J. Austral. Math. Soc. (*Series A*) **47** (1989), 466–482

ON THE PRODUCT OF DISTANCES TO A POINT SET ON A SPHERE

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(Received 14 January 1988; revised 17 May 1988)

Communicated by J. H. Loxton

Abstract

Let S be the surface of the unit sphere in three-dimensional euclidean space, and let $\omega_N = (x_1, x_2, \dots, x_N)$ be an N-tuple of points on S. We consider the product of mutual distances $\rho(\omega_N) = \prod_{j \neq k} |x_j - x_k|$ and, for the variable point x on S, the product of distances $p(x, \omega_N) = \prod_{j=1}^N |x - x_j|$ from x to the points of ω_N. We obtain essentially best possible bounds for $\max_{\omega_N} \rho(\omega_N)$ and for $\min_{\omega_N} \max_{x \in S} p(x, \omega_N)$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 11 K 38, 52 A 40.

1. Introduction

Let S be the surface of the unit sphere in three-dimensional euclidean space. Given an N-tuple of points $\omega_N = (x_1, x_2, \dots, x_N)$ on S, we define a function $p_N(x) = p(x, \omega_N)$ to be the product of the euclidean distances from the variable point $x \in S$ to the points $x_1, x_2, \dots, x_N$:

$$(1) \qquad p_N(x) = p(x, \omega_N) = \prod_{j=1}^N |x - x_j|.$$

What can be said about the maximum value attained by $p_N(x)$ on the surface S? Instead of (1), consider the function

$$q_N(x) = q(x, \omega_N) = \sum_{j=1}^N \left(\log |x - x_j| + \log \frac{\sqrt{e}}{2} \right).$$

https://doi.org/10.1017/S1446788700033206 Published online by Cambridge University Press

It is easy to verify that the integral of $q_N(x)$ over the surface S (with respect to the usual surface measure σ) vanishes. This means for the original function $p_N(x)$ that the following inequality holds trivially: $\max_{x\in S} p_N(x) > (2/\sqrt{e})^N$. We prove more.

THEOREM 1. *Let ω_N be an N-tuple of points on S. Then with some numerical constant $c_1 > 0$, the following inequality holds*:

$$\max_{x\in S} p_N(x) \geq (1+c_1)(2/\sqrt{e})^N.$$

This result is in a sense best possible. We prove

THEOREM 2. *For each N there exists an N-tuple of points ω_N on S such that the inequality*

$$\max_{x\in S} p_N(x) \leq (1+c_2)(2/\sqrt{e})^N$$

holds with some absolute constant $c_2 > 0$.

Note that Theorems 1 and 2 are true for the unit circle U (instead of the unit sphere) with $c_1 = c_2 = 1$ and the normalizing factor $(2/\sqrt{e})^N$ omitted. For in this case, the product $p_N(x)$ is the modulus of a polynomial on U with its zeros on U. Theorem 1 and 2 now follow from the fact that the unit root polynomials are "minimal polynomials" for U.

In the case of the unit circle the author proved in a previous paper [7] that Theorem 2 is no longer true if the N-tuples $\omega_N = (x_1, x_2, \dots, x_N)$ are sections of a given *infinite* sequence $\omega = (x_1, x_2, \dots)$. A similar result can be proved for the unit sphere.

THEOREM 3. *Let $\omega = (x_1, x_2, \dots)$ be an infinite sequence of points on S. Let $A_N(\omega) = (\sqrt{e}/2)^N \max_{x\in S} \prod_{j=1}^N |x - x_j|$. Then, for some absolute constant $c > 0$ and infinitely many values of N, the following inequality holds*:

$$A_N(\omega) \geq e^{c\sqrt{\log N}} \tag{3}$$

The lower bound (3) is probably not best possible. As in the case of the unit circle, one might conjecture that (3) holds with $\log N$ instead of $\sqrt{\log N}$.

The problems considered up to this point have a natural counterpart concerning the product of *mutual* distances between the points of a set. For an N-tuple of points $\omega_N = (x_1, x_2, \dots, x_N)$ on the unit circle it is known that $\prod_{j<k} |x_j - x_k| \leq N^{N/2}$ is true. In the case of the unit sphere we prove a similar upper bound with the exponent $(N/4)$ instead of $(N/2)$.

THEOREM 4. *For each N-tuple $\omega_N = (x_1, x_2, \ldots, x_N)$ of points on S the following inequality holds*:

$$\prod_{j<k} |x_j - x_k| \le (2/\sqrt{e})^{N(N-1)/2} \cdot N^{N/4+o(N)}.$$

Again the result of Theorem 4 is best possible, apart from the exact order of the error term $o(N)$ in the exponent. The point set ω_N, which will be constructed in order to prove Theorem 2, also satisfies the relation

$$\prod_{j<k} |x_j - x_k| \ge (2/\sqrt{e})^{N(N-1)/2} \cdot N^{N/4+o(N)}.$$

2. Construction of a good point set

We begin by proving Theorem 2. It is sufficient to find, for each N, an N-tuple $\omega_N^0 = (x_1, \ldots, x_N)$ on S such that $\max_{x \in S} q(x, \omega_N^0) \le O(1)$ holds for the function $q(x, \omega_N^0)$ defined by (2).

Let N ($\ge N_0$) be given. Set $M = [\sqrt{N}]$. We introduce spherical coordinates θ $(0 \le \theta \le \pi)$ and φ $(0 \le \varphi < 2\pi)$ on S. We define a partition of S into spherical zones in the following way.

Choose numbers θ_μ, $0 = \theta_0 < \theta < \cdots < \theta_{M-1} < \theta_M = \pi$ in such a way that $(N/2)(\cos\theta_\mu - \cos\theta_{\mu+1})$ $(\mu = 0, 1, \ldots, M-1)$ is a positive integer and that the differences $(\theta_{\mu+1} - \theta_\mu)$ are bounded from above and from below by $\mathrm{const}/\sqrt{N}$. Set $N_\mu = (N/2)(\cos\theta_\mu - \cos\theta_{\mu+1})$. On each of the M circles of latitude $\theta = \xi_\mu = (\theta_\mu + \theta_{\mu+1})/2$ $(\mu = 0, 1, \ldots, M-1)$ we place N_μ points; at the vertices of a regular N_μ-gon. The position of the regular N_μ-gon on the circle is arbitrary. In this way, $\sum_{\mu=0}^{M-1} N_\mu = \frac{N}{2} \sum_{\mu=0}^{M-1} (\cos\theta_\mu - \cos\theta_{\mu+1}) = N$ points are distributed over the surface S.

Denote the point set defined in this way by ω_N^0, its points by $x_{\mu\alpha}$ $(\mu = 0, \ldots, M-1;\ \alpha = 0, 1, \ldots, N_\mu - 1)$, and their coordinates by $(\xi_\mu, \varphi_{\mu\alpha})$ where we choose $\varphi_{\mu\alpha} = 2\pi\alpha/N_\mu$ for convenience.

We shall prove

$$\max_{x \in S} q(x, \omega_N^0) \le O(1). \tag{4}$$

As a first step, we split the distance function $(x = (\theta, \varphi), x_0 = (\theta_0, \varphi_0))$ $d(x, x_0) = \log|x - x_0| - \log(2/\sqrt{e})$ into two parts. Let

$$e(\theta, \theta_0) = e_-(\theta, \theta_0) := \log(e/4) + \log(1 + \cos\theta) + \log(1 - \cos\theta_0) \qquad \text{for } 0 < \theta \le \theta_0 < \pi \tag{5}$$

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and

$$e(\theta,\theta_0) = e_+(\theta,\theta_0) := \log(e/4) + \log(1-\cos\theta) + \log(1+\cos\theta_0) \quad \text{for } 0 < \theta \le \theta_0 < \pi.$$

Furthermore, let

(6) $$f(\theta,\varphi;\theta_0,\varphi_0) = \log|1 - \rho e^{i(\varphi-\varphi_0)}|$$

where

$$\rho = \rho_- = \left(\tan\frac{\theta}{2}\Big/\tan\frac{\theta_0}{2}\right) \text{ for } 0 < \theta \le \theta_0 < \pi$$

and

$$\rho = \rho_+ = \left(\tan\frac{\theta_0}{2}\Big/\tan\frac{\theta}{2}\right) \text{ for } 0 < \theta_0 \le \theta < \pi.$$

An easy calculation shows that $d(x,x_0) = \frac{1}{2}e(\theta,\theta_0) + f(\theta,\varphi;\theta_0,\varphi_0)$. Given the set ω_N^0, let $e_\mu(\theta) = e(\theta,\xi_\mu)$ and $f_{\mu\alpha}(\theta,\varphi) = f(\theta,\varphi;\xi_\mu,\varphi_{\mu\alpha})$ ($\mu = 0,\dots, M-1; \alpha = 0,\dots,N_\mu - 1$).

In order to prove (4), it is sufficient to show that the two functions

(7) $$E(\theta) = \sum_{\mu=0}^{M-1} N_\mu - e_\mu(\theta), \quad F(\theta,\varphi) = \sum_{\mu,\alpha} f_{\mu\alpha}(\theta,\varphi)$$

are bounded from above.

LEMMA 1. $E(\theta) \le O(1)$.

PROOF. Let $\theta \in (0,\pi)$. Choose $m \in \{0,1,\dots,M-1\}$ such that $\xi_{m-1} < \theta \le \xi_m$ holds. The case when $0 < \theta < \xi_0$ or $\xi_{M-1} < \theta < \pi$ is settled in a similar way. By (5), (7), and the construction of ω_N^0, we have

$$E(\theta) = \sum_{\mu=0}^{M-1} N_\mu e(\theta,\xi_\mu) = \frac{N}{2}\sum_{\mu=0}^{M-1}(\cos\theta_\mu - \cos\theta_{\mu+1})\cdot e(\theta,\xi_\mu)$$
$$= \frac{N}{2}\sum_{\mu=0}^{M-1}\Delta\theta_\mu\cdot\sin\xi_\mu\cdot(1 + O((\Delta\theta_\mu)^2))\cdot e(\theta,\xi_\mu).$$

Here we set $\Delta\theta_\mu = \theta_{\mu+1} - \theta_\mu$. Using the relation $\Delta\theta_\mu \ll N^{-1/2}$, we obtain $E(\theta) = \frac{N}{2}\sum_{\mu=0}^{M-1}\Delta\theta_\mu\cdot\sin\xi_\mu\cdot e(\theta,\xi_\mu) + O(\max_\mu|\sin\xi_\mu\cdot e(\theta,\xi_\mu)|)$. The term $\sin\xi_\mu\cdot e(\theta,\xi_\mu)$ is bounded, uniformly in ξ_μ and θ, as can be seen from the definition (5) and the inequality $|t\cdot\log t| \le e^{-1}$, valid for $0 < t \le 1$. Hence we get

$$E(\theta) = \frac{N}{2}\sum_{\mu=0}^{M-1}\Delta\theta_\mu\cdot\sin\xi_\mu\cdot e(\theta,\xi_\mu) + O(1) = E^*(\theta) + O(1).$$

It remains to prove that $E^*(\theta)$ is bounded from above. We have

$$E^*(\theta) = \frac{N}{2}\sum_{\mu=0}^{m-1} \Delta\theta_\mu \cdot \sin\xi_\mu \cdot e_+(\theta,\xi_\mu) + \frac{N}{2}\sum_{\mu=m}^{M-1} \Delta\theta_\mu \cdot \sin\xi_\mu \cdot e_-(\theta,\xi_\mu).$$

We interpret the first sum as a Riemann sum for the integral $\frac{N}{2}\int_0^{\theta_m} \sin t \cdot e_+(\theta,t)\,dt$ and obtain (see [4], Part II, Problem 11)

$$\begin{aligned}\frac{N}{2}&\sum_{\mu=0}^{m-1} \Delta\theta_\mu \cdot \sin\theta_\mu \cdot e_+(\theta,\xi_\mu)\\ &= \frac{N}{2}\int_0^{\theta_m} \sin t \cdot e_+(\theta,t)\,dt - \frac{N}{48}\sum_{\mu=0}^{m-1}(\Delta\theta_\mu)^3(\sin t \cdot e_+(\theta,t))''_{t=\eta_\mu}\end{aligned}$$

where $\theta_\mu < \eta_\mu < \theta_{\mu+1}$. A calculation shows that

$$\begin{aligned}-&\frac{d^2}{dt^2}(\sin t \cdot e_+(\theta,t))\\ &= \sin t\left(\log\frac{e^2}{4} + \frac{\cos t}{1+\cos t} + \log(1-\cos\theta) + \log(1+\cos t)\right),\end{aligned}$$

and this expression is bounded from above for $\theta, t \in (0,\pi)$. Hence

$$(8) \qquad \frac{N}{2}\sum_{\mu=0}^{m-1} \Delta\theta_\mu \cdot \sin\xi_\mu \cdot e_+(\theta,\xi_\mu) \le \int_0^{\theta_m} \sin t \cdot e_+(\theta,t)\,dt + O(1).$$

A similar estimate shows that

$$(9) \qquad \frac{N}{2}\sum_{\mu=0}^{M-1} \Delta\theta_\mu \cdot \sin\xi_\mu \cdot e_-(\theta,\xi_\mu) \le \int_{\theta_m}^{\pi} \sin t \cdot e_-(\theta,t)\,dt + O(1).$$

Adding (8) and (9), and carrying out the integrations, we obtain

$$\begin{aligned}E^*(\theta) &\le O(1) + \frac{N}{2}(1-\cos\theta_m)(\log(1-\cos\theta) - \log(1-\cos\theta_m))\\ &\quad + \frac{N}{2}(1+\cos\theta_m)(\log(1+\cos\theta) - \log(1+\cos\theta_m))\\ &= O(1) + R(\theta,\theta_m).\end{aligned}$$

The term $R(\theta,\theta_m)$ assumes its maximum value 0 at $\theta = \theta_m$, and is negative elsewhere. This proves Lemma 1.

LEMMA 2. $F(\theta,\varphi) \le O(1)$.

PROOF. Let again $\xi_{m-1} < \theta \le \xi_m$, $0 \le \varphi < 2\pi$. Note that the points of ω_N^0 have coordinates $(\xi_\mu, \varphi_{\mu\alpha})$ $(\mu = 0,\dots,M-1; \alpha = 0,\dots,N_\mu - 1)$ where

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$\theta_{\mu\alpha} = 2\pi\alpha/N_\mu$. For $\mu \le m-1$ fixed we have

$$\sum_{\alpha=0}^{N_\mu-1} f_{\mu\alpha}(\theta,\varphi) = \sum_{\alpha=0}^{N_\mu-1} \log|1-\rho_\mu \cdot e^{i(\varphi-\varphi_{\mu\alpha})}| = \log|1-\rho_\mu^{N_\mu} \cdot e^{iN_\mu\varphi}| \tag{10}$$

where $\rho_\mu = \tan(\xi_\mu/2)/\tan(\theta/2)$. Similarly, we get for $\mu \ge m$

$$\sum_{\alpha=0}^{N_\mu-1} f_{\mu\alpha}(\theta,\varphi) = \log|1-\rho_\mu'^{N_\mu} \cdot e^{iN_\mu\varphi}| \tag{11}$$

where $\rho'_\mu = \tan(\theta/2)/\tan(\xi_\mu/2)$. Relations (10) and (11) imply the following inequality:

$$\begin{aligned} F(\theta,\varphi) &\le \sum_{\mu=0}^{m-1} \log(1+\rho_\mu^{N_\mu}) + \sum_{\mu=m}^{M-1} \log(1+\rho_\mu'^{N_\mu}) \\ &\ll \sum_{\mu=0}^{m-1} \rho_\mu^{N_\mu} + \sum_{\mu=m}^{M-1} \rho_\mu'^{N_\mu}. \end{aligned} \tag{12}$$

We may assume that $0 < \theta \le \pi/2$. In order to obtain an estimate for the first sum in (12), note that $N_\mu \ge K_1 \cdot \mu$ and $\tan(\xi_\mu/2)/\tan(\theta/2) \le 1 - \frac{K_2}{\theta\sqrt{N}}(m-\mu-1)$ holds for $\mu = 0, 1, \dots, m-1$ and suitably chosen absolute constants $K_1 > 0$ and $K_2 > 0$. In addition, it follows from the assumption $\xi_{m-1} < \theta \le \xi_m$ that $\theta \le K_3 m/\sqrt{N}$, and hence $\rho_\mu \le 1 - K_4(m-\mu-1)/m$ where $K_4 = K_2/K_3$. We obtain

$$\begin{aligned} \sum_{\mu=0}^{m-1} \rho_\mu^{N_\mu} &\le \sum_{\mu=0}^{m-1} \left(1 - \frac{K_4}{m}(m-\mu-1)\right)^{K_1\mu} \le \sum_{\mu=0}^{m-1} e^{-K\cdot\frac{\mu}{m}(m-\mu-1)} \\ &\le 2\sum_{\mu=0}^{[\frac{m-1}{2}]} e^{-K\cdot\frac{\mu}{m}(m-\mu-1)} \ll 2\sum_{\mu=0}^{\infty} e^{-\frac{K}{2}\mu} \ll 1. \end{aligned} \tag{13}$$

Proceeding in quite a similar way as before, we get the corresponding estimate for the second sum in (12):

$$\sum_{\mu=m}^{M-1} \rho_\mu'^{N_\mu} \ll 1. \tag{14}$$

From (13) and (14) the assertion follows.

Lemma 1 and 2 together imply the validity of Theorem 2.

It seems remarkable that from the point of view of uniform distribution the point set ω_N^0 constructed above is not a very good one. Denoting by κ

an arbitrary spherical cap of S, by $F(\kappa)$ its area, and by $Z_N(\kappa)$ the number of points in κ, we certainly have

$$(15) \qquad \sup_{\kappa}\left|Z_N(\kappa) - \frac{1}{4\pi}N \cdot F(\kappa)\right| \gg N^{1/2}.$$

It is known, however (see [1]), that there exist point distributions for which the left-hand side of (15) is $\ll N^{1/4}\cdot \log^{1/2} N$. It seems that a point set ω_N for which the function $q(x, \omega_N)$ possesses a small maximum, must have properties somewhat distinct from those of a point set for which (15) is small.

As mentioned already, the point set ω_N^0 can also be used to prove that the upper bound in Theorem 4 is best possible. As the proof is even more computational in nature than that of Theorem 2, we shall not give all the details and restrict ourselves to a brief sketch.

Let $\omega_N^0 = (x_{\mu\alpha})$ ($\mu = 0, \dots, M-1$; $\alpha = 0, \dots, N_\mu - 1$) be the point set contructed above, where $x_{\mu\alpha} = (\xi_\mu, \varphi_{\mu\alpha})$. It is sufficient to prove that the sum

$$(16) \qquad \sum_{(\mu,\alpha)\neq(\nu,\beta)} \log|x_{\mu\alpha} - x_{\nu\beta}| - N(N-1)\cdot \log\frac{2}{\sqrt{e}}$$

is bounded from below by $(N/2)\cdot\log N + O(N)$. Using the functions e and f defined by (5) and (6), we obtain the following representation of the expression (16):

$$\begin{aligned} &= \sum_{\mu=0}^{M-1}\sum_{\alpha\neq\beta} f(\xi_\mu, \varphi_{\mu\alpha}; \xi_\mu, \varphi_{\mu\beta}) + \sum_{\mu\neq\nu}\sum_{\alpha,\beta} f(\xi_\mu, \varphi_{\mu\alpha}; \xi_\nu, \varphi_{\nu\beta}) \\ &\quad + \frac{1}{2}\sum_{\mu\neq\nu} N_\mu N_\nu e(\xi_\mu, \xi_\nu) + \frac{1}{2}\sum_{\mu}(N_\mu - 1)e(\xi_\mu, \xi_\mu) \\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}). \end{aligned}$$

The first sum (I) constitutes the main term and can be evaluated as follows:

$$\begin{aligned} (\mathrm{I}) &= \sum_{\mu=0}^{M-1} N_\mu \log N_\mu = N\log\sqrt{N} + \sum_{\mu=0}^{M-1} N_\mu \log\frac{N_\mu}{\sqrt{N}} \\ &= \frac{N}{2}\log N + \sqrt{N}\sum_{\mu=0}^{M-1}\frac{N_\mu}{\sqrt{N}}\log\frac{N_\mu}{\sqrt{N}} = \frac{N}{2}\log N + O(N). \end{aligned}$$

The second sum (II) is bounded by $O(N)$. In order to prove this, we may proceed as in the proof of Lemma 2, noting however that this time we need a lower bound instead of an upper bound. Finally, following the idea of the proof of Lemma 1, we can also show that the sums (III) and (IV) are bounded by $O(N)$.

3. Proof of Theorem 1

The proof of Theorem 1 is based on the remarkable fact that the distance function $d(x,x_0) = \log|x-x_0| + \log(\sqrt{e}/2)$ is superharmonic on S. As the Laplace operator Δ is rotation invariant, it is sufficient to consider $\Delta d(x,x_0)$ where x_0 is the north pole on S. In this case,

$$d(x,x_0) = \log\left(2\sin\frac{\theta}{2}\right) + \log\frac{\sqrt{e}}{2}$$

and

$$\Delta d(x,x_0) = \Delta\left(\log\sin\frac{\theta}{2}\right) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\log\sin\frac{\theta}{2}\right)\right) = -\frac{1}{2}.$$

Let $\omega_N = (x_1, x_2, \dots, x_N)$ be an arbitrary N-tuple of points on S. It follows from the preceding remark that $\Delta q(x,\omega_N) = -N/2$ holds for all points $x \neq x_j$ $(j = 1,2,\dots,N)$ on S.

In accordance with the point set ω_N, we construct a test function $T(\theta,\varphi)$ in the following way. Consider the domain $D \subset S$ defined by

$$D = \left\{(\theta,\varphi): 0 \le \frac{\pi}{2} = \omega - \theta \le \frac{\pi}{6}, 0 \le \varphi \le \frac{\pi}{2}\right\}.$$

(The choice of D is rather arbitrary.) Let r be an integer satisfying $2N \le 4^r < 8N$. We decompose the domain D into 4^r "squares"

$$B_{\mu\nu} = \left\{(\theta,\varphi): \frac{\pi}{6}(\nu-1)\cdot 2^{-r} \le \frac{\pi}{2} - \theta \le \frac{\pi}{6}\cdot\nu\cdot 2^{-r};\right.$$
$$\left.\frac{\pi}{6}(\mu-1)\cdot 2^{-r} \le \varphi \le \frac{\pi}{6}\mu\cdot 2^{-r}\right\}$$

with μ,ν both running from 1 to 2^r. There are two kinds of squares $B_{\mu\nu}$: squares of the first kind, denoted by $B'_{\mu\nu}$, that contain some point x_j in their interior, and squares of the second kind, denoted by $B''_{\mu\nu}$, that are free from such points. Note that the total area of squares $B''_{\mu\nu}$ satisfies the relation $\sum\sigma(B''_{\mu\nu}) \gg 1$ (σ = area measure). On each $B_{\mu\nu}$ we define a function $\tau_{\mu\nu}(\theta,\varphi)$ as follows. If $B_{\mu\nu}$ is of the first kind, let $\tau_{\mu\nu}(\theta,\varphi) \equiv 0$. If $B_{\mu\nu}$ is of the second kind, let $\tau_{\mu\nu}(\theta,\varphi) = \sin^2(6\cdot 2^r\theta)\cdot\sin^2(6\cdot 2^r\varphi)$. Note that the normal derivative of $\tau_{\mu\nu}$ with respect to the boundary of $B_{\mu\nu}$ vanishes, and that

$$|\Delta\tau_{\mu\nu}(\theta,\varphi)| = \left|\sin^{-2}\theta\cdot\frac{\partial^2\tau_{\mu\nu}}{\partial\varphi^2} + \sin^{-1}\theta\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial\tau_{\mu\nu}}{\partial\theta}\right)\right| \ll 4^r \ll N \tag{17}$$

holds everywhere. Now define the test function $T(\theta,\varphi)$ on S by $T(\theta,\varphi) = \Delta\tau_{\mu\nu}(\theta,\varphi)$ for $(\theta,\varphi) \in B_{\mu\nu}$, and $T(\theta,\varphi) = 0$ for $(\theta,\varphi) \notin D$. Using Green's second formula for domains on curved surfaces (see, for example, [2, Section 91]), we obtain the following estimate:

$$\begin{aligned}\int_S |q_N(x)| d\sigma(x) \cdot \sup_{x\in S} |T(x)| &\geq \left| \int_S q_N(x) T(x) d\sigma(x) \right| \\ &= \left| \sum_{\mu,\nu} \int_{B_{\mu\nu}} q_N(x) \cdot \Delta\tau_{\mu\nu}(x) d\sigma(x) \right| = \left| \sum_{\mu,\nu} \int_{B_{\mu\nu}} \Delta q_N(x) \tau_{\mu\nu}(x) d\sigma(x) \right| \\ &= \frac{N}{2} \sum_{\mu,\nu} \int_{B''_{\mu\nu}} \tau_{\mu\nu}(x) d\sigma(x) \gg \frac{N}{2} \sum \sigma(B''_{\mu\nu}) \gg N.\end{aligned}$$

Recalling inequality (17), we find that $\sup_{x\in S} |T(x)| \ll N$ holds, and that finally

$$\int_S |q_N(x)| d\sigma(x) \gg 1. \tag{18}$$

In view of the relation $\int_S q_N(x) d\sigma(x) = 0$, (18) is even stronger than the assertion of Theorem 1. This finishes the proof of Theorem 1.

4. Proof of Theorem 3

In order to prove Theorem 3 we make use of two ideas. The first one is due to K. F. Roth [6] and consists in replacing the dynamic problem by a static one. The second idea is due to G. Halasz [3]: different test functions are combined in the form of a Riesz product to obtain a lower bound for the maximum in question.

Let $\omega = (x_1, x_2, \dots)$ be an infinite sequence on S. Define numbers $a_n(\omega)$ by

$$a_n(\omega) = \max_{x\in S} \left(\sum_{j=1}^{n} \log|x - x_j| + \log\frac{\sqrt{e}}{2} \right).$$

Consider the section $\omega_N = (x_1, x_2, \dots, x_N)$ of ω with $N = 4^{l^2}$ where l is a fixed positive integer. We shall prove that

$$\max_{1\leq n\leq N} a_n(\omega) \gg l, \tag{19}$$

from which the assertion follows. Denote by X the box $S \times [0,1) = \{(\theta,\varphi,t)\colon 0 < \theta < \pi, 0 \leq \varphi < 2\pi, 0 \leq t < 1\}$. Let X be endowed with the product measure $v = \sigma \times \lambda$, where σ is the usual area measure on S, and λ is the one-dimensional Lebesgue measure. To each point $x_j = (\theta_j, \varphi_j)$ of the section

ω_N, we assign as a counterpart on X the point $y_j = (\theta_j, \varphi_j, (j-1)/N)$ ($j = 1, 2, \dots, N$). Define a function $Q(y) = Q(\theta, \varphi, t)$ on X by

$$Q(\theta, \varphi, t) = q_{[Nt]+1}(x) = \sum_{j=1}^{[Nt]+1} \left(\log|x - x_j| + \log \frac{\sqrt{e}}{2} \right).$$

Following Roth's idea, it is sufficient to prove the lower estimate

$$\max_X Q(\theta, \varphi, t) \gg l,$$

from which (19) follows. In the sequel, we may assume without loss of generality that $\int_X |Q| dv \ll l$, otherwise the assertion follows in view of the relation $\int_X Q dv = 0$.

Now we use the method of Halasz. We construct a test function $T(y) = \prod_{\alpha=1}^{l} (1 + \rho R_\alpha(y))$ on X, where $0 < \rho < 1/2$ is a suitably chosen parameter, and the functions $R_\alpha(y)$ possess the following properties:

(1) $|R_\alpha(y)| \leq 1$ for $y \in X$ and $\alpha = 1, 2, \dots, l$;

and

(2) $\int_X R_{\alpha_1}(y) \cdot R_{\alpha_2}(y) \cdots R_{\alpha_s}(y) dv(y) = 0$ for $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq l$.

These two properties imply that $\int_X T(y) dv(y) = \int_X 1 \cdot dv(y) = 4\pi$ holds. We have

$$(20) \quad \max_{y \in X} Q(y) = \frac{1}{4\pi} \max_{y \in X} Q(y) \cdot \int_X T(y) dv(y) \geq \frac{1}{4\pi} \int_X Q(y) T(y) dv(y).$$

The functions $R_\alpha(y)$ are chosen in such a way that the linear terms $\rho \cdot R_\alpha$ in the expansion of $\prod_{\alpha=1}^{l} (1 + \rho R_\alpha(y))$ give the main contribution to the integral on the right of (20), whereas the mixed terms $\rho^s R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_s}$ ($s \geq 2$) produce error terms dominated by the main contribution.

Let $X^* \subset X$ be the subdomain $X^* = D \times [0, 1)$, where D is the domain introduced in Section 3. Let $\alpha \in \{1, 2, \dots, l\}$ and define $n_\alpha = 2^{\alpha^2}$. Note that $N = 4^{l^2} = n_l^2$. Let P_α be a partition of X^* into boxes $A_{\mu\nu,\lambda}^{(\alpha)}$, where $A_{\mu\nu,\lambda}^{(\alpha)}$ is the cartesian product of the box

$$B_{\mu\nu}^{(\alpha)} = \Big\{ (\theta, \varphi) \colon \frac{\pi}{6} \cdot (\nu - 1) n_\alpha^{-1} \leq \theta \leq \frac{\pi}{6} \cdot \nu \cdot n_\alpha^{-1}; \\ \frac{\pi}{6} \cdot (\mu - 1) n_\alpha^{-1} \leq \varphi \leq \frac{\pi}{6} \cdot \mu \cdot n_\alpha^{-1} \Big\} \subset D$$

with the interval $C_\lambda^{(\alpha)} = \{t \colon (\lambda - 1) \cdot \frac{n_\alpha^2}{2N} \leq t \leq \lambda \cdot \frac{n_\alpha^2}{2N}\}$. The indices μ and ν run from 1 to n_α, the index λ runs from 1 to $2N/n_\alpha^2$. Hence, for each α, $1 \leq \alpha \leq l$, we have a partition of X^* into $2N$ boxes $A_{\mu\nu,\lambda}^{(\alpha)}$. Again, there exist two kinds of boxes: boxes of the first kind that contain some point y_j as an interior point, and boxes of the second kind that are free from such points.

On each box of the second kind we define a function $u^{(\alpha)}_{\mu\nu,\lambda}(y) = u^{(\alpha)}_{\mu\nu,\lambda}(\theta, \varphi, t)$ by

$$u^{(\alpha)}_{\mu\nu,\lambda}(\theta, \varphi, t) = \Delta\tau_{\mu\nu}(\theta, \varphi) \cdot \operatorname{sign} t,$$

where sign $= -1$ for $(\lambda - 1)\frac{n_\alpha^2}{2N} \leq t < (\lambda - \frac{1}{2})\frac{n_\alpha^2}{2N}$, $\operatorname{sign} t = +1$ for $(\lambda - \frac{1}{2})\frac{n_\alpha^2}{2N} \leq t < \lambda \cdot \frac{n_\alpha^2}{2N}$, and $\tau_{\mu\nu}(\theta, \varphi) = -c \cdot n_\alpha^{-2} \cdot \sin^2 6n_\alpha\theta \cdot \sin^2 6n_\alpha\varphi$. Here $c > 0$ is an absolute constant ($c = 1/1000$ will do) chosen in such a way that the spherical Laplacian derivative $\Delta\tau_{\mu\nu}(\theta, \varphi)$ is bounded by 1 in absolute value.

On boxes of the first kind, set $u^{(\alpha)}_{\mu\nu,\lambda}(y) = 0$. Finally, define $R_\alpha(y) = u^{(\alpha)}_{\mu\nu,\lambda}(y)$ for $y \in A^{(\alpha)}_{\mu\nu,\lambda}$, $R_\alpha(y) = 0$ for $y \in X\backslash X^*$, and let $T(y) = \prod_{\alpha=1}^{l}(1 + \rho \cdot R_\alpha(y))$. The parameter ρ, $0 < \rho < 1/2$, will be chosen later on.

First of all, we shall prove that the R_α's form a system of functions completely orthogonal with respect to the measure v.

LEMMA 3. *Let* $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq l$. *Then the following orthogonality relation holds*: $\int_X R_{\alpha_1}(y)R_{\alpha_2}(y)\cdots R_{\alpha_s}(y)dv(y) = 0$.

PROOF. For fixed θ, φ consider the set $\{(\theta, \varphi, t): (\lambda - 1)n_{\alpha_1}^2/2N \leq t < \lambda n_{\alpha_1}^2/2N\}$. The functions $R_{\alpha_2}, \ldots, R_{\alpha_s}$ are of constant value on this interval, whereas R_{α_1} is of constant absolute value but changes sign in the middle of the interval. Integrating first with respect to dt, then with respect to $\sin\theta\, d\theta\, d\varphi$, proves the assertion.

COROLLARY 1. $\int_X T(y)dv(y) = \int_X dv(y) = 4\pi$.

LEMMA 4. *For each box* $A^{(\alpha)}_{\mu\nu,\lambda}$ *of the second kind, the following inequality holds*:

$$\int_{A^{(\alpha)}_{\mu\nu,\lambda}} Q(y)R_\alpha(y)dv(y) \gg N^{-1}. \tag{21}$$

PROOF. Fix t_0, $(\lambda-1)n_\alpha^2/2N \leq t_0 < (\lambda-\frac{1}{2})n_\alpha^2/2N$, and set $t_1 = t_0+\frac{1}{2}n_\alpha^2/2N$. Note that $R_\alpha(\theta, \varphi, t_0) = -R_\alpha(\theta, \varphi, t_1)$. We obtain the integral in question by first integrating $Q(\theta, \varphi, t_0) \cdot R_\alpha(\theta, \varphi, t_0) + Q(\theta, \varphi, t_1) \cdot R_\alpha(\theta, \varphi, t_1)$ with respect to $d\sigma(\theta, \varphi)$ over $B^{(\alpha)}_{\mu\nu}$, using Green's formula as in the proof of Theorem 1, and then with respect to dt_1 over the interval $(\lambda - \frac{1}{2})n_\alpha^2/2N \leq t_1 \leq \lambda n_\alpha^2/2N$.

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We have

$$(22)\qquad \begin{aligned} &\int_{B_{\mu\nu}^{(\alpha)}}(Q(\theta,\varphi,t_0)\cdot R_\alpha(\theta,\varphi,t_0)+Q(\theta,\varphi,t_1)\cdot R_\alpha(\theta,\varphi,t_1))\,d\sigma\\ &=\int_{B_{\mu\nu}^{(\alpha)}}R_\alpha(\theta,\varphi,t_1)(Q(\theta,\varphi,t_1)-Q(\theta,\varphi,t_0))d\sigma\\ &=\int_{B_{\mu\nu}^{(\alpha)}}\Delta\tau_{\mu\nu}^{(\alpha)}(\theta,\varphi)\cdot\sum_{[Nt_0]+2}^{[Nt_1]+1}\left(\log|x-x_j|+\log\frac{\sqrt{e}}{2}\right)d\sigma(\theta,\varphi)\\ &\gg N(t_1-t_0)\cdot\sigma(B_{\mu\nu}^{(\alpha)})\cdot n_\alpha^{-2}\gg n_\alpha^2\cdot\sigma(B_{\mu\nu}^{(\alpha)})\cdot n_\alpha^{-2}=\sigma(B_{\mu\nu}^{(\alpha)}).\end{aligned}$$

Integrating (22) with respect to dt_1, and noting that $v(A_{\mu\nu,\lambda}^{(\alpha)})\gg N^{-1}$, we obtain the desired result. This proves Lemma 4.

Summing (21) over all boxes $A_{\mu\nu\lambda}^{(\alpha)}$ of the second kind, we get

COROLLARY 2. $\sum_{\alpha=1}^{l}\rho\int_X Q(y)R_\alpha(y)dv(y)\gg\rho\cdot l$.

In a final step we show that the remaining terms in the expansion of $\prod_{\alpha=1}^{l}(1+\rho R_\alpha(y))$ give a contribution which is dominated by $\rho\cdot l$.

LEMMA 5. *The following inequality holds*:

$$\left|\sum_{\alpha_1<\alpha_2}\rho^2\int_X QR_{\alpha_1}R_{\alpha_2}dv+\sum_{\alpha_1<\alpha_2<\alpha_3}\rho^3\int_X QR_{\alpha_1}R_{\alpha_2}R_{\alpha_3}dv+\cdots\right|\ll\rho^2\cdot l.$$

PROOF. Let $1\le\alpha_1<\alpha_2<\cdots<\alpha_s\le l$. Consider a box of the form $B_{\mu\nu}^{(\alpha_s)}\times C_\lambda^{(\alpha_1)}$. Denote by I_1, I_2 the two halves of the interval $C_\lambda^{(\alpha_1)}$. For fixed θ,φ, the functions $R_{\alpha_2},\dots,R_{\alpha_s}$ are constant in the variable t on $B_{\mu\nu}^{(\alpha_s)}\times C_\lambda^{(\alpha_1)}$, whereas R_{α_1} is constant in absolute value, but changes sign in the middle of $C_\lambda^{(\alpha_1)}$. We split the product $R_{\alpha_1}(y)\cdot R_{\alpha_2}(y)\cdot\dots\cdot R_{\alpha_{s-1}}(y)$ on $B_{\mu\nu}^{(\alpha_s)}\times I_2$ into two parts: $R_{\alpha_1}(y)\cdot\dots\cdot R_{\alpha_{s-1}}(y)=m+\varepsilon(y)$, where m is the mean value on $B_{\mu\nu}^{(\alpha_s)}\times I_2$ with respect to v, and $\varepsilon(y)$ is the error term. Similarly, on $B_{\mu\nu}^{(\alpha_s)}\times I_1$ we get $R_{\alpha_1}(y)\cdot\dots\cdot R_{\alpha_{s-1}}(y)=-m+\varepsilon(y)$. In view of $|R_\alpha(y)|\le 1$, we have the inequalities $|m|\le 1$ and

$$|\varepsilon(y)|\le\frac{\pi}{6}\cdot n_{\alpha_s}^{-1}\cdot\sup_X\left(\left|\frac{\partial}{\partial\theta}R_{\alpha_1}\cdots R_{\alpha_{s-1}}\right|+\left|\frac{\partial}{\partial\varphi}R_{\alpha_1}\cdots R_{\alpha_{s-1}}\right|\right)\ll n_{\alpha_{s-1}}/n_{\alpha_s}.$$

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Integrating over $B_{\mu\nu}^{(\alpha_s)} \times C_\lambda^{(\alpha_1)} = B_{\mu\nu}^{(\alpha_1)} = B_{\mu\nu}^{(\alpha_s)} \times (I_1 \cup I_2)$, we obtain the following inequality:

$$\begin{aligned}
&\left| \int_{B_{\mu\nu}^{(\alpha_s)} \times C_\lambda^{(\alpha_1)}} Q(y) R_{\alpha_1}(y) \cdot \dots \cdot R_{\alpha_s}(y) dv(y) \right| \\
&\quad = \left| \int_{B_{\mu\nu}^{(\alpha_s)} \times I_1} (-m + \varepsilon(y)) Q(y) R_{\alpha_s}(y) dv(y) \right. \\
&\qquad\qquad \left. + \int_{B_{\mu\nu}^{(\alpha_s)} \times I_2} (m + \varepsilon(y)) Q(y) R_{\alpha_s}(y) dv(y) \right| \qquad (23) \\
&\quad \ll \left| \int_{B_{\mu\nu}^{(\alpha_s)} \times I_2} Q(y) R_{\alpha_s}(y) dv(y) - \int_{B_{\mu\nu}^{(\alpha_s)} \times I_1} Q(y) R_{\alpha_s}(y) dv(y) \right| \\
&\qquad + (n_{\alpha_{s-1}}/n_{\alpha_s}) \cdot \int_{B_{\mu\nu}^{(\alpha_s)} \times C_\lambda^{(\alpha_1)}} |Q(y)| dv(y) \\
&\quad \ll (n_{\alpha_1}/n_{\alpha_s})^2 \cdot v(B_{\mu\nu}^{(\alpha_s)} \times C_\lambda^{(\alpha_1)}) + (n_{\alpha_{n-1}}/n_{\alpha_s}) \cdot \int_{B_{\mu\nu}^{(\alpha_s)} \times C_\lambda^{(\alpha_1)}} |Q(y)| dv(y).
\end{aligned}$$

In order to obtain the first term in (23), we may proceed as in the proof of Lemma 4, replacing the symbol $\gg$ by $\ll$, and noting that $N(t_1 - t_0) \ll n_{\alpha_1}^2$ in our case.

Summing (23) over all boxes $B_{\mu\nu}^{(\alpha_s)} \times C_\lambda^{(\alpha_1)}$, and noting that $\int_X |Q(y)| dy(y) \ll l$ without loss of generality, we get

$$\begin{aligned}
&\left| \int_X Q(y) R_{\alpha_1}(y) \cdot \dots \cdot R_{\alpha_s}(y) dv(y) \right| \\
&\quad \ll n_{\alpha_1}^2 / n_{\alpha_s}^2) + l \cdot (n_{\alpha_{s-1}}/n_{\alpha_s}) \qquad (24) \\
&\quad \ll l \cdot (n_{\alpha_{s-1}}/n_{\alpha_s}).
\end{aligned}$$

Summing (24) over all s-tuples $1 \le \alpha_1 < \alpha_2 < \dots < \alpha_s \le l$ with $s \ge 2$, we get

$$\begin{aligned}
&\left| \sum_{\alpha_1 < \alpha_2} \rho^2 \int_X Q R_{\alpha_1} R_{\alpha_2} dv + \sum_{\alpha_1 < \alpha_2 < \alpha_3} \rho^3 \int_X Q R_{\alpha_1} R_{\alpha_2} R_{\alpha_3} dv + \cdots \right| \\
&\quad \ll \rho^2 l \sum_{\alpha_1 < \alpha_2} (n_{\alpha_1}/n_{\alpha_2}) + \rho^3 l \sum_{\alpha_1 < \alpha_2 < \alpha_3} (n_{\alpha_2}/n_{\alpha_3}) + \cdots \\
&\quad \ll \rho^2 l \sum_{\alpha_1 < \alpha_2} 2^{-\alpha_2} + j \rho^3 l \sum_{\alpha_1 < \alpha_2 < \alpha_3} 2^{-\alpha_3} + \cdots \\
&\quad = \rho^2 l \left(\sum_{\alpha=2}^{l} \binom{\alpha - 1}{1} 2^{-\alpha} + \rho \sum_{\alpha=3}^{l} \binom{\alpha - 1}{2} 2^{-\alpha} + \cdots \right) \\
&\quad = \rho \cdot l \sum_{\alpha=2}^{l} 2^{-\alpha} \left((1 + \rho)^{\alpha - 1} - 1 \right) \le \rho^2 \cdot l \sum_{\alpha=2}^{l} (\alpha - 1)(1 + \rho)^{\alpha - 2} 2^{-\alpha} \ll \rho^2 l.
\end{aligned}$$

This proves Lemma 5.

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Corollary 2 and Lemma 5 imply the following final inequality:

$$\begin{aligned}\int_X Q(y)T(y)dv(y) &= \int_X Q(y)dv(y) + \rho^l \sum_{\alpha=1}^{l} \int_X R_\alpha(y)Q(y)dv(y) \\ &\quad + \sum_{s=2}^{l} \rho^s \sum_{\alpha_1<\cdots<\alpha_s} \int_X Q(y)R_{\alpha_1}(y)\cdot\dots\cdot R_{\alpha_s}(y)dv(y) \\ &\geq c_1\rho l - c_2\rho^2 l,\end{aligned}$$

where c_1 and c_2 are positive absolute constants. Choosing $\rho < c_1/2c_2$, the assertion of Theorem 3 follows in view of relation (20).

5. On the product of mutual distances

Let $\omega_N = (x_1, x_2, \dots, x_N)$ be a point set on S, and let θ_j, φ_j be the spherical coordinates of the points x_j. We shall derive an upper bound for the product of mutual distances between the points of ω_N. The first proof is more elegant and gives a slightly better remainder term, whereas the idea of the second proof is more general and applicable both to higher dimensions and various types of similar kernels.

First proof. Set

$$a_j = \cos(\theta_j/2)\cdot\exp(i\varphi_j/2) \quad \text{and} \quad b_j = \sin(\theta_j/2)\cdot\exp(-i\varphi_j/2),$$

and note that

$$|x_j - x_k| = 2\cdot|a_j b_k - a_k b_j|. \tag{25}$$

Relation (25) allows us to express the product of mutual distances by means of a determinant of the Vandermonde type. Consider the (N, N)-matrix $P = (p_{\mu\nu})$ $(\mu, \nu = 1, 2, \dots, N)$ where $p_{\mu\nu} = a_\mu^{N-\nu}\cdot b_\mu^{\nu-1}$. It is easy to prove that

$$\prod_{j<k} |x_j - x_k| = 2^{N(N-1)/2}\cdot|\operatorname{Det} P|.$$

Now replace the matrix P by a new matrix $P' = (p'_{\mu\nu})$ where

$$p'_{\mu\nu} = p_{\mu\nu}\cdot\binom{N-1}{\nu-1}^{1/2} \qquad (\mu, \nu = 1, 2, \dots, N).$$

Note that $|\operatorname{Det} P| = |\operatorname{Det} P'|\cdot\prod_{\nu=1}^{N}\binom{N-1}{\nu-1}^{-1/2}$ from which

$$\prod_{j<k}|x_j - x_k| = 2^{N(N-1)/2}\prod_{\nu=1}^{N}\binom{N-1}{\nu-1}^{-1/2}\cdot|\operatorname{Det} P'|$$

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follows. We obtain an upper bound for $|\operatorname{Det} P'|$ by applying Hadamard's inequality. The length of the μth row vector of P' is equal to

$$\left(\sum_{\nu=1}^{N}|p'^2_{\mu\nu}|\right)^{1/2} = \left(\sum_{\nu=1}^{N}\binom{N-1}{\nu-1}|a^2_\mu|^{N-\nu}|b^2_\mu|^{\nu-1}\right)^{1/2} = (|a^2_\mu|+|b^2_\mu|)^{(N-1)/2}$$
$$= (\sin^2(\theta_\mu/2)+\cos^2(\theta_\mu/2))^{(N-1)/2} = 1.$$

Hence $|\operatorname{Det} P'| \le 1$ and

$$\prod_{j<k}|x_j - x_k| \le 2^{N(N-1)/2}\prod_{\nu=1}^{N}\binom{N-1}{\nu-1}^{-1/2}. \tag{26}$$

It suffices to evaluate $\prod_{\nu=1}^{k}\binom{N-1}{\nu-1} = \prod_{\nu=1}^{N-1}(\nu^\nu/\nu!)$. Using Stirling's formula, we get

$$\prod_{\nu=1}^{N-1}(\nu^\nu/\nu!) = e^{(1/2)N(N-1)}\cdot N^{-(N/2)+o(N)}. \tag{27}$$

From (26) and (27) the upper estimate

$$\prod_{j<k}|x_j - x_k| \le (2/\sqrt{e})^{N(N-1)/2}\cdot N^{N/4+o(N)}$$

follows.

SECOND PROOF. Let x and y be points on S, and let γ be the angle between the radius vectors pointing from the center of the sphere to x and y, respectively. We have $\log|x-y| = (1/2)\log(2-2\cos\gamma)$. Define a new distance function $d_r(x,y)$ on S by setting $d_r(x,y) = (1/2)\log(1+r^2-2r\cos\gamma)$. The parameter $r\in(0,1)$ will be chosen later on. Note that the mean value m_r of $d_r(x,y)$ on S is

$$m_r = \frac{1}{2}\int_0^\pi \frac{1}{2}\log(1+r^2-2r\cos\theta)\sin\theta\,d\theta$$
$$= -\frac{1}{2}+\frac{(1+r)}{4r}\log(1+r) - \frac{(1-r)^2}{4r}\log(1-r).$$

For short, let $D(x,y) = \log|x-y| - \log(2/\sqrt{e})$ and $D_r(x,y) = d_r(x,y) - m_r$. Consider the following trivial inequality:

$$\sum_{j\neq k}\log|x_j-x_k| - N(N-1)\log(2/\sqrt{e}) = \sum_{j\neq k}D(x_j,x_k)$$
$$= \sum_{j,k}D_r(x_j,x_k) - \sum_j D_r(x_j,x_j) + \sum_{j\neq k}(D(x_j,x_k)-D_r(x_j,x_k))$$
$$\le \sum_{j,k}D_r(x_j,x_k) - N\log(1-r) + N\cdot m_r$$
$$+ N(N-1)\max_{x,y\in S}(D(x,y)-D_r(x,y)).$$

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A calculation shows that

$$\max_{x,y\in S}(D(x,y) - D_r(x,y)) = ((1-r)^2/4r)\log((1+r)/(1-r)).$$

Now choose $r = 1 - 1/\sqrt{N\log N}$. We obtain

$$\begin{aligned} &\sum_{j\neq k}\log|x_j - x_k| - N(N-1)\log(2/\sqrt{e}) \\ &\quad \leq \sum_{j,ik} D_r(x_j,x_k) + \frac{N}{2}\log N + O(N\log\log N). \end{aligned} \tag{28}$$

We shall prove that the first term on the right of (28) is negative. Let $\sum_{n=1}^{\infty} a_n(r)P_n(\cos\theta)$ be the expansion into spherical harmonics of the kernel $k_r(\theta) = (1/2)\log(1 + r^2 - 2r\cos\theta) - m_r$ $(0 < r < 1)$. It is known that all the coefficients $a_n(r)$ are negative (the proof works in exactly the same manner as the proof of [5, Hilfssatz 6, page 36]). Consider the new kernel $\kappa_r(\theta) = \sum_{n=1}^{\infty}(-a_n(r)\cdot\frac{2n+1}{2\pi})^{1/2}P_n(\cos\theta)$, which is a convolution root of $-k_r(\theta)$. Let $\delta_r(x,y)$ be the distance function generated by $\kappa_r(\theta)$ on S. Then Legendre's addition formula for spherical harmonics implies the following relation:

$$\sum_{j,k} D_r(x_j,x_k) = -\int_S\left(\sum_j \delta_r(x,x_j)\right)^2 d\sigma(x) \leq 0. \tag{29}$$

From (28) and (29) the assertion of Theorem 4 follows a second time.

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