

PRIME NUMBERS OF THE FORM $[n^c]$

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Abstract. In 1953, Pjateckiĭ-Šapiro has proved that there are infinitely many primes of the form $[n^c]$ for $1 < c < \frac{12}{11}$ (with an asymptotic result). This range, which measures our progress in the technique of exponential sums, has been improved by many authors. In this paper we obtain $1 < c \leq \frac{243}{205}$.

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1. Introduction. One of the most odd problems in prime number theory is to represent infinitely many prime numbers by polynomials. Though this problem has been completely solved for linear polynomials by the Dirichlet theorem on primes in arithmetic progressions, until now it remains open for polynomials of degree greater than 1 and it seems to be extremely difficult. Iwaniec [6] has proved that there exists an infinity of integers n such that $n^2 + 1$ has at most two prime factors. As another direction of approaching this problem, Pjateckiĭ-Šapiro [9] has proposed to investigate the prime numbers of the form $[n^c]$, where $c > 1$ and $[t]$ denotes the integer part of t . Clearly $[n^c]$ can be regarded as “polynomials of degree c ”. Define

$$\pi_c(x) := |\{n \leq x : [n^c] \text{ is prime number}\}|.$$

Pjateckiĭ-Šapiro [9] has shown for $1 < c < \frac{12}{11} = 1.\overline{09}$ that

$$\pi_c(x) \sim x/\log x \quad (x \rightarrow \infty). \tag{1.1}$$

This range has been improved by many authors. A detailed history can be found in [10]. The best record known is due to Rivat and Sargos [11]: $1 < c < \frac{2817}{2426} \approx 1.16117$.

Jia and Rivat have proved, independently, that

$$\pi_c(x) \gg x/\log x \quad (x \geq x_0) \tag{1.2}$$

for $1 < c < \frac{7}{6} = 1.\overline{16}$ (Rivat [10]) and $1 < c \leq \frac{20}{17} \approx 1.17647$ (Jia [6, I], and independently Baker, Harman and Rivat [1]). This result has been improved to $1 < c < \frac{13}{11} = 1.\overline{18}$ by Jia [6, II] and to $1 < c \leq \frac{45}{38} \approx 1.18421$ by Kumchev [8]. The admissible range for c provides a measure of our progress towards the quadratic case, and it is interesting to extend it further.

We shall establish the following result.

THEOREM 1. *The inequality (1.2) is true for $1 < c \leq \frac{243}{205} \approx 1.18536$ and $x \geq x_0(c)$.*

We shall prove Theorem 1 by sieving the following two sequences

$$\mathcal{A} := \{m : m = [n^c], X \leq n^c < 2X\}, \quad \mathcal{B} := \{n : X \leq n < 2X\}.$$

For this it is necessary to present some standard notations in the sieve. For any finite set $\mathcal{E} \subset \mathbb{N}$, $d \in \mathbb{N}$ and $z \geq 2$, we define $\mathcal{E}_d := \{n \in \mathcal{E} : n \equiv 0 \pmod{d}\}$ and $S(\mathcal{E}, z) := |\{n \in \mathcal{E} : p|n \Rightarrow p \geq z\}|$. In order to prove Theorem 1, it suffices to show that

$$S(\mathcal{A}, (2X)^{1/2}) \gg \frac{\delta X}{\log X} \quad (X \geq X_0), \quad (1.3)$$

where and in the sequel we put $\gamma := 1/c$ and $\delta := X^{\gamma-1}(2^\gamma - 1)$.

Following Baker–Harman–Rivat [1], we shall repeatedly use the Buchstab identity

$$S(\mathcal{E}, z_1) = S(\mathcal{E}, z_1) - \sum_{z_2 \leq p < z_1} S(\mathcal{E}_p, p) \quad (z_2 \leq z_1) \quad (1.4)$$

together with asymptotic formulas in Lemmas 3.4 and 3.5 below. For some of them we can obtain asymptotic formulas by these Lemmas and the rest may be discarded; (see (4.7) below). In order to get such asymptotic formulas, we must treat the following exponential sums:

$$S_I := \sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} b_n e(h(mn)^\gamma) \right|,$$

$$S_{II} := \sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(h(mn)^\gamma) \right|,$$

where $e(t) := e^{2\pi i t}$, $|a_m| \leq 1$, $|b_n| \leq 1$, $H = X^{1-\gamma+\varepsilon}$, $MN = X$, ε is an arbitrarily small positive number and $m \sim M$ means $M < m \leq 2M$. In all previous works ([1], [7], [8], [10]), these two sums were estimated by the method of Fouvry–Iwaniec [2] and by the method of Heath-Brown [5].

Here we shall combine the method of Rivat–Sargos [11] with a ‘‘Weyl shift’’ (Lemma 2.5 of [3]) to improve that of Heath-Brown [5]; (see Theorem 2 below). This allows us to extend $X^{1-\gamma+\varepsilon} \leq N \leq X^{5\gamma-4-\varepsilon}$ of ([1], [7], [8], [10]) to $X^{1-\gamma+\varepsilon} \leq N \leq X^{(61\gamma-49)/11-\varepsilon}$, which is the key to proving Theorem 1. In addition the work of Sargos and Wu [12, Theorem 7] concerning the method of Fouvry–Iwaniec enables us to extend the range $\frac{16}{19} < \gamma < 1$ in Lemmas 3 and 4 of [8] to $\frac{37}{44} < \gamma < 1$. Finally we shall obtain (see (4.16) below)

$$S(\mathcal{A}, (2X)^{1/2}) \geq \{F(\gamma) + O(\varepsilon)\} \frac{\delta X}{\log X},$$

where $F(\gamma)$ is a very complicated function (for the definition of $F(\gamma)$, see (4.17) below). If we use γ_0 to denote the zero of $F(\gamma)$, then $c_0 := 1/\gamma_0$ will be our desired value. It seems quite tedious to find the exact value of γ_0 , but using Maple we have been able to obtain a good approximation guaranteed by a sufficient precision. We shall provide the details of these numerical computations in a separate section.

Throughout the paper (except for § 2), we suppose that $\frac{45}{38} < c \leq \frac{243}{205}$ and write $\gamma := 1/c$. We use ε to denote an arbitrarily small positive number and put $\eta := \varepsilon^2$. Finally we define $\mathcal{L} := \log X$. The notations of § 2 are independent of other sections.

2. Estimates on exponential sums. This section is devoted to investigating multiple exponential sums. The notations of this section are independent. Firstly we combine the method of Rivat–Sargos [11] with a “Weyl shift” [3, Lemma 2.5] to treat the following triple exponential sum of type II:

$$S := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_{m,n} e\left(X \frac{h^\alpha m^\beta n^\gamma}{H^\alpha M^\beta N^\gamma}\right). \tag{2.1}$$

THEOREM 2. *Let $k \geq 2$ and $K := 2^{k+1} - 2$. Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha(\alpha - 1) \beta \gamma (\gamma - 1) \neq 0$ and $(\gamma - 1)/(1 - \alpha) \neq 1, \dots, k$. Let $X > 0, H \geq 1, M \geq 1, N \geq 1, \mathcal{L} := \log(2 + XHMN), |a_h| \leq 1$ and $|b_{m,n}| \leq 1$. Let S be defined as in (2.1). If $X \leq H^2$, then*

$$S \ll \left\{ (X^K H^{4K-2k+2} M^{5K-k} N^{5K-3k})^{1/(6K-2k)} + (X^2 H^6 M^7 N^9)^{1/10} + (XH^2 M^4 N^2)^{1/4} + H(MN)^{1/2} + H^{3/4} MN + X^{-1/2} HMN \right\} \mathcal{L}.$$

Proof. In view of Theorem 2 in [2], we can suppose that $X \geq MN$. Let c_0, c_1, \dots be some suitable constants depending on α, β, γ at most and $\Xi := c_0 M^\beta N^\gamma$. Let $Q \geq 10$ be a parameter to be chosen later. By the Cauchy inequality and by Lemma 5 in [11] (or [13, Lemma 2.5]) with $x_i = m^\beta n^\gamma / \Xi$ and $z_i = b_{m,n} e(Xh^\alpha m^\beta n^\gamma / H^\alpha M^\beta N^\gamma)$, we have

$$\begin{aligned} |S|^2 &\ll H \sum_{h \sim H} \left| \sum_{m \sim M} \sum_{n \sim N} b_{m,n} e\left(X \frac{h^\alpha m^\beta n^\gamma}{H^\alpha M^\beta N^\gamma}\right) \right|^2 \\ &\ll HQ \sum_{m \sim M} \sum_{\tilde{m} \sim M} \sum_{n \sim N} \sum_{\tilde{n} \sim N} \left| \sum_{h \sim H} e\left(\frac{Xh^\alpha u}{H^\alpha M^\beta N^\gamma}\right) \right|, \end{aligned}$$

$|u| \leq \Xi/Q$

where $u = u_{\mathbf{m}}(\mathbf{n}) := m^\beta n^\gamma - \tilde{m}^\beta \tilde{n}^\gamma, \mathbf{m} := (m, \tilde{m})$ and $\mathbf{n} := (n, \tilde{n})$.

According to Lemma 1 in [2], the contribution of $|u| \leq \Xi \mathcal{L} / MN$ is $O(H^2 MNQ\mathcal{L})$. Thus after a splitting-up argument, we find, for some $\Delta \in (L/MN, 1/Q]$, that

$$|S|^2 \ll HQ\mathcal{L} \sum_{m \sim M} \sum_{\tilde{m} \sim M} \sum_{n \sim N} \sum_{\tilde{n} \sim N} \left| \sum_{h \sim H} e\left(\frac{Xh^\alpha u}{H^\alpha M^\beta N^\gamma}\right) \right| + H^2 MNQ\mathcal{L}. \tag{2.2}$$

$\Delta < |u|/\Xi \leq 2\Delta$

If $H' := X\Delta/H \leq \varepsilon$, by Theorem 2.1 in [3] and Lemma 1 in [2], the first term on the right-hand side of (2.2) is $\ll X^{-1} (HMN)^2 Q\mathcal{L} \ll H^2 MNQ\mathcal{L}$. Next we suppose that $H' \geq \varepsilon$. By applying Lemma 2.2 in [12], the innermost sum of (2.2) is equal to

$$(X\Delta)^{-1/2} H\tilde{b}_u \sum_{h' \in I'} \tilde{a}_{h'} e(f(h')) + O(\mathcal{L} + \Psi_1(u) + \Psi_2(u)),$$

where $\Psi_i(u) := \min \{ (X\Delta)^{-1/2} H, 1/\|c_i Xu/H\Xi\| \}$, $\|t\| := \min_{n \in \mathbb{Z}} |t - n|$, $|\tilde{a}_{h'}| \leq 1$, $|\tilde{b}_u| \leq 1$, $f(h') := c_3 X\Delta(u/\Delta\Xi)^{1/(1-\alpha)} (h'/H)^{\alpha/(\alpha-1)}$ and I' is a subinterval of $[H', 2H']$. Inserting into (2.2), we deduce, for some $\Delta \in (\mathcal{L}/MN, 1/Q]$ and some $h' \asymp H'$, the following inequality:

$$|S|^2 \ll (X\Delta)^{-1/2} H^2 Q S(\Delta) + HQ\{T_1(\Delta) + T_2(\Delta)\} + \{H^2 MNQ + H(MN)^2\} \mathcal{L}, \tag{2.3}$$

where

$$S(\Delta) := \sum_{m \sim M} \sum_{\tilde{m} \sim M} \sum_{n \sim N} \sum_{\tilde{n} \sim N} \left| \sum_{h' \in I'} \tilde{a}_{h'} e(f(h')) \right|, \quad T_i(\Delta) := \sum_{m \sim M} \sum_{\tilde{m} \sim M} \sum_{n \sim N} \sum_{\tilde{n} \sim N} \Psi_i(u).$$

$\Delta < |u|/\Xi \leq 2\Delta$ $\Delta < |u|/\Xi \leq 2\Delta$

According to (2.13) in [13], we have

$$T_i(\Delta) \ll \{X^{1/2} \Delta M^2 N + X^{1/2} MN + \Delta(MN)^2\} \mathcal{L}^2. \tag{2.4}$$

Next we shall apply the method of [11] to treat $S(\Delta)$. For this we must smooth the coefficient of u . By the Cauchy inequality and a ‘‘Weyl shift’’ [3, Lemma 2.5], we have, for any $R \in (0, \varepsilon H']$, that

$$|S(\Delta)|^2 \ll (\Delta H')^2 (MN)^4 R^{-1} + \Delta H' (MN)^2 R^{-1} \sum_{1 \leq r \leq R} \sum_{h' \sim H'} |S_{h',r}(\Delta)|, \tag{2.5}$$

where

$$S_{h',r}(\Delta) := \sum_{m \sim M} \sum_{\tilde{m} \sim M} \sum_{n \sim N} \sum_{\tilde{n} \sim N} e(g(u)),$$

$\Delta < |u|/\Xi \leq 2\Delta$

$$g(u) := Yu^{1/(1-\alpha)} \text{ and } Y := c_3 X\Delta \{ (h' + r)^{\alpha/(\alpha-1)} - h'^{\alpha/(\alpha-1)} \} / H'^{\alpha/(\alpha-1)} (\Delta\Xi)^{1/(1-\alpha)}.$$

Clearly $c_4 \leq (m/\tilde{m})^{\beta/\gamma} \leq c_5$ for $m \sim M$ and $\tilde{m} \sim M$. We split the interval $I := [c_4, c_5]$ into $J := \lceil |I|/\varepsilon\Delta \rceil + 1$ subintervals I_j of length $\delta := |I|/J \asymp \varepsilon\Delta$. For each $d \in \mathbb{N}$, we put

$$\mathcal{M}_j(d) := \{ \mathbf{m} : m \sim M, \tilde{m} \sim M, (m, \tilde{m}) = d, (m/\tilde{m})^{\beta/\gamma} \in I_j \},$$

and for each $\mathbf{m} \in \mathcal{M}_j(d)$ we define $\mathcal{N}(\mathbf{m}) := \{ \mathbf{n} : n \sim N, \tilde{n} \sim N, \Delta < |u|/\Xi \leq 2\Delta \}$. Thus

$$S_{h',r}(\Delta) = \sum_{d \leq 2M} \sum_{j \leq J} \sum_{\mathbf{m} \in \mathcal{M}_j(d)} |S_{h',r}^{(j)}(\Delta, \mathbf{m})|, \quad S_{h',r}^{(j)}(\Delta, \mathbf{m}) := \sum_{\mathbf{n} \in \mathcal{N}(\mathbf{m})} e(g(u)). \tag{2.6}$$

We pick $a_j/b_j \in I_j \cup \mathbb{Q}$ with b_j least and subject to $b_j \geq \delta^{-1/2}$. For $d_j \in \mathbb{Z}$ we define $\Gamma_{d_j} := \{ (x, y) \in \mathbb{R}^2 : y = (a_j x - d_j)/b_j \}$. Clearly when d_j runs over \mathbb{Z} , $\mathbb{Z}^2 \cap \Gamma_{d_j}$ constitutes a partition of \mathbb{Z}^2 . Hence we can write

$$S_{h',r}^{(j)}(\Delta, \mathbf{m}) \ll \sum_{d_j \in \mathbb{Z}} \left| \sum_{\mathbf{n} \in \mathcal{N}(\mathbf{m}) \cap \Gamma_{d_j}} e(g(u)) \right|. \tag{2.7}$$

Let (n_0, \tilde{n}_0) be the starting point of $\mathcal{N}(\mathbf{m}) \cap \Gamma_{d_j} \neq \emptyset$. Then $b_j \tilde{n}_0 = a_j n_0 - d_j$. Now we write the parameter equation of this straight line as follows:

$$n = b_j l + n_0, \quad \tilde{n} = a_j l + \tilde{n}_0 \quad (0 \leq l \ll N/b_j \asymp N/a_j).$$

Define

$$u(t) := m^\beta (b_j t + n_0)^\gamma - \tilde{m}^\beta (a_j t + \tilde{n}_0)^\gamma, \quad F(t) := g(u(t)).$$

From (2.7) and Lemma 2.7 in [13], we deduce that

$$S_{h',r}^{(j)}(\Delta, \mathbf{m}) \ll \begin{cases} \Delta N b_j \left| \sum_{l \ll N/b_j} e(F(l)) \right| & \text{if } b_j \leq N, \\ \Delta N b_j & \text{if } b_j > N. \end{cases} \tag{2.8}$$

Similarly to Lemma 8 in [11] (or [13, Lemma 2.8]), we easily prove the following result.

LEMMA 2.1. *If $(\gamma - 1)/(1 - \alpha) \neq 0, 1, \dots, k - 1$, then*

$$u(t) = A(b_j t + n_0)^{\gamma-1} \{1 + O(\varepsilon + \Delta)\}, \tag{2.9}$$

$$F^{(i)}(t) = B(b_j t + n_0)^{(\gamma-1)/(1-\alpha)-k} b_j^k \{1 + O_k(\varepsilon + \Delta)\} \quad (0 \leq i \leq k), \tag{2.10}$$

where

$$A := \gamma \tilde{m}^\beta (a_j/b_j)^{\gamma-1} (d_j/b_j) \asymp \Delta M^\beta N, \\ B := \prod_{i=0}^{k-1} ((\gamma - 1)/(1 - \alpha) - i) Y A^{1/(1-\alpha)} \asymp Hr N^{-(\gamma-1)/(1-\alpha)}.$$

Applying Theorem 2.9* in [3] to the sum over l in (2.8) with $(F, N) = (Hr, N/b_j)$ yields

$$S_{h',r}^{(j)}(\Delta, \mathbf{m}) \ll \Delta N^2 \Theta_j,$$

where

$$\Theta_j := \begin{cases} \min \{1, \Theta(b_j)\} & \text{if } b_j \leq N, \\ b_j/N & \text{if } b_j > N, \end{cases}$$

and $\Theta(b_j) := (Hr N^{-k} b_j^k)^{2/(K-2)}$. Inserting into (2.6) leads to

$$S_{h',r}(\Delta) \ll \Delta N^2 \mathcal{L} \sum_{d \leq 2M} \sum_{j \leq J, b_j \leq N} \sum_{\mathbf{m} \in \mathcal{M}_j(d)} \Theta_j + \Delta N^2 \mathcal{L} \sum_{d \leq 2M} \sum_{j \leq J, b_j > N} \sum_{\mathbf{m} \in \mathcal{M}_j(d)} \Theta_j. \tag{2.11}$$

By using Lemma 2.6 in [13], we have

*Clearly the condition $I \subset [N, 2N]$ in this theorem can be relaxed to $I \subset (1, 2N]$.

$$\sum_{j \leq J, b_j > N} \Theta_j \ll \sum_{0 \leq i \ll \mathcal{L}} \sum_{j \leq J, 1 \leq b_j/2^i N \leq 2} b_j/N \ll \sum_{0 \leq i \ll \mathcal{L}} 2^{-i} (N\delta)^{-2} \ll (N\delta)^{-2}.$$

In addition since the points in $\mathcal{M}_j(d)$ are spaced by $\gg (d/M)^2$, we have $|\mathcal{M}_j(d)| \ll \delta(M/d)^2 + 1$. Thus the second member on the right-hand side of (2.11) is

$$\ll \Delta N^2 (\delta M^2 + M) (N\delta)^{-2} \mathcal{L} \ll (M^2 + \Delta^{-1} M) \mathcal{L}. \tag{2.12}$$

Similarly Lemma 2.6 in [13] allows us to deduce that

$$\sum_{j \leq J, b_j \leq N} \Theta_j \ll \sum_{0 \leq i \ll \mathcal{L}} \sum_{j \leq J, 1 \leq b_j/2^i \delta^{-1/2} \leq 2} \Theta(b_j) \ll \sum_{i \geq 0} 4^{-i} \delta^{-1} \Theta(2^i \delta^{-1/2}) \ll \delta^{-1} \Theta(\delta^{-1/2}).$$

Thus the first member on the right-hand side of (2.11) is

$$\begin{aligned} &\ll \Delta N^2 \mathcal{L} \sum_{d \leq \sqrt{\delta} M} \sum_{j \leq J, b_j \leq N} \sum_{\mathbf{m} \in \mathcal{M}_j(d)} \Theta(b_j) + \Delta N^2 \mathcal{L} \sum_{d > \sqrt{\delta} M} \sum_{j \leq J, b_j \leq N} \sum_{\mathbf{m} \in \mathcal{M}_j(d)} 1 \\ &\ll \Delta N^2 \mathcal{L} \sum_{d \leq \sqrt{\delta} M} \delta(M/d)^2 \sum_{j \leq J, b_j \leq N} \Theta(b_j) + \Delta N^2 \mathcal{L} \sum_{d > \sqrt{\delta} M} (M/d)^2 \\ &\ll \Delta (MN)^2 \Theta(\delta^{-1/2}) \mathcal{L} + \Delta \delta^{-1/2} MN^2 \mathcal{L} \\ &\ll \Delta (MN)^2 \Theta(\Delta^{-1/2}) \mathcal{L} + \Delta^{1/2} MN^2 \mathcal{L}. \end{aligned} \tag{2.13}$$

Inserting (2.12) and (2.13) into (2.11), we obtain

$$S_{H',r}(\Delta) \ll \{(\Delta^{K-k-2} H^2 M^{2K-4} N^{2K-2k-4} r^2)^{1/(K-2)} + \Delta^{1/2} MN^2 + M^2 + \Delta^{-1} M\} \mathcal{L}.$$

Inserting into (2.5) leads to the inequality

$$\begin{aligned} |S(\Delta)|^2 &\ll \{(\Delta^{2K-k-4} H'^{2K-4} H^2 M^{4K-8} N^{4K-2k-8} R^2)^{1/(K-2)} \\ &\quad + (\Delta H')^2 (MN)^4 R^{-1} + (\Delta^3 H'^4 M^6 N^8)^{1/2} + \Delta H'^2 M^4 N^2 + H'^2 M^3 N^2\} \mathcal{L}. \end{aligned}$$

Optimizing R over $(0, \varepsilon H']$ we get

$$\begin{aligned} |S(\Delta)|^2 &\ll \{(\Delta^{2K-k} H'^{2K} H^2 M^{4K} N^{4K-2k})^{1/K} \\ &\quad + \Delta^2 H' (MN)^4 + (\Delta^3 H'^4 M^6 N^8)^{1/2} + \Delta H'^2 M^4 N^2 + H'^2 M^3 N^2\} \mathcal{L}. \end{aligned}$$

Replacing H' by $X\Delta/H$ yields

$$\begin{aligned} |S(\Delta)|^2 &\ll \{(X^{2K} \Delta^{4K-k} H^{-2K+2} M^{4K} N^{4K-2k})^{1/K} + X \Delta^3 H^{-1} (MN)^4 \\ &\quad + (X^4 \Delta^7 H^{-4} M^6 N^8)^{1/2} + X^2 \Delta^3 H^{-2} M^4 N^2 + X^2 \Delta^2 H^{-2} M^3 N^2\} \mathcal{L}. \end{aligned} \tag{2.14}$$

Inserting (2.14) and (2.4) into (2.3) and using $\Delta \leq 1/Q$, we have

$$\begin{aligned} |S|^2 &\ll \{(X^K H^{2K+2} M^{4K} N^{4K-2k} Q^{-(K-k)})^{1/2K} \\ &\quad + (H^3 M^4 N^4)^{1/2} + (X^2 H^4 M^6 N^8 Q^{-1})^{1/4} + (XH^2 M^4 N^2)^{1/2} \\ &\quad + (XH^2 M^3 N^2 Q)^{1/2} + (XH^2 M^2 N^2 Q^2)^{1/2} + H^2 MNQ + H(MN)^2\} \mathcal{L}^2. \end{aligned}$$

Since $X \leq H^2$, we have $(XH^2M^2N^2Q^2)^{1/2} \leq H^2MNQ$. If $Q \geq Q_0 := \max\{10, XH^{-2}M\}$, then $(XH^2M^3N^2Q)^{1/2} \leq H^2MNQ$. Thus the previous estimation can be simplified as follows:

$$|S|^2 \ll \left\{ (X^K H^{2K+2} M^{4K} N^{4K-2k} Q^{-(K-k)})^{1/2K} + (X^2 H^4 M^6 N^8 Q^{-1})^{1/4} + H^2 MNQ + H^{3/2} M^2 N^2 + X^{1/2} HM^2 N + H(MN)^2 \right\} \mathcal{L}^2.$$

Now optimizing Q over $[Q_0, \infty)$ and noticing that $(XM^2N)^{1/2} \leq (XH^2M^4N^2)^{1/4}$, we obtain the required result. This completes the proof. \square

We also need another result on triple exponential sum of type II; see [12, Theorem 7].

LEMMA 2.2. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha\beta\gamma(\alpha - 1)(\alpha - 2) \neq 0$, $X > 0$, $H \geq 1$, $M \geq 1$, $N \geq 1$, $\mathcal{L} := \log(2 + XHMN)$, $|a_h| \leq 1$ and $|b_{m,n}| \leq 1$. Let S be defined as in (2.1). Then*

$$S \ll \left\{ (X^4 H^{15} M^{22} N^{22})^{1/26} + (XH^2 M^3 N^3)^{1/4} + H^{11/18} MN + H(MN)^{3/4} + X^{-1/2} HMN \right\} \mathcal{L}^2.$$

From Theorem 2 and Lemma 2.2, we immediately deduce the following result.

COROLLARY 1. *Let $\frac{37}{44} < \gamma < \frac{61}{71}$, $H = X^{1-\gamma+4\eta}$, $MN = X$, $|a_m| \leq 1$ and $|b_n| \leq 1$. If one of the following three conditions*

- (i) $X^{3-3\gamma+\varepsilon} \leq N \leq X^{3\gamma-2-\varepsilon}$,
- (ii) $X^{1-\gamma+\varepsilon} \leq N \leq X^{(61\gamma-49)/11-\varepsilon}$,
- (iii) $X^{(60-61\gamma)/11+\varepsilon} \leq N \leq X^{\gamma-\varepsilon}$

holds, then

$$\sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(h(mn)^\gamma) \right| \ll X^{1-5\eta}. \tag{2.15}$$

Proof. By using Lemma 2.2 with $(H, M, N) = (X/N, H_0, N)$ ($H_0 \leq H$), we see that (2.15) holds provided $\frac{36}{43} < \gamma < 1$ and $X^{3-3\gamma+\varepsilon} \leq N \leq X^{(22\gamma-15)/7-\varepsilon}$. Interchanging X/N and N , we find $X^{22(\gamma-1)/7+\varepsilon} \leq N \leq X^{3\gamma-2-\varepsilon}$. Since $22(\gamma - 1)/7 \leq 1/2 \leq (22\gamma - 15)/7$ for $\gamma > \frac{37}{44}$, this proves (2.15) provided (i) holds and $\frac{37}{44} < \gamma < 1$.

If (ii) holds and $\frac{5}{6} < \gamma < \frac{61}{71}$, we can get (2.15) by using Theorem 2 with $k = 5$ and $(H, M, N) = (X/N, H_0, N)$ ($H_0 \leq H$). Similarly if (iii) holds and $\frac{5}{6} < \gamma < \frac{61}{71}$, we can get (2.15) by using Theorem 2 with $k = 5$ and $(H, M, N) = (N, H_0, X/N)$ ($H_0 \leq H$). \square

We also need a result on triple exponential sums of type I [12, Theorem 8].

LEMMA 2.3. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha\beta\gamma(\gamma - 2)(\alpha + \gamma - 1)(\alpha + 2\gamma - 2) \neq 0$, $X > 0$, $H \geq 1$, $M \geq 1$, $N \geq 1$, $\mathcal{L} := \log(2 + XHMN)$, $|a_h| \leq 1$ and $|b_m| \leq 1$. Define*

$$S := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_m e\left(X \frac{h^\alpha m^\beta n^\gamma}{H^\alpha M^\beta N^\gamma}\right).$$

Then

$$S \ll \{(X^{13} H^{15} M^{22} N^4)^{1/26} + (X^2 H^2 M^3 N)^{1/4} + (X H^4 M^3 N)^{1/4} + (X^9 H^{11} M^{18})^{1/18} + HM + X^{-1} HMN\} \mathcal{L}^2.$$

COROLLARY 2. Let $\frac{37}{44} < \gamma < \frac{28}{33}$, $H = X^{1-\gamma+4\eta}$, $MN = X$, $|b_n| \leq 1$. If $N \leq X^{3\gamma-2-\varepsilon}$, then

$$\sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} b_n e(h(mn)^\gamma) \right| \ll X^{1-5\eta}. \tag{2.16}$$

Proof. By Lemma 2.3 with $(H, M, N) = (N, H_0, X/N)$ ($H_0 \leq H$), we see that (2.16) holds provided $N \leq X^{(22\gamma-13)/11-\varepsilon}$ and $\frac{9}{11} < \gamma < \frac{28}{33}$. Combining this and Corollary 1(i) and noticing that $(22\gamma - 13)/11 > 3 - 3\gamma$ for $\gamma > \frac{46}{55}$, we deduce the desired result. \square

3. An alternative sieve. The aim of this section is to describe simply the alternative sieve of Baker–Harman–Rivat [1]. We insert new results on exponential sums in § 2 into their alternative sieve. This allows us to improve all results there. Since the proof is very similar, we shall state our results. For more details see [1], [8] and [4].

Let $\frac{37}{44} < \gamma < \frac{28}{33}$, $\theta_0 := 6\gamma - 5$, $\theta_1 := 1 - \gamma$, $\theta_2 := (61\gamma - 49)/11$, $\theta_3 := 3 - 3\gamma$, $\theta_4 := 3\gamma - 2$, $\theta_5 := (60 - 61\gamma)/11$, $\theta_6 := \gamma$, $\phi_i := X^{\theta_i}$ for $i = 1, \dots, 6$ and $z := (2X)^{1/2}$. Define $I_i(\gamma) := [\phi_{2i-1} X^\varepsilon, \phi_{2i} X^{-\varepsilon}]$ for $i = 1, 2, 3$.

LEMMA 3.1. Let $|b_n| \ll X^\eta$ and $\frac{37}{44} < \gamma < \frac{28}{33}$. If $N \leq X^{\theta_4-\varepsilon}$, then

$$\sum_{n \leq N} b_n |\mathcal{A}_n| = \delta X \sum_{n \leq N} b_n/n + O(\delta X^{1-3\eta}).$$

Proof. We can proceed as in Lemma 6 in [1], by replacing Lemma 5 in [1] by Corollary 2. \square

LEMMA 3.2. Let $\frac{37}{44} < \gamma < \frac{61}{11}$, $|a_m| \ll X^{\eta/2}$ and $|b_n| \ll X^{\eta/2}$. If N lies in one of $I_i(\gamma)$ ($i = 1, 2, 3$), then

$$\sum_{\substack{m \sim M \\ mn \in \mathcal{A}}} \sum_{n \sim N} a_m b_n = \sum_{\substack{m \sim M \\ mn \in \mathcal{B}}} \sum_{n \sim N} a_m b_n \gamma (mn)^{\gamma-1} + O(\delta X^{1-3\eta}).$$

Proof. We can proceed as in Lemma 7 in [1], by replacing Lemma 4 in [1] by Corollary 1. \square

LEMMA 3.3. Suppose that $\frac{37}{44} < \gamma < \frac{28}{33}$ and $N \leq X^{\theta_4-\varepsilon}$. Then we have

$$\sum_{n \sim N} b_n S(\mathcal{A}_n, \phi_0) = \delta \sum_{n \sim N} b_n S(\mathcal{B}_n, \phi_0) \{1 + O(1/\mathcal{L})\} + O(\delta X^{1-2\eta}),$$

where $0 \leq b_n \ll X^\eta$ and $b_n = 0$ unless $p|n \Rightarrow p \geq \phi_0$.

Proof. This can be proved in the same way as in Lemma 12 in [1], by using Lemmas 3.2 and 3.3 in place of Lemmas 7 and 9 in [1]. □

LEMMA 3.4. *Let k be a given positive integer and let $\mathcal{D} \subset \{1, \dots, k\}$. Assuming that N lies in one of $I_i(\gamma)$ ($i = 1, 2, 3$), then*

$$\sum_{p_1} \cdots \sum_{p_k}^* S(\mathcal{A}_{p_1 \cdots p_k}, p_1) = \delta \sum_{p_1} \cdots \sum_{p_k}^* S(\mathcal{B}_{p_1 \cdots p_k}, p_1) \{1 + O(1/\mathcal{L})\} + O(\delta X^{1-2\eta}),$$

where $*$ means that (p_1, \dots, p_k) satisfy $X^\varepsilon \leq p_1 < p_2 < \dots < p_k$, $\prod_{j \in \mathcal{D}} p_j \asymp N$ together with no more than ε^{-1} further conditions which take the form $R \leq \prod_{j \in \mathcal{J}} p_j \leq S$.

Proof. We can proceed as in Lemma 11 in [1], by replacing Lemma 7 in [1] by Lemma 3.2. □

4. Decomposition of $S(\mathcal{A}, z)$. As described in § 1, we repeatedly use the Buchstab identity (1.4) together with asymptotic formulas in Lemmas 3.4 and 3.5 to decompose $S(\mathcal{A}, z)$ such that the loss (i.e. the sums discarded) is as small as possible. Henceforth the letters p, q, r, s always denote prime numbers.

By (1.4), we first write

$$\begin{aligned} S(\mathcal{A}, z) &= S(\mathcal{A}, \phi_0) - \sum_{\phi_0 \leq p < \phi_1} S(\mathcal{A}_p, p) - \sum_{\phi_1 \leq p < \phi_2} S(\mathcal{A}_p, p) \\ &\quad - \sum_{\phi_2 \leq p < \phi_3} S(\mathcal{A}_p, p) - \sum_{\phi_3 \leq p < z} S(\mathcal{A}_p, p) \\ &=: S_1 - S_2 - S_3 - S_4 - S_5, \end{aligned} \tag{4.1}$$

where S_1 can be evaluated asymptotically by Lemma 3.3 and S_3, S_5 by Lemma 3.4.

Applying three times (1.4) to S_2 yields

$$\begin{aligned} S_2 &= \sum_{\phi_0 \leq p < \phi_1} S(\mathcal{A}_p, \phi_0) - \sum_{\phi_0 \leq q < p < \phi_1} \sum S(\mathcal{A}_{pq}, \phi_0) + \sum_{\phi_0 \leq r < q < p < \phi_1} \sum \sum S(\mathcal{A}_{pqr}, \phi_0) \\ &\quad - \sum_{\phi_0 \leq s < r < q < p < \phi_1} \sum \sum \sum S(\mathcal{A}_{pqrs}, s) - \sum_{\phi_0 \leq s < r < q < p < \phi_1} \sum \sum \sum S(\mathcal{A}_{pqrs}, s) \\ &\quad \quad \quad (\nabla_4) \quad \quad \quad (\nabla_4') \\ &=: S_6 - S_7 + S_8 - S_9 - S_{10}, \end{aligned} \tag{4.2}$$

where

- (∇_4) some subproducts of pqr s lie in one of the intervals $I_i(\gamma)$ ($i = 1, 2, 3$),
- (∇_4') no subproduct of pqr s lies in one of the intervals $I_i(\gamma)$ ($i = 1, 2, 3$).

We have asymptotic formulas for S_6, S_7, S_8 by Lemma 3.3 and for S_9 by Lemma 3.4.

In order to treat S_4 in (4.1), we first write, by (1.4),

$$\begin{aligned}
 S_4 &= \sum_{\phi_2 \leq p < \phi_3} S(\mathcal{A}_p, \phi_0) - \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_0 \leq q < \phi_1} S(\mathcal{A}_{pq}, q) \\
 &\quad - \sum_{\phi_1 \leq q < \phi_2 \leq p < \phi_3} S(\mathcal{A}_{pq}, q) - \sum_{\phi_2 \leq q < p < \phi_3} S(\mathcal{A}_{pq}, q) \\
 &=: S_{11} - S_{12} - S_{13} - S_{14}.
 \end{aligned}
 \tag{4.3}$$

Lemma 3.3 provides asymptotic formulae for S_{11} and Lemma 3.4 for S_{13} .

For S_{12} , we split the range of summation into three parts:

$$\begin{aligned}
 S_{12} &= \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 > \phi_4, pq \in [\phi_3, \phi_4]}} \sum_{\phi_0 \leq q < \phi_1} S(\mathcal{A}_{pq}, q) + \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 > \phi_4, pq \notin [\phi_3, \phi_4]}} \sum_{\phi_0 \leq q < \phi_1} S(\mathcal{A}_{pq}, q) + \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 \leq \phi_4}} \sum_{\phi_0 \leq q < \phi_1} S(\mathcal{A}_{pq}, q) \\
 &=: S_{15} + S_{16} + S_{17}.
 \end{aligned}$$

According to Lemma 3.4, we have asymptotic formulas for S_{15} .

By applying (1.4), we have

$$\begin{aligned}
 S_{16} &= \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 > \phi_4, pq \in [\phi_3, \phi_4]}} \sum_{\phi_0 \leq q < \phi_1} S(\mathcal{A}_{pq}, (2X/pq)^{1/2}) + \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 > \phi_4, pq \notin [\phi_3, \phi_4], (\nabla_3)}} \sum_{\phi_0 \leq q < \phi_1} \sum_{q \leq r < (2X/pq)^{1/2}} S(\mathcal{A}_{pqr}, r) \\
 &\quad + \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 > \phi_4, pq \in [\phi_3, \phi_4], (\nabla_3)'}} \sum_{\phi_0 \leq q < \phi_1} \sum_{q \leq r < (2X/pq)^{1/2}} S(\mathcal{A}_{pqr}, r) \\
 &=: S_{18} + S_{19} + S_{20},
 \end{aligned}$$

where

- (∇_3) some subproducts of pqr lie in one of the intervals $I_i(\gamma)$ ($i = 1, 2, 3$),
- (∇_3)' no subproduct of pqr lies in one of the intervals $I_i(\gamma)$ ($i = 1, 2, 3$).

We can evaluate asymptotically S_{19} by Lemma 3.3.

By applying (1.4) twice, we have

$$\begin{aligned}
 S_{17} &= \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 \leq \phi_4}} \sum_{\phi_0 \leq q < \phi_1} S(\mathcal{A}_{pq}, \phi_0) - \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 \leq \phi_4}} \sum_{\phi_0 \leq r < q < \phi_1} \sum_{\phi_0 \leq s < r} S(\mathcal{A}_{pqr}, \phi_0) \\
 &\quad + \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 \leq \phi_4, (\nabla_4)}} \sum_{\phi_0 \leq s < r < q < \phi_1} \sum_{\phi_0 \leq t < s} \sum_{\phi_0 \leq u < t} S(\mathcal{A}_{pqrs}, s) + \sum_{\substack{\phi_2 \leq p < \phi_3 \\ pq^2 \leq \phi_4, (\nabla_4)'}} \sum_{\phi_0 \leq s < r < q < \phi_1} \sum_{\phi_0 \leq t < s} \sum_{\phi_0 \leq u < t} S(\mathcal{A}_{pqrs}, s) \\
 &=: S_{21} - S_{22} + S_{23} + S_{24}.
 \end{aligned}$$

Lemma 3.3 provides asymptotic formulae for S_{21} , S_{22} and Lemma 3.4 for S_{23} .

Combining these we obtain

$$S_{12} = S_{15} + S_{18} + S_{19} + S_{20} + S_{21} - S_{22} + S_{23} + S_{24}, \tag{4.4}$$

where S_{15} , S_{19} , S_{21} , S_{22} , S_{23} can be evaluated asymptotically.

It remains to treat S_{14} in (4.3). We have

$$\begin{aligned}
 S_{14} &= \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq < \phi_3}} S(\mathcal{A}_{pq}, q) + \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq \in [\phi_3, \phi_4]}} S(\mathcal{A}_{pq}, q) + \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq > \phi_4}} S(\mathcal{A}_{pq}, q) \tag{4.5} \\
 &= \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq < \phi_3}} S(\mathcal{A}_{pq}, (2X/pq)^{1/2}) + \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq < \phi_3, (\nabla_3)}} \sum_{q \leq r < (2X/pq)^{1/2}} S(\mathcal{A}_{pq}, q) \\
 &\quad + \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq < \phi_3, (\nabla_3)'}} \sum_{q \leq r < (2X/pq)^{1/2}} S(\mathcal{A}_{pqr}, r) + \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq \in [\phi_3, \phi_4]}} S(\mathcal{A}_{pq}, q) + \sum_{\substack{\phi_2 \leq q < p < \phi_3 \\ pq > \phi_4}} S(\mathcal{A}_{pq}, q) \\
 &=: S_{25} + S_{26} + S_{27} + S_{28} + S_{29},
 \end{aligned}$$

where S_{26} and S_{28} can be evaluated asymptotically by Lemma 3.4.

Inserting (4.4) and (4.5) into (4.3) yields

$$\begin{aligned}
 S_4 &= S_{11} - S_{15} - S_{18} - S_{19} - S_{20} - S_{21} + S_{22} \tag{4.6} \\
 &\quad - S_{23} - S_{24} - S_{13} - S_{25} - S_{26} - S_{27} - S_{28} - S_{29},
 \end{aligned}$$

where S_j can be evaluated asymptotically, except for $j = 18, 20, 24, 25, 27, 29$.

Finally from (4.1), (4.2) and (4.6), we deduce that

$$\begin{aligned}
 S(\mathcal{A}, z) &= S_1 - S_6 + S_7 - S_8 + S_9 + S_{10} - S_3 - S_{11} + S_{15} + S_{18} + S_{19} + S_{20} \\
 &\quad + S_{21} - S_{22} + S_{23} + S_{24} + S_{13} + S_{25} + S_{26} + S_{27} + S_{28} + S_{29} - S_5,
 \end{aligned}$$

where S_j can be evaluated asymptotically, except for $j = 10, 18, 20, 24, 25, 27, 29$.

Obviously the same decomposition also holds for $S(\mathcal{B}, z)$; that is

$$\begin{aligned}
 S(\mathcal{B}, z) &= S'_1 - S'_6 + S'_7 - S'_8 + S'_9 + S'_{10} - S'_3 - S'_{11} + S'_{15} + S'_{18} + S'_{19} + S'_{20} \\
 &\quad + S'_{21} - S'_{22} + S'_{23} + S'_{24} + S'_{13} + S'_{25} + S'_{26} + S'_{27} + S'_{28} + S'_{29} - S'_5,
 \end{aligned}$$

where S'_j is defined similarly to S_j with the only difference that \mathcal{A} is replaced by \mathcal{B} .

Since $S_j = \delta S'_j \{1 + O(\varepsilon)\}$ except for $j = 10, 18, 20, 24, 25, 27, 29$, we can obtain

$$\begin{aligned}
 S(\mathcal{A}, z) &= \delta \{S(\mathcal{B}, z) - S'_{10} - S'_{18} - S'_{20} - S'_{24} - S'_{25} - S'_{27} - S'_{29}\} \{1 + O(\varepsilon)\} \\
 &\quad + S_{10} + S_{18} + S_{20} + S_{24} + S_{25} + S_{27} + S_{29},
 \end{aligned}$$

from which we deduce, in view of $S_j \geq 0$ for $j = 10, 18, 20, 24, 25, 27, 29$,

$$S(\mathcal{A}, z) \geq \delta \{S(\mathcal{B}, z) - S'_{10} - S'_{18} - S'_{20} - S'_{24} - S'_{25} - S'_{27} - S'_{29}\} \{1 + O(\varepsilon)\}. \tag{4.7}$$

By the prime number theorem, we have

$$S(\mathcal{B}, z) \sim \frac{X}{\log X}. \tag{4.8}$$

Next we shall give upper bounds for S'_j for $j = 10, 18, 20, 24, 25, 27, 29$. Define α_j by the relations $p := X^{\alpha_1}$, $q := X^{\alpha_2}$, $r := X^{\alpha_3}$ and $s := X^{\alpha_4}$. Let $\omega(t)$ be the Buchstab function.

A. Estimation of S'_{10} .

According to Lemma 5 in [8], we deduce that

$$S'_{10} = \{1 + O(\varepsilon)\} \sum_{\substack{\phi_0 \leq s < r < q < p < \phi_1 \\ (\nabla_4)'}} \sum \sum \sum \sum \frac{X}{pqrs \log s} \omega\left(\frac{\log(X/pqrs)}{\log s}\right).$$

By using the standard procedure for replacing sums over primes by integrals, we can prove that

$$S'_{10} = \frac{X}{\log X} \int_{\mathbb{D}_{10}} \omega\left(\frac{1 - \alpha_1 - \dots - \alpha_4}{\alpha_4}\right) \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\alpha_1 \alpha_2 \alpha_3 \alpha_4^2} \{1 + O(\varepsilon)\},$$

where

$$\mathbb{D}_{10} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \theta_0 \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1 \leq \theta_1, (C)_4\}$$

and

$$(C)_4 \quad \text{there is no subsum of } \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \text{ in } [\theta_1, \theta_2], [\theta_3, \theta_4].$$

Since $\omega(t) < 0.5672$ for $t \geq 2$ and $(1 - \alpha_1 - \dots - \alpha_4)/\alpha_4 \geq 2$ for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10}$, we find that

$$S'_{10} \leq f_{10}(\gamma) \frac{X}{\log X}, \tag{4.9}$$

where

$$f_{10}(\gamma) := 0.5672 \int_{\mathbb{D}_{10}} \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\alpha_1 \alpha_2 \alpha_3 \alpha_4^2}.$$

B. Estimation of S'_{18} .

According to the definition of S'_{18} , we have

$$S'_{18} = \sum_{\substack{\phi_2 \leq p < \phi_3, \phi_0 \leq q < \phi_1 \\ pq^2 > \phi_4, pq \notin [\phi_3, \phi_4]}} \sum_{X/pq < r \leq 2X/pq} \sum 1 = \sum_{\substack{\phi_2 \leq p < \phi_3, \phi_0 \leq q < \phi_1 \\ pq^2 > \phi_4, pq \notin [\phi_3, \phi_4]}} \sum \frac{\{1 + O(\varepsilon)\}X}{pq \log(X/pq)},$$

from which we deduce that

$$S'_{18} = \{f_{18}(\gamma) + O(\varepsilon)\} \frac{X}{\log X}, \tag{4.10}$$

where

$$f_{18}(\gamma) := \int_{\mathbb{D}_{18}} \frac{d\alpha_1 d\alpha_2}{\alpha_1 \alpha_2 (1 - \alpha_1 - \alpha_2)}$$

and

$$\mathbb{D}_{18} := \{(\alpha_1, \alpha_2) : \theta_2 \leq \alpha_1 \leq \theta_3, \theta_0 \leq \alpha_2 \leq \theta_1, \alpha_1 + 2\alpha_2 > \theta_4, \alpha_1 + \alpha_2 \notin [\theta_3, \theta_4]\}.$$

C. Estimation of S'_{20} .

Similarly to **A**, we have

$$S'_{20} \leq f_{20}(\gamma) \frac{X}{\log X}, \quad (4.11)$$

where

$$f_{20}(\gamma) := 0.5672 \int_{\mathbb{D}_{20}} \frac{d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1 \alpha_2 \alpha_3^2},$$

$$\mathbb{D}_{20} := \{(\alpha_1, \alpha_2, \alpha_3) : \theta_2 \leq \alpha_1 \leq \theta_3, \theta_0 \leq \alpha_2 \leq \theta_1, \alpha_2 \leq \alpha_3 \leq \frac{1}{2}(1 - \alpha_1 - \alpha_2), \alpha_1 + 2\alpha_2 \geq \theta_4, (C)_3\}$$

and

$$(C)_3 \quad \text{there is no subsum of } \alpha_1 + \alpha_2 + \alpha_3 \text{ in } [\theta_1, \theta_2], [\theta_3, \theta_4], [\theta_5, \theta_6].$$

D. Estimation of S'_{24} .

Similarly to **A**, we have

$$S'_{24} \leq f_{24}(\gamma) \frac{X}{\log X}, \quad (4.12)$$

where

$$f_{24}(\gamma) := 0.5672 \int_{\mathbb{D}_{24}} \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\alpha_1 \alpha_2 \alpha_3 \alpha_4^2},$$

$$\mathbb{D}_{24} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \theta_2 \leq \alpha_1 \leq \theta_3, \theta_0 \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \theta_1, \alpha_1 + 2\alpha_2 \leq \theta_4, (C)'_4\}$$

and

$$(C)'_4 \quad \text{there is no subsum of } \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \text{ in } [\theta_1, \theta_2], [\theta_3, \theta_4], [\theta_5, \theta_6].$$

E. Estimation of S'_{25} .

Similarly to **{B}**, we have

$$S'_{25} \leq \{f_{25}(\gamma) + O(\varepsilon)\} \frac{X}{\log X}, \quad (4.13)$$

where

$$f_{25}(\gamma) := \int_{\mathbb{D}_{25}} \frac{d\alpha_1 d\alpha_2}{\alpha_1 \alpha_2 (1 - \alpha_1 - \alpha_2)}$$

and

$$\mathbb{D}_{25} := \{(\alpha_1, \alpha_2) : \theta_2 \leq \alpha_2 \leq \alpha_1 \leq \theta_3, \alpha_1 + \alpha_2 \leq \theta_3\}.$$

F. Estimation of S'_{27} .

Similarly to **B**, we have

$$S'_{27} \leq \{f_{27}(\gamma) + O(\varepsilon)\} \frac{X}{\log X}, \tag{4.14}$$

where

$$f_{27}(\gamma) := \int_{\mathbb{D}_{27}} \frac{d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1 \alpha_2 \alpha_3 (1 - \alpha_1 - \alpha_2 - \alpha_3)},$$

$$\mathbb{D}_{27} := \{(\alpha_1, \alpha_2, \alpha_3) : \theta_2 \leq \alpha_2 \leq \alpha_1 \leq \theta_3, \alpha_2 \leq \alpha_3 \leq \frac{1}{2}(1 - \alpha_1 - \alpha_2), \alpha_1 + \alpha_2 \leq \theta_3, (C'_3)\}$$

and

$$(C'_3) \quad \text{there is no subsum of } \alpha_1 + \alpha_2 + \alpha_3 \text{ in } [\theta_3, \theta_4].$$

G. Estimation of S'_{29} .

Similarly to **A**, we have

$$S'_{29} \leq \{f_{29}(\gamma) + O(\varepsilon)\} \frac{X}{\log X}, \tag{4.15}$$

where

$$f_{29}(\gamma) := \int_{\mathbb{D}_{29}} \omega\left(\frac{1 - \alpha_1 - \alpha_2}{\alpha_2}\right) \frac{d\alpha_1 d\alpha_2}{\alpha_1 \alpha_2^2}$$

and

$$\mathbb{D}_{29} := \{(\alpha_1, \alpha_2) : \theta_2 \leq \alpha_2 \leq \alpha_1 \leq \theta_3, \alpha_1 + \alpha_2 \geq \theta_4, \alpha_1 + 2\alpha_2 \leq 1\}.$$

Inserting (4.8)–(4.15) into (4.7), we obtain

$$S(\mathcal{A}, z) \geq \{F(\gamma) + O(\varepsilon)\} \frac{\delta X}{\log X}, \tag{4.16}$$

where

$$F(\gamma) := 1 - f_{10}(\gamma) - f_{18}(\gamma) - f_{20}(\gamma) - f_{24}(\gamma) - f_{25}(\gamma) - f_{27}(\gamma) - f_{29}(\gamma). \tag{4.17}$$

5. End of the proof of Theorem 1: numerical calculations. The problem of computing γ_0 with a sufficient precision remains to compute multiple integrals over a simplex domain with sufficient precision. In general, this is a difficult question of numerical analysis. The Monte Carlo method provides an approximation but the precision is not guaranteed. Hence we choose a safer method: symbolic integration and numerical integral in 1 variable (which is safe using Simpson or another method), using Maple. Since the use of min and max is not safe with Maple, we shall manually split the domains of integration in order to simplify them. (We have slightly enlarged them, by the way.)

We have

$$\mathbb{D}_{10} \subset \mathbb{D}_{10}^A \cup \mathbb{D}_{10}^B \cup \mathbb{D}_{10}^C \cup \mathbb{D}_{10}^D \cup \mathbb{D}_{10}^E,$$

where

$$\begin{aligned} \mathbb{D}_{10}^A &:= \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 > \theta_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \theta_4\}, \\ \mathbb{D}_{10}^B &:= \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 > \theta_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < \theta_3\}, \\ \mathbb{D}_{10}^C &:= \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 < \theta_1, \alpha_1 + \alpha_2 > \theta_2, \alpha_2 + \alpha_3 > \theta_2\}, \\ \mathbb{D}_{10}^D &:= \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 < \theta_1, \alpha_1 + \alpha_2 > \theta_2, \alpha_2 + \alpha_3 < \theta_1\}, \\ \mathbb{D}_{10}^E &:= \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 < \theta_1, \alpha_1 + \alpha_2 < \theta_1\}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{D}_{10}^A &\subset \{\frac{1}{4}\theta_4 < \alpha_1 < \theta_1, \frac{1}{3}(\theta_4 - \theta_1) < \alpha_2 < \alpha_1, \frac{1}{2}\theta_2 < \alpha_3 < \alpha_2, \theta_2 - \alpha_3 < \alpha_4 < \alpha_3\}, \\ \mathbb{D}_{10}^B &\subset \{\frac{1}{2}\theta_2 < \alpha_1 < \theta_3 - \frac{3}{2}\theta_2, \frac{1}{2}\theta_2 < \alpha_2 < \theta_3 - \theta_2 - \alpha_1, \frac{1}{2}\theta_2 < \alpha_3 < \alpha_2, \theta_2 - \alpha_3 < \alpha_4 < \alpha_3\}, \\ \mathbb{D}_{10}^C &\subset \{\theta_2 - \frac{1}{2}\theta_1 < \alpha_1 < \theta_1, \theta_2 - \theta_1 + \theta_0 < \alpha_2 < \alpha_1, \theta_0 < \alpha_4 < \frac{1}{2}\theta_1, \alpha_4 < \alpha_3 < \theta_1 - \alpha_4\}, \\ \mathbb{D}_{10}^D &\subset \{\theta_2 - \frac{1}{2}\theta_1 < \alpha_1 < \theta_1, \frac{1}{3}\theta_2 < \alpha_2 < \theta_1 - \theta_0, \theta_2 - \theta_1 < \alpha_4 < \frac{1}{2}\theta_1, \alpha_4 < \alpha_3 < \frac{1}{2}\theta_1\}, \\ \mathbb{D}_{10}^E &\subset \{\frac{1}{3}\theta_2 < \alpha_1 < \theta_1 - \frac{1}{3}\theta_2, \frac{1}{3}\theta_2 < \alpha_2 < \alpha_1, \theta_2 - \theta_1 < \alpha_3 < \frac{1}{2}\theta_1, \theta_2 - \theta_1 < \alpha_4 < \alpha_3\}. \end{aligned}$$

We have

$$\mathbb{D}_{18} = \mathbb{D}_{18}^A \cup \mathbb{D}_{18}^B \cup \mathbb{D}_{18}^C,$$

where

$$\begin{aligned} \mathbb{D}_{18}^A &:= \{(\alpha_1, \alpha_2) \in \mathbb{D}_{18} : \alpha_1 + \alpha_2 < \theta_3, \alpha_2 < \frac{1}{2}(\theta_4 - \theta_2)\}, \\ \mathbb{D}_{18}^B &:= \{(\alpha_1, \alpha_2) \in \mathbb{D}_{18} : \alpha_1 + \alpha_2 < \theta_3, \alpha_2 > \frac{1}{2}(\theta_4 - \theta_2)\}, \\ \mathbb{D}_{18}^C &:= \{(\alpha_1, \alpha_2) \in \mathbb{D}_{18} : \alpha_1 + \alpha_2 > \theta_4\}. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathbb{D}_{18}^A &= \{\theta_0 < \alpha_2 < \frac{1}{2}(\theta_4 - \theta_2), \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_2\}, \\ \mathbb{D}_{18}^B &= \{\frac{1}{2}(\theta_4 - \theta_2) < \alpha_2 < \theta_1, \theta_2 < \alpha_1 < \theta_3 - \alpha_2\}, \\ \mathbb{D}_{18}^C &= \{\theta_0 < \alpha_2 < \theta_1, \theta_4 - \alpha_2 < \alpha_1 < \theta_3\}. \end{aligned}$$

We have

$$\mathbb{D}_{20} = \mathbb{D}_{20}^A \cup \mathbb{D}_{20}^B \cup \mathbb{D}_{20}^C \cup \mathbb{D}_{20}^D \cup \mathbb{D}_{20}^E \cup \mathbb{D}_{20}^F \cup \mathbb{D}_{20}^G \cup \mathbb{D}_{20}^H \cup \mathbb{D}_{20}^I \cup \mathbb{D}_{20}^J \cup \mathbb{D}_{20}^K \cup \mathbb{D}_{20}^L \cup \mathbb{D}_{20}^M \cup \mathbb{D}_{20}^N,$$

where

$$\begin{aligned} \mathbb{D}_{20}^A &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 < \frac{1}{3}(\theta_1 - \theta_3 + \theta_4)\}, \\ \mathbb{D}_{20}^B &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 > \frac{1}{3}(\theta_1 - \theta_3 + \theta_4)\}, \\ \mathbb{D}_{20}^C &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 > \theta_4\}, \\ \mathbb{D}_{20}^D &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 < \theta_3, \alpha_3 < \frac{1}{3}(2\theta_2 + \theta_3 - \theta_4)\}, \\ \mathbb{D}_{20}^E &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 < \theta_3, \alpha_3 > \frac{1}{3}(2\theta_2 + \theta_3 - \theta_4)\}, \\ \mathbb{D}_{20}^F &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 > \theta_4, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 < \frac{1}{2}\theta_1\}, \\ \mathbb{D}_{20}^G &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 > \theta_4, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 > \frac{1}{2}\theta_1\}, \\ \mathbb{D}_{20}^H &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 > \theta_4, \alpha_1 + \alpha_2 > \theta_4\}, \\ \mathbb{D}_{20}^I &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 < \theta_3, \alpha_1 + \alpha_3 < \theta_3\}, \\ \mathbb{D}_{20}^J &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 < \theta_3, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 < \frac{1}{2}\theta_2\}, \\ \mathbb{D}_{20}^K &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 < \theta_3, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 > \frac{1}{2}\theta_2, \alpha_1 < \theta_4 - \theta_2\}, \\ \mathbb{D}_{20}^L &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 < \theta_3, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 > \frac{1}{2}\theta_2, \alpha_1 > \theta_4 - \theta_2\}, \\ \mathbb{D}_{20}^M &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 > \theta_4, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 < 1 - 2\theta_2 - \theta_3\}, \\ \mathbb{D}_{20}^N &:= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 > \theta_4, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 > 1 - 2\theta_2 - \theta_3\}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{D}_{20}^A &\subset \{\theta_0 < \alpha_2 < \frac{1}{3}(\theta_1 - \theta_3 + \theta_4), \alpha_2 < \alpha_3 < 2\alpha_2 + \theta_3 - \theta_4, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\ \mathbb{D}_{20}^B &\subset \{\frac{1}{3}(\theta_1 - \theta_3 + \theta_4) < \alpha_2 < \frac{1}{2}\theta_1, \alpha_2 < \alpha_3 < \theta_1 - \alpha_2, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\ \mathbb{D}_{20}^C &\subset \{\theta_0 < \alpha_2 < \frac{1}{2}\theta_1, \alpha_2 < \alpha_3 < \theta_1 - \alpha_2, \theta_4 - \alpha_2 < \alpha_1 < \theta_3\}, \\ \mathbb{D}_{20}^D &\subset \{\frac{1}{2}\theta_2 < \alpha_3 < \frac{1}{3}(2\theta_2 + \theta_3 - \theta_4), \theta_2 - \alpha_3 < \alpha_2 < \alpha_3, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\ \mathbb{D}_{20}^E &\subset \{\frac{1}{3}(2\theta_2 + \theta_3 - \theta_4) < \alpha_3 < \theta_1, \frac{1}{2}(\alpha_3 + \theta_4 - \theta_3) < \alpha_2 < \alpha_3, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\ \mathbb{D}_{20}^F &\subset \{\theta_0 < \alpha_2 < \frac{1}{2}\theta_1, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_2, \frac{1}{2}(\theta_4 - \theta_3 + \theta_2) < \alpha_3 < \theta_1\}, \\ \mathbb{D}_{20}^G &\subset \{\frac{1}{2}\theta_1 < \alpha_2 < \theta_1 + \theta_3 - \theta_4, \theta_4 - \theta_1 < \alpha_1 < \theta_3 - \alpha_2, \frac{1}{2}(\theta_4 - \theta_3 + \theta_2) < \alpha_3 < \theta_1\}, \\ \mathbb{D}_{20}^H &\subset \{\frac{1}{2}\theta_2 < \alpha_3 < \theta_1, \theta_2 - \alpha_3 < \alpha_2 < \alpha_3, \theta_4 - \alpha_2 < \alpha_1 < \theta_3\}, \\ \mathbb{D}_{20}^I &\subset \{\theta_2 < \alpha_1 < \theta_3 - \theta_2, \frac{1}{2}(\theta_4 - \alpha_1) < \alpha_2 < \theta_1, \theta_2 < \alpha_3 < \theta_3 - \alpha_1\}, \\ \mathbb{D}_{20}^J &\subset \{\theta_0 < \alpha_2 < \frac{1}{2}\theta_2, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_2, \theta_2 < \alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\ \mathbb{D}_{20}^K &\subset \{\frac{1}{2}\theta_2 < \alpha_2 < \theta_1, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_4 - \theta_2, \theta_4 - \alpha_1 < \alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\ \mathbb{D}_{20}^L &\subset \{\frac{1}{2}\theta_2 < \alpha_2 < \theta_1, \theta_4 - \theta_2 < \alpha_1 < \theta_3 - \alpha_2, \theta_2 < \alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\ \mathbb{D}_{20}^M &\subset \{\theta_0 < \alpha_2 < 1 - 2\theta_2 - \theta_3, \theta_4 - \alpha_2 < \alpha_1 < \theta_3, \theta_2 < \alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\ \mathbb{D}_{20}^N &\subset \{1 - 2\theta_2 - \theta_3 < \alpha_2 < \theta_1, \theta_4 - \alpha_2 < \alpha_1 < 1 - 2\theta_2 - \alpha_2, \theta_2 < \alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}. \end{aligned}$$

We have

$$\mathbb{D}_{24} \subset \mathbb{D}_{24}^A \cup \mathbb{D}_{24}^B \cup \mathbb{D}_{24}^C \cup \mathbb{D}_{24}^D,$$

where

$$\mathbb{D}_{24}^A := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{24} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 + \alpha_4 > \theta_2\},$$

$$\mathbb{D}_{24}^B := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{24} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 + \alpha_4 < \theta_1\},$$

$$\mathbb{D}_{24}^C := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{24} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \theta_4\},$$

$$\mathbb{D}_{24}^D := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{24} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < \theta_3\}.$$

Hence

$$\mathbb{D}_{24}^A \subset \{\frac{1}{2}\theta_2 < \alpha_3 < \frac{1}{2}(\theta_3 - \theta_2), \alpha_3 < \alpha_2 < \theta_3 - \frac{3}{2}\theta_2, \theta_2 < \alpha_1 < \theta_3 - \theta_2, \frac{1}{2}(3\theta_2 - \theta_3) < \alpha_4 < \alpha_3\},$$

$$\mathbb{D}_{24}^B \subset \{\theta_2 < \alpha_1 < \theta_3 - \theta_2, \theta_2 - \theta_1 < \alpha_4 < \frac{1}{2}\theta_1, \theta_2 - \alpha_4 < \alpha_2 < \theta_1, \alpha_4 < \alpha_3 < \theta_1 - \alpha_4\},$$

$$\mathbb{D}_{24}^C \subset \{\theta_0 < \alpha_4 < \alpha_3 < \frac{1}{2}\theta_1, \frac{1}{3}\theta_2 < \alpha_2 < \theta_1 - \alpha_3, \theta_4 - \frac{3}{2}(\alpha_2 + \alpha_3) < \alpha_1 < \theta_3 - \alpha_2 - \alpha_3\},$$

$$\mathbb{D}_{24}^D \subset \{\theta_2 - \theta_1 < \alpha_3 < \frac{1}{2}\theta_1, \frac{1}{3}\theta_2 < \alpha_2 < \theta_1 - \alpha_3, \theta_2 < \alpha_1 < \theta_3 - \theta_2, \theta_2 - \theta_1 < \alpha_4 < \alpha_3\}.$$

We have

$$\mathbb{D}_{25} = \{(\alpha_1, \alpha_2) : \theta_2 < \alpha_2 < \frac{1}{2}\theta_3, \alpha_2 < \alpha_1 < \theta_3 - \alpha_2\}.$$

Also, we have

$$\mathbb{D}_{27} = \{(\alpha_1, \alpha_2, \alpha_3) : \theta_2 < \alpha_2 < \frac{1}{2}\theta_3, \alpha_2 < \alpha_1 < \theta_3 - \alpha_2, \alpha_2 < \alpha_3 < \theta_3 - \alpha_1\}.$$

Finally

$$\mathbb{D}_{29} \subset \mathbb{D}_{29}^A \cup \mathbb{D}_{29}^B \cup \mathbb{D}_{29}^C \cup \mathbb{D}_{29}^D \cup \mathbb{D}_{29}^E \cup \mathbb{D}_{29}^F,$$

where

$$\mathbb{D}_{29}^A := \{(\alpha_1, \alpha_2) : \frac{1}{2}\theta_4 < \alpha_1 < \frac{1}{2}(3\theta_4 - 1), \theta_4 - \alpha_1 < \alpha_2 < \alpha_1\},$$

$$\mathbb{D}_{29}^B := \{(\alpha_1, \alpha_2) : \frac{1}{2}(3\theta_4 - 1) < \alpha_1 < 1 - 3\theta_2, \frac{1}{3}(1 - \alpha_1) < \alpha_2 < \alpha_1\},$$

$$\mathbb{D}_{29}^C := \{(\alpha_1, \alpha_2) : 1 - 3\theta_2 < \alpha_1 < \frac{1}{3}, \theta_2 < \alpha_2 < \alpha_1\},$$

$$\mathbb{D}_{29}^D := \{(\alpha_1, \alpha_2) : \frac{1}{3} < \alpha_1 < \theta_3, \theta_2 < \alpha_2 < \frac{1}{2}(1 - \alpha_1)\},$$

$$\mathbb{D}_{29}^E := \{(\alpha_1, \alpha_2) : \frac{1}{2}(3\theta_4 - 1) < \alpha_1 < \theta_4 - \theta_2, \theta_4 - \alpha_1 < \alpha_2 < \frac{1}{3}(1 - \alpha_1)\},$$

$$\mathbb{D}_{29}^F := \{(\alpha_1, \alpha_2) : \theta_4 - \theta_2 < \alpha_1 < 1 - 3\theta_2, \theta_2 < \alpha_2 < \frac{1}{3}(1 - \alpha_1)\}.$$

(For the integrals on \mathbb{D}_{29}^E and \mathbb{D}_{29}^F we make use of $(t\omega(t))' = \omega(t - 1)$ and $\omega(t) \leq 0.5644$ for $t \geq 3$.)

Using these domains we obtain $F(\frac{205}{243}) > 0.0003$. Hence it suffices to choose $\gamma \geq \frac{205}{243}$, i.e. $c \leq \frac{243}{205}$.

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REFERENCES

1. R. C. Baker, G. Harman and J. Rivat, Primes of the form $[n^c]$, *J. Number Theory* **50** (1995), 261–277.
2. E. Fouvry and H. Iwaniec, Exponential sums with monomials, *J. Number Theory* **33** (1989), 311–333.
3. S. W. Graham and G. Kolesnik, *Van der Corput's method of exponential sums* (Cambridge University Press, 1991).
4. G. Harman, On the distribution of αp modulo one II, *Proc. London Math. Soc.* (3) **72** (1996), 241–260.
5. D. R. Heath-Brown, The Pjateckiĭ–Šapiro prime number theorem, *J. Number Theory* **16** (1983), 242–266.
6. H. Iwaniec, On the presentation of almost-primes by quadratic polynomials, *Invent. Math.* **47** (1978), 171–188.
7. C. H. Jia, On Piatetskiĭ–Šapiro's prime number theorem I, *Chinese Ann. Math. Ser. B* **15** (1994), 9–22; II, *Science in China* **36** (1993), 913–926.
8. A. Kumchev, On the distribution of prime numbers of the form $[n^c]$, *Glasgow Math. J.* **41** (1999), 85–102.
9. I. I. Pjateckiĭ–Šapiro, On the distribution of prime numbers in sequences of the form $[f(n)]$, *Mat. Sb.* **33** (1953), 559–566.
10. J. Rivat, *Autour d'un théorème de Pjateckiĭ–Šapiro: (nombres premiers dans la suite $[n^c]$)*, Thèse de Doctorat (Université Paris–Sud, 1992).
11. J. Rivat and P. Sargos, Nombres premiers de la forme $[n^c]$, Preprint.
12. P. Sargos and J. Wu, Multiple exponential sums with monomials and their applications in number theory, *Acta Math. Hungar.* **87** (4) (2000), 333–354.
13. J. Wu, The square-free and cube-free divisor problems, Preprint.