THE FITTING SUBGROUP OF A LINEAR SOLVABLE GROUP

JOHN D. DIXON

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1. Introduction

Let G be a group. The *Fitting subgroup* $F(G)$ of G is defined to be the set union of all normal nilpotent subgroups of G. Since the product of two normal nilpotent subgroups is again a normal nilpotent subgroup (see [10] p. 238), $F(G)$ is the unique maximal normal, locally nilpotent subgroup of G. In particular, if G is finite, then $F(G)$ is the unique maximal normal nilpotent subgroup of G. If G is a nontrivial solvable group, then clearly $F(G) \neq 1$.

The principal result of this paper is the following theorem.

THEOREM 1. *Let G be a completely reducible solvable subgroup of the general linear group $GL(n, \mathscr{F})$ over an algebraically closed field $\mathscr{F}$. Then*

$$|G : F(G)| \leqq a^{-1}b^{n}$$

where $a = 2.3^{\frac{1}{3}} = 2.88 \cdots$ and $b = 2.3^{\frac{2}{3}} = 4.16 \cdots$.

Moreover the bound is attained by some finite solvable group over the complex field whenever $n = 2.4^{k}$ $(k = 0, 1, \cdots)$.

NOTE. When G is finite and $\mathscr{F}$ is perfect, then the condition "algebraically closed" is unnecessary since the field may be extended without affecting the complete reducibility. See [3] Theorem (70.15).

A theorem of A. I. Mal'cev shows that there is a bound β_n such that each completely reducible linear solvable group of degree n has a normal abelian subgroup of index at most β_n. (See [5].) The previous estimates for β_n are by no means precise, but Theorem 1 allows us to give quite a precise estimate of the corresponding bound for a *subnormal* abelian subgroup.

THEOREM 2. *There is a constant c such that each completely reducible solvable subgroup G of $GL(n, \mathscr{F})$ (over an algebraically closed field $\mathscr{F}$) has a subnormal abelian subgroup A with*

$$|G : A| \leqq (2a)^{-1}c^{n}.$$

The best value for c lies between $\sqrt{2b}$ and $2b$. (a and b are defined in Theorem 1).

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The proofs of these theorems require the following result which is of interest in itself.

THEOREM 3. *Let $\sigma(n)$ and $\nu(n)$ denote the largest orders of subgroups of the symmetric group S_n which are solvable and nilpotent, respectively. Then, for each $n \geqq 1$,*

$$\sigma(n) \leqq a^{n-1} \quad and \quad \nu(n) \leqq 2^{n-1}.$$

The bounds are attained in the former case whenever $n = 4^k$, and in the latter case whenever $n = 2^k$ $(k = 0, 1, \cdots)$.

NOTE. It will be clear from the proof how to prove the corresponding bound $\alpha(n)$ for abelian subgroups of S_n. We have $\alpha(n) \leqq 3^{n/3}$ with equality whenever $n = 3k$ $(k = 1, 2, \cdots)$. (See [2]).

2. The proof of theorem 3

The proofs of the inequalities in Theorem 3 are similar in the two cases. We shall give the proof for $\sigma(n)$, and it will be evident how to modify this proof (with substantial simplifications in (iii)) to give one for $\nu(n)$.

We proceed by induction, and note that the result holds if $n = 1$. Let G be a solvable subgroup of order $\sigma(n)$ in S_n $(n \geqq 2)$, and consider three cases.

(i) *Suppose G is intransitive.* If $\Omega_1, \cdots, \Omega_k$ are the orbits of G, then $G_i = \{x\,|\Omega_i|\, x \in G\}$ is a solvable permutation group on $|\Omega_i| = n_i$ symbols $(i = 1, \cdots, k)$. The mapping

$$x \to (x\,|\Omega_1, \cdots, x|\,\Omega_k) \qquad (x \in G)$$

is an isomorphism of G onto a subgroup of $G_1 \times \cdots \times G_k$. Hence, by the induction hypothesis,

$$|G| \leqq \prod_{i=1}^{k} |G_i| \leqq \prod_{i=1}^{k} a^{n_i-1} < a^{n-1}$$

since $n = n_1 + \cdots + n_k$.

(ii) *Suppose G is transitive but imprimitive.* Then $n = md$ where G has d blocks (sets of impritivity) Γ_i $(i = 1, \cdots, d)$ each containing m symbols. The set of elements in G which map each block into itself is a normal subgroup H of G, and G/H is isomorphic to a subgroup of S_d. Since H is intransitive, we have (as in (i))

$$|G| = |G/H|\,|H| \leqq \sigma(d)\sigma(m)^d \leqq a^{d-1}(a^{m-1})^d = a^{n-1}.$$

(iii) *Suppose G is primitive. Let A be a minimal normal subgroup of* G. Since G is solvable, A is an elementary abelian p-subgroup of order p^k, say, for some prime p. If G_1 is a stabilizer of G (i.e. the subgroup fixing a given symbol), then G_1 is maximal because G is primitive. Since G is transitive, G_1 contains no nontrivial normal subgroup of G, and so $G = G_1 A$. The centralizer $C(A)$ of A in G is normal in G, so $C(A) \cap G_1$ is normalized by both A and G_1. Thus $C(A) \cap G_1 = 1$. Thus $C(A) = A$ and $n = |G : G_1| = |A| = p^k$. It now follows that $G_1 \cong G/C(A)$ which is isomorphic to a subgroup of the automorphism group $Aut\ A$ (See [7] p. 50). Since A is elementary abelian

$$|Aut\ A| = (p^k-1)(p^k-p)\cdots(p^k-p^{k-1}) < p^{k^2}$$

(see [10] p. 112), and so

$$\sigma(p^k) = |G| = p^k|G_1| \leqq p^k\ |Aut\ A|.$$

Finally, by direct calculation

$$\begin{aligned}
&\sigma(p^k) < p^{k+k^2} \leqq a^{p^k-1} && (k \geqq 3, p^k \neq 2^3),\\
&\sigma(2^3) \leqq 2^3(2^3-1)(2^3-2)(2^3-2^2) < a^{2^3-1} && (p^k = 2^3),\\
&\sigma(p^2) \leqq p^2(p^2-1)(p^2-p) \leqq a^{p^2-1} && (k = 2),\\
\text{and}\quad &\sigma(p) \leqq p(p-1) \leqq a^{p-1} && (k = 1).
\end{aligned}$$

This concludes the proof that $\sigma(n) \leqq a^{n-1}$. (For the nilpotent case (iii) is trivial because a primitive nilpotent group is cyclic of prime order).

We conclude the proof of Theorem 3 by showing that the bounds are attained. This is easy for $\nu(n)$ because the Sylow 2-groups of S_{2^k} have order 2^{2^k-1}. In the solvable case we proceed as follows. Put $H_0 = 1$, and for $k \geqq 1$ partition $\{1, 2, \cdots, 4^k\}$ into four blocks $\Gamma_i = \{4s+i \mid s = 1, 2, \cdots, 4^{k-1}\}$ $(i = 1, 2, 3, 4)$. By induction on k we may suppose that there is a solvable subgroup H_{k-1} of order $a^{4^{k-1}-1}$ in $S_{4^{k-1}}$. We construct isomorphic copies $H_{k-1}^{(i)}$ of H_{k-1} acting on the blocks Γ_i. Then the group $H_k = S_4 \prod_{i=1}^{4} H_{k-1}^{(i)}$ is a solvable subgroup of order $24(a^{4^{k-1}-1})^4 = a^{4^k-1}$ in S_{4^k}.

3. The proof of the direct part of theorem 1

We shall use the following observations about the Fitting subgroup. If H is a normal subgroup of a group G, then the characteristic subgroup $F(H)$ of H is normal in G, and so $F(H) \subseteqq F(G)$. On the other hand, if G is contained in a group K, then $F(K) \cap G \subseteqq F(G)$ and so $|G : F(G)| \leqq |K : F(K)|$. Finally, if $G = G_1 \times \cdots \times G_k$, then $F(G) = F(G_1) \times \cdots \times F(G_k)$, and so $|G : F(G)| = \prod_{i=1}^{k} |G_i : F(G_i)|$.

The proof of the direct part of Theorem 1 will consist of two main steps. The first is to reduce the problem to the case of a primitive group, and from that to a problem on solvable subgroups of the finite symplectic groups, (Lemma 1). The second step is the proof of Lemma 1.

Using induction on the degree n we first show that we may take G to be primitive (and irreducible). First of all, if G is reducible, then the underlying vector space $\mathscr{V} = \mathscr{U}_1 \oplus \mathscr{U}_2$ where $\mathscr{U}_1$, $\mathscr{U}_2$ are nontrivial invariant subspaces for G of degrees n_1 and n_2, say. Then the group $G_i = \{x|\mathscr{U}_i| \, x \in G\}$ is a solvable subgroup of $GL(n_i, \mathscr{F})$ for $i = 1, 2$. The mapping

$$x \to (x\,|\mathscr{U}_1, x|\,\mathscr{U}_2) \qquad (x \in G)$$

is an isomorphism of G onto a subgroup of $G_1 \times G_2$. Therefore, by the observations above and the induction hypothesis,

$$|G : F(G)| \leqq |G_1 : F(G_1)|\,|G_2 : F(G_2)| \leqq (a^{-1}b^{n_1})(a^{-1}b^{n_2}) < a^{-1}b^{n}.$$

Similarly, if G is imprimitive (but transitive), then $n = md$ $(d > 1)$ where the underlying vector space $\mathscr{V} = \mathscr{U}_1 \oplus \cdots \oplus \mathscr{U}_d$ where the $\mathscr{U}_i$ are nontrivial invariant subspaces for G transitively permuted under the action of G. Each $\mathscr{U}_i$ has dimension m. The set of elements of G mapping each $\mathscr{U}_i$ into itself is a normal subgroup N of G, and G/N is isomorphic to a subgroup of S_d. (See [3] Theorem (50.2).) Since N is intransitive, and N restricted to $\mathscr{U}_i$ has degree m, we find (as in the case above)

$$\begin{aligned} |G : F(G)| &\leqq |G : N|\,|N : F(N)| \leqq |G : N| \;\; (a^{-1}b^{m-1})^d \\ &\leqq a^{d-1}\,(a^{-1}b^{m-1})^d \quad \text{(by Theorem 3)} \\ &< a^{-1}b^n. \end{aligned}$$

Thus we may suppose that G is a primitive solvable group. We now apply some results of Suprunenko [8]. Consider a normal series

$$G \supseteqq A \supseteqq Z \supseteqq 1$$

of G where Z is the centre of G and A/Z is a maximal normal abelian subgroup of G/Z. Since G is primitive it has no noncentral normal abelian subgroup ([8] Lemma 7), and so Z is the unique maximal abelian subgroup of G. Therefore, by [8] Theorem 11 (with $F = Z$, $V = G$), G/A has the following form:

There is a divisor d of n whose cannonical decomposition into primes is $d = q_1^{l_1} \cdots q_k^{l_k}$ such that G/A is isomorphic to a subgroup of the direct product of k symplectic groups $Sp(2l_i, q_i)$ $(i = 1, \cdots, k)$.

Since $A' \subseteqq Z \subseteqq Z(A)$, $A \subseteqq F(G)$, and so $|G : F(G)| \leqq |G : A|$. Therefore Theorem 1 will follow from Suprunenko's theorem when we have proved the following lemma.

Lemma 1. *If q is a prime, then the largest order $s(q^l)$ of a solvable subgroup of Sp $(2l, q)$ is at most $a^{-1}b^{q^l}$ $(= 1, 2, \cdots)$.*

Proof. $|Sp(2l, q)| = (q^{2l}-1)(q^{2l-2}-1) \cdots (q^2-1)\, q^{l^2} < q^{l(2l+1)}$ (see [1] page 147). We consider several cases.

(a) *The cases* $q^l \neq 2^2, 2^3$ *or* 2^4. Direct calculation shows that

$$s(q^l) < q^{l(2l+1)} < 4^{q^l}/3 < a^{-1}b^{q^l} \; (q^l > 4,\, q^l \neq 2^3, 2^4)$$
$$s(3) \leqq (3^2-1)3 < a^{-1}b^3 = 72/a,$$

and
$$s(2) \leqq (2^2-1)2 = 6 = a^{-1}b^2$$

(b) *The cases* $q = 2^2$ *or* 2^4. In the former case, $Sp\,(4,2) \cong S_6$ ([4] Theorem 118 where Sp is denoted by SA). An analysis similar to that of (i) and (ii) of the proof of Theorem 3 then shows that $s(2^2) = \sigma(6) = 72 < a^{-1}b^{2^2}$ as required. In the second case we use the fact that $Sp\,(2l, 2)$ is simple when $l \geqq 3$ ([1] page 177). Suppose that H is a solvable subgroup of index h in $Sp(8, 2)$. Since $h > 1$, there is a representation of $Sp(8, 2)$ as a permutation group on the set of h cosets of H. Since $Sp(8, 2)$ is simple, the representation is faithful and so $|Sp(8,2)| = 2^{16}.3^4.5^2.7.51$ divides $|S_h| = h!$. Hence $h \geqq 51$, and so

$$s(2^4) \leqq |Sp(8,2)|/51 < 4^{16}/3 < a^{-1}b^{16}.$$

(c) *The final case* $q^l = 2^3$ requires a deeper analysis of the symplectic group. I am indebted to Professor G. E. Wall who first supplied the proof of this case.

By definition $Sp(6,2)$ is the set of all nonsingular linear operators on a 6-dimensional vector space $\mathscr{V}$ over $GF(2)$ with the property that they leave invariant a certain *alternate metric* $(\ ,\ )$ on $\mathscr{V}$. If $\mathscr{W}$ is a subspace of $\mathscr{V}$, then $\mathscr{W}^* = \{v \in \mathscr{V} \mid (v, w) = 0 \;(\forall w \in \mathscr{W})\}$ is the perpendicular subspace, and $\dim \mathscr{W} + \dim \mathscr{W}^* = \dim \mathscr{V} = 6$ (see [1] p. 117). Consider two cases.

If $\mathscr{W} \cap \mathscr{W}^* = 0$, then $\mathscr{V} = \mathscr{W} \oplus \mathscr{W}^*$ and the restriction of the alternate metric to $\mathscr{W}$ and $\mathscr{W}^*$ is nondegenerate. In this case $\dim \mathscr{W}$ is even, and (if $\mathscr{W} \neq 0$ or $\mathscr{V}$) the subgroup of $Sp(6,2)$ leaving $\mathscr{W}$ (and hence $\mathscr{W}^*$) invariant is isomorphic to $Sp(4,2) \times Sp(2,2)$ which has order $2^5.3^3.5$.

If $\mathscr{W} \subseteqq \mathscr{W}^*$, then from above $\dim \mathscr{W} = d \leqq 3$, and the alternate metric vanishes identically on $\mathscr{W}$, i.e. $\mathscr{W}$ is *totally isotropic*. Witt's theorem ([1] p. 121) shows that $Sp(6,2)$ permutes transitively the set of all totally isotropic subspaces of $\mathscr{V}$ of given dimension, and so the index in $Sp(6,2)$ of the subgroup leaving $\mathscr{W}$ invariant equals the number of totally isotropic subspaces of dimension d. This is

$$(2^6-1)/(2-1) = 3^2.7 \qquad \text{for } d = 1$$
$$(2^6-1)(2^5-2)/(2^2-1)(2^2-2) = 3.5.7 \qquad \text{for } d = 2$$
$$(2^6-1)(2^5-2)(2^4-2^2)/(2^3-1)(2^3-2)(2^3-2^2) = 3^3.5 \quad \text{for } d = 3.$$

Thus in all cases considered the order of the subgroup consisting of all elements of $Sp(6,2)$ which leave $\mathscr{W}$ invariant is at most $2^9.3^2.5$.

Let G be a solvable subgroup of order $s(2^3)$ in $Sp(6,2)$. If G is reducible, then a minimal invariant subspace $\mathscr{W}$ for G satisfies either $\mathscr{W} \cap \mathscr{W}^* = 0$ or $\mathscr{W} \subseteqq \mathscr{W}^*$, and so from the result above $|G| \leqq 2^9.3^2.5 \leqq 4^8/3 < a^{-1}b^8$. On the other hand, if G is irreducible, then we may apply a result of Huppert ([6] Satz 14) which states that any completely reducible solvable linear group of degree n over $GF(2^f)$ has its Sylow 2-groups of order at most $2^{f(n-1)}$. In our case this means that $|G|$ divides $2^5.3^4.5.7$. Since the latter number is less than $5a^{-1}b^8$, it will be sufficient to show, in order to prove this final case, that $35 \nmid |G|$. Suppose, on the contrary that $35 \mid |G|$. Since G is solvable, it has a Hall subgroup of order 35, and since each group of order 35 is cyclic (from the Sylow theorems), therefore G has an element x of order 35. Let $m(X)$ be the minimal polynomial for x. Then $m(X)$ divides the cyclotomic polynomial $\Phi_{35}(x)$. But $\Phi_{35}(x)$ is the product of two irreducible factors of degree 12 (over $GF(2)$), and so $m(X)$ has degree at least 12. But x acts on a 6-dimensional space and so the degree of $m(X)$ is at most 6. This contradiction shows that $35 \nmid |G|$ and completes the proof of this case.

This completes the proof of Lemma 1 and hence the proof of the direct part of Theorem 1.

NOTE. The second half of (c) above is different from the original proof of Professor Wall which was longer but more direct.

4. The limiting cases in theorem 1

We shall now show that the bound given in Theorem 1 is attained when $\mathscr{F}$ is the field of complex numbers and $n = 2.4^k$ $(k = 0, 1, \cdots)$. For $n = 2$ this is implied by the next lemma.

LEMMA 2. *The matrix group G generated by*

$$x = \frac{1}{\sqrt{2}}\begin{pmatrix} -\zeta & -\zeta \\ \zeta^{-1} & -\zeta^{-1} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}$$

(where $\zeta = (i+1)/\sqrt{2}$) is a solvable group of order 48 whose Frattini subgroup has order 8.

PROOF. Clearly $x^3 = y^8 = 1$. Putting $z = yx$ we find that $F = \langle y^2, z^2 \rangle$ is a normal subgroup of G because $y^{-1}z^2y = xz^2x^{-1} = (xy)^2 = z^2y^2$. F has order 8, and $G/F \cong \langle X, Y | X^3 = Y^3 = (XY)^2 = 1 \rangle \cong S_3$. Thus F is the largest normal 2-subgroup of G, and G is solvable of order 48. Since the Sylow 3-groups (of order 3) in G are not normal, $F = F(G)$.

COROLLARY. The matrix group G_0 generated by the elements of G together with any 2×2 scalar matrix $\alpha 1 \neq 0$ has $|G_0 : F(G_0)| = 6 = a^{-1}b^2$.

In the general case $n = 2.4^k$ we proceed as follows. Choose α in the preceding corollary as a primitive pth root of unity for a prime $p \neq 2$ or 3. Let N be the group of all diagonal block matrices diag $(x_1, \cdots, x_{4^k})$ with each $x_i \in G_0$. Let H be a group of $n \times n$ block permutation matrices (with blocks of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

such that H is isomorphic to a solvable permutation group of degree $n/2 = 4^k$ and order a^{4^k-1}. (See Theorem 3.) Then we assert that $G_k = HN$ is a solvable matrix group of degree n such that $|G_k : F(G_k)| = |G_k : F(N)| = a^{-1}b^n$.

We first note that $F(G_k) \supseteqq F(N)$ and

$$\begin{aligned} |G_k : F(N)| &= |G_k : N|\, |N : F(N)| = |H|\, |G_0 : F(G_0)|^{4^k} \\ &= a^{-1}b^n \ \ (\text{because } N \cong G_0 \times \cdots \times G_0 \ \ (4^k \text{ times})). \end{aligned}$$

Since G_k is obviously solvable, there only remains to show that $F(G_k) \subseteqq F(N)$. Let $x \in G_k$, $x \notin N$. Then the subgroup A of N consisting of all matrices diag $(\alpha_1 1, \cdots, \alpha_{4^k} 1)$ (with each α_i a pth root of unity) is clearly not centralized by x.

Thus the (abelian) Sylow p-group of the group B generated by A and x is not in the centre $Z(B)$. In particular, B is not a direct product of its Sylow subgroups, and so B is not nilpotent. Therefore $B \nsubseteq F(G_k)$, but $A \subseteqq F(N) \subseteqq F(G_k)$, and so $x \notin F(G_k)$. This proves that $F(G_k) \subseteqq N$, and so, by the definition of $F(N)$, $F(G_k) \subseteqq F(N)$ as required.

NOTE. In the same way we can construct examples over any algebraically closed field whose characteristic is not 2 or 3. In the exceptional cases it may be possible to sharpen the bound in Theorem 1.

5. The proof of theorem 2

From its definition $F(G)$ is locally nilpotent. By Clifford's theorem ([3] Theorem (49.2)) and the complete reducibility of G, $F(G)$ is completely reducible, and so $F(G)$ corresponds to a group of monomial matrices over a suitable basis for the underlying vector space. (See [8] Lemma 38.) The elements of $F(G)$ which correspond to diagonal matrices form a normal abelian subgroup A of $F(G)$, and the factor group $F(G)/A$ is isomorphic to a subgroup of S_n. (This latter isomorphism corresponds to the natural homomorphism of the group of monomial matrices onto the group of permutation matrices.) Since $F(G)$ is locally nilpotent, $F(G)/A$ is nilpotent and so $|F(G) : A| \leqq \nu(n) \leqq 2^{n-1}$ by Theorem 3. Therefore, by Theorem 1,

$$|G : A| = |G : F(G)|\, |F(G) : A| \leqq a^{-1}b^n 2^{n-1} = (2a)^{-1}(2b)^n.$$

This proves the existence of c and gives the upper bound for c.

On the other hand every subnormal nilpotent subgroup of G is contained in $F(G)$ (by the observations at the beginning of § 3.) Thus each subnormal abelian subgroup A_0 of G is in $F(G)$. In the group G_k defined in § 4 a largest subnormal abelian subgroup A_0 of $F(G_k) = F(N)$ has index 2^{4^k} in $F(G_k)$. But A_0 is a largest subnormal abelian subgroup of G_k, and

$$|G_k : A_0| = a^{-1} b^n 2^{n/2} = a^{-1} (\sqrt{2}b)^n.$$

This gives the lower bound on c.

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University of New South Wales
Kensington, N.S.W.

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