that

$$(1.11)\qquad (|u|-M)^+=\max(|u|-M,0)\in L_p\big(0,T;\overset{0}{W}{}^1_p(G)\big),$$

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$$(|u| - M)^+|_{t=0} = 0,$$

then for $p > 1$, the generalized solution $u \in C\big(0,T;L_2(G)\big) \cap L_p\big(0,T;W^1_p(G)\big)$ of (1.1) is globally bounded on Q (even for the case of $1 < p \le \frac{2n}{n+2}$, any additional integrability for u is unnecessary).

In this paper, we extend the above results to parabolic equations with anisotropic growth conditions. In the definition of the generalized solution, we require $u \in L_p(Q)$, which is different from the isotropoic case because it does not ensure

$$L_p\big(0,T;W^1_{(p_i)}(G)\big) \hookrightarrow L_p(Q),$$

is smooth sufficiently but the boundary of G. In addition, we restrict $p < n$ because the embedding inequality for $p < n$ is different from the one for $p \ge n$.

In Section 2, we give some lemmas as preliminaries. In Section 3, we prove the local boundedness of solutions and the global boundedness is proved in Section 4.

2. **Preliminaries.**

LEMMA 1. *Let $u \in C\big(0,T;L_2(G)\big) \cap L_p\big(0,T;\overset{0}{W}{}^1_{(p_i)}(G)\big)$. Then there is a constant $C > 0$, depending only on n and p_i, such that*

$$\Big(\iint_Q |u|^l \,dx\,dt\Big)^{\frac{n}{n+p}} \le C\Big\{\operatorname*{ess\,sup}_{t\in(0,T)} \int_G |u|^2\,dx + \iint_Q \sum_i |u_{x_i}|^{p_i}\,dx\,dt\Big\} \tag{2.1}$$

PROOF. Clearly, $p^* = \frac{np}{n-p} > p(1+\frac{2}{n}) = l > 2$, for $p > \frac{2n}{n+2}$, and $p^* < l < 2$ for $1 < p < \frac{2n}{n+2}$. So setting $\alpha = \frac{n}{n+2} \in (0,1)$, we get from the interpolation inequality and (1.8) that

$$\begin{aligned}\int_G |u|^l\,dx &\le \Big(\int_G |u|^2\,dx\Big)^{\frac{(1-\alpha)l}{2}} \Big(\int_G |u|^{p^*}\,dx\Big)^{\frac{\alpha l}{p^*}}\\ &\le C\Big(\int_G |u|^2\,dx\Big)^{\frac{(1-\alpha)l}{2}} \int_G \sum_i |u_{xi}|^{p_i}\,dx;\end{aligned}$$

hence

$$\iint_Q |u|^l\,dx\,dt \le C\Big(\operatorname*{ess\,sup}_{t\in(0,T)} \int_G |u|^2\,dx\Big)^{\frac{p}{n}} \iint_Q \sum_i |u_{x_i}|^{p_i}\,dx\,dt. \tag{2.2}$$

If $p = \frac{2n}{n+2}$, then $l = p(1+\frac{2}{n}) = 2$. By taking $\alpha = \frac{n}{n+2}$, we have

$$\begin{aligned}\int_G |u|^l\,dx &= \Big(\int_G |u|^2\,dx\Big)^{1-\alpha} \Big(\int_G |u|^2\,dx\Big)^{\alpha}\\ &\le C\Big(\int_G |u|^2\,dx\Big)^{1-\alpha} \int_G \sum_i |u_{x_i}|^{p_i}\,dx,\end{aligned}$$

which implies (2.2) again. By using Young's inequality and (2.2), we deduce (2.1). The proof of Lemma 1 is completed.

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LEMMA 2. *Suppose $f(x_1, x_2, \ldots, x_n, t)$ is a nonnegative and bounded function. If for any $\rho_i^1 \le y_i < z_i \le \rho_i^0$ and $t_{-1} \le \tau_0, < \tau_1 \le t_0$, there holds*

$$f(y_1, y_2, \ldots, y_n, \tau_1) \le \theta f(z_1, z_2, \ldots, z_n, \tau_0) + \sum_i \frac{A_i}{(z_i - y_i)^{p_i}} + \frac{B}{\tau_i - \tau_0} + D \tag{2.3}$$

where $\theta \in (0,1)$, $p_i > 1$, $A_i, B, D \ge 0$ are constants, then there exists a constant C, independent of $y_1, \ldots, y_n, z_1, \ldots, z_n, \tau_0, \tau_1$, such that

$$f(y_1, y_2, \ldots, y_n, \tau_1) \le C\Big(\sum_i \frac{A_i}{(z_i - y_i)^{p_i}} + \frac{B}{\tau_i - \tau_0} + D\Big) \quad \forall \quad \rho_i^1 \le y_i < z_i \le \rho_i^0, \quad t_1 \le \tau_0 < \tau_1 \le t_1. \tag{2.4}$$

PROOF. Let $\lambda \in (\frac{1}{2}, 1)$ satisfy

$$\theta\lambda^{-p_i} < 1, \quad i = 1, 2, \ldots, n; \quad \text{and} \quad \theta\lambda^{-1} < 1. \tag{2.5}$$

Setting $r_i^{(m)} = z_i - \lambda^m (z_i - y_i)$, $t_i^{(m)} = \tau_0 + \lambda^m(\tau_1 - \tau_0)$, we have

$$\begin{aligned} f(y_1, y_2, \ldots, y_n, \tau_1) &= f\big(r_1^{(0)}, \ldots, r_n^{(0)}, t^{(0)}\big) \\ &\le \theta f\big(r_1^{(1)}, \ldots, r_n^{(1)}, t^{(1)}\big) + \sum_i \frac{A_i}{\big(r_i^{(1)} - r_i^{(0)}\big)^{p_i}} + \frac{B}{t^{(0)} - t^{(1)}} + D \\ &\le \cdots \\ &\le \theta^{m+1} f(r_1^{(m+1)}, \ldots, r_n^{(m+1)}, t^{(m+1)}) \\ &\quad + \sum_i \frac{A_i}{(z_i - y_i)^{p_i}} \sum_{j=0}^m (\theta\lambda^{-p_i})^j (1-\lambda)^{-1} \\ &\quad + \frac{B}{\tau_1 - \tau_0} \sum_{j=0}^m (\theta\lambda^{-1})^j (1-\lambda)^{-1} + D\sum_{j=0}^m \theta^m. \end{aligned}$$

By (2.5) and letting $m \longrightarrow \infty$, we get (2.4). The proof is completed.

3. **Local boundedness of solutions.**

THEOREM 1. *Let conditions (1.2)–(1.8) and $\frac{2n}{n+2} < p < n$ be satisfied. Let $u \in C\big(0, T; L_2(G)\big) \cap L_p\big(0, T; W^1_{(p_i)}(G)\big) \cap L_p(Q)$ be a generalized solution of equation (1.1). Then u is bounded locally in Q.*

PROOF. Let $K(\rho_i) = \{|x_i| < \rho_i, i = 1, 2, \ldots, n\}$, $\rho_i = \rho^{\frac{p}{p_i}}$, and $0 < \rho < 1$ be so small that

$$K(\rho_i) \times (t_0 - \rho^p, t_0) \subset Q.$$

We claim that u is bounded in $K(\frac{1}{2}\rho_i) \times (t_0 - \frac{1}{2}\rho^p, t_0)$. To prove this, take $\rho_i^1, \rho_i^0, \tau_0$ and τ_1 such that

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$$\frac{1}{2}\rho_i \le \rho_i^1 < \rho_i^0 \le \rho_i, \quad t_0 - \rho^p \le \tau_0 < \tau_1 \le t_0 - \frac{1}{2}\rho^p.$$

Let $\zeta_i(x_i)$ and $\psi(t)$ be piecewise linear continuous functions of x_i and t respectively and satisfy

$$\zeta_i(x_i) = \begin{cases} 1, & \text{for } |x_i| \le \rho_i^1 \\ 0, & \text{for } |x_i| \ge \rho_i^0, \end{cases} \qquad \psi(t) = \begin{cases} 1, & \text{for } t \ge \tau_1 \\ 0, & \text{for } t \le \tau_0 \end{cases}$$

Then, we have

$$0 \le |\zeta_i'| \le \frac{1}{\rho_i^0 - \rho_i^1}, \quad 0 \le \psi'(t) \le \frac{1}{\tau_1 - \tau_0}.$$

Let $k > 0$, $q > \frac{p_i}{p_i - 1}$ $(i = 1, 2, \ldots, n)$, and set

$$v = \zeta^q \psi^q (u-k)^+, \quad \zeta(x) = \prod_i \zeta_i(x_i). \tag{3.1}$$

For convenience, we assume $u_t \in L_2(Q)$ (otherwise, we may substitute the Sleklov time-average of v for v and deal with its similarly). Thus we may take v as a test function. Inserting it into (1.1′)and integrating by parts with respect to t, we obtain

$$\begin{aligned} 0 &= \int_0^t \int_G \Big(v u_t + \sum_i v_{x_i} A_i(x,t,u,\nabla u) + vB(x,t,u,\nabla u)\Big)\,dx\,dt \\ &\ge \frac{1}{2}\int_G \zeta^q\psi^q |(u-k)^+|^2\,dx - \frac{q}{2}\int_0^t\int_G \zeta^q \psi^{q-1}\psi' |(u-k)^+|^2\,dx\,dt \\ &\quad + \int_0^t \int_G \zeta^q\psi^q \Big\{ \sum_i |u_{x_i}|^{p_i} \\ &\qquad - (u-k)^+ \Big(\sum_i c_i(x,t)|u_{x_i}|^{\gamma_i} + K|u|^{l-1} + f(x,t)\Big)\Big\}\,dx\,dt \\ &\quad - \sum_i \kappa_1 q \int_0^t\int_G \zeta^{q-1}\psi^q (u-k)^+ |\nabla\zeta| \Big(\sum_j |u_{x_j}|^{p_j}\Big)^{1-\frac{1}{p_i}}\,dx\,dt. \end{aligned}$$

By using Young's inequality and taking the supremum for $t \in (0, t_0)$, we have

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(0,t_0)} \int_G \zeta^q\psi^q |(u-k)^+|^2\,dx + \iint_{K(k,\rho_i^0,\tau_0)} \zeta^q\psi^q \sum_i |u_{x_i}|^{p_i}\,dx\,dt \\ &\quad \le C\Big\{ \frac{1}{\tau_1 - \tau_0} \iint_{K(k,\rho_i^0,\tau_0)} |u-k|^2\,dx\,dt \\ &\qquad + \sum_i \frac{1}{(\rho_i^0 - \rho_i^1)^{p_i}} \iint_{K(k,\rho_i^0,\tau_0)} (u-k)^{p_i}\,dx\,dt \\ &\qquad + \iint_{K(k,\rho_i^0,\tau_0)} (u-k)\Big(\sum_i c_i(x,t)|u_{x_i}|^{\gamma_i} + \kappa|u|^{l-1} + f(x,t)\Big)\,dx\,dt\Big\}, \end{aligned} \tag{3.2}$$

where $K(k, \rho_i^0, \tau_0) = \{K(\rho_i^0) \times (\tau_0, t_0)\} \cap \{u > k\}$ is the effective domain of the integrations, and the constant $C > 0$ is independent of $k, \rho_i^0, \rho_i^1, \tau_0$, and τ_1. By the Hölder inequality, we have

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$$
\begin{aligned}
&\iint_{K(k,\rho_i^0,\tau_0)} (u-k)\sum_i c_i(x,t)|u_{x_i}|^{\gamma_i}\,dx\,dt \\
&\quad\le \sum_i \Big(\iint_{K(k,\rho_i^0,\tau_0)} (u-k)^l\,dx\,dt\Big)^{\frac{1}{l}} \Big(\iint_{K(k,\rho_i^0\tau_0)} \sum_i |u_{x_i}|^{p_i}\,dx\,dt\Big)^{\frac{\gamma_i}{p_i}} \epsilon_i(k) \\
&\quad\le \frac{1}{2C}\iint_{K(k,\rho_i^0\tau_0)} \sum_i |u_{x_i}|^{p_i}\,dx\,dt \\
&\qquad + C\sum_i \epsilon_i(k)^{\frac{p_i}{p_i-\gamma_i}} \Big(\iint_{K(k,\rho_i^0,\tau_0)} (u-k)^l\,dx\,dt\Big)^{\frac{1}{l}(\frac{p_i}{p_i-\gamma_i})},
\end{aligned}
\tag{3.3}
$$

where C is the constant in (3.2),

$$
\epsilon_i(k) = \|c_i(x,t)\|_{L_{r_i}\left(K(k,\rho_i^0,\gamma_0)\right)} \tag{3.4}
$$

Combining (3.2), (3.3) and Lemma 2, we get

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,t_0)} \int_G \zeta^q\psi^q |(u-k)^+|^2\,dx + \iint_{K(k,\rho_i^1,\tau_1)} \sum_i |u_{x_i}|^{p_i}\,dx\,dt \\
&\quad\le C\Big\{ \frac{1}{\tau_1-\tau_0}\iint_{K(k,\rho_i^0,\tau_0)} (u-k)^2\,dx\,dt \\
&\qquad + \sum_i \frac{1}{(\rho_i^0-\rho_i^1)^{p_i}} \iint_{K(k,\rho_i^0,\tau_0)} (u-k)^{p_i}\,dx\,dt \\
&\qquad + \sum_i \epsilon_i(k)^{\frac{p_i}{p_i-\gamma_i}} \Big(\iint_{K(k,\rho_i^0,\tau_0)} (u-k)^l\,dx\,dt\Big)^{\frac{1}{l}\frac{p_i}{(p_i-\gamma_i)}} \\
&\qquad + \iint_{K(k,\rho_i^0,\tau_0)} (u-k)\big(|u|^{l-1}+f(x,t)\big)\,dx\,dt \Big\}
\end{aligned}
\tag{3.5}
$$

It follows from Lemma 1 that

$$
\begin{aligned}
&\Big(\iint_{K(k,\rho_i^2,\tau_2)} (u-k)^l\,dx\,dt\Big)^{\frac{n}{n+p}} \\
&\quad\le C\Big\{ \sum_i \frac{1}{(\rho_i^1-\rho_i^2)^{p_i}} \iint_{K(k,\rho_i^1,\tau_1)} (u-k)^{p_i}\,dx\,dt \\
&\qquad + \operatorname*{ess\,sup}_{t\in(0,t_0)} \int_{K(\rho_i^1)} |(u-k)^+|^2 dx + \iint_{K(k,\rho_i^1,\tau_1)} \sum_i |u_{x_i}|^{p_i}\,dx\,dt \Big\} \\
&\qquad\quad \forall\ \frac{1}{2}\rho_i \le \rho_i^2 < \rho_i^1 < \rho_i^0 \le \rho_i, \quad t_0-\rho^p \le \tau_0 < \tau_1 < \tau_2 \le t_0.
\end{aligned}
\tag{3.6}
$$

Since the constant C of (3.6) is independent of $\rho_i^1, \rho_i^2, \tau_1$, and τ_2, combining with (3.5), (3.6) and taking $\rho_i^0-\rho_i^1 = \rho_i^1-\rho_i^2$, $\tau_1-\tau_0 = \tau_2-\tau_1$,we have

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$$
\begin{aligned}
&\Big(\iint_{K(k,\rho_i^2,\tau_2)}(u-k)^l\,dx\,dt\Big)^{\frac{n}{n+p}}\\
&\qquad\le C\Big\{\frac{1}{\tau_2-\tau_0}\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^2\,dx\,dt\\
&\qquad\quad+\sum_i\frac{1}{(\rho_i^0-\rho_i^2)^{p_i}}\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^{p_i}\,dx\,dt \qquad (3.7)\\
&\qquad\quad+\sum_i\epsilon_i(k)^{\frac{p_i}{p_i-\gamma_i}}\Big(\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt\Big)^{\frac{1}{l}(\frac{p_i}{p_i-\gamma_i})}\\
&\qquad\quad+\iint_{K(k,\rho_i^0,\tau_0)}(u-k)\big(|u|^{l-1}+f(x+t)\big)\,dx\,dt\Big\}.
\end{aligned}
$$

Let $|e|$ denote the $n+1$-dimensional Lebesgue measure of set e. By virtue of

$$|K(k,\rho_i^0,\tau_0)|\le\frac{1}{k^p}\int_{t_0-\rho^p}^{t}\int_{K(\rho_i)}|u|^p\,dx\,dt\longrightarrow 0,\qquad \text{as } \kappa\longrightarrow\infty,$$

and the absolute continuity of a Lebesgue integral and (3.4), we have $\epsilon_i(k)\longrightarrow 0$ as $k\longrightarrow\infty$. Observing (1.7), there holds

$$\frac{1}{l}\Big(\frac{p_i}{p_i-\gamma_i}\Big)\ge\frac{1}{l}\Big(\frac{p(n+2)}{n+p}\Big)=\frac{n}{n+p},$$

then

$$
\begin{aligned}
&\sum_i\epsilon_i(k)^{\frac{p_i}{p_i-\gamma_i}}\Big\{\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt\Big\}^{\frac{1}{l}\frac{p_i}{p_i-\gamma_i}}\\
&\qquad\qquad=\epsilon(k)\Big(\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt\Big)^{\frac{n}{n+p}}, \qquad (3.8)
\end{aligned}
$$

where

$$\epsilon(k)=\sum_i\epsilon_i(k)^{\frac{p_i}{p_i-\gamma_i}}\Big(\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt\Big)^{\frac{1}{l}(\frac{p_i}{p_i-\gamma_i})-\frac{n}{n+p}}\longrightarrow 0 \text{ as } k\longrightarrow\infty.$$

Hence, if $k\ge k_0$ ($k_0>0$ is large enough), by Lemma 2, it follows that

$$
\begin{aligned}
&\Big(\iint_{K(k,\rho_i^2,\tau_2)}(u-k)^l\,dx\,dt\Big)^{\frac{n}{n+p}}\\
&\quad\le C\Big\{\frac{1}{\tau_2-\tau_0}\Big(\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt\Big)^{\frac{2}{l}}|K(k,\rho_i^0,\tau_0)|^{1-\frac{2}{l}}\\
&\quad\quad+\sum_i\frac{1}{(\rho_i^0-\rho_i^2)^{p_i}}\Big(\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt\Big)^{\frac{p_i}{l}}|K(k,\rho_i^0,\tau_0)|^{1-\frac{p_i}{l}}\\
&\quad\quad+\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt+k^l|K(k,\rho_i^0,\tau_0)| \qquad (3.9)\\
&\quad\quad+\Big(\iint_{K(k,\rho_i^0,\tau_0)}(u-k)^l\,dx\,dt\Big)^{\frac{1}{l}}\|f\|_{L_s(Q)}|K(k,\rho_i^0,\tau_0)|^{1-\frac{1}{l}-\frac{1}{s}}\Big\}.
\end{aligned}
$$

If $k > h$, we have

$$|K(k,\rho_i^0,\tau_0)| \le \iint_{K(k,\rho_i^0,\tau_0)} \left|\tfrac{u-h}{k-h}\right|^l \,dx\,dt \le \iint_{K(h,\rho_i^0,\tau_0)} \left|\tfrac{u-h}{k-h}\right|^l \,dx\,dt,$$

and (3.9) can be rewritten as

$$\begin{aligned}
&\Big(\iint_{K(k,\rho_i^2,\tau_2)} (u-k)^l\,dx\,dt\Big)^{\frac{n}{n+p}} \\
&\quad\le C\Big\{\frac{1}{\tau_2-\tau_0}(k-h)^{2-l}\iint_{K(h,\rho_i^0,\tau_0)}(u-h)^l\,dx\,dt \\
&\qquad+\sum_i \frac{1}{(\rho_i^0-\rho_i^2)^{p_i}}(k-h)^{p_i-l}\iint_{K(h,\rho_i^0,\tau_0)}(u-h)^l\,dx\,dt \\
&\qquad+\Big(1+(\frac{k}{k-h})^l\Big)\iint_{K(h,\rho_i^0,\tau_0)}(u-h)^l\,dx\,dt \\
&\qquad+(k-h)^{-l(1-\frac{1}{l}-\frac{1}{s})}\Big(\iint_{K(h,\rho_i^0,\tau_0)}(u-h)^l\,dx\,dt\Big)^{1-\frac{1}{s}}\Big\}, \\
&\qquad\forall\, k>h\ge k_0,\quad \frac{1}{2}\rho_i \le \rho_i^2 < \rho_i^0 \le \rho_i,\quad t_0-\rho^p\le\tau_0<\tau_2\le t_0-\frac{1}{2}\rho^p.
\end{aligned}\tag{3.10}$$

Let $\epsilon > 0$ be determinated. Considering the absolute continuity of a Lebesgue integral, we take $H > k_0$ large enough such that

$$\int_{t_0-\rho^p}^{t_0}\int_{K(\rho_i)} |(u-H)^+|^l\,dx\,dt \le \epsilon\rho^{n+p}\tag{3.11}$$

For $m = 0, 1, 2, \ldots$ set

$$k_m = 2H - \frac{H}{2m},\quad \rho_i^{(m)} = \Big(\frac{1}{2}+\frac{1}{2^m+1}\Big)\rho^{\frac{p}{p_i}},$$

$$\tau_m = t_0 - \frac{1}{2}\rho^p - \frac{1}{2^{m+1}}\rho^p,\quad J_m = \iint_{K(k_m,\rho_i^{(m)},\tau_m)} (u-k_m)^l\,dx\,dt.$$

Since the constant C in (3.10) is independent of $k, h, \rho_i^0, \rho_i^2, \tau_0$ and τ_2, substituting k_m, k_{m+1}, $\rho_i^{(m)}$, $\rho_i^{(m+1)}$, τ_m and τ_{m+1} for $h, k, \rho_i^0, \rho_i^2, \tau_0$ and τ_2 respectively, we have

$$\begin{aligned}
J_{m+1}^{\frac{n}{n+p}} &\le C\Big\{\frac{2^{m+2}}{\rho^p}(\frac{2^{m+1}}{H})^{l-2}J_m + \sum_i \frac{2^{(m+2)p}}{\rho^p}(\frac{2^{m+1}}{H})^{l-p_i}J_m \\
&\qquad+\Big(1+2^{(m+2)l}\Big)J_m(\frac{2^{m+1}}{H})^{l(1-\frac{1}{l}-\frac{1}{s})}J_m^{1-\frac{1}{s}}\Big\},\qquad m=0,1,2,\ldots
\end{aligned}\tag{3.12}$$

Noting that $H > 1$ and changing correspondingly the constant C in (3.12), we can simplify (3.12) as

$$J_{m+1}^{\frac{n}{n+p}} \le CJ_m^{\frac{n}{n+p}}\Big\{\frac{2^{(1+p)m}}{\rho^p}J_m^{\frac{p}{n+p}} + 2^{lm}J_m^{\frac{p}{n+p}-\frac{1}{s}}\Big\},\qquad m=0,1,2,\ldots\tag{3.13}$$

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and since (3.11) implies $J_0 \le \epsilon\rho^{n+p}$, we can prove by induction for suitable $\delta \in (0,1)$ that

$$J_m \le \delta^m \epsilon \rho^{n+p}, \quad m = 0, 1, 2, \ldots. \tag{3.14}$$

In fact, assume that (3.14) holds for m. It follows by combining (3. 13) with (3. 14) that

$$J_{m+1}^{\frac{n}{n+p}} \le C J_m^{\frac{n}{n+p}} \Big(2^{(l+p)m} \delta^{\frac{pm}{n+p}} \frac{p}{\epsilon^{n+p}} + 2^{lm} \delta^{(\frac{p}{n+p}-\frac{1}{s})m} \big(\epsilon\rho^{n+p}\big)^{\frac{p}{n+p}-\frac{1}{s}} \Big). \tag{3.15}$$

In view of $0 < \rho < 1$, if at the beginning, we let ϵ, δ satisfy

$$\begin{aligned} C(\epsilon^{\frac{p}{n+p}} + \epsilon^{\frac{p}{n+p}-\frac{1}{s}}) &\le \quad \delta^{\frac{n}{n+p}}, \\ 2^{l+p}\delta^{\frac{p}{n+p}} \le 1; \qquad 2^l \delta^{\frac{p}{n+p}-\frac{1}{s}} &\le 1, \end{aligned}$$

it is easy to see from (3.15) that (3.14) holds for $m+1$. By induction, (3.14) holds for all m. Thus,

$$0 = \lim_{m\to\infty} J_m = \iint_{K(2H, \frac{1}{2}\rho^{\frac{p}{p_i}}, t_0 - \frac{1}{2}\rho^p)} (u-2H)^l \, dx\, dt,$$

i.e.

$$\operatorname*{ess\,sup}_{K(\frac{1}{2}\rho^{\frac{p}{p_i}})\times(t_0-\frac{1}{2}\rho^p, t_0)} u \le 2H.$$

So, we have proved that u is locally bounded above in Q. And moreover , substituting $-u$ for u, we obtain similarly that u is locally bounded below. The proof of Theorem 1 is completed.

THEOREM 2. *Suppose (1.2)–(1.8) hold and* $1 < p \le \frac{2n}{n+2}$. *Let* $u \in C\big(0, T; L_2(G)\big) \cap L_p\big(0, T; W^1_{(p_i)}(G)\big) \cap L_p(Q)$ *be a generalized solution of (1.1). Then if (1.10) holds, u is locally bounded in Q.*

PROOF. We can deduce similarly that (3.7) holds for $1 < p \le \frac{2n}{n+2}$, and simplify (3.7) and (3.8), that is, we also have for $1 < p \le \frac{2n}{n+2}$

$$\begin{aligned} &\Big(\iint_{K(k,\rho_i^2,\tau_2)} (u-k)^l \, dx\, dt \Big)^{\frac{n}{n+p}} \\ &\quad \le C \Big\{ \frac{1}{\tau_2 - \tau_0} \iint_{K(k,\rho_i^0,\tau_0)} (u-k)^2 \, dx\, dt \\ &\qquad + \sum_i \frac{1}{(\rho_i^0 - \rho_i^2)^{p_i}} \iint_{K(k,\rho_i^0,\tau_0)} (u-k)^{p_i} \, dx\, dt \\ &\qquad + \iint_{K(k,\rho_i^0,\tau_0)} (u-k)\big(|u|^{l-1} + f(x,t)\big) \, dx\, dt \Big\} \end{aligned} \tag{3.16}$$

If $1 < p < \frac{2n}{n+2}$, then $l = p(1+\frac{2}{n}) < 2$. Although we can not deal with it as in Theorem 1, by condition (1.10) and the interpolation inequality, we have

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$$\iint_K (u-k)^2\,dx\,dt \le \Big(\iint_K (u-k)^l\,dx\,dt\Big)^{\frac{2\alpha}{l}} \Big(\iint_K (u-k)^{\tilde l}\,dx\,dt\Big)^{\frac{2(1-\alpha)}{\tilde l}}, \tag{3.17}$$

where $\alpha \in (0,1)$ satisfies

$$1 = \frac{2\alpha}{l} + \frac{2(1-\alpha)}{\tilde l} \tag{3.18}$$

Thus by (3.10) it follows that

$$\begin{aligned} &\Big(\iint_{K(k,\rho_i^2,\tau_2)} (u-k)^l\,dx\,dt\Big)^{\frac{n}{n+p}} \\ &\quad\le C\Big\{\frac{1}{\tau_2-\tau_0}\Big(\iint_{K(h,\rho_i^0,\tau_0)} (u-h)^l\,dx\,dt\Big)^{\frac{2\alpha}{l}} \\ &\qquad \cdot \Big(\iint_{K(h,\rho_i^0,\tau_0)} (u-k)^{\tilde l}\,dx\,dt\Big)^{\frac{2(1-\alpha)}{\tilde l}} \\ &\qquad + \sum_i \frac{1}{(\rho_i^0-\rho_i^2)^{p_i}}(k-h)^{p_i-l}\iint_{K(h,\rho_i^0,\tau_0)} (u-h)^l\,dx\,dt \\ &\qquad + \Big(1+(\frac{k}{k-h})^l\Big)\iint_{K(h,\rho_i^0,\tau_0)} (u-h)^l\,dx\,dt \\ &\qquad + (k-h)^{-l(1-\frac{1}{l}-\frac{1}{s})}\Big(\iint_{K(h,\rho_i^0,\tau_0)} (u-h)^l\,dx\,dt\Big)^{1-\frac{1}{s}}\Big\}, \\ &\forall k > h \ge k_0, \quad \frac{1}{2}\rho_i \le \rho_i^2 < \rho_i^0 \le \rho_i, \quad t_0-\rho^p \le \tau_0 < \tau_2 \le t_0 - \frac{1}{2}\rho^p. \end{aligned} \tag{3.19}$$

Let $\epsilon > 0$. We can take $H > k_0$ large enough such that

$$\int_{t_0-\rho^p}^{t_0}\int_{K(\rho_i)} |(u-H)^+|^{\tilde l}\,dx\,dt \le \epsilon\rho^{n+p}. \tag{3.20}$$

Similar to Theorem 1, we get

$$\begin{aligned} J_{m+1}^{\frac{n}{n+p}} \le CJ_m^{\frac{n}{n+p}}\Big\{&\frac{2^m}{\rho^p}J_m^{\frac{2\alpha}{l}-\frac{n}{n+p}}(\epsilon\rho^{n+p})^{\frac{2(1-\alpha)}{\tilde l}} \\ &+ \frac{2(l+p)m}{\rho^p}J_m^{\frac{n}{n+p}} + 2^{lm}J_m^{\frac{p}{n+p}-\frac{1}{s}}\Big\}, \quad m = 0,1,2,\ldots. \end{aligned} \tag{3.21}$$

(3.20) implies

$$J_0 \le (\epsilon\rho^{n+p})^{\frac{l}{\tilde l}}(\omega\rho^{n+p})^{1-\frac{l}{\tilde l}}, \tag{3.22}$$

where ω is the unit-ball volume in E^n. (1.10) and (3.18) yield

$$\frac{2\alpha}{l} = 1 - \frac{2-1}{\tilde l - l} > \frac{n}{n+2}. \tag{3.23}$$

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Combining (3.18) with (3.21)–(3.23), for suitable δ, ϵ, we have

$$J_m \leq \delta^m \epsilon^{\frac{1}{l}} \omega^{1-\frac{1}{l}} \rho^{n+p}, \quad m = 0, 1, 2, \ldots. \tag{3.24}$$

(3.24) implies that u is locally bounded for the case of $1 < p < \frac{2n}{n+2}$.

If $p = \frac{2n}{n+2}$, then $l = p(1 + \frac{2}{n}) = 2 < \frac{n(2-p)}{p} = \tilde{l}$. Taking $(0,1) \ni \alpha > \frac{n}{n+p}$, we have

$$\begin{aligned}
&\iint_{K(k,\rho_i^0,\tau_0)} (u-k)^2\,dx\,dt \\
&\quad \leq \Big(\iint_{K(k,\rho_i^0,\tau_0)} (u-k)^l\,dx\,dt\Big)^{\alpha} \Big(\iint_{K(k,\rho_i^0\tau_0)} (u-k)^l\,dx\,dt\Big)^{1-\alpha} \\
&\quad \leq \Big(\iint_{K(k,\rho_i^0,\tau_0)} (u-k)^l\,dx\,dt\Big)^{\alpha} \Big((k-h)^{\tilde{l}-l} \iint_{K(k,\rho_i^0,\tau_0)} (u-k)^{\tilde{l}}\,dx\,dt\Big)^{1-\alpha}, \\
&\qquad\qquad \forall k > h \geq k_0.
\end{aligned}$$

As with (3.21), we have

$$\begin{aligned}
J_{m+1}^{\frac{n}{n+p}} \leq C J_m^{\frac{n}{n+p}} \Big\{ &\frac{2^m}{\rho^p} (\frac{2^m}{H})^{(\tilde{l}-l)(1-\alpha)} J_m^{\alpha-\frac{n}{n+p}} (\epsilon \rho^{n+p})^{1-\alpha} \\
&+ \frac{2^{(l+p)m}}{\rho^p} J_m^{\frac{n}{n+p}} + 2^{lm} J_m^{\frac{p}{n+p}-\frac{1}{s}}, \Big\} \qquad m = 0, 1, 2, \ldots.
\end{aligned} \tag{3.25}$$

According to (3.22) and (3.25), we can prove the local boundedness of u in Q for the case of $p = \frac{2n}{n+2}$. The proof of Theorem 2 is completed.

4. **Global boundedness of solutions.**

THEOREM 3. *Suppose conditions (1.2)–(1.8) hold and* $1 < p < n$. *Let* $u \in C\big(0, T; L_2(G)\big) \cap L_p\big(0, T; W^1_{(p_i)}(G)\big) \cap L_p(Q)$ *is a generalized solution of (1.1). If there exists a constant* $M > 0$, *such that*

$$(u-M)^+ \in L_p\big(0, T; \overset{0}{W}{}^1_{(p_i)}(G)\big) \quad \text{and} \quad (u-M)^+|_{t=0} = 0, \tag{4.1}$$

then u is globally bounded on Q.

PROOF. Let $k > M$. Substituting k for M, (4.1) still holds. Let $u_t \in L_2(Q)$, and take $\upsilon = (u-k)^+$ as a test function. Then repeating the deduction process similarly as in Theorem 1, we get correspondingly

$$\begin{aligned}
&\Big(\iint_{A(k)} (u-k)^l\,dx\,dt\Big)^{\frac{n}{n+p}} \\
&\quad \leq C\Big\{ \sum_i \epsilon_i(k)^{\frac{p_i}{p_i-\gamma_i}} \Big(\iint_{A(k)} (u-k)^{p_i}\,dx\,dt\Big)^{\frac{1}{l}(\frac{p_i}{p_i-\gamma_i})} dx\,dt \\
&\qquad + \iint_{A(k)} (u-k)\big(|u|^{l-1} + f(x,t)\big)\,dx\,dt \Big\},
\end{aligned} \tag{4.2}$$

where

$$A(k) = Q \cap \{u > k\}, \quad \epsilon_i(k) = \|c(x,t)\|_{L_{r_i}(A(k))}.$$

Then the rest of proof is similar to that of Theorem 1. Noticing that the right side of (4.2) does not appear the integral term with $(u-k)^2$, we do not need any additional integrability of u even if $1 < p \le \frac{2n}{n+2}$.

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