SEMI-NORMAL OPERATORS ON UNIFORMLY SMOOTH BANACH SPACES

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1. Introduction. In this paper we shall examine the relationship between the numerical ranges and the spectra for semi-normal operators on uniformly smooth spaces.

Let X be a complex Banach space. We denote by X^* the dual space of X and by $B(X)$ the space of all bounded linear operators on X. A linear functional F on $B(X)$ is called *state* if $\|F\| = F(I) = 1$. When $x \in X$ with $\|x\| = 1$, we denote

$$D(x) = \{f \in X^* : \|f\| = f(x) = 1\}.$$

Let us set

$$\Pi = \{(x, f) \in X \times X^* : \|f\| = f(x) = \|x\| = 1\}.$$

The *spatial numerical range* $V(T)$ and the *numerical range* $V(B(X), T)$ of $T \in B(X)$ are defined by

$$V(T) = \{f(Tx) : (x, f) \in \Pi\}$$

and

$$V(B(X), T) = \{F(T) : F \text{ is a state on } B(X)\},$$

respectively.

If $V(T) \subset \mathbb{R}$, then T is called *hermitian*. An operator $T \in B(X)$ is called *hyponormal* (*co-hyponormal*) if there are hermitian operators H and K such that $T = H + iK$ and $C = i(HK - KH) \geq 0$ (≤ 0).

An operator $T \in B(X)$ is called *semi-normal* if T is hyponormal or co-hyponormal.

An operator T is called *normal* if there are hermitians H and K such that $T = H + iK$ and $HK = KH$.

For an operator $T \in B(X)$, the spectrum, the approximate point spectrum, the point spectrum, the kernel and the dual operator of T are denoted by $\sigma(T)$, $\sigma_\pi(T)$, $\sigma_p(T)$, $\mathrm{Ker}(T)$ and T^*, respectively.

The following results are well-known:

(1) $\overline{\mathrm{co}}\, V(T) = V(B(X), T)$, where $\overline{\mathrm{co}}\, E$ is the closed convex hull of E.

(2) $\mathrm{co}\, \sigma(T) \subset \overline{V(T)}$, where $\mathrm{co}\, E$ and $\bar{E}$ are the convex hull and the closure of E, respectively.

(3) $V(T) \subset V(T^*) \subset \overline{V(T)}$.

(4) If T is normal, then $\sigma(T) = \sigma_\pi(T)$ and $\mathrm{co}\, \sigma(T) = \overline{V(T)} = V(B(X), T)$.

Remark 1. From (3), if T is hyponormal or co-hyponormal, then T^* is co-hyponormal or hyponormal, respectively.

We set, for $t > 0$:

$$\rho(t) = \sup\{\tfrac{1}{2}(\|x + y\| + \|x - y\|) - 1;\ \|x\| = 1,\ \|y\| \leq t\}.$$

A Banach space X is called *uniformly smooth* if

$$\frac{\rho(t)}{t} \to 0 \quad \text{as} \quad t \to 0.$$

REMARK 2. A Banach space X is uniformly smooth iff X^* is uniformly convex. See **[3]** for details.

We recall from **[1]** and **[2]** the construction of a larger space X^0 from a given Banach space X. Then the mapping $T \to T^0$ is an isometric isomorphism of $B(X)$ onto a closed subalgebra of $B(X^0)$. Let Lim be fixed Banach limit on the space of all bounded sequences of complex numbers with the norm $\|\{\lambda_n\}\| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$. Let $\tilde{X}$ be the space of all bounded sequences $\{x_n\}$ of X. Let N be the subspace of $\tilde{X}$ consisting of all bounded sequences $\{x_n\}$ with $\text{Lim} \|x_n\|^2 = 0$. The space X^0 is defined as the completion of the quotient space $\tilde{X}/N$ with respect to the norm $\|\{x_n\} + N\| = (\text{Lim} \|x_n\|^2)^{1/2}$. Then the following results hold:

$$\sigma(T) = \sigma(T^0), \qquad \sigma_\pi(T) = \sigma_\pi(T^0) = \sigma_p(T^0) \quad \text{and} \quad \overline{\text{co}}\, V(T) = V(T^0).$$

See **[1]** and **[2]** for details.

We need the following results.

THEOREM A [**2**, Theorem 4]. *X is uniformly convex iff X^0 is uniformly convex.*

THEOREM B [**5**, Lemma 20.3 and Corollary 20.10]. *If H is hermitian and $Hx = 0$ with $\|x\| = 1$, then there exists $f \in X^*$ such that $(x, f) \in \Pi$ and $H^*f = 0$.*

2. Semi-normal operators on uniformly smooth spaces.

THEOREM 1. *Let X be uniformly smooth. Let $T = H + iK$ be semi-normal on X.*

(1) If $a \in \sigma(H)$, then there is a real number b such that $b \in \sigma(K)$ and $a + ib \in \sigma(T)$.

(2) If $b' \in \sigma(K)$, then there is a real number a' such that $a' \in \sigma(H)$ and $a' + ib' \in \sigma(T)$.

Proof. (1) Since H is hermitian, there exists a sequence $\{x_n\}$ of unit vectors in X such that $(H - a)x_n \to 0$. Since X^* is uniformly convex, by Theorem 3.11 in Mattila **[11]** it follows that $(H^* - a)f_n \to 0$, where $f_n \in D(x_n)$. Consider the larger space X^{*0} of X^*. Then $\text{Ker}(H^{*0} - a)$ is a non-zero subspace of X^{*0}. If $f_0 \in \text{Ker}(H^{*0} - a)$ such that $\|f_0\| = 1$, then by Theorem B there is $\varphi \in X^{*0*}$ such that $\|\varphi\| = \varphi(f_0) = 1$ and $(H^{*0*} - a)\varphi = 0$. We may assume that $C = i(HK - KH) \geq 0$. Then $C^* = i(K^*H^* - H^*K^*) \geq 0$ and

$$\varphi(C^{*0}f_0) = i\varphi(K^{*0}(H^{*0} - a)f_0) - i\hat{f}_0(K^{*0*}(H^{*0*} - a)\varphi) = 0,$$

where $\hat{f}_0$ is the Gel'fand representation of f_0. Since, by Theorem A, the space X^{*0} is uniformly convex and $C^{*0} \geq 0$, it follows that $C^{*0}f_0 = 0$ by Theorem 2.1 in **[12]**. Therefore, we have that

$$(H^{*0} - a)K^{*0}f_0 = 0.$$

It is easy to see that $\text{Ker}(H^{*0} - a)$ is invariant for K^{*0}. Hence, there exist a real number b and non-zero vector g_0 in $\text{Ker}(H^{*0} - a)$ such that $K^{*0}g_0 = bg_0$. It follows that $b \in \sigma_p(K^{*0})$ and $a + ib \in \sigma_p(T^{*0})$. And we have that $b \in \sigma(K^*) = \sigma(K)$ and $a + ib \in \sigma(T^*) = \sigma(T)$.

(2) is proved in the same way as (1).

THEOREM 2. *Let X be uniformly smooth. Let $T = H + iK$ be semi-normal. Then*

$$\text{co}\, \sigma(T) = \overline{V(T)} = V(B(X), T).$$

Proof. We assume that $\mathrm{Re}\,\sigma(T) \subset \mathbb{R}^+$. Then by Theorem 1 it follows that $\sigma(H) \subset \mathbb{R}^+$. Since $\mathrm{co}\,\sigma(H) = \overline{V(H)} = V(B(X), H)$, it follows that $\mathrm{Re}\,V(B(X), T) \subset \mathbb{R}^+$. Since $\alpha T + \beta$ is semi-normal for every $\alpha, \beta \in \mathbb{C}$, it follows that $\mathrm{co}\,\sigma(T) = \overline{V(T)} = V(B(X), T)$.

THEOREM 3. *Let X be uniformly smooth. Let $T = H + iK$ be co-hyponormal on X. If $a + ib \in \sigma(T)$, then $a \in \sigma(H)$ and $b \in \sigma(K)$.*

Proof. If $a + ib \in \sigma(T)$, then $a + ib \in \sigma(T^*)$. Thus there exists $b' \in \mathbb{R}$ such that $a + ib'$ belongs to the boundary of $\sigma(T^*)$. Therefore there exists a sequence $\{f_n\}$ of unit vectors in X^* such that $(T^* - (a + ib'))f_n \to 0$. Since X^* is uniformly convex and T^* is hyponormal on X^*, by Theorem 2.7 in **[12]** we have that $(H^* - a)f_n \to 0$ and $(K^* - b')f_n \to 0$. It follows that $a \in \sigma(H)$.

$b \in \sigma(K)$ is proved analogously.

COROLLARY 4. *Let X be uniformly smooth. Let $T = H + iK$ be co-hyponormal on X. Then* $\mathrm{Re}\,\sigma(T) = \sigma(H)$ *and* $\mathrm{Im}\,\sigma(T) = \sigma(K)$.

Proof. The proof follows easily from Theorems 1 and 3.

PROBLEM. Does Theorem 3 hold for a hyponormal operator?

REMARK 3. The following theorem holds, which corresponds to Theorem 10.6 in **[4]**. Let X be uniformly smooth. Then

$$\{\lambda \in \overline{V(T)} : |\lambda| = \|T\|\} \subset \sigma_\pi(T).$$

It follows from the uniform convexity of X^* and $\overline{V(T)} = \overline{V(T^*)}$.

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