*J. Aust. Math. Soc.* **103** (2017), 70–90
doi:10.1017/S1446788716000446

# MONOTONE OPERATORS AND THE PROXIMAL POINT ALGORITHM IN COMPLETE CAT(0) METRIC SPACES

**HADI KHATIBZADEH✉ and SAJAD RANJBAR**

(Received 20 May 2015; accepted 18 June 2016; first published online 23 September 2016)

Communicated by J. Borwein

**Abstract**

In this paper, we generalize monotone operators, their resolvents and the proximal point algorithm to complete CAT(0) spaces. We study some properties of monotone operators and their resolvents. We show that the sequence generated by the inexact proximal point algorithm $\Delta$-converges to a zero of the monotone operator in complete CAT(0) spaces. A strong convergence (convergence in metric) result is also presented. Finally, we consider two important special cases of monotone operators and we prove that they satisfy the range condition (see Section 4 for the definition), which guarantees the existence of the sequence generated by the proximal point algorithm.

2010 *Mathematics subject classification*: primary 47H05; secondary 47J05, 47J20, 65K05.

*Keywords and phrases*: Hadamard space, monotone operator, proximal point algorithm, $\Delta$-convergence, subdifferential, nonexpansive mapping.

## 1. Introduction

One of the most important parts of nonlinear and convex analysis is monotone operator theory. It has an essential role in convex analysis, optimization, variational inequalities, semigroup theory and evolution equations. A zero of a monotone operator is a solution of a variational inequality associated to the monotone operator, an equilibrium point of an evolution equation governed by the monotone operator and a solution of a minimization problem for a convex function when the monotone operator is the subdifferential of the convex function. Therefore the existence and approximation of zeros of monotone operators are central considerations of many recent researchers. The most popular method for the approximation of a zero of a monotone operator is the proximal point algorithm, which was introduced by Martinet [20] and Rockafellar [23]. Rockafellar [23] showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal

The authors are grateful to the referees for valuable comments and suggestions. The second author was supported by Higher Education Center of Eghlid in Iran.

monotone operator in Hilbert spaces. Güler's counterexample [15] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper and lower semicontinuous function. For some generalizations and modified versions of the proximal point algorithm in Hilbert spaces, the reader can consult [7, 12, 15, 23].

In this paper, we consider the proximal point algorithm in a nonlinear version of Hilbert spaces (that is, complete CAT(0) spaces). By using the duality theory introduced in [3], we extend monotone operators, their resolvents and some of their properties to CAT(0) spaces. Our results extend the previous results in Hilbert spaces as well as the recent results on Hadamard manifolds (see, for example, [1, 18] and references therein) to complete CAT(0) spaces.

The paper has been organized as follows. In Section 2, we give some preliminaries of CAT(0) spaces, monotone operators and the proximal point algorithm. In Section 3, we define monotone operators, their resolvents and Yosida approximations in CAT(0) spaces. Then we study some of their properties. Section 4 is devoted to the proximal point algorithm in complete CAT(0) spaces. In this section, we prove that the proximal point algorithm $\Delta$-converges to a zero of the maximal monotone operator in complete CAT(0) spaces. Also in this section, we prove a strong convergence result when the operator is strongly monotone. In the two final sections of the paper, two important special cases of monotone operators are studied. In Sections 5, when the monotone operator is the subdifferential of a convex function, we prove the range condition (defined in Section 4), which implies the existence of the sequence generated by the proximal point algorithm, in this case. Section 6 is devoted to the other special case when the monotone operator is in form $I$–$T$, where $I$ and $T$ are, respectively, identity and nonexpansive mappings. We prove the range condition, in this case, in CAT(0) spaces. We also show that, in this case, in spite of Hilbert spaces, the monotone operator in form $I$–$T$ is not necessarily maximal monotone in arbitrary CAT(0) spaces.

## 2. Preliminaries

Let $(X, d)$ be a metric space and let $x, y \in X$. A geodesic path joining $x$ to $y$ is an isometry $c : [0, d(x, y)] \longrightarrow X$ such that $c(0) = x, c(d(x, y)) = y$. The image of a geodesic path joining $x$ to $y$ is called a geodesic segment between $x$ and $y$. The metric space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$.

A geodesic space $(X, d)$ is a CAT(0) space if it satisfies the following inequality.

*CN-inequality*: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$, then

$$d^2(x, y_0) \leq \tfrac{1}{2}d^2(x, y_1) + \tfrac{1}{2}d^2(x, y_2) - \tfrac{1}{4}d^2(y_1, y_2).$$

A complete CAT(0) space is called a *Hadamard* space. It is known that a CAT(0) space is an uniquely geodesic space.

https://doi.org/10.1017/S1446788716000446 Published online by Cambridge University Press

For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [8, 9, 14, 16]. For all $x$ and $y$ belonging to a CAT(0) space $X$, we write $(1-t)x \oplus ty$ for the unique point $z$ in the geodesic segment joining from $x$ to $y$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1-t)d(x, y)$. Set $[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}$. a subset $C$ of $X$ is called convex if $[x, y] \subseteq C$ for all $x, y \in C$. In CAT(0) spaces, the following technical lemma is well known.

Lemma 2.1 [8]. *A geodesic space $(X, d)$ is a* CAT(0) *space if and only if, for all $x, y, z, w \in X$ and all $t \in [0, 1]$,*

$$d^2(tx \oplus (1-t)y, z) \le td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y). \tag{2.1}$$

*In this case:*

(1) $d(tx \oplus (1-t)y, z) \le td(x, z) + (1-t)d(y, z)$; *and*

(2) $d(tx \oplus (1-t)y, tz \oplus (1-t)w) \le td(x, z) + (1-t)d(y, w)$.

A Hadamard space $X$ is called a flat Hadamard space if and only if the inequality in (2.1) is an equality. Every closed convex subset of a Hilbert space is a flat Hadamard space.

A kind of convergence in complete CAT(0) spaces, called $\Delta$-convergence, was introduced by Lim [19], which has the following definition.

Let $(x_n)$ be a bounded sequence in a complete CAT(0) space $(X, d)$ and let $x \in X$. Set $r(x, (x_n)) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $(x_n)$ is given by $r((x_n)) = \inf\{r(x, (x_n)) : x \in X\}$ and the asymptotic center of $(x_n)$ is the set $A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}$. It is known that in the complete CAT(0) spaces, $A((x_n))$ consists of exactly one point (see [17]). A sequence $(x_n)$ in the complete CAT(0) space $(X, d)$ is said to be $\Delta$-convergent to $x \in X$ if $A((x_{n_k})) = \{x\}$ for every subsequence $(x_{n_k})$ of $(x_n)$. The concept of $\Delta$-convergence has been studied by many authors (see, for example, [11, 13]).

Berg and Nikolaev [6] introduced the concept of *quasilinearization* for a CAT(0) space $X$. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle\cdot,\cdot\rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \tfrac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \quad (a, b, c, d \in X).$$

It can be easily verified that $\langle\overrightarrow{ab}, \overrightarrow{ab}\rangle = d^2(a, b)$, $\langle\overrightarrow{ba}, \overrightarrow{cd}\rangle = -\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle\overrightarrow{ae}, \overrightarrow{cd}\rangle + \langle\overrightarrow{eb}, \overrightarrow{cd}\rangle$ are satisfied for all $a, b, c, d, e \in X$. Also, we can formally add compatible vectors, more precisely, $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$ for all $a, b, c \in X$. We say that $X$ satisfies the Cauchy–Schwarz inequality if

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle \le d(a, b)d(c, d) \quad (a, b, c, d \in X).$$

It is known [6, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwarz inequality. By using quasilinearization, Ahmadi Kakavandi [2] proved that $(x_n)$ $\Delta$-converges to $x \in X$ if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$. Throughout the paper, we denote $\Delta$-convergence by $\rightharpoonup$ and metric convergence by $\to$.

Ahmadi Kakavandi and Amini [3] have introduced the concept of dual space of a complete CAT(0) space $X$, based on a work of Berg and Nikolaev [6], as follows.

Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax}\rangle \quad (t \in \mathbb{R}, a, b, x \in X),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on $X$. Then the Cauchy–Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz seminorm $L(\Theta(t, a, b)) = |t|d(a, b)$ $(t \in \mathbb{R}, a, b \in X)$, where $L(\varphi) = sup\{(\varphi(x) - \varphi(y))/d(x, y) : x, y \in X, x \neq y\}$ is the Lipschitz seminorm for any function $\varphi : X \to \mathbb{R}$. A pseudometric $D$ on $\mathbb{R} \times X \times X$ is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)) \quad (t, s \in \mathbb{R}, a, b, c, d \in X).$$

For a Hadamard space $(X, d)$, the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$. It is obtained [3, Lemma 2.1] that $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ for all $x, y \in X$. Thus, $D$ can impose an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $(t, a, b)$ is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b), (s, c, d))$, which is called the dual space of $(X, d)$. It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. Fix $o \in X$; we write $\mathbf{0} = [\overrightarrow{oo}]$ as the zero of the dual space. In [3], it is shown that the dual of a closed and convex subset of Hilbert space $H$ with nonempty interior is $H$ and $t(b - a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R}, a, b \in H$. Note that $X^*$ acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy}\rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle \quad (x^* = [t\overrightarrow{ab}] \in X^*, x, y \in X).$$

We also use the following notation in the subsequent work.

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy}\rangle := \alpha\langle x^*, \overrightarrow{xy}\rangle + \beta\langle y^*, \overrightarrow{xy}\rangle \quad (\alpha, \beta \in \mathbb{R}, x, y \in X, x^*, y^* \in X^*).$$

In the final part of this section, we give a brief review of monotone operators and the proximal point algorithm in Hilbert spaces. Let $H$ be a real Hilbert space. The multivalued operator $A : D(A) \subset H \to 2^H$ with $\mathbb{D}(A) := \{x \in X : Ax \neq \emptyset\}$ is called monotone if and only if

$$\langle x - y, x^* - y^*\rangle \geq 0 \quad (\forall x, y \in D(A), \forall x^* \in Ax, \forall y^* \in Ay).$$

The multivalued monotone operator $A : H \to 2^H$ is maximal if there exists no monotone operator $B : H \to 2^H$ such that gra$(B)$ properly contains gra$(A)$. It is well

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known that maximality of monotone operators is equivalent to surjectivity of $I + A$, where $I$ is the identity operator (see [21]). The proximal point algorithm introduced by Rockafellar [23] is defined as

$$x_{n-1} - x_n \in \lambda_n A(x_n) \quad x_0 \in H, \tag{2.2}$$

where $(\lambda_n)$ is a sequence of positive real numbers. In fact, Rockafellar [23] proved that the sequence generated by the proximal point algorithm is weakly convergent to a zero of the monotone operator $A$, provided $\lambda_n \geq \lambda > 0$ for all $n \geq 1$. The condition on the control sequence $\lambda_n$ was improved by Brézis and Lions [7]. In this paper, as well as the definition of monotone operators, their resolvents and verifying some of their properties, we extend some of the previous results on the convergence of the proximal point algorithm to complete CAT(0) spaces.

## 3. Monotone operators

Let $X$ be a Hadamard space with dual $X^*$ and let $A : X \to 2^{X^*}$ be a multivalued operator with domain $\mathbb{D}(A) := \{x \in X : Ax \neq \emptyset\}$, range $\mathbb{R}(A) := \bigcup_{x\in X} Ax$, $A^{-1}(x^*) := \{x \in X : x^* \in Ax\}$ and graph $\mathrm{gra}(A) := \{(x, x^*) \in X \times X^* : x \in \mathbb{D}(A), x^* \in Ax\}$.

Definition 3.1. Let $X$ be a Hadamard space with dual space $X^*$. The multivalued operator $A : X \to 2^{X^*}$ is:

(i) monotone if and only if, for all $x, y \in \mathbb{D}(A)$, $x^* \in Ax$ and $y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0;$$

(ii) strictly monotone if and only if, for all $x, y \in \mathbb{D}(A)$, $x \neq y$, $x^* \in Ax$ and $y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx}\rangle > 0;$$

(iii) $\alpha$-strongly monotone for $\alpha > 0$ if and only if, for all $x, y \in \mathbb{D}(A)$, $x^* \in Ax$ and $y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq \alpha d^2(x, y).$$

It is clear that every $\alpha$-strongly monotone operator for $\alpha > 0$ is strictly monotone and every strictly monotone operator is monotone.

Proposition 3.2. *If $A : X \to 2^{X^*}$ is strictly monotone, then $A^{-1}(\mathbf{0})$ is singleton.*

Proof. It is clear by the definition. □

Definition 3.3. Let $X$ be a Hadamard space with dual $X^*$. The multivalued monotone operator $A : X \to 2^{X^*}$ is maximal if there exists no monotone operator $B : X \to 2^{X^*}$ such that $\mathrm{gra}(B)$ properly contains $\mathrm{gra}(A)$ (that is, for any $(y, y^*) \in X \times X^*$, the inequality $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$ for all $(x, x^*) \in \mathrm{gra}(A)$ implies that $y^* \in Ay$).

In the subsequent work, we define resolvent and Yosida approximation of a monotone operator in CAT(0) spaces. We extend some facts on the resolvent operators and Yosida approximations to CAT(0) spaces.

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DEFINITION 3.4. Let $X$ be a Hadamard space with dual $X^*$, $\lambda > 0$ and let $A : X \to 2^{X^*}$ be a multivalued operator. The resolvent and Yosida approximation of $A$ of order $\lambda$ are the multivalued mappings $J_\lambda : X \to 2^X$ and $A_\lambda : X \to 2^{X^*}$ defined, respectively, by $J_\lambda(x) := \{z \in X \mid [(1/\lambda)\overrightarrow{zx}] \in Az\}$ and $A_\lambda(x) := \{[(1/\lambda)\overrightarrow{yx}] \mid y \in J_\lambda(x)\}$.

DEFINITION 3.5. Let $X$ be a Hadamard space with dual $X^*$ and let $T : C \subset X \to X$ be a mapping. We say that $T$ is firmly nonexpansive if $d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle$ for any $x, y \in C$.

By the definition and Cauchy–Schwarz inequality, it is clear that any firmly nonexpansive mapping $T$ is nonexpansive.

PROPOSITION 3.6. *Let $X$ be a Hadamard space with dual $X^*$. The mapping $T : C \subset X \to X$ is firmly nonexpansive if and only if*

$$\langle \overrightarrow{TxTy}, \overrightarrow{(Tx)x} \rangle + \langle \overrightarrow{TyTx}, \overrightarrow{(Ty)y} \rangle \leq 0 \quad \forall x, y \in C.$$

PROOF.

$$\begin{aligned}
&2\langle \overrightarrow{TxTy}, \overrightarrow{(Tx)x} \rangle + 2\langle \overrightarrow{TyTx}, \overrightarrow{(Ty)y} \rangle \\
&\quad = d^2(Tx, Ty) + d^2(Tx, x) - d^2(x, Ty) + d^2(Tx, Ty) + d^2(Ty, y) - d^2(Tx, y) \\
&\quad = 2d^2(Tx, Ty) - 2\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle.
\end{aligned}$$

Hence

$$\langle \overrightarrow{TxTy}, \overrightarrow{(Tx)x} \rangle + \langle \overrightarrow{TyTx}, \overrightarrow{(Ty)y} \rangle \leq 0 \quad \text{if and only if } d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle.$$

This concludes the proof. □

THEOREM 3.7 [10]. *Let $C$ be a nonempty convex subset of a* CAT*(0) space $X$, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if*

$$\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle \leq 0 \quad \forall y \in C.$$

COROLLARY 3.8. *The metric projection onto a closed convex subset $K \subset X$ is a firmly nonexpansive mapping.*

PROOF. By Theorem 3.7, for all $x, y \in X$, $\langle \overrightarrow{P_K x P_K y}, \overrightarrow{(P_K x)x} \rangle \leq 0$ because $P_K y \in K$ for all $y \in X$. Thus, by Proposition 3.6, we get the desired result. □

THEOREM 3.9. *Let $X$ be a* CAT*(0) space and let $A : X \to 2^{X^*}$. Suppose $J_\lambda$ and $A_\lambda$ are, respectively, resolvent and Yosida approximation of the operator $A$ of order $\lambda$.*

(i) *For any $\lambda > 0$, $\mathbb{R}(J_\lambda) \subset \mathbb{D}(A)$, $F(J_\lambda) = A^{-1}(\mathbf{0}) = A_\lambda^{-1}(\mathbf{0})$, where $\mathbb{R}(J_\lambda)$ and $F(J_\lambda)$ are, respectively, the range and the fixed points set of $J_\lambda$.*
(ii) *If $J_\lambda$ is single valued, then $A_\lambda$ is single-valued and $A_\lambda(x) \subset A(J_\lambda(x))$.*
(iii) *If $A$ is monotone, then $J_\lambda$ is a single-valued and firmly nonexpansive mapping.*

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(iv) *If $A$ is monotone, then $A_\lambda$ is a monotone operator.*

(v) *If $A$ is monotone and $0 < \lambda \le \mu$, then $d^2(J_\lambda x, J_\mu x) \le (\mu - \lambda)/(\mu + \lambda)d^2(x, J_\mu x)$, which implies that $d(x, J_\lambda x) \le 2d(x, J_\mu x)$.*

Proof. (i) Let $z$ not belong to $\mathbb{D}(A)$; then $A(z) = \varnothing$. Thus, for all $x \in X$, $[(1/\lambda)\overrightarrow{zx}]$ does not belong to $Az$. Hence $z$ does not belong to $J_\lambda(x)$ for all $x \in X$. Therefore, $\mathbb{R}(J_\lambda) \subset \mathbb{D}(A)$. Also, if $x \in X$,

$$\begin{aligned}\mathbf{0} \in A(x) &\Leftrightarrow \Big[\frac{1}{\lambda}\overrightarrow{xx}\Big] \in Ax \Leftrightarrow x \in J_\lambda(x)\\ &\Leftrightarrow x \in F(J_\lambda) \Leftrightarrow x \in J_\lambda(x)\\ &\Leftrightarrow \Big[\frac{1}{\lambda}\overrightarrow{xx}\Big] \in A_\lambda(x) \Leftrightarrow \mathbf{0} \in A_\lambda(x).\end{aligned}$$

(ii) It is clear, by Definition 3.4.

(iii) Let $x \in X$ and $z_1, z_2 \in J_\lambda(x)$. Then $[(1/\lambda)\overrightarrow{z_1x}] \in A(z_1)$ and $[(1/\lambda)\overrightarrow{z_2x}] \in A(z_2)$. Monotonicity of $A$ implies that

$$\begin{aligned}0 &\le \Big\langle\Big[\frac{1}{\lambda}\overrightarrow{z_1x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{z_2x}\Big], \overrightarrow{z_2z_1}\Big\rangle\\ &= \frac{1}{\lambda}(\langle\overrightarrow{z_1x}, \overrightarrow{z_2z_1}\rangle - \langle\overrightarrow{z_2x}, \overrightarrow{z_2z_1}\rangle)\\ &= \frac{1}{\lambda}(\langle\overrightarrow{z_1x}, \overrightarrow{z_2z_1}\rangle + \langle\overrightarrow{xz_2}, \overrightarrow{z_2z_1}\rangle)\\ &= \frac{1}{\lambda}\langle\overrightarrow{z_1z_2}, \overrightarrow{z_2z_1}\rangle = -\frac{1}{\lambda}d^2(z_1, z_2).\end{aligned}$$

Hence $d(z_1, z_2) = 0$, which implies $z_1 = z_2$. Therefore, $J_\lambda$ is single valued. Now we show that $J_\lambda$ is firmly nonexpansive. We know that $[(1/\lambda)\overrightarrow{J_\lambda(x)x}] \in A(J_\lambda(x))$ and $[(1/\lambda)\overrightarrow{J_\lambda(y)y}] \in A(J_\lambda(y))$. By monotonicity of $A$,

$$\begin{aligned}0 &\le 2\Big\langle\Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(y)y}\Big], \overrightarrow{J_\lambda(y)J_\lambda(x)}\Big\rangle\\ &= \frac{2}{\lambda}(\langle\overrightarrow{J_\lambda(x)x}, \overrightarrow{J_\lambda(y)J_\lambda(x)}\rangle - \langle\overrightarrow{J_\lambda(y)y}, \overrightarrow{J_\lambda(y)J_\lambda(x)}\rangle)\\ &= \frac{1}{\lambda}(d^2(J_\lambda(y), x) - d^2(J_\lambda(x), J_\lambda(y)) - d^2(J_\lambda(x), x)\\ &\qquad - d^2(J_\lambda(x), J_\lambda(y)) - d^2(J_\lambda(y), y) + d^2(J_\lambda(x), y)),\end{aligned}$$

which implies that

$$2d^2(J_\lambda(x), J_\lambda(y)) \le d^2(J_\lambda(y), x) - d^2(J_\lambda(x), x) - d^2(J_\lambda(y), y) + d^2(J_\lambda(x), y).$$

Hence

$$d^2(J_\lambda(x), J_\lambda(y)) \le \langle\overrightarrow{J_\lambda(x)J_\lambda(y)}, \overrightarrow{xy}\rangle.$$

Therefore, $J_\lambda$ is firmly nonexpansive.

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(iv) For any $x, y \in X$, $A_\lambda(x) = \{[(1/\lambda)\overrightarrow{J_\lambda(x)x}]\} \subset A(J_\lambda(x))$ and $A_\lambda(y) = \{[(1/\lambda)\overrightarrow{J_\lambda(y)y}]\} \subset A(J_\lambda(y))$. Thus, by monotonicity of $A$,

$$\begin{aligned}
\langle A_\lambda(x) - A_\lambda(y), \overrightarrow{yx}\rangle &= \Big\langle\Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(y)y}\Big], \overrightarrow{yx}\Big\rangle \\
&= \Big\langle\Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(y)y}\Big], \overrightarrow{yJ_\lambda(y)}\Big\rangle \\
&\quad + \Big\langle\Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(y)y}\Big], \overrightarrow{J_\lambda(y)J_\lambda(x)}\Big\rangle \\
&\quad + \Big\langle\Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(y)y}\Big], \overrightarrow{J_\lambda(x)x}\Big\rangle \\
&\geq \Big\langle\Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(y)y}\Big], \overrightarrow{yJ_\lambda(y)}\Big\rangle \\
&\quad + \Big\langle\Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{J_\lambda(y)y}\Big], \overrightarrow{J_\lambda(x)x}\Big\rangle \\
&= \frac{1}{\lambda}(\langle\overrightarrow{J_\lambda(x)x}, \overrightarrow{yJ_\lambda(y)}\rangle - \langle\overrightarrow{J_\lambda(y)y}, \overrightarrow{yJ_\lambda(y)}\rangle \\
&\quad + \langle\overrightarrow{J_\lambda(x)x}, \overrightarrow{J_\lambda(x)x}\rangle - \langle\overrightarrow{J_\lambda(y)y}, \overrightarrow{J_\lambda(x)x}\rangle) \\
&= \frac{1}{\lambda}(\langle\overrightarrow{J_\lambda(x)x}, \overrightarrow{yJ_\lambda(y)}\rangle + \langle\overrightarrow{yJ_\lambda(y)}, \overrightarrow{yJ_\lambda(y)}\rangle \\
&\quad + \langle\overrightarrow{J_\lambda(x)x}, \overrightarrow{J_\lambda(x)x}\rangle + \langle\overrightarrow{yJ_\lambda(y)}, \overrightarrow{J_\lambda(x)x}\rangle) \\
&= \frac{1}{\lambda}(d^2(x, J_\lambda(x)) + d^2(y, J_\lambda(y)) + 2\langle\overrightarrow{J_\lambda(x)x}, \overrightarrow{yJ_\lambda(y)}\rangle) \\
&\geq \frac{1}{\lambda}(d^2(x, J_\lambda(x)) + d^2(y, J_\lambda(y)) - 2d(x, J_\lambda(x))d(y, J_\lambda(y))) \\
&= \frac{1}{\lambda}(d(x, J_\lambda(x)) - d(y, J_\lambda(y)))^2 \geq 0.
\end{aligned}$$

Thus $A_\lambda$ is a monotone operator.

(v) By (iii), $J_\lambda x$ and $J_\mu x$ are single valued and, by the definition of resolvent, $[(1/\lambda)\overrightarrow{(J_\lambda x)x}] \in A(J_\lambda x)$ and $[(1/\mu)\overrightarrow{(J_\mu x)x}] \in A(J_\mu x)$. Thus monotonicity of $A$ implies that

$$0 \leq 2\Big\langle\Big[\frac{1}{\mu}\overrightarrow{(J_\mu x)x}\Big] - \Big[\frac{1}{\lambda}\overrightarrow{(J_\lambda x)x}\Big], J_\lambda x J_\mu x\Big\rangle.$$

It then follows that

$$2\langle\overrightarrow{(J_\lambda x)x}, \overrightarrow{J_\lambda x J_\mu x}\rangle \leq \frac{2\lambda}{\mu}\langle\overrightarrow{(J_\mu x)x}, \overrightarrow{J_\lambda x J_\mu x}\rangle.$$

That is,

$$d^2(J_\lambda x, J_\mu x) + d^2(x, J_\lambda x) - d^2(x, J_\mu x) \leq \frac{\lambda}{\mu}(d^2(x, J_\lambda x) - d^2(J_\lambda x, J_\mu x) - d^2(x, J_\mu x)),$$

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which implies that

$$d^2(J_\lambda x, J_\mu x) \le \frac{\mu - \lambda}{\mu + \lambda} d^2(x, J_\mu x) \le d^2(x, J_\mu x).$$

Thus we get

$$d(x, J_\lambda x) \le d(x, J_\mu x) + d(J_\mu x, J_\lambda x) \le 2d(x, J_\mu x),$$

which is the required result. □

REMARK 3.10. It is well known that if $T$ is a nonexpansive mapping on subset $C$ of CAT(0) space $X$, then $F(T)$ is closed and convex. Thus, if $A$ is a monotone operator on CAT(0) space $X$, then, by parts (i) and (iii) of Theorem 3.9, $A^{-1}(\mathbf{0})$ is closed and convex.

## 4. Proximal point algorithm

Let $X$ be a Hadamard space with dual $X^*$. The problem of finding a zero of a monotone operator $A : X \to 2^{X^*}$ can be formulated as

$$\text{Find } x \in X, \qquad \text{such that } \mathbf{0} \in A(x),$$

where $\mathbf{0}$ is the zero of dual space $X^*$. $A^{-1}(\mathbf{0})$ is called the set of singularity points of $A$. We say that $A$ satisfies the *range condition* if, for every $\lambda > 0$, $D(J_\lambda) = X$. It is known that if $A$ is a maximal monotone operator on a Hilbert space $H$, then $R(I + \lambda A) = H$ for all $\lambda > 0$, where $I$ is the identity operator. Thus every maximal monotone operator $A$ on a Hilbert space satisfies the *range condition*. Also as it has been shown in [18] if $A$ is a maximal monotone operator on a Hadamard manifold, then $A$ satisfies the *range condition*. For presenting some examples of monotone operators in CAT(0) spaces, in the next section of the paper, it has been proved that the subdifferential of a convex, proper and lower semicontinuous function (which is defined in [3] and recalled in the next section) satisfies the range condition. Also, in the last section of the paper, we study the range condition for the monotone operator $Az = [\overrightarrow{Tzz}]$, where $T$ is a nonexpansive mapping. We do not know whether every maximal monotone operator $A : X \to 2^{X^*}$ satisfies the *range condition* when $X$ is a Hadamard space. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator on a Hadamard space $X$ with dual $X^*$ that satisfies the *range condition* and let $(\lambda_n)$ be a sequence of positive real numbers. The proximal point algorithm for monotone operator $A$ in Hadamard space $X$ is the sequence generated by

$$\begin{cases} \left[\dfrac{1}{\lambda_n}\overrightarrow{x_n x_{n-1}}\right] \in A x_n, \\ x_0 \in X, \end{cases} \tag{4.1}$$

which, by the definition of the resolvent operator, is equivalent to

$$\begin{cases} x_n = J_{\lambda_n} x_{n-1}, \\ x_0 \in X. \end{cases} \tag{4.2}$$

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Note that the range condition and part (iii) of Theorem 3.9 guarantee existence and well definedness of the sequence $\{x_n\}$ in (4.1) or (4.2). Also (4.2) is in accordance with the proximal point algorithm (2.2) in Hilbert spaces.

The inexact version of (4.2) can be formulated as

$$\begin{cases} u_n = J_{\lambda_n} y_{n-1}, \\ d(u_n, y_n) \leq e_n, \\ y_0 \in X, \end{cases} \tag{4.3}$$

where $(e_n)$ is a sequence in $(0, \infty)$. In the following, we prove $\Delta$-convergence of the sequence generated by the proximal point algorithm (4.3) to an element of $A^{-1}(\mathbf{0})$ with the summability condition on the error sequence. To this purpose, we need the following lemmas. The first one is a generalization of the Opial lemma in CAT(0) spaces.

Lemma 4.1 [22, Lemma 2.1]. *Let $(X, d)$ be a* CAT*(0) space and let $(x_n)$ be a sequence in $X$. If there exists a nonempty subset $F$ of $X$ verifying:*

(i) *for every $z \in F$, $\lim_n d(x_n, z)$ exists; and*
(ii) *if a subsequence $(x_{n_j})$ of $(x_n)$ is $\Delta$-convergent to $x \in X$, then $x \in F$,*

*then there exists $p \in F$ such that $(x_n)$ $\Delta$-converges to $p$ in $X$.*

Lemma 4.2. *Let $X$ be a Hadamard space with dual $X^*$ and let $A : X \to 2^{X^*}$ be a multivalued monotone operator which satisfies the range condition and $A^{-1}(\mathbf{0}) \neq \emptyset$. Suppose that $x_0 = y_0 \in X$. Assume that the sequences $(x_n)$ and $(y_n)$ are generated by the algorithms (4.2) and (4.3), respectively, and that $\sum_{n=1}^{\infty} e_n < +\infty$. In this case:*

(i) *if $(x_n)$ converges strongly to a singularity of $A$, then $(y_n)$ does; and*
(ii) *if $(x_n)$ $\Delta$-converges to a singularity of $A$, then $(y_n)$ does.*

Proof. For every fixed $k$, consider the sequence $(\xi_n(k))$ defined by $\xi_0(k) = y_k, \xi_1(k) = J_{\lambda_{k+1}}(\xi_0(k)), \xi_2(k) = J_{\lambda_{k+2}}(\xi_1(k)), \ldots, \xi_n(k) = J_{\lambda_{k+n}}(\xi_{n-1}(k))$. By part (iii) of Theorem 3.9, $J_\lambda$ is nonexpansive, so, if $p \in A^{-1}(\mathbf{0})$, then, by part (i) of Theorem 3.9, for any $n, k \in \mathbb{N}$,

$$\begin{aligned} d(\xi_n(k), p) &= d(J_{\lambda_{k+n}}(\xi_{n-1}(k)), p) \\ &\leq \cdots \leq d(y_k, p) \\ &\leq d(y_k, u_k) + d(u_k, p) \\ &\leq d(y_k, u_k) + d(y_{k-1}, p) \\ &\leq e_k + d(y_{k-1}, p) \\ &\leq \cdots \\ &\leq d(y_1, p) + \sum_{i=1}^{k} e_i < \infty, \end{aligned}$$

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which implies that $(\xi_n(k))$ is bounded. On the other hand,

$$\begin{aligned} d(\xi_n(k), \xi_{n+1}(k-1)) &= d(J_{\lambda_{k+n}}(\xi_{n-1}(k)), J_{\lambda_{k+n}}(\xi_n(k-1))) \\ &\le d(\xi_{n-1}(k), \xi_n(k-1)) \\ &\le \cdots \\ &\le d(\xi_0(k), \xi_1(k-1)) \\ &= d(y_k, J_{\lambda_k}(y_{k-1})) = d(y_k, u_k) \le e_k. \end{aligned}$$

Thus

$$d(\xi_n(k), \xi_{n+1}(k-1)) \le e_k. \tag{4.4}$$

Now we will prove (i). Assume that $(x_n)$ converges strongly to a singularity of $A$. Then, by the definition, $(\xi_n(k))$ converges strongly to some $\xi(k) \in A^{-1}(\mathbf{0})$ as $n \to \infty$. Thus (4.4) implies that $d(\xi(k), \xi(k-1)) \le e_k$. Therefore, by the assumptions, $(\xi(k))_{k\in\mathbb{N}}$ is a Cauchy sequence. Since, by Remark 3.10, $A^{-1}(\mathbf{0})$ is closed and $(\xi(k))$ converges to some $a \in A^{-1}(\mathbf{0})$. We show that $(y_n)$ converges to $a$. We know that

$$\begin{aligned} d(y_k, \xi_{n+1}(k-n-1)) \le\; & d(\xi_0(k), \xi_1(k-1)) + d(\xi_1(k-1), \xi_2(k-2)) \\ & + \cdots + d(\xi_n(k-n), \xi_{n+1}(k-n-1)) \\ \le\; & e_k + e_{k-1} + \cdots + e_{k-n} \\ =\; & \sum_{i=k-n}^{k} e_i, \end{aligned}$$

and hence

$$d(y_{k+n}, \xi_{n+1}(k-1)) \le \sum_{i=k}^{k+n} e_i.$$

It follows that

$$\begin{aligned} d(y_{k+n}, a) &\le d(y_{k+n}, \xi_{n+1}(k-1)) + d(\xi_{n+1}(k-1), \xi(k-1)) + d(\xi(k-1), a) \\ &\le \sum_{i=k}^{k+n} e_i + d(\xi_{n+1}(k-1), \xi(k-1)) + d(\xi(k-1), a). \end{aligned}$$

Taking limsup when $n \to \infty$ from both sides of this inequality, we get that

$$\limsup_{n\to\infty} d(y_{n+k}, a) \le \sum_{i=k}^{\infty} e_i + d(\xi(k-1), a).$$

Now part (i) is proved by letting $k \to \infty$.

To prove (ii), let $(x_n)$ $\Delta$-converge to a zero of $A$. Then, by the definition, $(\xi_n(k))$ $\Delta$-converges to some $\xi(k) \in A^{-1}(\mathbf{0})$ as $n \to \infty$. Hence $\limsup_n \langle \overrightarrow{\xi(k)\xi_n(k)}, \overrightarrow{\xi(k)y} \rangle \le 0$ for all $y \in X$. Thus by (4.4),

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$$\begin{aligned}
d^2(\xi(k),\xi(k-1)) &= \langle \overrightarrow{\xi(k)\xi(k-1)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle \\
&= \langle \overrightarrow{\xi(k)\xi_n(k)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle + \langle \overrightarrow{\xi_n(k)\xi_{n+1}(k-1)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle \\
&\quad + \langle \overrightarrow{\xi_{n+1}(k-1)\xi(k-1)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle \\
&\le \langle \overrightarrow{\xi(k)\xi_n(k)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle + d(\xi_n(k),\xi_{n+1}(k-1))d(\xi(k),\xi(k-1)) \\
&\quad + \langle \overrightarrow{\xi_{n+1}(k-1)\xi(k-1)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle \\
&\le \langle \overrightarrow{\xi(k)\xi_n(k)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle + e_k d(\xi(k),\xi(k-1)) \\
&\quad + \langle \overrightarrow{\xi_{n+1}(k-1)\xi(k-1)}, \overrightarrow{\xi(k)\xi(k-1)}\rangle.
\end{aligned}$$

Taking limsup when $n \to \infty$, we get $d(\xi(k),\xi(k-1)) \le e_k$. Thus, by the assumptions, $(\xi(k))$ is a Cauchy sequence. Since $A^{-1}(\mathbf{0})$ is closed, $(\xi(k))$ converges to some $a \in A^{-1}(\mathbf{0})$. We show that $(y_n)$ $\Delta$-converges to $a$. Using a method similar to that of part (i), we get that

$$d(y_{k+n},\xi_{n+1}(k-1)) \le \sum_{i=k}^{k+n} e_i. \tag{4.5}$$

For all $z \in X$,

$$\begin{aligned}
\langle \overrightarrow{y_{n+k}a}, \overrightarrow{za}\rangle &= \langle \overrightarrow{y_{n+k}\xi_{n+1}(k-1)}, \overrightarrow{za}\rangle + \langle \overrightarrow{\xi_{n+1}(k-1)\xi(k-1)}, \overrightarrow{z\xi(k-1)}\rangle \\
&\quad + \langle \overrightarrow{\xi_{n+1}(k-1)\xi(k-1)}, \overrightarrow{\xi(k-1)a}\rangle + \langle \overrightarrow{\xi(k-1)a}, \overrightarrow{za}\rangle \\
&\le d(y_{n+k},\xi_{n+1}(k-1))d(z,a) + \langle \overrightarrow{\xi_{n+1}(k-1)\xi(k-1)}, \overrightarrow{z\xi(k-1)}\rangle \\
&\quad + d(\xi(k-1),a)d(\xi_{n+1}(k-1),\xi(k-1)) + d(z,a)d(\xi(k-1),a),
\end{aligned}$$

which, by (4.5), implies that

$$\begin{aligned}
\langle \overrightarrow{y_{n+k}a}, \overrightarrow{za}\rangle &\le \sum_{i=k}^{n+k} e_i d(z,a) + \langle \overrightarrow{\xi_{n+1}(k-1)\xi(k-1)}, \overrightarrow{z\xi(k-1)}\rangle \\
&\quad + d(\xi(k-1),a)d(\xi_{n+1}(k-1),\xi(k-1)) + d(z,a)d(\xi(k-1),a).
\end{aligned}$$

First, taking limsup when $n \to \infty$ and then taking limsup when $k \to \infty$ from both sides of this inequality, we get $\limsup_k \limsup_n \langle \overrightarrow{y_{n+k}a}, \overrightarrow{za}\rangle \le 0$, which implies that $\limsup_n \langle \overrightarrow{y_n a}, \overrightarrow{za}\rangle \le 0$: that is, $\Delta - \lim_n y_n = a$. □

In the following theorem, we prove $\Delta$-convergence of the sequence given by (4.2) to a zero of $A$. This theorem extends all related results in the literature for convergence of the proximal point algorithm in Hilbert spaces and Hadamard manifolds.

Theorem 4.3. *Let $X$ be a Hadamard space with dual $X^*$ and let $A : X \to 2^{X^*}$ be a multivalued monotone operator which satisfies the range condition and $A^{-1}(\mathbf{0}) \neq \emptyset$, where $\mathbf{0} \in X^*$ is the zero of the dual space. Let $(\lambda_n)$ be a sequence of positive real numbers such that $\lambda_n \ge \lambda > 0$. Then the sequence generated by the proximal point algorithm (4.2) is $\Delta$-convergent to a point $p \in A^{-1}(\mathbf{0})$. Hence, by part (ii) of Lemma 4.2, the sequence generated by (4.3) is also $\Delta$-convergent to a zero of $A$.*

Proof. Let $x \in A^{-1}(\mathbf{0})$. By (4.1), $[(1/\lambda_n)\overrightarrow{x_n x_{n-1}}] \in Ax_n$ for all $n \in \mathbb{N}$. Monotonicity of $A$ implies that

$$\begin{aligned}0 &\leq 2\Big\langle\Big[\frac{1}{\lambda_n}\overrightarrow{x_n x_{n-1}}\Big] - \mathbf{0}, \overrightarrow{x x_n}\Big\rangle \\ &= \frac{2}{\lambda_n}\langle\overrightarrow{x_n x_{n-1}}, \overrightarrow{x x_n}\rangle \\ &= \frac{1}{\lambda_n}(d^2(x, x_{n-1}) - d^2(x, x_n) - d^2(x_n, x_{n-1})),\end{aligned}$$

and so

$$0 \leq d^2(x_n, x_{n-1}) \leq d^2(x, x_{n-1}) - d^2(x, x_n).$$

Thus $(d(x, x_n))$ is convergent for all $x \in A^{-1}(\mathbf{0})$. Hence $(x_n)$ is bounded and $d(x_{n+1}, x_n) \to 0$. Thus, by (4.2), $d(x_n, J_{\lambda_n} x_n) \to 0$. On the other hand, by part (v) of Theorem 3.9, we get

$$d(x_n, J_\lambda x_n) \leq 2d(x_n, J_{\lambda_n} x_n),$$

which implies that $d(x_n, J_\lambda x_n) \to 0$. Now, if subsequence $(x_{n_j})$ of $(x_n)$ is $\Delta$-convergent to $q \in X$, $\Delta$-demicloseness of nonexpansive mappings (see [17]) implies that $q \in A^{-1}(\mathbf{0})$. Therefore, we have proved that:

(1) for every $x \in A^{-1}(\mathbf{0})$, $\lim_n d(x_n, x)$ exists; and
(2) if the subsequence $(x_{n_j})$ of $(x_n)$ is $\Delta$-convergent to $q \in X$, then $q \in A^{-1}(\mathbf{0})$.

Hence Lemma 4.1 completes the proof. □

In the following theorem, we prove the strong convergence of the proximal point algorithm (4.2) to the unique element of $A^{-1}(\mathbf{0})$ with strong monotonicity of the operator $A$.

Theorem 4.4. *Let $X$ be a Hadamard space with dual $X^*$ and let $A : X \to 2^{X^*}$ be a multivalued $\alpha$-strongly monotone operator which satisfies the range condition and $A^{-1}(\mathbf{0}) \neq \emptyset$, where $\mathbf{0} \in X^*$ is the zero of dual space. Suppose that $(\lambda_n)$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then the sequence generated by the proximal point algorithm (4.2) converges strongly to the single element $x$ of $A^{-1}(\mathbf{0})$. Hence, by part (i) of Lemma 4.2, the sequence generated by (4.3) is also strongly convergent to the unique zero of $A$.*

Proof. Clearly, $A$ is a strictly monotone operator. Thus, by Theorem 3.2, let $A^{-1}(\mathbf{0}) = \{x\}$. By (4.1), $[(1/\lambda_n)\overrightarrow{x_n x_{n-1}}] \in Ax_n$ for all $n \in \mathbb{N}$. $\alpha$-strong monotonicity of $A$ implies that

$$\begin{aligned}\alpha d^2(x_n, x) &\leq 2\Big\langle\Big[\frac{1}{\lambda_n}\overrightarrow{x_n x_{n-1}}\Big] - \mathbf{0}, \overrightarrow{x x_n}\Big\rangle \\ &= \frac{2}{\lambda_n}\langle\overrightarrow{x_n x_{n-1}}, \overrightarrow{x x_n}\rangle \\ &= \frac{1}{\lambda_n}(d^2(x, x_{n-1}) - d^2(x, x_n) - d^2(x_n, x_{n-1})),\end{aligned}$$

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from which it follows that

$$\alpha\lambda_n d^2(x, x_n) \le d^2(x, x_{n-1}) - d^2(x, x_n). \tag{4.6}$$

Summing from $n = 1$ to $n = k$ and letting $k \to +\infty$, we get

$$\sum_{n=1}^{+\infty} \lambda_n d^2(x, x_n) < +\infty,$$

which, by the assumption on $\{\lambda_n\}$, implies that $\liminf_{n\to+\infty} d(x_n, x) = 0$. Since by (4.6), $\lim_n d(x_n, x)$ exists, $x_n \to x$ as $n \to +\infty$. This is the desired result. □

## 5. Subdifferential case

A well known result in Hilbert spaces claims that the subdifferential of any convex, proper and lower semicontinuous function is maximal monotone, and therefore it satisfies the range condition. In this section, we recall the definition of the subdifferential operator in Hadamard spaces from [3]. We show that it satisfies the range condition. Then we show that the approach of Bačák [4] for the proximal point algorithm in Hadamard spaces for convex functions is equivalent to (4.2) when the monotone operator is subdifferential of a convex function.

Definition 5.1 [3]. Let $X$ be a Hadamard space with dual $X^*$ and let $f : X \to ]-\infty, +\infty]$ be a proper function with efficient domain $D(f) := \{x : f(x) < +\infty\}$. Then the subdifferential of $f$ is the multivalued function $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \ge \langle x^*, \overrightarrow{xz}\rangle \ (z \in X)\},$$

when $x \in D(f)$ and $\partial f(x) = \emptyset$, otherwise.

The following theorem has essentially been proved in [3]. It is given for sake of completeness.

Theorem 5.2 [3, Theorem 4.2]. *Let $f : X \to ]-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space $X$ with dual $X^*$. Then*

(i) *$f$ attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial f(x)$;*
(ii) *$\partial f : X \to 2^{X^*}$ is a monotone operator; and*
(iii) *for any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha\overrightarrow{xy}] \in \partial f(x)$.*

Proof. For (iii), fix $y \in X$ and $\alpha > 0$. Set $g(x) = f(x) + (\alpha/2)d^2(x, y)$. By a proof similar to that of Theorem 4.2 of [3], there exists a point $x \in X$ such that $[\alpha\overrightarrow{xy}] \in \partial f(x)$. To prove uniqueness, if there exists $x, z \in X$ such that $[\alpha\overrightarrow{xy}] \in \partial f(x)$ and $[\alpha\overrightarrow{zy}] \in \partial f(z)$, then, by part (ii),

$$\begin{aligned} 0 &\le 2\langle [\alpha\overrightarrow{xy}] - [\alpha\overrightarrow{zy}], \overrightarrow{zx}\rangle \\ &= 2\alpha\langle \overrightarrow{xy}, \overrightarrow{zx}\rangle - 2\alpha\langle \overrightarrow{zy}, \overrightarrow{zx}\rangle \\ &= -2\alpha d^2(x, z), \end{aligned}$$

which implies that $x = z$. □

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Part (iii) of Theorem 5.2 shows that the subdifferential of a convex, proper and lower semicontinuous function satisfies the range condition. Therefore the existence of the sequence $\{x_n\}$ given by (4.2) is guaranteed. Convergence of the proximal point algorithm for convex functions in Hadamard spaces has been investigated by Bačák [4] by studying the algorithm

$$\begin{cases} x_0 \in X \text{ and } (\lambda_n) \subset (0, +\infty), \\ x_{n+1} = \operatorname{Argmin}_{z\in X}\Big\{f(z) + \dfrac{1}{2\lambda_n} d^2(z, x_n)\Big\}. \end{cases} \tag{5.1}$$

In the following proposition, we show that algorithm (5.1) is equivalent to

$$\begin{cases} x_{n+1} = J_{\lambda_n}^{\partial f} x_n, \\ x_0 \in X, \end{cases}$$

which is the proximal point algorithm (4.2) when $A = \partial f$.

Proposition 5.3. *Let $f : X \to ]-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space $X$ with dual $X^*$. Then*

$$J_\lambda^{\partial f} x = \operatorname{Argmin}_{z\in X}\Big\{f(z) + \frac{1}{2\lambda} d^2(z, x)\Big\}, \tag{5.2}$$

*for all $\lambda > 0$ and $x \in X$.*

Proof. Let $\lambda > 0$ and $x \in X$. Set $y = \operatorname{Argmin}_{z\in X}\{f(z) + (1/2\lambda)d^2(z, x)\}$. Then

$$f(y) + \frac{1}{2\lambda} d^2(y, x) \le f(z) + \frac{1}{2\lambda} d^2(z, x) \quad (z \in X).$$

Set $z = ty \oplus (1 - t)u$, where $u$ is an arbitrary element in $X$ and $t \in [0, 1)$. Then

$$\begin{aligned} f(y) + \frac{1}{2\lambda} d^2(y, x) &\le f(ty \oplus (1 - t)u) + \frac{1}{2\lambda} d^2(ty \oplus (1 - t)u, x) \\ &\le t f(y) + (1 - t) f(u) + \frac{1}{2\lambda}(t d^2(y, x) + (1 - t) d^2(u, x) \\ &\quad - t(1 - t) d^2(y, u)). \end{aligned}$$

That is,

$$(1 - t)(f(y) - f(u)) \le \frac{1 - t}{2\lambda}(d^2(u, x) - t d^2(y, u) - d^2(y, x)),$$

which implies that

$$f(y) - f(u) \le \frac{1}{2\lambda}(d^2(u, x) - t d^2(y, u) - d^2(y, x)).$$

Now, when $t \to 1$, we get

$$f(y) - f(u) \le \frac{1}{2\lambda}(d^2(u, x) - d^2(y, u) - d^2(y, x)).$$

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Thus

$$f(y) - f(u) \leq \left\langle \left[\frac{1}{\lambda}\overrightarrow{yx}\right], \overrightarrow{uy} \right\rangle,$$

for all $u \in X$.

Therefore, by the definition of subdifferential, $[(1/\lambda)\overrightarrow{yx}] \in \partial f(y)$. Hence, by the definition of resolvent, $y \in J_\lambda^{\partial f} x$. On the other hand, by Theorem 5.2, $\partial f$ is monotone and hence, by part (iii) of Theorem 3.9, $J_\lambda^{\partial f}$ is single valued. This implies that $y = J_\lambda^{\partial f} x$. □

REMARK 5.4. Bačák [5] defined the resolvent of a convex function by (5.2). Proposition 5.3 shows that his definition is equivalent to our definition of resolvent when the monotone operator is the subdifferential of a convex function. Therefore our definition of resolvent of a monotone operator extends the definition of Bačák.

## 6. Nonexpansive case

In Hilbert spaces, it is well known that if $T$ is a nonexpansive self-mapping, then $I$–$T$ is maximal monotone and hence it satisfies the range condition, where $I$ is the identity mapping. In this section, we consider maximality and the range condition for the operator $I$–$T$ in Hadamard spaces.

LEMMA 6.1. *Let $X$ be a* CAT*(0) space and let $T : X \to X$ be an arbitrary nonexpansive mapping. Then the operator $Az = [\overrightarrow{Tzz}]$ is monotone. In the other words, for all $x, y \in X$,*

$$\langle \overrightarrow{Txx} - \overrightarrow{Tyy}, \overrightarrow{yx} \rangle \geq 0.$$

PROOF. Let $x, y \in X$. Then

$$\begin{aligned}
\langle \overrightarrow{Txx} - \overrightarrow{Tyy}, \overrightarrow{yx} \rangle &= \langle \overrightarrow{TxTy} + \overrightarrow{Tyy} + \overrightarrow{yx} - \overrightarrow{Tyy}, \overrightarrow{yx} \rangle \\
&= \langle \overrightarrow{TxTy}, \overrightarrow{yx} \rangle + \langle \overrightarrow{yx}, \overrightarrow{yx} \rangle \\
&\geq d^2(x, y) - d(Tx, Ty)d(x, y) \\
&= d(x, y)(d(x, y) - d(Tx, Ty)) \\
&\geq 0,
\end{aligned}$$

which is the desired result. □

LEMMA 6.2. *Let $X$ be a Hadamard space and let $T : X \to X$ be an arbitrary nonexpansive mapping. For every $x \in X$ and every $\lambda > 0$, there exists a unique $y \in X$ such that*

$$\left\langle [\overrightarrow{Tyy}] - \left[\frac{1}{\lambda}\overrightarrow{yx}\right], \overrightarrow{yz} \right\rangle \geq 0 \quad \forall z \in X. \tag{6.1}$$

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Proof. Let $x \in X$ and $\lambda > 0$ be fixed. Define $F : X \to X$ with $Fu = (1/(1+\lambda))x \oplus (\lambda/(1+\lambda))Tu$. Then

$$\begin{aligned} d(Fu, Fv) &= d\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu, \frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tv\Big) \\ &\leq \frac{\lambda}{1+\lambda}d(Tu, Tv) \leq \frac{\lambda}{1+\lambda}d(u, v). \end{aligned}$$

Thus $F$ is a contraction. Hence, by Banach contraction principle, $F$ has a unique fixed point. Let $Fy = y$. Therefore $y = (1/1+\lambda)x \oplus (\lambda/1+\lambda)Ty$. For all $z \in X$,

$$\begin{aligned} 2\Big\langle [\overrightarrow{Tyy}] - \Big[\frac{1}{\lambda}\overrightarrow{yx}\Big], \overrightarrow{yz}\Big\rangle &= 2\Big\langle \overrightarrow{Ty\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)}, \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)z}\Big\rangle \\ &\quad - 2\frac{1}{\lambda}\Big\langle \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)x}, \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)z}\Big\rangle \\ &= d^2(Ty, z) - d^2\Big(Ty, \frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big) \\ &\quad - d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty, z\Big) - \frac{1}{\lambda}\Big(d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty, z\Big) \\ &\quad + d^2\Big(x, \frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big) - d^2(x, z)\Big) \\ &= d^2(Ty, z) - \frac{1}{1+\lambda}d^2(Ty, x) + \frac{1}{\lambda}d^2(x, z) \\ &\quad - \frac{1+\lambda}{\lambda}d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty, z\Big) \\ &\geq d^2(Ty, z) - \frac{1}{1+\lambda}d^2(Ty, x) + \frac{1}{\lambda}d^2(x, z) \\ &\quad - \frac{1+\lambda}{\lambda}\frac{1}{1+\lambda}d^2(x, z) - \frac{1+\lambda}{\lambda}\frac{\lambda}{1+\lambda}d^2(Ty, z) \\ &\quad + \frac{1+\lambda}{\lambda}\frac{1}{1+\lambda}\frac{\lambda}{1+\lambda}d^2(Ty, x) \\ &= 0. \end{aligned}$$

Thus

$$\langle \overrightarrow{Tyy} - \frac{1}{\lambda}\overrightarrow{yx}, \overrightarrow{yz}\rangle \geq 0 \quad \forall z \in X.$$

Now, suppose that $y_1, y_2 \in X$ are solutions of (6.1). Then

$$\Big\langle [\overrightarrow{Ty_1y_1}] - \Big[\frac{1}{\lambda}\overrightarrow{y_1x}\Big], \overrightarrow{y_1y_2}\Big\rangle \geq 0$$

and

$$\Big\langle [\overrightarrow{Ty_2y_2}] - \Big[\frac{1}{\lambda}\overrightarrow{y_2x}\Big], \overrightarrow{y_2y_1}\Big\rangle \geq 0.$$

A simple computation by Lemma 6.1 implies that $y_1 = y_2$. □

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PROPOSITION 6.3. *Let $X$ be a Hadamard space and let $T : X \to X$ be an arbitrary nonexpansive mapping. If the monotone operator $Az = [\overrightarrow{Tzz}]$ is maximal, then $Az = [\overrightarrow{Tzz}]$ satisfies the range condition.*

PROOF. By Lemma 6.2, for every $x \in X$ and $\lambda \in (0, \infty)$, there exists $y \in X$ such that

$$\left\langle [\overrightarrow{Tyy}] - \left[\frac{1}{\lambda}\overrightarrow{yx}\right], \overrightarrow{yz} \right\rangle \geq 0 \quad \forall z \in X.$$

On the other hand, by Lemma 6.1,

$$\langle [\overrightarrow{Tzz}] - [\overrightarrow{Tyy}], \overrightarrow{yz} \rangle \geq 0 \quad \forall z \in X.$$

Hence, for every $x \in X$ and $\lambda \in (0, \infty)$, there exists $y \in X$ such that, for all $z \in X$,

$$\left\langle [\overrightarrow{Tzz}] - \left[\frac{1}{\lambda}\overrightarrow{yx}\right], \overrightarrow{yz} \right\rangle = \langle [\overrightarrow{Tzz}] - [\overrightarrow{Tyy}], \overrightarrow{yz} \rangle + \left\langle [\overrightarrow{Tyy}] - \left[\frac{1}{\lambda}\overrightarrow{yx}\right], \overrightarrow{yz} \right\rangle \geq 0,$$

which, by maximal monotonicity of $A$, implies that $[(1/\lambda)\overrightarrow{yx}] = Ay = [\overrightarrow{Tyy}]$. Thus, for every $x \in X$ and $\lambda \in (0, \infty)$, there exists $y \in X$ such that $[(1/\lambda)\overrightarrow{yx}] = Ay = [\overrightarrow{Tyy}]$. Consequently, the operator $Az = [\overrightarrow{Tzz}]$ satisfies the range condition. □

We have just proved that a monotone operator of nonexpansive type (that is, of the form $x \mapsto [\overrightarrow{Txx}]$) satisfies the range condition if it is maximal monotone. We note that, in Hilbert spaces, each monotone operator of nonexpansive type is maximal monotone and therefore satisfies the range condition. In the following proposition, we show that it is only in flat CAT(0) spaces that a monotone operator of nonexpansive type satisfies the range condition. Therefore the maximality condition in Proposition 6.3 is necessary.

PROPOSITION 6.4. *Let $X$ be a Hadamard space. For every nonexpansive mapping $T : X \to X$, the operator $Az = [\overrightarrow{Tzz}]$ satisfies the range condition if and only if*

$$d^2(\alpha x \oplus (1-\alpha)y, z) = \alpha d^2(x, z) + (1-\alpha)d^2(y, z) - \alpha(1-\alpha)d^2(x, y)$$

*for all $x, y, z \in X$.*

PROOF. If, for every nonexpansive mapping $T : X \to X$, the operator $Az = [\overrightarrow{Tzz}]$ satisfies the range condition, then, for every nonexpansive mapping $T$, $x \in X$ and $\lambda \in (0, \infty)$, there exists $u \in X$ such that $[1/\lambda\overrightarrow{ux}] = [\overrightarrow{Tuu}]$. Thus we get $0 = \langle [\overrightarrow{Tuu}] - [1/\lambda\overrightarrow{ux}], \overrightarrow{uz} \rangle$ for all $z \in X$, which, by Lemma 6.2, implies that $u = 1/1 + \lambda x \oplus \lambda/1 + \lambda Tu$. Hence,

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for all $z \in X$, we obtain

$$\begin{aligned}
0 &= 2\Big\langle [\overrightarrow{Tuu}] - \Big[\frac{1}{\lambda}\overrightarrow{ux}\Big], \overrightarrow{uz}\Big\rangle \\
&= 2\Big\langle \overrightarrow{Tu\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu\Big)}, \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu\Big)z}\Big\rangle \\
&\quad - 2\frac{1}{\lambda}\Big\langle \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu\Big)x}, \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu\Big)z}\Big\rangle \\
&= d^2(Tu,z) - d^2\Big(Tu, \frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu\Big) - d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu, z\Big) \\
&\quad - \frac{1}{\lambda}\Big(d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu, z\Big) + d^2\Big(x, \frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu\Big) - d^2(x,z)\Big) \\
&= d^2(Tu,z) - \frac{1}{1+\lambda}d^2(Tu,x) + \frac{1}{\lambda}d^2(x,z) - \frac{1+\lambda}{\lambda}d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu, z\Big).
\end{aligned}$$

Therefore, for every nonexpansive mapping $T$, $x \in X$ and $\lambda \in (0,\infty)$, there exists $u \in X$ such that, for all $z \in X$,

$$d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Tu, z\Big) = \frac{1}{1+\lambda}d^2(x,z) + \frac{\lambda}{1+\lambda}d^2(Tu,z) - \frac{\lambda}{(1+\lambda)^2}d^2(Tu,x). \tag{6.2}$$

Now suppose that $y \in X$ and $\alpha \in (0,1)$ are arbitrary elements. If we apply equality (6.2) for $\lambda = (1-\alpha)/\alpha$ and the constant mapping $Tz = y$ for all $z \in X$, we obtain

$$d^2(\alpha x \oplus (1-\alpha)y, z) = \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y) \quad \forall x,y,z \in X.$$

Now suppose the inverse. Let the nonexpansive mapping $T : X \to X$ be arbitrary and let $x \in X$ and $\lambda > 0$ be fixed. Define $F : X \to X$ with $Fu = (1/1+\lambda)x \oplus (\lambda/1+\lambda)Tu$. By the proof of Lemma 6.1, $F$ is a contraction. Hence, by Banach contraction principle, $F$ has a unique fixed point. Let $Fy = y$. Therefore $y = (1/1+\lambda)x \oplus (\lambda/1+\lambda)Ty$. By the assumptions, for all $z \in X$,

$$\begin{aligned}
2\Big\langle [\overrightarrow{Tyy}] - \Big[\frac{1}{\lambda}\overrightarrow{yx}\Big], \overrightarrow{yz}\Big\rangle &= 2\Big\langle \overrightarrow{Ty\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)}, \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)z}\Big\rangle \\
&\quad - 2\frac{1}{\lambda}\Big\langle \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)x}, \overrightarrow{\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big)z}\Big\rangle \\
&= d^2(Ty,z) - d^2\Big(Ty, \frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big) \\
&\quad - d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty, z\Big) - \frac{1}{\lambda}\Big(d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty, z\Big) \\
&\quad + d^2\Big(x, \frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty\Big) - d^2(x,z)\Big) \\
&= d^2(Ty,z) - \frac{1}{1+\lambda}d^2(Ty,x) + \frac{1}{\lambda}d^2(x,z) \\
&\quad - \frac{1+\lambda}{\lambda}d^2\Big(\frac{1}{1+\lambda}x \oplus \frac{\lambda}{1+\lambda}Ty, z\Big)
\end{aligned}$$

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$$
\begin{aligned}
&= d^2(Ty,z) - \frac{1}{1+\lambda}d^2(Ty,x) + \frac{1}{\lambda}d^2(x,z) \\
&\quad - \frac{1+\lambda}{\lambda}\frac{1}{1+\lambda}d^2(x,z) - \frac{1+\lambda}{\lambda}\frac{\lambda}{1+\lambda}d^2(Ty,z) \\
&\quad + \frac{1+\lambda}{\lambda}\frac{1}{1+\lambda}\frac{\lambda}{1+\lambda}d^2(Ty,x) \\
&= 0.
\end{aligned}
$$

Thus

$$\Big\langle [\overrightarrow{Tyy}] - \Big[\frac{1}{\lambda}\overrightarrow{yx}\Big], \overrightarrow{yz}\Big\rangle = 0 \quad \forall z \in X,$$

which, by the definition of the equivalence relation, implies that

$$\Big[\frac{1}{\lambda}\overrightarrow{yx}\Big] = [\overrightarrow{Tyy}].$$

Hence the operator $Az = [\overrightarrow{Tzz}]$ satisfies the range condition. □

By Proposition 6.4, if $X$ is a Hadamard space such that there exist $x, y, z \in X$, which satisfy

$$d^2(\alpha x \oplus (1-\alpha)y, z) < \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y),$$

then there exists a nonexpansive mapping $T$ such that the operator $Az = [\overrightarrow{Tzz}]$ does not satisfy the range condition and hence, by Proposition 6.3, it is not maximal monotone.

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HADI KHATIBZADEH, Department of Mathematics,
University of Zanjan, Zanjan, PO Box 45195-313, Iran
e-mail: hkhatibzadeh@znu.ac.ir

SAJAD RANJBAR, Department of Mathematics,
University of Zanjan, Zanjan, PO Box 45195-313, Iran
e-mail: sranjbar74@yahoo.com
and
Department of Mathematics, College of Sciences,
Higher Education Center of Eghlid, Eghlid, Iran
e-mail: sranjbar@eghlid.ac.ir

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