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# A DISCRETE ANALOGUE OF THE PALEY-WIENER THEOREM FOR BOUNDED ANALYTIC FUNCTIONS IN A HALF PLANE

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In this note we prove a discrete analogue to the following Paley–Weiner theorem: *Let $f$ be the restriction to the line of a bounded analytic function in the upper half plane; then the spectrum of $f$ is contained in* $[0,\infty)$. The discrete analogue of complex analysis is the theory of discrete analytic functions invented by Lelong–Ferrand (1944) and developed by Duffin (1956) and others. A function $f$ on a subset $M$ of the two-dimensional lattice $Z^2$ is said to be *discrete analytic* there if, for $(m,n)\in M$,

$$[f(m+1,n+1)-f(m,n)]/1+i=[f(m,n+1)-f(m+1,n)]/i-1$$

which is equivalent to the requirement

$$f(m,n)+i\,f(m+1,n)-f(m+1,n+1)-i\,f(m,n+1)=0. \tag{1}$$

Given a bounded sequence $\{c_n\}_{-\infty}^{\infty}$, its spectrum is the support of the distribution $\Sigma_{-\infty}^{\infty}\, c_n e^{-int}$ on the circle $T$. It is well known that it coincides with the Fourier–Carleman spectrum which is the complement of the set through which

$$\sum_{0}^{\infty} c_n z^{-n}(|z|>1) \quad \text{and} \quad -\sum_{-\infty}^{-1} c_n z^{-n} \qquad (|z|<1)$$

can be analytically continued to each other.

Now, we are in a position to prove the following theorem.

THEOREM: *Let $f(m,n)$ be discrete analytic and bounded in the upper half lattice, $n\geqq 0$. The spectrum of $\{f(m,0)\}_{-\infty}^{\infty}$ is contained in* $[0,\pi)$.

PROOF: The theorem is obviously true for $f$ constant, for then $\Sigma_{-\infty}^{\infty}\, Ce^{-int}=C\delta$, $\delta$ being the Dirac delta measure, which is supported at $\{0\}$.

So without loss of generality we may assume that $f(0,0) = 0$. The theorem will be proven if we show that

$$\sum_{0}^{\infty} f(m,0)\,z^{-m} \qquad (|z|>1) \quad \text{and} \quad -\sum_{-\infty}^{-1} f(m,0)\,z^{-m} \qquad (|z|<1)$$

can be analytically continued to each other through the lower unit semi-circle; or, equivalently that $\varphi_+(z) = \Sigma_0^\infty f(m,0)z^m$ $(|z|<1)$ and $-\varphi_-(z) = -\Sigma_{-\infty}^{-1} f(m,0)z^m$ $(|z|>1)$ are analytic continuations of each other through the upper unit semi-circle.

Let

$$F(z,w) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m,n)\,z^m w^n, \quad \psi(w) = \sum_{n=0}^{\infty} f(0,n)w^n.$$

Since $f$ is bounded, $F$, $\varphi_+$, $\psi$, are all analytic in the polydisc $\{|z|<1\}\times\{|w|<1\}$. By (1),

$$(1 + iz - zw - iw)\,F(z,w) = (1+iz)\varphi_+(z) + (1-iw)\psi(w) \tag{2}$$

and it follows that the r.h.s. vanishes for $|z|<1$, $|w|<1$, $w = (1+iz)/(z+i)$. Therefore, for $|z|<1$, $\left|\dfrac{z-i}{z+i}\right|<1$,

$$\varphi_+(z) = \frac{2iz}{z^2+1}\,\psi\left(\frac{1+iz}{z+i}\right). \tag{3}$$

Since $\psi(0) = 0$, the r.h.s. of (3) has no pole at $z = i$ and is an analytic function in the upper half plane $\left\{\left|\dfrac{z-i}{z+i}\right|<1\right\}$. It follows that $\varphi_+(z)$ can be analytically continued through the upper semi-circle to the whole upper half plane. A similar reasoning for the left upper quarter lattice shows that

$$\varphi_-(z) = \frac{-2iz}{z^2+1}\,\psi\left(\frac{1+iz}{z+i}\right) \qquad \left(|z|>1,\ \left|\frac{z-i}{z+i}\right|<1\right).$$

Thus $\varphi_-(z)$ can be analytically continued through the upper unit semi-circle to the whole upper half plane and we have that $\varphi_+(z)$ and $-\varphi_-(z)$ are analytic continuations of each other through the upper unit semi-circle and the theorem follows.

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