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Kaisa Matomäki and Xuancheng Shao

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Abstract

In this paper we prove two results concerning Vinogradov's three primes theorem with primes that can be called almost twin primes. First, for any m, every sufficiently large odd integer N can be written as a sum of three primes p_1, p_2 and p_3 such that, for each $i \in \{1, 2, 3\}$, the interval $[p_i, p_i + H]$ contains at least m primes, for some $H = H(m)$. Second, every sufficiently large integer $N \equiv 3 \pmod 6$ can be written as a sum of three primes p_1, p_2 and p_3 such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most two prime factors.

1. Introduction

The Hardy–Littlewood prime tuples conjecture says that, for any admissible set of k integers $\mathcal{H} = \{h_1, \ldots, h_k\}$, there are infinitely many values of n such that $n + h_1, \ldots, n + h_k$ are all prime. Here $\mathcal{H}$ is said to be admissible if it misses at least one residue class modulo p for every prime p. In particular, the twin prime conjecture is the special case when $\mathcal{H} = \{0, 2\}$.

Using an elaboration of the linear sieve method, Chen [Che73] proved that there are infinitely many primes p such that $p+2$ is the product of at most two primes (this property is traditionally denoted by $p + 2 = P_2$). If one insists on prime values, it is only recently that Zhang [Zha14], and subsequently Maynard [May15], made the breakthrough showing that there are infinitely many values of n for which at least two of $n + h_1, \ldots, n + h_k$ are prime, provided that k is large enough but fixed. Indeed, Maynard's argument shows that one can find m primes among $n + h_1, \ldots, n + h_k$ for any m, provided that k is large enough in terms of m. This result was proved independently by Tao in an unpublished work. We refer the reader to the excellent survey article [Gra15] for the main ideas behind these works.

Since the introduction of the Hardy–Littlewood circle method, there have been a flurry of results about solving linear equations in prime variables, by analyzing exponential sums over primes. In 1937, Vinogradov showed that all sufficiently large odd positive integers can be written as a sum of three primes. This establishes the ternary version of the Goldbach conjecture. In this paper, we prove the analogous statement for the special types of almost twin primes mentioned above.

Theorem 1.1. *For any positive integer m, there exist positive constants $H = H(m)$ and $N_0 = N_0(m)$ such that every odd integer $N \geqslant N_0$ can be written in the form $N = p_1 + p_2 + p_3$, where, for $i = 1, 2, 3$, p_i are primes such that the interval $[p_i, p_i + H]$ contains at least m primes.*

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In view of recent work of Helfgott [Hel15], one can in fact take $N_0 = 7$ above (after possibly increasing H).

Theorem 1.2. *Every large enough integer $N \equiv 3 \pmod 6$ can be written in the form $N = p_1 + p_2 + p_3$, where, for $i = 1, 2, 3$, p_i are primes such that $p_i + 2$ is a product of at most two primes.*

Related problems have been considered before. Green and Tao [GT06] showed that there are infinitely many three-term arithmetic progressions in the almost twin primes considered in Theorem 1.2, and this has been generalized in [Zho09] to handle k-term progressions for any fixed k. See [Pin15] for analogous results for the almost twin primes considered in Theorem 1.1. As we discuss in the next section, since the equation $N = p_1 + p_2 + p_3$ is not translation-invariant, for subsets of the primes the ternary Goldbach problem involves additional complications compared to the problem of finding three-term arithmetic progressions. For the ternary Goldbach problem, Matomäki [Mat09] previously showed that $N = p_1 + p_2 + p_3$ is solvable in primes with $p_1 + 2 = P_2$, $p_2 + 2 = P_2'$, and $p_3 + 2 = P_7$.

It is worth mentioning that a vast generalization of Vinogradov's theorem has been proved by Green and Tao [GT10], with a crucial ingredient from the work of Green, Tao, and Ziegler [GTZ12]. They introduced the concept of higher-order Fourier analysis, which allows one to handle all linear systems of finite complexity (that excludes the twin prime or the binary Goldbach case). We plan to return to a generalization of Theorem 1.1 in this direction in a future work.

2. Outline of proof

In this section we describe the main ingredients in the proofs of Theorems 1.1 and 1.2. The general strategy for proving both theorems follows closely the transference principle initiated in [Gre05]. Let f be the (weighted) indicator function of the considered subset of the primes, and let ν be a sieve majorant so that $f \leqslant \nu$ and that f has positive density in ν. The Fourier analytic transference principle in [Gre05] produces a dense model $\widetilde{f}$ of f, such that $0 \leqslant \widetilde{f} \leqslant 1$ and that $\widetilde{f}$ has positive average. Moreover,

$$\sum_{\substack{1 \leqslant n_1, n_2, n_3 \leqslant N \\ n_1 + n_2 + n_3 = N}} f(n_1) f(n_2) f(n_3) \approx \sum_{\substack{1 \leqslant n_1, n_2, n_3 \leqslant N \\ n_1 + n_2 + n_3 = N}} \widetilde{f}(n_1) \widetilde{f}(n_2) \widetilde{f}(n_3). \tag{2.1}$$

If we are instead looking for solutions of a homogeneous linear equation such as $n_1 + n_2 = 2n_3$, then the right-hand side above is bounded from below by Roth's theorem. In this way one can find arithmetic progressions in subsets of primes [GT06, Zho09, Pin15]. In our current case, the right-hand side above could vanish if, for example, $\widetilde{f}$ is supported on $[1, N/4]$ or if, writing $\|x\|$ for the distance from the nearest integer, we had $\|\sqrt{2}N\| > 3/10$ and $\widetilde{f}$ is supported on numbers n for which $\|\sqrt{2}n\| < 1/10$.

To get around this issue, we need to know more about the structure of $\widetilde{f}$. Examining the proof of the transference principle, one may observe that $\widetilde{f}$ is the convolution of f with a Bohr set. If we ensure that $\widetilde{f}$ is bounded below pointwise, then the right-hand side of (2.1) is certainly bounded below as well. This pointwise lower bound translates to the requirement that primes from the considered subset can be found in Bohr sets.

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2.1 Smooth Bohr cutoff

Given a cyclic group $G = \mathbb{Z}/N\mathbb{Z}$, a subset $\Omega \subseteq G$ and $\eta \in (0, 1/2]$, define the Bohr set

$$B = \mathrm{Bohr}(\Omega, \eta) = \{n \in G : \|\xi n/N\| \leqslant \eta \text{ for all } \xi \in \Omega\}.$$

For technical reasons, it is more convenient to study a smooth version of 1_B, whose Fourier spectrum has bounded size.

For $\eta \in (0, 1/2]$ and a positive integer D, let $S^+_{D,\eta}(x) : \mathbb{R}/\mathbb{Z} \to [0, 2]$ be the Selberg polynomial of degree D that majorizes the interval $[-\eta, \eta]$. The definition can be bound in [Mon94, ch. 1, formula 21^+] and is given in (3.1) below. The Selberg polynomial has a Fourier expansion

$$S^+_{D,\eta}(x) = \sum_{|k| \leqslant D} \widehat{S}^+_{D,\eta}(k) e(kx)$$

with $|\widehat{S}^+_{D,\eta}(k)| \leqslant 1/(D + 1) + \min\{2\eta, 1/|k|\}$ by [Mon94, ch. 1, formula (22)].

Definition 2.1 (Smooth Bohr cutoff). Given a cyclic group $G = \mathbb{Z}/N\mathbb{Z}$, a subset $\Omega \subseteq G$ and $\eta \in (0, 1/2]$, let $D = \lceil 4/\eta \rceil^{2|\Omega|}$ and define the smooth Bohr cutoff $\chi = \chi_{\Omega,\eta} : G \to \mathbb{R}_{\geqslant 0}$ by

$$\chi(n) := \prod_{\xi \in \Omega} S^+_{D,\eta}(\xi n/N).$$

Note that since $S^+_{D,\eta}(x)$ is a majorant of $1_{\|x\| \leqslant \eta}(x)$, we have the lower bound $\chi(n) \geqslant 1$ for $n \in \mathrm{Bohr}(\Omega, \eta)$.

Remark 2.2. Using the Selberg polynomials $S^+_{D,\eta}$ is not essential here, one could replace them for instance by the function $(\cos \pi x)^D$ for some large even D depending on η and $|\Omega|$. This way $\chi(n)$ would no longer be at least one in the Bohr set, but one could easily prove good enough variants of the lemmas we need.

2.2 A transference type result

Let $G = \mathbb{Z}/N\mathbb{Z}$. We use the standard notation $\mathbb{E}_{n \in G}$ to denote the average $N^{-1} \sum_{n \in G}$. For a function $f : G \to \mathbb{C}$, its Fourier transform is defined by

$$\widehat{f}(\xi) = \mathbb{E}_{n \in G} f(n) e\left(-\frac{\xi n}{N}\right),$$

and its L^1-norm is defined by

$$\|f\|_1 = \mathbb{E}_{n \in G} |f(n)|.$$

For two functions $f, g : G \to \mathbb{C}$, their convolution is defined by

$$f * g(t) = \mathbb{E}_{n \in G} f(n) g(t - n).$$

In § 4 we prove the following transference type result. It says that we can handle a non-homogeneous linear equation if we have some additional hypotheses about averages in Bohr sets.

THEOREM 2.3. *Let $G = \mathbb{Z}/N\mathbb{Z}$ for some large N, and let $f_1 : G \to \mathbb{R}_{\geqslant 0}$ be a function. Let $K \geqslant 1$ and $\delta > 0$ be parameters. There exists a Bohr cutoff $\chi = \chi_{\Omega,\eta}$ (depending on f_1) with $|\Omega| \ll_{K,\delta} 1$, $1 \in \Omega$, and $\eta = \eta(K,\delta) \in (0, 0.05)$, such that the following statement holds. Let $f_2, f_3 : G \to \mathbb{R}_{\geqslant 0}$ be functions satisfying*

$$f_i * \chi(t) \geqslant \delta \|\chi\|_1, \tag{2.2}$$

for every $t \in [N/4, N/2)$ and $i \in \{2, 3\}$. Suppose that

$$\sum_{0.1N \leqslant n \leqslant 0.4N} f_1(n) \geqslant \delta N, \tag{2.3}$$

and that

$$\sum_{\xi \in G} |\widehat{f}_i(\xi)|^{5/2} \leqslant K \tag{2.4}$$

for every $i \in \{1, 2, 3\}$. Then $f_1 * f_2 * f_3(N) \geqslant \delta^3/200$.

The artificial requirement $1 \in \Omega$ and the assumption that (2.2) holds only for $t \in [N/4, N/2)$ come from the way Theorem 2.3 will be applied. To avoid wrapping around issues, we will apply Theorem 2.3 with each f_i supported on $[N/4, N/2)$. If $1 \in \Omega$ and $\eta < 0.1$, then $B(\Omega, \eta) \subset (-0.1N, 0.1N)$, so that (2.2) can be expected to hold when $t \in [N/4, N/2)$.

We will see that the condition (2.4) for the types of almost twin primes we consider follows easily from the work of Green and Tao [GT06].

2.3 Almost twin primes in Bohr sets

To apply Theorem 2.3 to prove Theorems 1.1 and 1.2 in § 5, we need to verify the hypothesis (2.2) for the indicator functions of the types of almost twin primes we consider. This is achieved in Theorems 2.5 and 2.6, in statements of which we use the following definition.

DEFINITION 2.4. For a function $\chi : \mathbb{Z} \to \mathbb{C}$, we say that it has Fourier complexity at most M if χ can be written as a linear combination of at most M exponential phases:

$$\chi(n) = \sum_{i=1}^{M} b_i e(\alpha_i n),$$

for some $|b_i| \leqslant M$, and $\alpha_i \in \mathbb{R}/\mathbb{Z}$.

Note that since we do not request b_i to be non-zero, if χ is of Fourier complexity at most M, then it is of Fourier complexity at most M' for any $M' \geqslant M$. Note also that the smooth Bohr cutoff $\chi_{\Omega,\eta}$ in Definition 2.1 (extended to $\mathbb{Z}$ in the obvious manner) has Fourier complexity at most $O_{|\Omega|,\eta}(1)$.

THEOREM 2.5. *For any positive integer m, there exist a positive integer $k = k(m)$ and positive constants $\delta_0 = \delta_0(m)$ and $\rho = \rho(m)$ such that the following holds. Let $\chi : \mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ be a function with Fourier complexity at most M for some $M \geqslant 1$, and let $\varepsilon > 0$ be given. Let $W = \prod_{p \leqslant w} p$ with w large enough in terms of m, M and ε, and let $(b, W) = 1$. There exist non-zero distinct integers $h_1, \ldots, h_{k-1} = O_{m,M,\varepsilon}(1)$ with h_j positive for $j = 1, \ldots, m-1$, and a positive integer $N_0 = N_0(m, M, \varepsilon, w)$ such that, for every $N \geqslant N_0$ and $|t| \leqslant 5N$,*

$$\sum_{\substack{N \leqslant n < 2N \\ Wn+b \in \mathbb{P} \\ Wn+b+Wh_i \in \mathbb{P} \text{ for } i=1,\ldots,m-1 \\ p \mid \prod_{i=m}^{k-1}(Wn+b+Wh_i) \implies p \geqslant N^\rho}} \chi(t-n) \geqslant \delta_0 \frac{1}{(\log N)^k} \frac{W^k}{\varphi(W)^k} \left(\sum_{N \leqslant n < 2N} \chi(t-n) - \frac{N}{w^{1/3}} + O_m(\varepsilon N) \right).$$

Theorem 2.6. *There exists a positive constant δ_1 such that the following holds. Let $\chi : \mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ be a function with Fourier complexity at most M for some $M \geqslant 1$. Let $W = \prod_{p \leqslant w} p$ with w large enough in terms of M, and let $(b, W) = 1$. There exists a positive constant $N_0 = N_0(M, w)$ such that, for every $N \geqslant N_0$ and $|t| \leqslant 5N$,*

$$\sum_{\substack{N \leqslant n < 2N \\ Wn+b \in \mathbb{P} \\ Wn+b+2 = P_2 \\ p \mid Wn+b+2 \implies p \geqslant N^{1/100}}} \chi(t-n) \geqslant \delta_1 \frac{1}{(\log N)^2} \frac{W^2}{\varphi(W)^2} \Bigg(\sum_{N \leqslant n < 2N} \chi(t-n) - \frac{N}{w^{1/3}} \Bigg).$$

Let us briefly discuss the proofs of these results. In § 6 we shall state the results of Maynard and Chen saying that one can find almost twin primes in sets that are equidistributed in arithmetic progressions in certain precise senses. Bohr sets in general are not equidistributed but in § 7 we will show that it is enough to show variants of Theorems 2.5 and 2.6 that are more apt for applications of Maynard's and Chen's theorems. Then in §§ 9 and 10 we shall prove these variants using the Fourier expansion of the smooth Bohr cutoff discussed in § 3 as well as exponential sum estimates which we will state in § 8.

3. Smooth Bohr cutoff and its Fourier expansion

In this section we discuss a few basic properties of the Bohr cutoff $\chi = \chi_{\Omega,\eta}$ from Definition 2.1.

Lemma 3.1. *Given a cyclic group $G = \mathbb{Z}/N\mathbb{Z}$, a subset $\Omega \subseteq G$ and $\eta \in (0, 1/2]$, the smooth Bohr cutoff $\chi = \chi_{\Omega,\eta}$ has the following properties:*

(i) *we have the lower bound*
$$\|\chi\|_1 \geqslant (\eta/2)^{|\Omega|};$$

(ii) *if $n \notin \mathrm{Bohr}(\Omega, 2\eta)$, then*
$$|\chi(n)| \leqslant (\eta^2/8)^{|\Omega|}.$$

Proof. Part (i) follows from the observation that $\chi(n) \geqslant 1$ when $n \in \mathrm{Bohr}(\Omega, \eta)$, together with the lower bound $|\mathrm{Bohr}(\Omega, \eta)| \geqslant (\eta/2)^{|\Omega|} N$ from a standard pigeon-holing argument (see, e.g., [TV10, Lemma 4.20]). For part (ii) we can clearly assume that $\eta \leqslant 1/4$. Let us first give the precise definition of $S^+_{D,\eta}(x)$. For an integer $K \geqslant 1$, write $\Delta_K(x)$ for the Fejér kernel

$$\Delta_K(x) := \sum_{|k| \leqslant K} \left(1 - \frac{|k|}{K}\right) e(kx) = \frac{1}{K} \left(\frac{\sin \pi K x}{\sin \pi x}\right)^2.$$

Then Vaaler's polynomial $V_D(x)$ is defined as the trigonometric polynomial of degree D with

$$\begin{aligned} V_D(x) := {} & \frac{1}{D+1} \sum_{k=1}^{D} \left(\frac{k}{D+1} - \frac{1}{2}\right) \Delta_{D+1}\left(x - \frac{k}{D+1}\right) \\ & + \frac{1}{2\pi(D+1)} \sin 2\pi(D+1)x - \frac{1}{2\pi} \Delta_{D+1}(x) \sin 2\pi x. \end{aligned}$$

Finally,

$$S^+_{D,\eta}(x) := 2\eta + V_D(x-\eta) + V_D(-x-\eta) + \frac{1}{2D+2}(\Delta_{D+1}(x-\eta) + \Delta_{D+1}(-x-\eta)). \tag{3.1}$$

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Note that, writing $s(x)$ for the sawtooth function (so that $s(x) = \{x\} - 1/2$ if $x \notin \mathbb{Z}$ and $s(x) = 0$ if $x \in \mathbb{Z}$),

$$1_{\|x\| \leqslant \eta}(x) = 2\eta + s(x - \eta) + s(-x - \eta),$$

except when $x = \eta$ or $x = -\eta$. By a result of Vaaler [Vaa85, Theorem 18], we know that, for any x,

$$|V_D(x) - s(x)| \leqslant \frac{1}{2D+2} \Delta_{D+1}(x).$$

Hence,

$$\begin{aligned} |S^+_{D,\eta}(x) - 1_{\|x\| \leqslant \eta}(x)| &\leqslant \frac{2}{2D+2} (\Delta_{D+1}(x - \eta) + \Delta_{D+1}(-x - \eta)) \\ &\leqslant \frac{1}{(D+1)^2} \left(\frac{1}{(\sin \pi \|x - \eta\|)^2} + \frac{1}{(\sin \pi \|-x - \eta\|)^2} \right). \end{aligned}$$

If $\|x\| \geqslant 2\eta$, then we get

$$|S^+_{D,\eta}(x)| \leqslant \frac{2}{(D+1)^2} \cdot \frac{1}{(\sin \pi\eta)^2} \leqslant \frac{2}{(D+1)^2} \cdot \frac{1}{(2\eta)^2} \leqslant \frac{1}{\eta^2 D^2}.$$

Now, if $n \notin \mathrm{Bohr}(\Omega, 2\eta)$, then $\|\xi_0 n/N\| \geqslant 2\eta$ for some $\xi_0 \in \Omega$. Thus,

$$|\chi(n)| = |S^+_{D,\eta}(\xi_0 n/N)| \prod_{\xi \in \Omega \setminus \{\xi_0\}} |S^+_{D,\eta}(\xi n/N)| \leqslant \frac{2^{|\Omega|}}{\eta^2 D^2}.$$

The conclusion then follows by our choice $D = \lceil 4/\eta \rceil^{2|\Omega|}$. □

The following lemma gives the Fourier expansion of a function of bounded Fourier complexity in a convenient form. In particular, it allows us to separate the phases giving 'major arc' contribution from those giving 'minor arc' contribution.

LEMMA 3.2. *Let $A, M \geqslant 1$, and let $B = A(3M)^M$. Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a function with Fourier complexity at most M, and let W be a positive integer. Then for any large N, we may write*

$$\chi(n) = \sum_{i=1}^{M} b_i e\left(\left(W \frac{a_i}{q_i} + \beta_i \right) n \right)$$

for some $|b_i| \leqslant M$, $0 \leqslant a_i < q_i \leqslant N/(\log N)^{100B}$, $(a_i, q_i) = 1$, and $|\beta_i| \leqslant W(\log N)^{100B}/(q_i N)$. Moreover, there exists a positive integer $Q \leqslant (\log N)^B$ such that, for each $1 \leqslant i \leqslant M$, either $q_i \mid Q$ or $q_i/(q_i, Q^2) > (\log N)^A$.

Proof. By the definition of Fourier complexity in Definition 2.4, we may write

$$\chi(n) = \sum_{i=1}^{M} b_i e(\alpha_i n),$$

for some $|b_i| \leqslant M$ and $\alpha_i \in \mathbb{R}/\mathbb{Z}$. By the Dirichlet approximation theorem, for each $1 \leqslant i \leqslant M$, there exist integers $q_i \in [1, N/(\log N)^{100B}]$ and a_i such that $(a_i, q_i) = 1$ and

$$\left| \frac{\alpha_i}{W} - \frac{a_i}{q_i} \right| \leqslant \frac{(\log N)^{100B}}{q_i N}.$$

This gives the desired Fourier expansion of χ, apart from the existence of Q mentioned in the last sentence of the statement.

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To define Q, let $\mathcal{Q} = \{q_1, \dots, q_M\}$. Take $Q_0 = 1$ and for $i \geqslant 0$ define

$$Q_{i+1} = \prod_{\substack{q \in \mathcal{Q} \\ q/(q, Q_i^2) \leqslant (\log N)^A}} q.$$

There is some $I \leqslant |\mathcal{Q}| = M$ such that $Q_{I+1} = Q_I$. We claim that $Q = Q_I$ satisfies the desired properties. Indeed, for $q \in \mathcal{Q}$, if $q \nmid Q$, then $q \nmid Q_{I+1}$ so that $q/(q, Q_I^2) > (\log N)^A$ by the definition of Q_{I+1}. Furthermore, it is easy to see from the construction that

$$Q_{i+1} \leqslant (Q_i^2 (\log N)^A)^M.$$

Thus, a simple induction reveals that $Q_i \leqslant (\log N)^{A \cdot 3^i M^i}$, so that $Q \leqslant (\log N)^B$. □

This lemma can be thought of as a very special case of the general factorization theorem for nilsequences [GT12, Theorem 1.19].

4. The transference-type result

In this section we prove Theorem 2.3. Let $\eta, \varepsilon > 0$ be small enough depending on K and δ, and take

$$\Omega = \{\xi \in G : |\widehat{f_1}(\xi)| \geqslant \varepsilon\} \cup \{1\}.$$

By (2.4), we have $|\Omega| \leqslant \varepsilon^{-5/2} K + 1$. Let $\chi = \chi_{\Omega, \eta}$ be the smooth Bohr cutoff from Definition 2.1. For $i \in \{2, 3\}$, define $g_i, h_i : G \to \mathbb{R}$ by setting

$$g_i = \frac{1}{\|\chi\|_1} f_i * \chi, \quad h_i = f_i - g_i.$$

Hence,

$$\widehat{g_i} = \frac{1}{\|\chi\|_1} \widehat{f_i} \cdot \widehat{\chi} \quad \text{and} \quad \widehat{h_i} = \widehat{f_i} \left(1 - \frac{\widehat{\chi}}{\|\chi\|_1}\right). \tag{4.1}$$

In particular, using the trivial bound $|\widehat{\chi}(\xi)| \leqslant \|\chi\|_1$ we obtain

$$\sum_{\xi \in G} |\widehat{g_i}(\xi)|^{5/2} \leqslant K \quad \text{and} \quad \sum_{\xi \in G} |\widehat{h_i}(\xi)|^{5/2} \leqslant 2^{5/2} K. \tag{4.2}$$

We write

$$f_1 * f_2 * f_3(N) = f_1 * g_2 * g_3(N) + f_1 * g_2 * h_3(N) + f_1 * h_2 * g_3(N) + f_1 * h_2 * h_3(N).$$

By the assumption (2.2) we have, for $i \in \{1, 2\}$ the pointwise lower bound $g_i(t) \geqslant \delta$ for all $t \in [N/4, N/2)$. Thus,

$$f_1 * g_2 * g_3(N) \geqslant \frac{1}{N^2} \sum_{n_1} f_1(n_1) \sum_{\substack{N/4 \leqslant n_2, n_3 < N/2 \\ n_1 + n_2 + n_3 = N}} \delta^2 \geqslant \frac{\delta^2}{100N} \sum_{0.1N \leqslant n_1 \leqslant 0.4N} f_1(n_1) \geqslant \frac{1}{100} \delta^3$$

by the assumption (2.3).

To conclude the proof, it remains to show that

$$|f_1 * h_2 * h_3(N)| \leqslant \tfrac{1}{1000} \delta^3, \tag{4.3}$$

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and the same bound with either h_2 replaced by g_2 or h_3 replaced by g_3. We have

$$|f_1 * h_2 * h_3(N)| \leqslant \sum_{\xi \in G} |\widehat{f_1}(\xi)\widehat{h_2}(\xi)\widehat{h_3}(\xi)|. \tag{4.4}$$

First we bound the contribution of summands with $\xi \notin \Omega$. By the definition of Ω we have $|\widehat{f_1}(\xi)| < \varepsilon$ for $\xi \notin \Omega$. Thus,

$$\sum_{\xi \in G \setminus \Omega} |\widehat{f_1}(\xi)\widehat{h_2}(\xi)\widehat{h_3}(\xi)| < \varepsilon^{1/2} \sum_{\xi \in G} |\widehat{f_1}(\xi)|^{1/2} |\widehat{h_2}(\xi)\widehat{h_3}(\xi)|.$$

By Hölder's inequality, this is bounded by

$$\varepsilon^{1/2}\bigg(\sum_{\xi \in G} |\widehat{f_1}(\xi)|^{5/2}\bigg)^{1/5}\bigg(\sum_{\xi \in G} |\widehat{h_2}(\xi)|^{5/2}\bigg)^{2/5}\bigg(\sum_{\xi \in G} |\widehat{h_3}(\xi)|^{5/2}\bigg)^{2/5} \leqslant 4K\varepsilon^{1/2},$$

by (2.4) and (4.2). This is acceptable if ε is small enough. To bound the contribution to the right-hand side of (4.4) of summands with $\xi \in \Omega$, it suffices to show that $|\widehat{h_2}(\xi)| \leqslant 30\eta K^{2/5}$ for $\xi \in \Omega$ (the rest of the argument follows just as above). Since, by (2.4), $|\widehat{f_2}(\xi)| \leqslant K^{2/5}$, by (4.1) it suffices to show that

$$\left|1 - \frac{\widehat{\chi}(\xi)}{\|\chi\|_1}\right| \leqslant 30\eta$$

for $\xi \in \Omega$. We may write

$$1 - \frac{\widehat{\chi}(\xi)}{\|\chi\|_1} = \frac{1}{N\|\chi\|_1} \sum_{n \in G} \chi(n)(1 - e(\xi n/N)).$$

If $n \in \mathrm{Bohr}(\Omega, 2\eta)$, then $|1 - e(\xi n/N)| \leqslant 20\eta$. If $n \notin \mathrm{Bohr}(\Omega, 2\eta)$, then by Lemma 3.1 we have $|\chi(n)| \leqslant \eta\|\chi\|_1$. Combining these together we obtain

$$\left|1 - \frac{\widehat{\chi}(\xi)}{\|\chi\|_1}\right| \leqslant \frac{1}{N\|\chi\|_1}\bigg(20\eta \sum_{n \in G} \chi(n) + \sum_{n \in G} 2\eta\|\chi\|_1\bigg) \leqslant 30\eta,$$

as desired. This completes the proof of (4.3) and the cases where either h_2 is replaced by g_2 or h_3 is replaced by g_3 follow completely similarly. Hence, Theorem 2.3 follows.

Remark 4.1. Theorem 2.3 in particular says that if, for a positive density subset of the primes, the ternary Goldbach does not hold for all large odd N, then there must be some sort of Bohr set obstruction (including, as special cases, local obstructions modulo primes), since the condition (2.4) holds in this case by the work of Green and Tao [GT06]. On the other hand, as mentioned in § 2, such obstructions may indeed prevent the ternary Goldbach from holding.

Remark 4.2. The condition (2.2) should be compared with the usual hypotheses needed in carrying out the circle method. In a traditional application of the circle method, one requires the set to be equidistributed in Bohr sets so that the minor arc contributions are negligible, leading to an asymptotic formula for the number of solutions. In Theorem 2.3, with a weaker assumption (2.2) about distribution in Bohr sets, we deduce a lower bound for the number of solutions (of the correct order of magnitude).

5. Proof of Theorems 1.1 and 1.2 assuming Theorems 2.5 and 2.6

In this section we deduce Theorems 1.1 and 1.2 from the transference principle, Theorems 2.5 and 2.6 and the work of Green and Tao [GT06]. Let us first record the consequence of [GT06] we shall need. Here and later, we call a set of linear forms $\mathcal{L} = \{L_1, \dots, L_k\}$ admissible if they are distinct and $\prod_{i=1}^{k} L_i(n)$ has no fixed prime divisors. In this case, we define the singular series

$$\mathfrak{S}(\mathcal{L}) = \prod_{p \in \mathbb{P}} \left(1 - \frac{|\{n \in \mathbb{Z}/p\mathbb{Z} : p \mid L_1(n) \cdots L_k(n)\}|}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}. \tag{5.1}$$

Proposition 5.1. *Let $\rho \in (0, 1/2)$ be real and let $k \geqslant 1$ be an integer. Let $\mathcal{L} = \{L_1, \dots, L_k\}$ be an admissible set of k linear functions $L_i(n) = a_i n + b_i$ with $|a_i|, |b_i| \leqslant N$. Write*

$$X = \left\{ n \leqslant N : p \;\middle|\; \prod_{i=1}^{k} L_i(n) \implies p \geqslant N^{\rho} \right\},$$

and let $\mathfrak{S}(\mathcal{L})$ be defined as in (5.1). Let $G = \mathbb{Z}/N\mathbb{Z}$ and let $f : G \to \mathbb{R}_{\geqslant 0}$ be such that

$$f(n) \leqslant \begin{cases} (\log N)^k/\mathfrak{S} & \text{if } n \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Here we naturally identified G with $\{1, 2, \dots, N\}$. Then

$$\sum_{\xi \in G} |\widehat{f}(\xi)|^{5/2} \leqslant K,$$

for some positive constant $K = K(k, \rho)$.

Proof. Let $F = L_1 L_2 \cdots L_k$, $R = N^{\rho/2}$, and let $\beta_R(n)$ be the enveloping sieve given by [GT06, Proposition 3.1], so that $\beta_R(n) \gg_{k,\rho} f(n)$. Applying [GT06, Proposition 4.2] with $a_n = f(n)/\beta_R(n)$ if $\beta_R(n) \neq 0$ and $a_n = 0$ otherwise, we obtain that

$$\left(\sum_{\xi \in G} |\widehat{f}(\xi)|^{5/2}\right)^{2/5} \ll_k (\mathbb{E}_{n \leqslant N} a_n^2 \beta_R(n))^{1/2} \ll_{k,\rho} (\mathbb{E}_{n \leqslant N} \beta_R(n))^{1/2} \ll_{k,\rho} 1,$$

where the last inequality follows from [GT06, Lemma 4.1]. □

Proof of Theorem 1.1. Let $k = k(m)$, $\delta_0 = \delta_0(m)$ and $\rho = \rho(m)$ be as in Theorem 2.5, and let $K = K(k, \rho/2)$, where $K(k, \rho/2)$ is as in Proposition 5.1. Let $\varepsilon > 0$ be small enough depending on m, let w be large enough depending on ε and m, and let $W = \prod_{p \leqslant w} p$.

Let N' be an odd positive integer, sufficiently large in terms of all the preceding quantities $m, k, \delta_0, \rho, K, \varepsilon, W$. Our goal is to find a representation

$$N' = p_1 + p_2 + p_3,$$

where, for $j = 1, 2, 3$, p_j are primes such that the interval $[p_j, p_j + H]$ contains at least m primes. For $j = 1, 2, 3$, let b_j be integers such that $1 \leqslant b_j \leqslant W$, $(b_j, W) = 1$, and $N' \equiv b_1 + b_2 + b_3 \pmod{W}$. Let

$$N = \frac{N' - b_1 - b_2 - b_3}{W}.$$

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Let $h_1^{(1)}, \ldots, h_{k-1}^{(1)} \ll_m 1$ be as in Theorem 2.5 with $\chi = 1$. We can assume that w is so large that $|h_i^{(1)}| < w/2$ for each i.

With these choices $w, b_1, h_j^{(1)}$ we define

$$X_1 = \Big\{ n \leqslant N : Wn + b_1 \in \mathbb{P}, Wn + b_1 + Wh_i^{(1)} \in \mathbb{P} \text{ for } i = 1, \ldots, m-1,$$
$$\text{and } p \,\Big|\, \prod_{i=m}^{k-1} (Wn + b_1 + Wh_i^{(1)}) \implies p \geqslant N^{\rho/2} \Big\}, \tag{5.2}$$

and let $f_1 : \mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ be defined by

$$f_1(n) = \begin{cases} (\log N)^k \dfrac{\varphi(W)^k}{W^k} & \text{if } n \in X_1 \cap [0.2N, 0.4N), \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Theorem 2.5 implies

$$\sum_{0.2N \leqslant n < 0.4N} f_1(n) \geqslant \frac{\delta_0}{10} N$$

whereas Proposition 5.1 applied with the linear forms

$$\mathcal{L} = \{Wn + b_1, Wn + b_1 + Wh_1^{(1)}, \ldots, Wn + b_1 + Wh_{k-1}^{(1)}\}$$

implies

$$\sum_{\xi \in G} |\widehat{f_1}(\xi)|^{5/2} \leqslant K$$

since $|h_j^{(1)}| \leqslant |w|/2$, so that $\mathfrak{S}(\mathcal{L}) \leqslant (W/\varphi(W))^k$.

Further, let $\chi = \chi_{\Omega,\eta}$ be the Bohr cutoff associated to f_1 with $\delta = \delta_0/40$ from Theorem 2.3, with $|\Omega| \ll_m 1$, $1 \in \Omega$, and $1 \ll_m \eta < 0.05$. For $j = 2, 3$, let $h_1^{(j)}, \ldots, h_{k-1}^{(j)} \ll_m 1$ be as in Theorem 2.5 with $b = b_j$ and this choice of χ. We can assume that w is so large that $|h_i^{(j)}| < w/2$. With these choices $w, b_j, h_i^{(j)}$ we define, for $j = 2, 3$, X_j and f_j analogously to (5.2) and (5.3), but with f_j now supported on $[N/4, N/2)$. For $t \in [N/4, N/2)$, Theorem 2.5 implies

$$\begin{aligned} \sum_{N/4 \leqslant n < N/2} f_j(n)\chi(t-n) &\geqslant \frac{\delta_0}{10} \Big(\sum_{N/4 \leqslant n < N/2} \chi(t-n) + O\Big(\frac{N}{w^{1/3}} + \varepsilon N\Big)\Big) \\ &\geqslant \frac{\delta_0}{30} \Big(\sum_{n \in G} \chi(n) + O\Big(\frac{N}{w^{1/3}} + \varepsilon N\Big)\Big), \end{aligned}$$

where the second inequality follows since χ is symmetric around zero and is essentially supported on $|n| \leqslant 0.1N$, in the sense that

$$\sum_{0.1N < n < 0.9N} \chi(n) \leqslant \eta \sum_{n \in G} \chi(n) \tag{5.4}$$

by Lemma 3.1. When w is large enough and ε is small enough in terms of m, η and $|\Omega|$ (the size of which depend only on m), this together with Lemma 3.1 implies that

$$f_j * \chi(t) \geqslant \frac{\delta_0}{40} \|\chi\|_1.$$

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Furthermore, Proposition 5.1 implies that, for $j=2,3$,

$$\sum_{\xi\in G}|\widehat{f_j}(\xi)|^{5/2}\leqslant K.$$

Hence, all the assumptions of Theorem 2.3 are satisfied, and thus $f_1*f_2*f_3(N)\gg\delta^3$. In particular, there exists n_1,n_2,n_3 lying in the support of f_1,f_2,f_3, respectively, such that $n_1+n_2+n_3\equiv 0 \pmod N$. By the definitions of f_1,f_2,f_3, we necessarily have $n_1+n_2+n_3=N$, and moreover for $i=1,2,3$, Wn_i+b_i are primes and so are $Wn_i+b_i+Wh_j^{(i)}$ for $1\leqslant j\leqslant m-1$. This gives the desired representation

$$N'=(Wn_1+b_1)+(Wn_2+b_2)+(Wn_3+b_3),$$

once H is large enough in terms of m. □

Proof of Theorem 1.2. Let $K=K(2,1/2000)$, where $K(k,\rho)$ is as in Proposition 5.1. Let w be a large parameter, and let $W=\prod_{p\leqslant w}p$.

Let $N'\equiv 3 \pmod 6$ be a positive integer, sufficiently large in terms of K,W. Our goal is to find a representation

$$N'=p_1+p_2+p_3,$$

where, for $j=1,2,3$, p_j+2 has at most two prime factors. For $j=1,2,3$, let b_j be integers such that $1\leqslant b_j\leqslant W$, $(b_j,W)=(b_j+2,W)=1$, and $N'\equiv b_1+b_2+b_3 \pmod W$. Let

$$N=\frac{N'-b_1-b_2-b_3}{W}.$$

For $j=1,2,3$, we define

$$X_j=\{n\leqslant N: Wn+b_j\in\mathbb{P}, Wn+b_j+2=P_2, p\mid Wn+b_j+2 \implies p\geqslant N^{1/1000}\},$$

and let $f_1:\mathbb{Z}\to\mathbb{R}_{\geqslant 0}$ be defined by

$$f_1(n)=\begin{cases}(\log N)^2\dfrac{\varphi(W)^2}{W^2} & \text{if } n\in X_1\cap[0.2N,0.4N),\\ 0 & \text{otherwise.}\end{cases}$$

Now Theorem 2.6 with $\chi=1$ implies that

$$\sum_{0.2N\leqslant n<0.4N}f_1(n)\geqslant\frac{\delta_1}{10}N.$$

Further, let $\chi=\chi_{\Omega,\eta}$ be the Bohr cutoff associated to f_1 with $\delta=\delta_1/40$ from Theorem 2.3, with $|\Omega|\ll 1$, $1\in\Omega$ and $1\ll\eta<0.05$. We define f_j for $j=2,3$ as f_1 but with support $[N/4,N/2)$. Now Theorem 2.6 implies that, for $j=2,3$, and $t\in[N/4,N/2)$,

$$\sum_{N/4\leqslant n<N/2}f_j(n)\chi(t-n)\geqslant\frac{\delta_1}{10}\left(\sum_{N/4\leqslant n<N/2}\chi(t-n)+O\left(\frac{N}{w^{1/3}}\right)\right)\geqslant\frac{\delta_1}{30}\left(\sum_{n\in G}\chi(n)+O\left(\frac{N}{w^{1/3}}\right)\right)$$

since $\chi(n)$ is essentially supported on $|n|\leqslant 0.1N$ (see (5.4)) and is symmetric around zero. When w is large enough in terms of η and Ω (sizes of which depend only on m), this and Lemma 3.1 imply that

$$f_j*\chi(t)\geqslant\frac{\delta_1}{40}\|\chi\|_1.$$

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Furthermore Proposition 5.1 implies that, for $j = 1, 2, 3$,

$$\sum_{\xi \in G} |\widehat{f}_j(\xi)|^{5/2} \leqslant K.$$

Hence, all the assumptions of Theorem 2.3 are satisfied, and thus $f_1 * f_2 * f_3(N) \gg \delta^3$. In particular, there exists n_1, n_2, n_3 lying in the support of f_1, f_2, f_3, respectively, such that $n_1 + n_2 + n_3 \equiv 0 \pmod N$. By the definitions of f_1, f_2, f_3, we necessarily have $n_1 + n_2 + n_3 = N$, and moreover for each $i = 1, 2, 3$, $Wn_i + b_i$ is a prime and $Wn_i + b_i + 2$ has at most two prime factors. This gives the desired representation

$$N' = (Wn_1 + b_1) + (Wn_2 + b_2) + (Wn_3 + b_3). \qquad \square$$

6. Weighted versions of Maynard's theorem and Chen's theorem

As discussed in the introduction, the celebrated result of Maynard [May15] (obtained independently by Tao in an unpublished work) states that, for each $m \geqslant 1$, there exists a constant $H = H(m)$ such that there exists infinitely many primes p for which the interval $[p, p + H]$ contains at least m primes. In a subsequent paper [May16], Maynard generalized the result to show that any subset of the primes which is well-distributed in arithmetic progressions (in a certain precise sense) contains many primes with bounded gaps, and also made an extension to linear forms representing primes.

In this section we state a slight variant of the main result of [May16] in the case when the underlying set is weighted with weights $\omega_n \geqslant 0$. We also carefully state the dependencies between different parameters.

For a linear function $L(n) = l_1 n + l_2$, we define $\varphi_L(q) = \varphi(|l_1|q)/\varphi(|l_1|)$. Let us first state the required hypotheses which correspond to [May16, Hypothesis 1].

HYPOTHESIS 6.1. For a sequence (ω_n), a set of k admissible linear forms $\mathcal{L}$ and real numbers $x \geqslant 2$, $\theta \in (0, 1)$ and $C_H > 0$, we formulate the following hypothesis.

(i) The sequence (ω_n) is well-distributed in arithmetic progressions: we have

$$\sum_{r \leqslant x^\theta} \max_c \left| \sum_{\substack{x \leqslant n < 2x \\ n \equiv c \pmod r}} \omega_n - \frac{1}{r} \sum_{x \leqslant n < 2x} \omega_n \right| \leqslant C_H \frac{\sum_{x \leqslant n < 2x} \omega_n}{(\log x)^{101k^2}}.$$

(ii) Primes represented by linear forms in $\mathcal{L}$ are well-distributed in arithmetic progressions: for any $L \in \mathcal{L}$, we have

$$\sum_{r \leqslant x^\theta} \max_{(L(c), r) = 1} \left| \sum_{\substack{x \leqslant n < 2x \\ n \equiv c \pmod r \\ L(n) \in \mathbb{P}}} \omega_n - \frac{1}{\varphi_L(r)} \sum_{\substack{x \leqslant n < 2x \\ L(n) \in \mathbb{P}}} \omega_n \right| \leqslant C_H \frac{\sum_{x \leqslant n < 2x} \omega_n}{(\log x)^{101k^2}}.$$

(iii) The sequence (ω_n) is not too concentrated in any arithmetic progression: for any $r \leqslant x^\theta$ and any c, we have

$$\sum_{\substack{x \leqslant n < 2x \\ n \equiv c \pmod r}} \omega_n \leqslant C_H \frac{1}{r} \sum_{x \leqslant n < 2x} \omega_n.$$

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The slight variant of Maynard's main theorem [May16, Theorem 3.1] can now be stated as follows.

THEOREM 6.2. *Let $\alpha > 0, \theta \in (0,1)$ and $C_H > 0$. There exist a constant $C = C(\alpha, \theta)$ such that, for any $k \geqslant C$ there exist positive constants $x_0 = x_0(\alpha, \theta, k, C_H), \delta_0 = \delta_0(\alpha, \theta, k)$ and $\rho = \rho(\alpha, \theta, k)$ such that the following holds.*

Let (ω_n) be a sequence of non-negative real numbers, let $\mathcal{L} = \{L_1, \dots, L_k\}$ be an admissible set of k linear functions and let $x \geqslant x_0$ be an integer. Assume that the coefficients of $L_i(n) = a_i n + b_i$ satisfy $1 \leqslant a_i, b_i \leqslant x^\alpha$ for all $1 \leqslant i \leqslant k$, and assume that $k \leqslant (\log x)^\alpha$.

If Hypothesis 6.1 holds and $\delta > 1/(\log k)$ is such that

$$\frac{1}{k} \sum_{L \in \mathcal{L}} \frac{\varphi(a_i)}{a_i} \sum_{\substack{x \leqslant n < 2x \\ L(n) \in \mathbb{P}}} \omega_n \geqslant \frac{\delta}{\log x} \sum_{x \leqslant n < 2x} \omega_n, \tag{6.1}$$

then

$$\sum_{\substack{x \leqslant n < 2x \\ \#(\{L_1(n), \dots, L_k(n)\} \cap \mathbb{P}) \geqslant C^{-1} \delta \log k \\ p \mid L_1(n) \cdots L_k(n) \implies p > x^\rho}} \omega_n \geqslant \delta_0 \frac{\mathfrak{S}(\mathcal{L})}{(\log x)^k \exp(Ck)} \sum_{x \leqslant n < 2x} \omega_n,$$

where $\mathfrak{S}(\mathcal{L})$ is defined as in (5.1).

Proof. The proof is the same as Maynard's [May16, Proof of Theorem 3.1]. Introducing the weights ω_n makes no difference once one replaces $\#\mathcal{A}(x)$ in [May16] by the weighted version $\sum_{x \leqslant n < 2x} \omega_n$, etc. Furthermore, to see that the constants δ_0 and ρ do not depend on C_H, note that Hypothesis 6.1(1, 2) imply [May16, Hypothesis 1(1, 2)] with implied constant one once x is large enough in terms of C_H. On the other hand, in [May16, Proof of Theorem 3.1], [May16, Hypothesis 1(3)] is only used together with [May16, Hypothesis 1(1) or (2)] to dispose of some divisor functions through the Cauchy–Schwarz inequality (see [May16, Formulas (9.2) and (9.3)] for a typical example). In these situations one also wins a power of $\log x$ and thus can take the implied constant in the resulting bounds to be one once x is large enough in terms of C_H. Hence, none of the implied constants in the proof of Maynard's theorem depend on C_H once x is large enough in terms of C_H. □

Next we formulate a similar general version of Chen's theorem. We will need the notion of a well-factorable function of level R by which we mean a function $\lambda : \mathbb{N} \cap [1, R] \to [-1, 1]$ such that, for any $S, T \geqslant 1$ with $ST = R$, we can write $\lambda = \gamma * \delta$ with 1-bounded functions γ and δ supported on $[1, S]$ and $[1, T]$, respectively.

HYPOTHESIS 6.3. For $\varepsilon \in (0, 0.1)$, a sequence (ω_n) of non-negative real numbers, a set of two admissible linear forms $\mathcal{L} = \{L_1, L_2\}$ with $L_i(n) = u_i n + v_i$, and a real number $x \geqslant 2$, we formulate the following hypotheses.

(i) Primes represented by L_1 are well-distributed in arithmetic progressions: we have

$$\sum_{\substack{r \\ (r, u_2(u_2 v_1 - u_1 v_2)) = 1}} \mu(r)^2 \lambda_r \Bigg(\sum_{\substack{x \leqslant n < 2x \\ r \mid L_2(n) \\ L_1(n) \in \mathbb{P}}} \omega_n - \frac{u_1}{\varphi(r u_1)} \sum_{x \leqslant n < 2x} \frac{\omega_n}{\log L_1(n)} \Bigg) \leqslant \frac{\sum_{x \leqslant n < 2x} \omega_n}{(\log x)^{10}}$$

whenever λ is a well-factorable function of level $x^{1/2-\varepsilon}$ or $\lambda = 1_{p \in [P, P')} * \lambda'$, where λ' is a well-factorable function of level $x^{1/2-\varepsilon}/P$ and $2P \geqslant P' \geqslant P \in [x^{1/10}, x^{1/3-\varepsilon}]$.

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(ii) Almost primes represented by L_2 are well-distributed in arithmetic progressions: we have, for $j=1,2$,

$$\sum_{\substack{r\\(r,u_1(u_1v_2-u_2v_1))=1}} \mu(r)^2\lambda_r\Bigg(\sum_{\substack{x\leqslant n<2x\\ r|L_1(n)\\ L_2(n)\in B_j}}\omega_n-\frac{1}{\varphi_{L_2}(r)}\sum_{\substack{x\leqslant n<2x\\ L_2(n)\in B_j}}\omega_n\Bigg)\leqslant\frac{\sum_{x\leqslant n<2x}\omega_n}{(\log x)^{10}}$$

whenever λ is a well-factorable function of level $x^{1/2-\varepsilon}$, where

$$B_1=\{n=p_1p_2p_3 \mid x^{1/10}\leqslant p_1<x^{1/3-\varepsilon}, x^{1/3-\varepsilon}\leqslant p_2\leqslant (L_2(2x)/p_1)^{1/2}, p_3\geqslant x^{1/10}\} \tag{6.2a}$$

and

$$B_2=\{n=p_1p_2p_3 \mid x^{1/3-\varepsilon}\leqslant p_1\leqslant p_2\leqslant (L_2(2x)/p_1)^{1/2}, p_3\geqslant x^{1/10}\}. \tag{6.2b}$$

(iii) The sequence (ω_n) is not concentrated in B_j: we have, for $j=1,2$,

$$\sum_{\substack{x\leqslant n<2x\\ L_2(n)\in B_j}}\omega_n\leqslant(1+o(1))\frac{|B_j\cap[L_2(x),L_2(2x))|}{\varphi(u_2)}\cdot\frac{1}{x}\sum_{x\leqslant n<2x}\omega_n,$$

Note that the factor $u_1/(\varphi(ru_1)\log L_1(n))$ in the first hypothesis is the probability that a randomly chosen $n\in[x,2x)$ satisfies $r\mid L_2(n)$ and $L_1(n)\in\mathbb{P}$. Note also that it is straightforward to find the density of B_j: if $u_2,v_2\leqslant x^{o(1)}$, then

$$|B_j\cap[L_2(x),L_2(2x))|=(\delta(B_j)+o(1))\frac{u_2x}{\log x},$$

where

$$\delta(B_1)=\int_{1/10}^{1/3-\varepsilon}\int_{1/3-\varepsilon}^{(1-\alpha_1)/2}\frac{d\alpha_2\,d\alpha_1}{\alpha_1\alpha_2(1-\alpha_1-\alpha_2)},\quad \delta(B_2)=\int_{1/3-\varepsilon}^{1/3}\int_{\alpha_1}^{(1-\alpha_1)/2}\frac{d\alpha_2\,d\alpha_1}{\alpha_1\alpha_2(1-\alpha_1-\alpha_2)}. \tag{6.3}$$

To see that the coprimality conditions $(r,u_2(u_2v_1-u_1v_2))=1$ and $(r,u_1(u_1v_2-u_2v_1))=1$ occur naturally, note that if r and u_i share a common prime divisor p, then $p\nmid L_i(n)$ for all n by the admissibility of L_i, and thus the sum over those n satisfying $r|L_i(n)$ is empty. Similarly, if $(r,u_i)=1$ but r and $u_2v_1-u_1v_2$ share a common prime divisor $p<x^{1/10}$, then $p|L_i(n)$ implies $p|L_j(n)$ (where $j=3-i$), and thus the sum over those n satisfying $r|L_i(n)$ and $L_j(n)\in B$ (or $L_j(n)\in\mathbb{P}$) is empty.

Theorem 6.4. *There exist positive constants δ_0,ε and x_0 such that the following holds. Let (ω_n) be a sequence of non-negative real numbers, $\mathcal{L}=\{L_1,L_2\}$ be an admissible set of two linear functions and let $x\geqslant x_0$. Assume that the coefficients of $L_i(n)=u_in+v_i$ satisfy $1\leqslant u_i,v_i\leqslant x^{o(1)}$, and that Hypothesis 6.3 holds. Then*

$$\sum_{\substack{x\leqslant n<2x\\ L_1(n)\in\mathbb{P}\\ L_2(n)=P_2\\ p|L_2(n)\implies p\geqslant x^{1/10}}}\omega_n\geqslant\delta_0\frac{\mathfrak{S}(\mathcal{L})}{(\log x)^2}\sum_{x\leqslant n<2x}\omega_n-O\Big(x^{0.9}\max_n\omega_n\Big),$$

where $\mathfrak{S}(\mathcal{L})$ is as in (5.1).

Since the proof is essentially Chen's sieving device written in general terms, we postpone its proof to Appendix A.

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7. Technical reductions

The conclusion of Maynard's theorem does not quite correspond to the conclusion we want in Theorem 2.5. However, we can quickly deduce Theorem 2.5 from the following variant which is more apt for an application of Maynard's theorem.

Proposition 7.1. *For any positive integer m, there exist a positive integer $k = k(m)$ and positive constants $\delta_1 = \delta_1(m)$ and $\rho = \rho(m)$ such that the following holds. Let $\chi : \mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ be a function with Fourier complexity at most M for some $M \geqslant 1$, let $W = \prod_{p \leqslant w} p$ and let $(b, W) = 1$. There exists a positive constant $N_0 = N_0(m, M, w)$ such that, for any distinct integers $h_1, \ldots, h_k$ with $|h_j| < w/2$, any $N \geqslant N_0$ and $|t| \leqslant 5N$,*

$$\sum_{\substack{N \leqslant n < 2N \\ |\{W(n+h_i)+b\} \cap \mathbb{P}| \geqslant m \\ p \mid \prod_{i=1}^{k}(W(n+h_i)+b) \implies p \geqslant N^{\rho}}} \chi(t-n) \geqslant \delta_1 \frac{1}{(\log N)^k} \frac{W^k}{\varphi(W)^k} \bigg(\sum_{N \leqslant n < 2N} \chi(t-n) + O\bigg(\frac{M^2 N}{w^{1/2}}\bigg)\bigg).$$

Proof that Proposition 7.1 implies Theorem 2.5. Let $k = k(m)$, $\delta_1 = \delta_1(m)$ and $\rho = \rho(m)$ be as in Proposition 7.1. Let $\alpha_1, \ldots, \alpha_M$ be the phases appearing in the Fourier expansion of χ. By the simultaneous version of the Dirichlet approximation theorem, we can find k distinct positive integers $h_j' \ll_{M,\varepsilon,m} 1$ such that

$$\|\alpha_i h_j'\| \leqslant \frac{\varepsilon}{M^2} \quad \text{for every } i = 1, \ldots, M \text{ and } j = 1, \ldots, k.$$

These choices ensure that, whenever $n - n' \in \{h_1', \ldots, h_k'\}$, we have

$$|\chi(n) - \chi(n')| \ll \varepsilon. \tag{7.1}$$

We can assume that w is so large in terms of M, ε and m that $|h_j'| < w/2$ for all j and $w^{1/6}$ is at least $2M^2$ times the implied constant in the conclusion of Proposition 7.1. By Proposition 7.1 we see that, for any $|t| \leqslant 5N$,

$$\sum_{\substack{N \leqslant n < 2N \\ |\{W(n+h_i')+b\} \cap \mathbb{P}| \geqslant m \\ p \mid \prod_{i=1}^{k}(W(n+h_i')+b) \implies p \geqslant N^{\rho}}} \chi(t-n) \geqslant \delta_1 \frac{1}{(\log N)^k} \frac{W^k}{\varphi(W)^k} \bigg(\sum_{N \leqslant n < 2N} \chi(t-n) - \frac{N}{2w^{1/3}}\bigg).$$

We get that, for some $\mathcal{J} \subseteq \{1, \ldots, k\}$ with $\#\mathcal{J} = m$,

$$\sum_{\substack{N \leqslant n < 2N \\ W(n+h_j')+b \in \mathbb{P} \text{ for each } j \in \mathcal{J} \\ p \mid \prod_{i=1}^{k}(W(n+h_i')+b) \implies p \geqslant N^{\rho}}} \chi(t-n) \geqslant \frac{\delta_1}{\binom{k}{m}} \cdot \frac{1}{(\log N)^k} \frac{W^k}{\varphi(W)^k} \bigg(\sum_{N \leqslant n < 2N} \chi(t-n) - \frac{N}{2w^{1/3}}\bigg).$$

Let $r \in \mathcal{J}$ be such that h_r' is the minimal among h_j' with $j \in \mathcal{J}$. We take $h_1, \ldots, h_{k-1}$ to be any choice (unique up to permutation) such that

$$\{h_i : i = 1, \ldots, m-1\} = \{h_i' - h_r' : i \in \mathcal{J} \backslash \{r\}\}$$

and

$$\{h_i : i = m, \ldots, k-1\} = \{h_i' - h_r' : i \in \{1, \ldots, k\} \backslash \mathcal{J}\}.$$

Substituting $n' = n + h'_r$, we see that

$$\sum_{\substack{N+h'_r \leqslant n' < 2N+h'_r \\ Wn'+b \in \mathbb{P} \\ Wn'+Wh_i+b \in \mathbb{P} \text{ for } i=1,\ldots,m-1 \\ p \mid \prod_{i=m}^{k-1}(Wn'+Wh_i+b) \implies p \geqslant N^\rho}} \chi(t-n'+h'_r) \geqslant \frac{\delta_1}{\binom{k}{m}} \cdot \frac{1}{(\log N)^k} \frac{W^k}{\varphi(W)^k} \Bigg(\sum_{N \leqslant n < 2N} \chi(t-n) - \frac{N}{2w^{1/3}} \Bigg). \tag{7.2}$$

By (7.1) we may replace the summand $\chi(t-n'+h'_r)$ above by $\chi(t-n')$ using a standard sieve bound for the number of elements counted on the left-hand side of (7.2), getting that

$$\sum_{\substack{N+h'_r \leqslant n' < 2N+h'_r \\ Wn'+b \in \mathbb{P} \\ Wn'+Wh_i+b \in \mathbb{P} \text{ for } i=1,\ldots,m-1 \\ p \mid \prod_{i=m}^{k-1}(Wn'+Wh_i+b) \implies p \geqslant N^\rho}} \chi(t-n') \geqslant \frac{\delta_1}{\binom{k}{m}} \cdot \frac{1}{(\log N)^k} \frac{W^k}{\varphi(W)^k} \Bigg(\sum_{N \leqslant n < 2N} \chi(t-n) - \frac{N}{2w^{1/3}} + O(\varepsilon N) \Bigg)$$

with the implied constant depending only on k and ρ and thus only on m. Theorem 2.5 follows with $\delta_0 = \delta_1/(2\binom{k}{m})$ through noting that the terms with $n' \in [2N, 2N+h'_r)$ on the left-hand side contribute at most $M^2 h'_r$. $\square$

Since Bohr sets (and, in general, functions with bounded Fourier complexity) are not equidistributed in arithmetic progressions, we cannot apply Maynard's theorem to the situation in Proposition 7.1 directly, but we need to be careful with our choice of the sequence ω_n to which we apply Maynard's theorem. In particular, the moduli $q_i \mid Q$ in the Fourier expansion of χ in Lemma 3.2 are problematic, and for this reason we will split into residue classes (mod Q).

In § 9 we shall use Maynard's theorem (Theorem 6.2) and exponential sum estimates (which we will state in § 8) to prove the following proposition.

PROPOSITION 7.2. *For any positive integer m, there exist a positive integer $k = k(m)$ and positive constants $\delta_1 = \delta_1(m)$, $\rho = \rho(m)$ and $A = A(m)$ such that the following holds. Let $\chi : \mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ be a function with Fourier complexity at most M for some $M \geqslant 1$. Let $W = \prod_{p \leqslant w} p$, let $(b, W) = 1$ and let $N \geqslant N_0(m, M, w)$ be large. Let Q be from Lemma 3.2 corresponding to A. Then, for any distinct integers $h_1, \ldots, h_k$ with $|h_j| < w/2$, any $|t| \leqslant 5N$ and $c_0 \in \mathcal{C}_M$,*

$$\sum_{\substack{N \leqslant n < 2N \\ n \equiv c_0 \pmod{Q} \\ |\{W(n+h_i)+b\} \cap \mathbb{P}| \geqslant m \\ p \mid \prod_{i=1}^{k}(W(n+h_i)+b) \implies p \geqslant N^\rho}} \chi(t-n) \geqslant \delta_1 \frac{1}{(\log N)^k} \frac{W^k}{\varphi(W)^k} \frac{Q}{|\mathcal{C}_M|} \Bigg(\sum_{\substack{N \leqslant n < 2N \\ n \equiv c_0 \pmod{Q}}} \chi(t-n) + O\bigg(\frac{N}{Qw^{10}}\bigg) \Bigg),$$

where

$$\mathcal{C}_M = \{c_0 \pmod{Q} : (Wc_0 + Wh_i + b, Q) = 1 \textit{ for every } i = 1, \ldots, k\}.$$

Note that, since $|h_i| < w/2$, by the Chinese reminder theorem

$$|\mathcal{C}_M| = Q \prod_{p \mid Q, p > w} \bigg(1 - \frac{k}{p}\bigg). \tag{7.3}$$

Let us next state a similar proposition that we shall prove using Chen's theorem (Theorem 6.4).

PROPOSITION 7.3. *Let $\chi:\mathbb{Z}\to\mathbb{R}_{\geqslant 0}$ be a function with Fourier complexity at most M for some $M\geqslant 1$. Let $W=\prod_{p\leqslant w}p$, let $(b,W)=(b+2,W)=1$ and let $N\geqslant N_0(M,w)$ be large. Let Q be from Lemma 3.2 corresponding to some large enough A. Then, for any $|t|\leqslant 5N$, and $c_0\in\mathcal{C}_C$,*

$$\sum_{\substack{N\leqslant n<2N\\ n\equiv c_0 \pmod Q\\ Wn+b\in\mathbb{P}\\ Wn+b+2=P_2\\ p\mid Wn+b+2\implies p\geqslant N^{1/100}}}\chi(t-n)\geqslant \delta_1\frac{1}{(\log N)^2}\frac{W^2}{\varphi(W)^2}\frac{Q}{|\mathcal{C}_C|}\Bigg(\sum_{\substack{N\leqslant n<2N\\ n\equiv c_0 \pmod Q}}\chi(t-n)+O\bigg(\frac{N}{Q(\log N)^{100}}\bigg)\Bigg),$$

for some absolute constant $\delta_1>0$, where

$$\mathcal{C}_C=\{c_0 \pmod Q),(Wc_0+b,Q)=(Wc_0+b+2,Q)=1\}.$$

To show that Propositions 7.2 and 7.3 imply Proposition 7.1 and Theorem 2.6, we use the following lemma allowing us to sum over all the residue classes in $\mathcal{C}_M$ and $\mathcal{C}_C$.

LEMMA 7.4. *Let χ be a function of Fourier complexity at most M for some $M\geqslant 1$, and let N,Q be positive integers with $N\geqslant 2Q^2$. Let also $\mathcal{Q}$ be a collection of residue classes modulo Q such that, for all $1\neq q\mid Q$ and $(a,q)=1$, one has*

$$\sum_{c_0\in\mathcal{Q}}e\left(\frac{a}{q}c_0\right)=O(\eta|\mathcal{Q}|), \tag{7.4}$$

for some $\eta>0$. Then

$$\frac{Q}{|\mathcal{Q}|}\sum_{c_0\in\mathcal{Q}}\sum_{\substack{N\leqslant n<2N\\ n\equiv c_0 \pmod Q}}\chi(t-n)\geqslant\sum_{N\leqslant n<2N}\chi(t-n)+O(\eta M^2N+QM^2N^{1/2}). \tag{7.5}$$

Proof. By Definition 2.4, we have the Fourier expansion

$$\chi(t-n)=\sum_{i=1}^{M}b_ie(\alpha_i(t-n)),$$

for some $|b_i|\leqslant M$ and $\alpha_i\in\mathbb{R}/\mathbb{Z}$. For each $1\leqslant i\leqslant M$, we may find integers $0\leqslant a_i<q_i\leqslant N^{1/2}$ with $(a_i,q_i)=1$ such that $|\alpha_i-a_i/q_i|\leqslant 1/(q_iN^{1/2})$.

Let us first consider the contribution of those i with $q_i=1$ to the left-hand side of (7.5). This contribution is, using Lemma B.1,

$$\begin{aligned}\Sigma_1 &:= \frac{Q}{|\mathcal{Q}|}\sum_{c_0\in\mathcal{Q}}\sum_{\substack{1\leqslant i\leqslant M\\ q_i=1}}b_i\sum_{\substack{N\leqslant n<2N\\ n\equiv c_0 \pmod Q}}e(\alpha_i(t-n))\\ &=\frac{Q}{|\mathcal{Q}|}\sum_{c_0\in\mathcal{Q}}\sum_{\substack{1\leqslant i\leqslant M\\ q_i=1}}b_i\left(\frac{1}{Q}\sum_{N\leqslant n<2N}e(\alpha_i(t-n))+O(N^{1/2})\right)\\ &=\sum_{\substack{1\leqslant i\leqslant M\\ q_i=1}}b_i\sum_{N\leqslant n<2N}e(\alpha_i(t-n))+O(QM^2N^{1/2}).\end{aligned}$$

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By Lemma B.2 we can extend the sum to go over all $1 \leqslant i \leqslant M$, at the cost of an error of size $M^2 \max_i q_i \ll M^2 N^{1/2}$, obtaining

$$\begin{aligned}\Sigma_1 &= \sum_{N \leqslant n < 2N} \sum_{1 \leqslant i \leqslant M} b_i e(\alpha_i(t-n)) + O(QM^2 N^{1/2}) \\ &= \sum_{N \leqslant n < 2N} \chi(t-n) + O(QM^2 N^{1/2}).\end{aligned}$$

Hence, we are finished if we can show that, for each i such that with $q_i > 1$, we have

$$\frac{Q}{|\mathcal{Q}|} \sum_{c_0 \in \mathcal{Q}} \sum_{\substack{N \leqslant n < 2N \\ n \equiv c_0 \pmod{Q}}} e(\alpha_i(t-n)) = O(\eta N + QN^{1/2}). \tag{7.6}$$

In case $q_i \nmid Q$, we have $q_i/(q_i, Q) > 1$ and, thus, by Lemma B.2, the left-hand side is $O(Qq_i) = O(QN^{1/2})$.

In case $q_i \mid Q$, writing $\alpha_i = a_i/q_i + \beta_i$, the sum over n on the left-hand side of (7.6) equals

$$\begin{aligned}& e\left(\frac{a_i}{q_i}(t-c_0)\right) \sum_{\substack{N \leqslant n < 2N \\ n \equiv c_0 \pmod{Q}}} e(\beta_i(t-n)) \\ &= e\left(\frac{a_i}{q_i}(t-c_0)\right) \cdot \frac{1}{Q}\left(\sum_{N \leqslant n < 2N} e(\beta_i(t-n))\right) + O(N^{1/2})\end{aligned}$$

by Lemma B.1. Hence, the left-hand side of (7.6) equals

$$\sum_{N \leqslant n < 2N} e\left(\beta_i(t-n) + \frac{a_i}{q_i} t\right) \frac{1}{|\mathcal{Q}|} \sum_{c_0 \in \mathcal{Q}} e\left(-\frac{a_i}{q_i} c_0\right) + O(QN^{1/2}),$$

and (7.6) follows from the assumption (7.4). □

To show that (7.4) holds for $\mathcal{Q} = \mathcal{C}_M$ and for $\mathcal{Q} = \mathcal{C}_C$, we shall use the following elementary lemma related to a certain modification of Ramanujan sums.

LEMMA 7.5. *Let q be a natural number, $(a, q) = 1$ and let $P(n)$ be a polynomial with integer coefficients. Write $\rho(n) = \#\{k \pmod{n} : P(k) \equiv 0 \pmod{n}\}$. Then*

$$\left| \sum_{\substack{n \pmod{q} \\ (P(n), q) = 1}} e\left(\frac{an}{q}\right) \right| \leqslant \rho(q).$$

Proof. By Möbius inversion,

$$\sum_{\substack{n \pmod{q} \\ (P(n), q) = 1}} e\left(\frac{an}{q}\right) = \sum_{d \mid q} \mu(d) \sum_{\substack{n \pmod{q} \\ P(n) \equiv 0 \pmod{d}}} e\left(\frac{an}{q}\right).$$

For a fixed $d \mid q$, write $x_1, \ldots, x_{\rho(d)}$ for the roots of $P(n) \pmod{d}$. Then

$$\sum_{\substack{n \pmod{q} \\ P(n) \equiv 0 \pmod{d}}} e\left(\frac{an}{q}\right) = \sum_{i=1}^{\rho(d)} \sum_{\substack{n \pmod{q} \\ n \equiv x_i \pmod{d}}} e\left(\frac{an}{q}\right) = \sum_{i=1}^{\rho(d)} e\left(\frac{ax_i}{q}\right) \sum_{k \pmod{q/d}} e\left(\frac{ak}{q/d}\right), \tag{7.7}$$

where we have written $n = x_i + kd$. The last sum vanishes unless $d = q$ in which case (7.7) has absolute value at most $\rho(q)$, and the claim follows. □

Proof that Proposition 7.2 implies Proposition 7.1. We may assume that w is large enough in terms of m, since otherwise the error term dominates and the claim is trivial. By Lemma 7.4 it remains to show (7.4) for $\mathcal{Q} = \mathcal{C}_M$ and $1 \neq q \mid Q$ with $\eta = w^{-1/2}$. Writing $R(n) = \prod_{i=1}^{k}(Wn + Wh_i + b)$, equation (7.4) reduces to

$$\sum_{\substack{c_0 \pmod{Q} \\ (R(c_0),Q)=1}} e\left(\frac{a}{q}c_0\right) = O\left(\frac{Q}{w^{1/2}} \prod_{\substack{p \mid Q \\ p > w}} \left(1 - \frac{k}{p}\right)\right). \tag{7.8}$$

We can uniquely decompose $Q = qq'Q'$, where $(Q', q) = 1$ and $p \mid q' \implies p \mid q$. Then, when c_1 and c_2 run through residue classes (mod $q'Q'$) and (mod q), respectively, $c_1 q + c_2 Q'$ runs through residue classes (mod Q). Writing c_0 in this form, the left-hand side of (7.8) becomes

$$\sum_{\substack{c_1 \pmod{q'Q'} \\ (R(c_1 q),Q')=1}} \sum_{\substack{c_2 \pmod{q} \\ (R(c_2 Q'),q)=1}} e\left(\frac{aQ'}{q}c_2\right). \tag{7.9}$$

Since $R(n)$ is always co-prime to W, Lemma 7.5 implies that the inner sum in (7.9) vanishes unless $(q, W) = 1$. Furthermore, in this case it has absolute value at most

$$\#\{c_2 \pmod{q} : R(c_2 Q') \equiv 0 \pmod{q}\} \leqslant k^{\Omega(q)} \leqslant q^{1/3},$$

since $p \mid q \implies p > w$ and w is large enough. Hence, we obtain that the absolute value of (7.9) is at most

$$\sum_{\substack{c_1 \pmod{q'Q'} \\ (R(c_1 q),Q')=1}} q^{1/3} = q' q^{1/3} \sum_{\substack{c_1 \pmod{Q'} \\ (R(c_1 q),Q')=1}} 1.$$

By the definition of $R(n)$, $R(n)$ is always co-prime to $W = \prod_{p \leqslant w} p$, and for every $p > w$, $R(n) \equiv 0 \pmod{p}$ has k incongruent solutions (mod p) (since $|h_i| < w/2$ for every i). Hence, the absolute value of (7.9) is at most

$$q^{1/3} q' Q' \prod_{p \mid Q', p > w} \left(1 - \frac{k}{p}\right) \leqslant \frac{Q}{q^{1/2}} \prod_{p \mid Q, p > w} \left(1 - \frac{k}{p}\right),$$

and (7.8) follows since $q > 1$ and $(q, W) = 1$, so that $q > w$. □

Proof that Proposition 7.3 implies Theorem 2.6. By Lemma 7.4 it remains to show (7.4) for $\mathcal{Q} = \mathcal{C}_C$ and $1 \neq q \mid Q$ with $\eta = w^{-1/2}$. This time we take $R(n) = (Wn + b)(Wn + b + 2)$, and the claim follows exactly as in the previous proof, with $k = 2$. □

8. Exponential sum estimates

In this section we state exponential sum estimates that we will use in the proofs of Propositions 7.2 and 7.3. Since the proofs closely follow previous works, we postpone them to Appendix B.

8.1 Major arc estimates

Lemma 8.1. *Let $C_1, C_2 \geqslant 1$ and $\varepsilon > 0$. There exists a constant $x_0 = x_0(C_1, C_2, \varepsilon)$ such that the following holds. Let $Q \leqslant (\log x)^{C_1}$ and let $q \geqslant 1$ and a be integers such that $q \mid Q$ and $(a, q) = 1$.*

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Assume that $|\alpha - a/q| \leqslant (\log x)^{C_1}/x$. *Then, for every* $x \geqslant x_0$,

$$\sum_{r \leqslant x^{1/2-\varepsilon}} \max_{(c,rQ)=1} \Bigg| \sum_{\substack{x \leqslant p < 2x \\ p \equiv c \pmod{rQ}}} e(\alpha p) - \frac{Q}{\varphi(rQ)} \sum_{\substack{x \leqslant n < 2x \\ n \equiv c \pmod{Q}}} \frac{e(\alpha n)}{\log n} \Bigg| \leqslant \frac{x}{Q(\log x)^{C_2}}.$$

Lemma 8.2. *Let* $C_1, C_2 \geqslant 1$ *and* $\varepsilon > 0$. *There exists a constant* $x_0 = x_0(C_1, C_2, \varepsilon)$ *such that the following holds. Let* $Q \leqslant (\log x)^{C_1}$ *and let* $q \geqslant 1$ *and* a *be integers such that* $q \mid Q$ *and* $(a,q) = 1$. *Assume that* $|\alpha - a/q| \leqslant (\log x)^{C_1}/x$. *Then, for every* $x \geqslant x_0$, *any bounded sequences* $\{a_m\}$ *and* $\{b_n\}$, *and any* $x^{1/4} \leqslant M \leqslant x^{3/4}$,

$$\sum_{r \leqslant x^{1/2-\varepsilon}} \max_{(c,rQ)=1} \Bigg| \sum_{\substack{x \leqslant mn < 2x \\ mn \equiv c \pmod{rQ} \\ M \leqslant m < 2M}} a_m b_n e(\alpha mn) - \frac{1}{\varphi(rQ)} \sum_{\substack{x \leqslant mn < 2x \\ (mn,rQ)=1 \\ M \leqslant m < 2M}} a_m b_n e(\alpha mn) \Bigg|$$
$$\leqslant \frac{x}{Q(\log x)^{C_2}}.$$

8.2 Minor arc estimates

Our minor arc estimates are close variants of those proved in earlier papers. In particular, we follow [Mat09] which, in turn, is based on ideas developed in [BP85, Mik00].

Lemma 8.3 (Type I estimate). *There exists* x_0 *such that the following holds. Let* $Q, q \geqslant 1$ *and* a *be integers such that* $(a,q) = 1$. *Let* $|a_m| \leqslant 1$. *Write* $h = (q,Q)$. *Assume that* α *is such that* $|\alpha - a/q| < 1/(Qq^2)$ *and that* $Q \leqslant x^{1/2}$. *Then, for every* $x \geqslant x_0$ *and any* $M \geqslant 1$,

$$\sum_{r \leqslant x^{1/2}} \max_{(c,rQ)=1} \Bigg| \sum_{\substack{x \leqslant mn < 2x \\ mn \equiv c \pmod{rQ} \\ M \leqslant m < 2M}} a_m e(\alpha mn) \Bigg| \leqslant \frac{x}{Q} \left(\left(\frac{h}{q}\right)^{1/2} + \left(\frac{MQ}{x^{1/2}}\right)^{1/2} + \left(\frac{q}{x/Q}\right)^{1/2} \right) (\log x)^4.$$

Lemma 8.4 (Type II estimate). *Let* $C \geqslant 1$. *There exists a constant* $x_0 = x_0(C)$ *such that the following holds. Let* $Q, q \geqslant 1$ *and* a *be integers such that* $(a,q) = 1$, *write* $h = (q, Q^2)$, *and assume that* $|\alpha - a/q| < 1/(4q^2Q^2(\log x)^{2C})$. *Let* $M \in [x^{1/2}, x^{3/4}], Q \leqslant x^{3/2}/(2M^2(\log x)^C)$, $D \leqslant x/(MQ(\log x)^C)$ *and* $R \leqslant M/x^{1/2}$, *and let* $c' \in \mathbb{Z}$.

Then, for every $x \geqslant x_0$ *and any* $|a_k|, |b_k| \leqslant \tau(k)$,

$$\sum_{D \leqslant d < 2D} \max_{(c,dQ)=1} \sum_{\substack{R \leqslant r < 2R \\ (r,c'dQ)=1}} \Bigg| \sum_{\substack{x \leqslant mn < 2x \\ mn \equiv c' \pmod{r} \\ mn \equiv c \pmod{dQ} \\ M \leqslant m < 2M}} a_m b_n e(\alpha mn) \Bigg|$$
$$\leqslant \frac{x}{Q} \cdot \left(\frac{(\log x)^{C/2}}{(q/h)^{1/8}} + (\log x)^{C/2} Q^{1/2} \frac{q^{1/8}}{x^{1/8}} + \frac{1}{(\log x)^{C/8}} \right) (\log x)^{10}.$$

Combining the type I and II estimates through Vaughan's identity we will obtain the following minor arc estimates for exponential sums over primes.

Lemma 8.5. *Let* $C \geqslant 1$. *There exists a constant* $x_0 = x_0(C)$ *such that the following holds. Let* $Q, q \geqslant 1$ *and* a *be integers such that* $(a,q) = 1$, *write* $h = (q, Q^2)$, *and assume that*

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$|\alpha - a/q| < 1/(4q^2Q^2(\log x)^{2C})$. *Then, for every* $x \geqslant x_0$, *and* $Q \leqslant x^{1/10}$,

$$\sum_{r \leqslant x^{1/8}} \max_{(c,rQ)=1} \Bigg| \sum_{\substack{x \leqslant p < 2x \\ p \equiv c \ (\mathrm{mod}\ rQ)}} e(\alpha p) \Bigg| \leqslant \frac{x}{Q} \cdot \left(\frac{(\log x)^{C/2}}{(q/h)^{1/8}} + (\log x)^{C/2} Q^{1/2} \frac{q^{1/8}}{x^{1/8}} + \frac{1}{(\log x)^{C/8}} \right) (\log x)^{15}.$$

Lemma 8.6. *Let* $C \geqslant 1$. *There exists a constant* $x_0 = x_0(C)$ *such that the following holds. Let* $Q, q \geqslant 1$ *and* a *be integers such that* $(a,q) = 1$, *write* $h = (q, Q^2)$, *and assume that* $|\alpha - a/q| < 1/(4q^2Q^2(\log x)^{2C})$. *Let* λ_r *be as in Hypothesis 6.3(i).*

Then, for every $x \geqslant x_0$, $Q \leqslant x^{\varepsilon/2}$, $(c,Q) = 1$ *and* $c' \in \mathbb{Z}$,

$$\Bigg| \sum_{\substack{r \leqslant x^{1/2-\varepsilon} \\ (r,c'Q)=1}} \mu(r)^2 \lambda_r \sum_{\substack{x \leqslant p < 2x \\ p \equiv c' \ (\mathrm{mod}\ r) \\ p \equiv c \ (\mathrm{mod}\ Q)}} e(\alpha p) \Bigg| \leqslant \frac{x}{Q} \cdot \left(\frac{(\log x)^{C/2}}{(q/h)^{1/8}} + (\log x)^{C/2} Q^{1/2} \frac{q^{1/8}}{x^{1/8}} + \frac{1}{(\log x)^{C/8}} \right) (\log x)^{15}.$$

9. Proof of Proposition 7.2

In this section we prove Proposition 7.2 using Maynard's theorem (Theorem 6.2). Let us start by choosing the sequence ω_n and other parameters to which we apply Theorem 6.2. Let $C = C(1/8, 1/8)$ be as in Theorem 6.2, $k = \max\{C, e^{4Cm}\}$, and let $\rho = \rho(k, 1/8, 1/8)$ be as in Theorem 6.2. We take $x = N/Q$,

$$(\omega_n) = (\chi(t - Qn - c_0)), \quad \text{and, for } i = 1, \ldots, k, \quad L_i(n) = W(Qn + c_0 + h_i) + b.$$

We can assume that $\sum_{x \leqslant n < 2x} \omega_n \geqslant x/w^{10}$ since otherwise Proposition 7.2 is trivial. With these choices, we shall show that, for any $i = 1, \ldots, k$,

$$\sum_{r \leqslant x^{1/8}} \max_c \Bigg| \sum_{\substack{x \leqslant n < 2x \\ n \equiv c \ (\mathrm{mod}\ r)}} \omega_n - \frac{1}{r} \sum_{x \leqslant n < 2x} \omega_n \Bigg| \ll_{M,w} \frac{x}{(\log x)^{105k^2}}, \tag{9.1}$$

$$\sum_{r \leqslant x^{1/8}} \max_{\substack{c \\ (W(Qc+c_0+h_i)+b,r)=1}} \Bigg| \sum_{\substack{x \leqslant n < 2x \\ n \equiv c \ (\mathrm{mod}\ r) \\ W(Qn+c_0+h_i)+b \in \mathbb{P}}} \omega_n - \frac{QW}{\varphi(QWr)} \sum_{x \leqslant n < 2x} \frac{\omega_n}{\log(W(Qn + c_0 + h_i) + b)} \Bigg|$$
$$\ll_{M,w} \frac{x}{(\log x)^{105k^2}}, \tag{9.2}$$

and that, for any $r \leqslant x^{1/8}$ and any c, we have

$$\sum_{\substack{x \leqslant n < 2x \\ n \equiv c \ (\mathrm{mod}\ r)}} \omega_n \ll_{M,w} \frac{x}{rw^{10}}. \tag{9.3}$$

Now (9.1) implies Hypothesis 6.1(i) and (9.3) implies Hypothesis 6.1(iii). Furthermore, looking only at the $r = 1$ summand, we see that (9.2) implies that

$$\Bigg| \sum_{\substack{x \leqslant n < 2x \\ W(Qn+c_0+h_i)+b \in \mathbb{P}}} \omega_n - \frac{QW}{\varphi(QW)} \sum_{x \leqslant n < 2x} \frac{\omega_n}{\log(W(Qn + c_0 + h_i) + b)} \Bigg| \ll_{M,w} \frac{x}{(\log x)^{105k^2}}, \tag{9.4}$$

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which implies (6.1) with $\delta = 1/2$ (say). Furthermore, multiplying (9.4) by $\varphi(QW)/\varphi(QWr)$ and summing over $r \leqslant x^{1/8}$, we see that

$$\begin{aligned}\sum_{r\leqslant x^{1/8}} &\max_{\substack{c \\ (W(Qc+c_0+h_i)+b,r)=1}} \\ &\times \left| \frac{\varphi(QW)}{\varphi(QWr)} \sum_{\substack{x\leqslant n<2x \\ W(Qn+c_0+h_i)+b\in\mathbb{P}}} \omega_n - \frac{QW}{\varphi(QWr)} \sum_{x\leqslant n<2x} \frac{\omega_n}{\log(W(Qn+c_0+h_i)+b)} \right| \\ &\ll_{M,w} \frac{x}{(\log x)^{103k^2}},\end{aligned}$$

which together with (9.2) implies Hypothesis 6.1(ii) through the triangle inequality.

Hence, assuming we can prove (9.1)–(9.3), recalling our choice of k, Maynard's theorem with $\delta = 1/2$ gives

$$\sum_{\substack{x\leqslant n<2x \\ \#(\{L_1(n),\ldots,L_k(n)\}\cap\mathbb{P})\geqslant m \\ p\mid L_1(n)\cdots L_k(n) \implies p>x^{\rho}}} \omega_n \gg_m \frac{\mathfrak{S}(\mathcal{L})}{(\log x)^k} \sum_{x\leqslant n<2x} \omega_n.$$

Here

$$\begin{aligned}\mathfrak{S}(\mathcal{L}) &= \prod_p \left(1 - \frac{\#\{1\leqslant n\leqslant p : p \mid \prod_{i=1}^k (W(Qn+c_0+h_i)+b)\}}{p}\right)\left(1-\frac{1}{p}\right)^{-k} \\ &\gg \frac{1}{\exp(O(k))} \cdot \left(\frac{QW}{\varphi(QW)}\right)^k = \frac{1}{\exp(O(k))} \cdot \left(\frac{W}{\varphi(W)}\right)^k \cdot \frac{Q}{|\mathcal{C}_M|}\end{aligned}$$

by (7.3).

Recalling the definitions of ω_n and $L_i(n)$, we obtain,

$$\sum_{\substack{N\leqslant n<2N \\ n\equiv c_0 \pmod{Q} \\ \#(\{W(n+h_i)+b\}\cap\mathbb{P})\geqslant m \\ p\mid\prod_{i=1}^k (W(n+h_i)+b) \implies p\geqslant N^{\rho/2}}} \chi(t-n) \gg_m \left(\frac{W}{\varphi(W)}\right)^k \cdot \frac{Q}{|\mathcal{C}_M|} \frac{1}{(\log x)^k} \sum_{\substack{N\leqslant n<2N \\ n\equiv c_0 \pmod{Q}}} \chi(t-n)$$

which was the claim.

Hence, it remains to show (9.1)–(9.3). By the Fourier expansion of $\chi(n)$ in Lemma 3.2, it is enough to show these with

$$\omega_n = e\left(\left(W\frac{a}{q}+\beta\right)Qn\right), \tag{9.5}$$

where $0 \leqslant a < q \leqslant N/(\log N)^{100B}$, $(a,q) = 1$, $|\beta| \leqslant W(\log N)^{100B}/(qN)$, and, moreover, either $q \mid Q$ or $q/(q,Q^2) \geqslant (\log N)^A$. In particular, (9.3) follows immediately from a trivial estimate.

We also note that when considering (9.1)–(9.2) with ω_n as in (9.5), in case $|\beta| \leqslant 1/(Qx(\log x)^{111k^2})$ we can assume that $\beta = 0$ since $|e(y+h) - e(y)| = O(h)$. On the other hand, if $|\beta| > 1/(Qx(\log x)^{111k^2})$, then this combined with the upper bound for $|\beta|$ implies that $|\beta| < 1/(4Q^2q^2(\log x)^{3200k^2})$. Hence, we can in any case assume that

$$|\beta| < \min\left\{\frac{1}{4Q^2q^2(\log x)^{3200k^2}}, \frac{(\log x)^{110B}}{x}\right\}. \tag{9.6}$$

9.1 Establishing (9.1)

For $q \mid Q$, the left-hand side of (9.1) with ω_n as in (9.5) equals

$$\sum_{r\leqslant x^{1/8}} \max_c \left| \sum_{\substack{x\leqslant n<2x\\ n\equiv c \pmod r}} e(\beta Qn) - \frac{1}{r}\sum_{x\leqslant n<2x} e(\beta Qn)\right| \ll \sum_{r\leqslant x^{1/8}} (|\beta|Qx+1) \ll x^{1/2}$$

by Lemma B.1.

For $q \nmid Q$, the left-hand side of (9.1) with ω_n as in (9.5) is by triangle inequality at most

$$\log x \sum_{r\leqslant x^{1/8}} \max_c \left| \sum_{\substack{x\leqslant n<2x\\ n\equiv c \pmod r}} e\left(\left(W\frac{a}{q}+\beta\right)Qn\right)\right|. \tag{9.7}$$

Recall (9.6) and that $q/(q, WQ) \geqslant (\log N)^A/W$, so that, by Lemma 8.3 with $M = Q = h = 1$ and $q/(q, QW)$ in place of q, we obtain that (9.7) is at most

$$x\left(\frac{W^{1/2}}{(\log N)^{A/2}} + \frac{1}{x^{1/4}} + \frac{N^{1/2}}{x^{1/2}(\log N)^{50B}}\right)(\log x)^4 \ll \frac{x}{(\log x)^{110k^2}}$$

once A is large enough in terms of k.

9.2 Establishing (9.2)

By changes of variables $p, n' = W(Qn + c_0 + h_i) + b$ and $c' = W(Qc + c_0 + h_i) + b$, the left-hand side of (9.2) with ω_n as in (9.5) is at most

$$\begin{aligned}\sum_{r\leqslant x^{1/8}} \max_{(c',QWr)=1} \Bigg| &\sum_{\substack{QWx\leqslant p<2QWx\\ p\equiv c' \pmod{QWr}}} e\left(\left(\frac{a}{q}+\frac{\beta}{W}\right)p\right)\\ &- \frac{QW}{\varphi(QWr)} \sum_{\substack{QWx\leqslant n'<2QWx\\ n'\equiv c' \pmod{QW}}} (\log n')^{-1} e\left(\left(\frac{a}{q}+\frac{\beta}{W}\right)n\right)\Bigg| + O(x^{1/2}).\end{aligned}$$

In case $q \mid Q$ this is $O(x/(\log x)^{200k^2})$ by Lemma 8.1 recalling (9.6).

In case $q \nmid Q$, note that $q/(q,(QW)^2) > (\log N)^A/W^2$ and recall (9.6). We use the triangle inequality and estimate the two terms corresponding to the two sums inside the absolute values separately. The contribution corresponding to the sum over n' can be satisfactorily estimated by Lemma 8.3 with $r = M = 1$ after partial summation. Furthermore, Lemma 8.5 with $C = 1600k^2$ implies

$$\begin{aligned}&\sum_{r\leqslant x^{1/8}} \max_{(c',QWr)=1} \left| \sum_{\substack{QWx\leqslant p<2QWx\\ p\equiv c' \pmod{QWr}}} e\left(\left(\frac{a}{q}+\frac{\beta}{W}\right)p\right)\right|\\ &\leqslant x\cdot\left(\frac{(\log x)^{800k^2}}{((\log N)^A/W^2)^{1/8}} + (\log x)^{800k^2}Q^{1/2}W\frac{q^{1/8}}{x^{1/8}} + \frac{1}{(\log x)^{200k^2}}\right)(\log x)^{15} \ll \frac{x}{(\log x)^{150k^2}}\end{aligned}$$

when A is large enough in terms of k.

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10. Proof of Proposition 7.3

In this section we prove Proposition 7.3 using Chen's theorem (Theorem 6.4). Let $\mathcal{L} = \{L_1, L_2\}$ be the collection of two linear forms $L_1(n) = W(Qn + c_0) + b$ and $L_2(n) = W(Qn + c_0) + b + 2$, and note that

$$\mathfrak{S}(\mathcal{L}) \asymp \prod_{p \mid QW} \left(1 - \frac{1}{p}\right)^{-2} = \left(\frac{QW}{\varphi(QW)}\right)^2 \asymp \frac{W^2}{\varphi(W)^2} \frac{Q}{|\mathcal{C}_C|}.$$

Let $x = N/Q$. Define the sequence (ω_n) for $x \leqslant n < 2x$ by

$$\omega_n = \chi(t - Qn - c_0).$$

Since χ has Fourier complexity at most M, we have $\omega_n \leqslant M^2$ for every n. Thus, the conclusion follows from Chen's theorem (Theorem 6.4), once we verify the hypotheses. We may assume that $\sum_{x \leqslant n < 2x} \omega_n \geqslant x/(\log x)^{100}$ since otherwise the conclusion is trivial. Under this assumption, it suffices to show that, for λ_r as in Hypothesis 6.3(i),

$$\sum_{\substack{r \\ (r,QW)=1}} \mu(r)^2 \lambda_r \Bigg(\sum_{\substack{x \leqslant n < 2x \\ r \mid W(Qn+c_0)+b+2 \\ W(Qn+c_0)+b \in \mathbb{P}}} \omega_n - \frac{QW}{\varphi(QWr)} \sum_{x \leqslant n < 2x} \frac{\omega_n}{\log(W(Qn + c_0) + b)} \Bigg) \ll \frac{x}{(\log x)^{200}} \tag{10.1}$$

and that, for B_j and λ_r as in Hypothesis 6.3(ii),

$$\sum_{\substack{r \\ (r,QW)=1}} \mu(r)^2 \lambda_r \Bigg(\sum_{\substack{x \leqslant n < 2x \\ r \mid W(Qn+c_0)+b \\ W(Qn+c_0)+b+2 \in B_j}} \omega_n - \frac{\varphi(QW)}{\varphi(QWr)} \sum_{\substack{x \leqslant n < 2x \\ W(Qn+c_0)+b+2 \in B_j}} \omega_n \Bigg) \ll \frac{x}{(\log x)^{200}} \tag{10.2}$$

and that, for $\delta(B_j)$ as in (6.3),

$$\sum_{\substack{x \leqslant n < 2x \\ W(Qn+c_0)+b+2 \in B_j}} \omega_n = \frac{\delta(B_j) + o(1)}{\log x} \cdot \frac{QW}{\varphi(QW)} \sum_{x \leqslant n < 2x} \omega_n. \tag{10.3}$$

By the Fourier expansion of $\chi(n)$ in Lemma 3.2, it is enough to show these with

$$\omega_n = e\left(\left(W\frac{a}{q} + \beta\right) Qn\right), \tag{10.4}$$

where $0 \leqslant a < q \leqslant N/(\log N)^{100B}$, $(a, q) = 1$, $|\beta| \leqslant W(\log N)^{100B}/(qN)$ and, moreover, either $q \mid Q$ or $q/(q, Q^2) \geqslant (\log N)^A$. Furthermore, arguing as before (cf. (9.6)), we can assume

$$|\beta| < \min\left\{\frac{1}{4Q^2 q^2 (\log x)^{40000}}, \frac{(\log x)^{110B}}{x}\right\}. \tag{10.5}$$

10.1 Establishing (10.1)

After changes of variables $p, n' = W(Qn + c_0) + b$, we can rewrite the left-hand side of (10.1) with ω_n as in (10.4) essentially as

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$$\sum_{\substack{r\\(r,QW)=1}} \mu(r)^2\lambda_r\Bigg(\sum_{\substack{QWx\leqslant p<2QWx\\ p\equiv -2 \pmod r\\ p\equiv Wc_0+b \pmod{QW}}} e\bigg(\bigg(\frac{a}{q}+\frac{\beta}{W}\bigg)p\bigg)$$
$$-\frac{QW}{\varphi(QWr)}\sum_{\substack{QWx\leqslant n'<2QWx\\ n'\equiv Wc_0+b \pmod{QW}}} (\log n')^{-1} e\bigg(\bigg(\frac{a}{q}+\frac{\beta}{W}\bigg)n'\bigg)\Bigg).$$

In case $q \mid Q$, this is $O(x/(\log x)^{200})$ by Lemma 8.1 recalling (10.5). In case $q \nmid Q$, note that $q/(q,(QW)^2) > (\log N)^A/W^2$ and recall (10.5). We estimate the two terms corresponding to the sums over p and n' separately. The contribution from the term corresponding to the sum over n' can be satisfactorily estimated by Lemma 8.3 with $r = M = 1$ after partial summation. For the term corresponding the sum over p, Lemma 8.6 with $C = 20000$ implies the desired bound once A and B are large enough.

10.2 Establishing (10.2)

By the definition of B_1 in (6.2) we can write

$$\mathbf{1}_{W(Qn+c_0)+b+2\in B_1} = \sum_{\substack{mp=W(Qn+c_0)+b+2\\ p\geqslant x^{1/10}}} a_m,$$

where $a_m = 1$ if $m = p_1p_2$ for some $x^{1/10} \leqslant p_1 < x^{1/3-\varepsilon}$ and $x^{1/3-\varepsilon} \leqslant p_2 < (L_2(2x)/p_1)^{1/2}$, and $a_m = 0$ otherwise. Note that a_m is supported on $m \in [x^{1/3}, x^{2/3}]$. After a dyadic division and changes of variables $mp = W(Qn + c_0) + b + 2$, to prove (10.2) with ω_n as in (10.4) it suffices to show that for $M \in [x^{1/3}, x^{2/3}]$,

$$\sum_{\substack{r\\(r,QW)=1}} \mu(r)^2\lambda_r\Bigg(\sum_{\substack{QWx\leqslant mp<2QWx\\ mp\equiv 2 \pmod r\\ mp\equiv Wc_0+b+2 \pmod{QW}\\ M\leqslant m<2M}} a_m e\bigg(\bigg(\frac{a}{q}+\frac{\beta}{W}\bigg)mp\bigg)$$
$$-\frac{\varphi(QW)}{\varphi(QWr)}\sum_{\substack{QWx\leqslant mp<2QWx\\ mp\equiv Wc_0+b+2 \pmod{QW}\\ M\leqslant m<2M}} a_m e\bigg(\bigg(\frac{a}{q}+\frac{\beta}{W}\bigg)mp\bigg)\Bigg) \ll \frac{x}{(\log x)^{210}}.$$

In case $q \mid Q$, this follows from Lemma 8.2 applied twice (once with the $r = 1$ term only), recalling (10.5) and noting that we may add the restriction $(mp, QWr) = 1$ in the second sum above at a negligible cost, since for each r there are $O(x^{0.9})$ values of mp with $(mp, QWr) > 1$. In case $q \nmid Q$, note that $q/(q,(QW)^2) > (\log N)^A/W^2$ and recall (10.5). We estimate the two sums separately. The easier second sum can be estimated by Lemma 8.3 with $r = 1$. The first sum can be estimated by Lemma 8.4 (after factorizing λ_r) with $C = 20000$ once A is large enough.

Hypothesis (10.2) for B_2 follows similarly noting that

$$\mathbf{1}_{W(Qn+c_0)+b+2\in B_2} = \sum_{\substack{mp=W(Qn+c_0)+b+2\\ p\geqslant x^{1/10}}} a_m,$$

where $a_m = 1$ if $m = p_1p_2$ for some $x^{1/3-\varepsilon} \leqslant p_1 \leqslant p_2 \leqslant (L_2(2x)/p_1)^{1/2}$ and $a_m = 0$ otherwise; thus, a_m is supported on $m \in [x^{2/3-2\varepsilon}, x^{2/3+o(1)}]$, so that our type II results (Lemmas 8.2 and 8.4) are still applicable.

10.3 Establishing (10.3)

In case $q \mid Q$, by partial summation it is enough to prove (10.3) in case $\beta = 0$ (strictly speaking one should consider the interval $n \in [x, x']$ instead of $n \in [x, 2x)$ but this makes no difference). Since $q \mid Q$, we have $\omega_n \equiv 1$. By a change of variables $n' = W(Qn + c_0) + b + 2$, it suffices to show that

$$\sum_{\substack{QWx \leqslant n' < 2QWx \\ n' \equiv Wc_0 + b + 2 \pmod{QW}}} \mathbf{1}_{n' \in B_j} = \frac{\delta(B_j) + o(1)}{\varphi(QW)} \cdot \frac{QWx}{\log x},$$

which follows easily from the prime number theorem in arithmetic progressions. In case $q \nmid Q$, both sides of (10.3) are easily shown to be small using the argument from the previous subsection: the left-hand side can be estimated by Lemma 8.3 and the right-hand side can be estimated by Lemma B.2.

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Appendix A. Proof of the generalized Chen's theorem

In this section we prove Theorem 6.4.

A.1 The linear sieve

For a (finitely supported) sequence $\mathcal{A} = (a_m)$ of non-negative numbers we write $|\mathcal{A}| = \sum_m a_m$ and $\mathcal{A}_d = (a_{dm})_m$. We also define a sieving function

$$S(\mathcal{A}, z) = \sum_{(m, P(z)) = 1} a_m,$$

where

$$P(z) = \prod_{p < z} p.$$

To bound $S(\mathcal{A}, z)$ we need some information about $\mathcal{A}$. We will assume that, for all square-free integers d, we have

$$|\mathcal{A}_d| = \frac{g(d)}{d} X + r(\mathcal{A}, d),$$

where $g(d)$ is multiplicative and X is independent of d. Further, let

$$V(z) = \prod_{p \mid P(z)} \left(1 - \frac{g(p)}{p}\right).$$

We will use the linear sieve with a well-factorable error term due to Iwaniec [Iwa80]. For the following statement, see [FI10, Theorems 12.19 and 12.20].

LEMMA A.1. *Let $2 \leqslant z \leqslant D^{1/2}$ and $s = \log D / \log z$. Let $\varepsilon > 0$ be small enough and let $L(\varepsilon) = e^{1/\varepsilon^3}$. Assume that, for some absolute constant $K > 1$,*

$$\prod_{z_1 \leqslant p < z_2} \left(1 - \frac{g(p)}{p}\right)^{-1} \leqslant K \frac{\log z_2}{\log z_1}$$

for all $z_2 \geqslant z_1 \geqslant 2$. *Then*

$$S(\mathcal{A}, z) \leqslant XV(z)(F(s) + O_K(\varepsilon)) + \sum_{l<L(\varepsilon)} \sum_{d|P(z)} \lambda_l^+(d) r(\mathcal{A}, d)$$

and

$$S(\mathcal{A}, z) \geqslant XV(z)(f(s) - O_K(\varepsilon)) - \sum_{l<L(\varepsilon)} \sum_{d|P(z)} \lambda_l^-(d) r(\mathcal{A}, d).$$

Here, for each l, $\lambda_l^\pm$ *are well-factorable functions of level* D, *and* $F, f : [1, \infty) \to \mathbb{R}_{\geqslant 0}$ *are the continuous solutions to the system*

$$\begin{cases} sF(s) = 2e^\gamma & \text{if } 1 \leqslant s \leqslant 3, \\ sf(s) = 0 & \text{if } 1 \leqslant s \leqslant 2, \\ (sF(s))' = f(s-1) & \text{if } s > 3, \\ (sf(s))' = F(s-1) & \text{if } s > 2. \end{cases}$$

A.2 Introducing Chen's weights

Write $\mathcal{A} = (a_m)$ for the sequence defined by

$$a_m = \begin{cases} \omega_n \cdot \mathbf{1}_{L_1(n) \in \mathbb{P}} & m = L_2(n) \text{ for some } x \leqslant n < 2x, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{A}$ is supported on $L_2(x) \leqslant m < L_2(2x)$.

Using a slight modification of the weighted sieve method of Chen, we consider

$$S = \sum_{\substack{m \\ (m, P(x^{1/10}))=1}} a_m \Bigg(1 - \frac{1}{2} \sum_{\substack{x^{1/10} \leqslant p_1 < x^{1/3-\varepsilon} \\ p_1 | m}} 1 - \frac{1}{2} \sum_{\substack{m = p_1 p_2 p_3 \\ x^{1/10} \leqslant p_1 < x^{1/3-\varepsilon} \\ x^{1/3-\varepsilon} \leqslant p_2 \leqslant (L_2(2x)/p_1)^{1/2} \\ p_3 \geqslant x^{1/10}}} 1 - \sum_{\substack{m = p_1 p_2 p_3 \\ x^{1/3-\varepsilon} \leqslant p_1 \leqslant p_2 \leqslant (L_2(2x)/p_1)^{1/2} \\ p_3 \geqslant x^{1/10}}} 1 \Bigg).$$

Observe that the quantity in the parenthesis above is positive only if $m = P_2$ or $p^2 \mid m$ for some $x^{1/10} \leqslant p < x^{1/3-\varepsilon}$. Since the number of those m of the latter type is $O(x^{0.9})$, it suffices to show that

$$S \gg \frac{\mathfrak{S}(\mathcal{L})}{(\log x)^2} \sum_{x \leqslant n < 2x} \omega_n.$$

Using the sieve notation, we can write

$$\begin{aligned} S &= S(\mathcal{A}, x^{1/10}) - \frac{1}{2} \sum_{x^{1/10} \leqslant p < x^{1/3-\varepsilon}} S(\mathcal{A}_p, x^{1/10}) \\ &\quad - \frac{1}{2} \sum_{\substack{p_1, p_2, p_3 \\ x^{1/10} \leqslant p_1 < x^{1/3-\varepsilon} \\ x^{1/3-\varepsilon} \leqslant p_2 \leqslant (L_2(2x)/p_1)^{1/2} \\ p_3 \geqslant x^{1/10}}} a_{p_1 p_2 p_3} - \sum_{\substack{p_1, p_2, p_3 \\ x^{1/3-\varepsilon} \leqslant p_1 \leqslant p_2 \leqslant (L_2(2x)/p_1)^{1/2} \\ p_3 \geqslant x^{1/10}}} a_{p_1 p_2 p_3} \\ &= S_1 - \frac{1}{2} S_2 - \frac{1}{2} T_1 - T_2, \end{aligned}$$

say.

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A.3 Handling S_1 and S_2

Write

$$X = \frac{u_1}{\varphi(u_1)} \sum_{x \leqslant n < 2x} \frac{\omega_n}{\log L_1(n)}$$

and let g_1 be the multiplicative function defined by

$$g_1(d) = \begin{cases} 0 & (d, u_2(u_2v_1 - u_1v_2)) > 1, \\ \dfrac{d\varphi(u_1)}{\varphi(u_1 d)} & (d, u_2(u_2v_1 - u_1v_2)) = 1. \end{cases}$$

Since $|\mathcal{A}_d| = 0$ whenever $(d, u_2(u_2v_1 - u_1v_2)) > 1$, we have, by Hypothesis 6.3,

$$\sum_{d|P(x^{1/10})} \lambda_d \left(|\mathcal{A}_d| - \frac{g_1(d)}{d} X \right) \ll (\log x)^{-10} \sum_{x \leqslant n < 2x} \omega_n,$$

for any well-factorable function λ of level $D = x^{1/2-\varepsilon}$.

Hence, by Lemma A.1 with $z = x^{1/10}$,

$$S_1 \geqslant XV_1(x^{1/10})(f(5 - 10\varepsilon) - o(1)) - O\left((\log x)^{-9} \sum_{x \leqslant n < 2x} \omega_n \right),$$

where

$$V_1(z) = \prod_{p|P(z)} \left(1 - \frac{g_1(p)}{p}\right) = \prod_{\substack{p<z \\ p|u_1, p\nmid u_2}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid u_1u_2(u_1v_2 - u_2v_1)}} \left(1 - \frac{1}{p-1}\right).$$

Similarly, for any $2P \geqslant P' \geqslant P \in [x^{1/10}, x^{1/3-\varepsilon}]$ and any well-factorable bounded function λ of level $x^{1/2-\varepsilon}/P$ we have, by Hypothesis 6.3,

$$\sum_{P \leqslant p < P'} \sum_{d|P(x^{1/10})} \lambda_d \left(|\mathcal{A}_{pd}| - \frac{g_1(d)}{d} \frac{g_1(p)}{p} X \right) \ll (\log x)^{-10} \sum_{x \leqslant n < 2x} \omega_n,$$

since $|\mathcal{A}_{pd}| = 0$ whenever $(d, u_2(u_2v_1 - u_1v_2)) > 1$ and also $(p, d) = 1$ whenever $d|P(x^{1/10})$.

By Lemma A.1 with $s = \log(x^{1/2-\varepsilon}/P)/\log x^{1/10} = 5 - 10\varepsilon - 10 \log P/\log x$, we obtain

$$S_2 \leqslant \sum_{x^{1/10} \leqslant p < x^{1/3-\varepsilon}} \frac{g_1(p)}{p} XV_1(x^{1/10})(F(5 - 10\varepsilon - 10 \log p/\log x) + o(1)) + O\left((\log x)^{-9} \sum_{x \leqslant n < 2x} \omega_n \right).$$

Using the fact that

$$X = \frac{u_1}{\varphi(u_1)} \cdot \frac{1 + o(1)}{\log x} \sum_{x \leqslant n < 2x} \omega_n$$

since $\log L_1(n) = (1 + o(1)) \log L_1(x) = (1 + o(1)) \log x$, we conclude that

$$S_1 - \frac{1}{2} S_2 \geqslant \frac{V(x^{1/10})}{\log x} \left(f(5 - 10\varepsilon) - \frac{1}{2} \int_{1/10}^{1/3-\varepsilon} F(5 - 10\varepsilon - 10t) \frac{dt}{t} \right) (1 - o(1)) \sum_{x \leqslant n < 2x} \omega_n,$$

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where

$$V(z)=V_1(z)\frac{u_1}{\varphi(u_1)}=\prod_{p\mid(u_1,u_2)}\frac{p}{p-1}\prod_{\substack{p\leqslant z\\ p\nmid u_1u_2(u_1v_2-u_2v_1)}}\left(1-\frac{1}{p-1}\right).$$

A.4 Handling T_1 and T_2

Let $j\in\{1,2\}$. For B_j defined as in (6.2), we write

$$X_j=\sum_{\substack{x\leqslant n<2x\\ L_2(n)\in B_j}}\omega_n$$

and let g_2 be the multiplicative function defined by

$$g_2(d)=\begin{cases}0 & (d,u_1(u_2v_1-u_1v_2))>1,\\ \dfrac{d\varphi(u_2)}{\varphi(u_2d)} & (d,u_1(u_2v_1-u_1v_2))=1.\end{cases}$$

We consider the sequence $\mathcal{B}^{(j)}=(b_m^{(j)})$ defined by

$$b_m^{(j)}=\begin{cases}\omega_n\cdot\mathbf{1}_{L_2(n)\in B_j} & m=L_1(n)\text{ for some }x\leqslant n<2x,\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\mathcal{B}^{(j)}$ is supported on $L_1(x)\leqslant m<L_1(2x)$, and that, for $j=1,2$,

$$T_j=\sum_{m\in\mathbb{P}}b_m^{(j)}\leqslant S(\mathcal{B}^{(j)},x^{1/6}).$$

Note also that, for $j=1,2$,

$$|\mathcal{B}_d^{(j)}|=\sum_{\substack{x\leqslant n<2x\\ d\mid L_1(n)}}\omega_n\mathbf{1}_{L_2(n)\in B_j}.$$

We may apply Hypothesis 6.3(ii) to obtain that

$$\sum_{d\mid P(x^{1/6})}\lambda_d\left(|\mathcal{B}_d^{(j)}|-\frac{g_2(d)}{d}X_j\right)\ll(\log x)^{-10}\sum_{x\leqslant n<2x}\omega_n$$

for any well-factorable function λ_d of level $D=x^{1/2-\varepsilon}$. Hence, by Lemma A.1 with $z=x^{1/6}$, we have

$$T_j\leqslant X_jV_2(x^{1/6})(F(3-6\varepsilon)+o(1))+O\left((\log x)^{-9}\sum_{x\leqslant n<2x}\omega_n\right),$$

where

$$V_2(z)=\prod_{\substack{p\leqslant z\\ p\mid u_2,p\nmid u_1}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leqslant z\\ p\nmid u_1u_2(u_1v_2-u_2v_1)}}\left(1-\frac{1}{p-1}\right).$$

By Hypothesis 6.3(iii) and using (6.3), we have

$$X_j\leqslant\frac{u_2}{\varphi(u_2)}\cdot\frac{\delta(B_j)+o(1)}{\log x}\sum_{x\leqslant n<2x}\omega_n.$$

Hence,

$$T_j\leqslant\frac{V(x^{1/6})}{\log x}F(3-6\varepsilon)\delta(B_j)(1+o(1))\sum_{x\leqslant n<2x}\omega_n,$$

since $V(z)=V_2(z)(u_2/\varphi(u_2))$.

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A.5 Final numerical work

We may write

$$V(z) = \biggl(\prod_{p|(u_1,u_2)} \frac{p}{p-1} \prod_{\substack{p>2 \\ p|u_1u_2(u_1v_2-u_2v_1)}} \frac{p-1}{p-2}\biggr) \prod_{2<p\leqslant z}\biggl(1-\frac{1}{p-1}\biggr),$$

and note that the two products in the parenthesis contribute $\gg \mathfrak{S}(\mathcal{L})$ by the definition of the singular series. Thus,

$$V(x^{1/6}) = \biggl(\frac{3}{5}+o(1)\biggr)V(x^{1/10}), \quad V(x^{1/10}) \gg \frac{\mathfrak{S}(\mathcal{L})}{\log x}.$$

Since all the bounds we have obtained are continuous in ε and the double integral in $\delta(B_2)$ from (6.3) tends to zero when $\varepsilon \to 0$, it suffices to verify that

$$f(5) - \frac{1}{2}\int_{1/10}^{1/3} F(5-10t)\frac{dt}{t} - \frac{1}{2}\cdot\frac{3}{5}F(3)\int_{1/10}^{1/3}\int_{1/3}^{(1-\alpha_1)/2} \frac{d\alpha_2\,d\alpha_1}{\alpha_1\alpha_2(1-\alpha_1-\alpha_2)} > 0$$

just like in Chen's work. This is shown, for instance, in [HR74, ch. 11].

Appendix B. Proof of the exponential sum estimates

In this appendix we prove a couple of very simple auxiliary lemmas as well as the exponential sum estimates stated in § 8.

Lemma B.1. *Let $N \geqslant Q \geqslant 1$ and c_0 be integers, and let $\beta \in \mathbb{R}$. Then*

$$\sum_{\substack{N\leqslant n<2N \\ n\equiv c_0 \pmod{Q}}} e(\beta n) = \frac{1}{Q}\sum_{N\leqslant n<2N} e(\beta n) + O(|\beta|N+1).$$

Proof. We can clearly assume that $0 \leqslant c_0 < Q$. Let us write $n = c_0 + kQ$, obtaining that

$$\begin{aligned}\sum_{\substack{N\leqslant n<2N \\ n\equiv c_0 \pmod{Q}}} e(\beta n) &= e(\beta c_0)\sum_{(N-c_0)/Q\leqslant k<(2N-c_0)/Q} e(\beta kQ) \\ &= (1+O(\beta Q))\sum_{N/Q\leqslant k<2N/Q} e(\beta kQ) + O(1) \\ &= \sum_{N/Q\leqslant k<2N/Q} e(\beta kQ) + O(|\beta|N+1).\end{aligned}$$

Since the last expression is independent of c_0, summing over $0 \leqslant c_0 < Q$, we see that

$$Q\sum_{N/Q\leqslant k<2N/Q} e(\beta kQ) = \sum_{N\leqslant n<2N} e(\beta n) + O((|\beta|N+1)Q),$$

and the claim follows. □

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Lemma B.2. *Let $Q, q \geqslant 1$ and a be integers such that $(a,q)=1$ and $(Q,q)<q$. Assume that $|\alpha - a/q| \leqslant 1/(2qQ)$ and let $c_0 \in \mathbb{Z}$. Then*

$$\Bigg| \sum_{\substack{N \leqslant n < 2N \\ n \equiv c_0 \pmod{Q}}} e(\alpha n) \Bigg| \ll \frac{q}{(Q,q)}.$$

Proof. Let us write $n = c_0 + kQ$, obtaining that

$$\Bigg| \sum_{\substack{N \leqslant n < 2N \\ n \equiv c_0 \pmod{Q}}} e(\alpha n) \Bigg| = \Bigg| \sum_{(N-c_0)/Q \leqslant k < (2N-c_0)/Q} e(\alpha k Q) \Bigg| \ll \frac{1}{\|\alpha Q\|} \leqslant \frac{1}{1/(2q/(Q,q))}. \qquad \square$$

B.1 Major arc estimates

Proof of Lemma 8.1. By partial summation, it is enough to prove the claim in case $\alpha = a/q$ (strictly speaking one should consider intervals $p, n \in [x, x']$ instead of $[x, 2x]$ but this makes no difference). Since $q \mid Q$, the left-hand side of the claim equals

$$\begin{aligned}
&\sum_{r \leqslant x^{1/2-\varepsilon}} \max_{(c,rQ)=1} \Bigg| \sum_{\substack{x \leqslant p < 2x \\ p \equiv c \pmod{rQ}}} 1 - \frac{Q}{\varphi(rQ)} \sum_{\substack{x \leqslant n < 2x \\ n \equiv c \pmod{Q}}} \frac{1}{\log n} \Bigg| \\
&\leqslant \sum_{r \leqslant x^{1/2-\varepsilon}} \max_{(c,rQ)=1} \Bigg| \sum_{\substack{x \leqslant p < 2x \\ p \equiv c \pmod{rQ}}} 1 - \frac{|\mathbb{P} \cap [x,2x)|}{\varphi(rQ)} \Bigg| + O(x(\log x)^{-C_1-C_2}) \\
&\leqslant \sum_{d \leqslant x^{1/2-\varepsilon/2}} \max_{(c,d)=1} \Bigg| \sum_{\substack{x \leqslant p < 2x \\ p \equiv c \pmod{d}}} 1 - \frac{|\mathbb{P} \cap [x,2x)|}{\varphi(d)} \Bigg| + O((\log x)^{-C_1-C_2}),
\end{aligned}$$

and the claim follows from the Bombieri–Vinogradov prime number theorem. $\square$

Proof of Lemma 8.2. Arguing similarly, Lemma 8.2 reduces to showing

$$\sum_{d \leqslant x^{1/2-\varepsilon/2}} \max_{(c,d)=1} \Bigg| \sum_{\substack{x \leqslant mn < 2x \\ mn \equiv c \pmod{d} \\ M \leqslant m < 2M}} a_m b_n - \frac{1}{\varphi(d)} \sum_{\substack{x \leqslant mn < 2x \\ (mn,d)=1 \\ M \leqslant m < 2M}} a_m b_n \Bigg| \ll \frac{x}{(\log x)^{2C_1+C_2+1}}$$

which follows from type II information used in the proof of the Bombieri–Vinogradov prime number theorem (see, e.g., [IK04, Theorem 17.4]). $\square$

B.2 Minor arc estimates for type I sums

Note that all the minor arc estimates are trivial if $q > x$, so that we can always assume that $q \leqslant x$. Lemma 8.3 follows easily from the following slight variant of a lemma usually used in type I estimates.

Lemma B.3. *Let $q \geqslant 1$ and a be integers such that $(a,q)=1$ and assume that $|\alpha - a/q| < 1/q^2$. For any $x \geqslant M \geqslant 1$ and any integer $k \geqslant 2$,*

$$\sum_{M \leqslant m < 2M} \tau_k(m) \min\left\{ \frac{x}{M}, \frac{1}{\|\alpha m\|} \right\} \ll_k \left(\frac{x}{q^{1/2}} + x^{1/2} M^{1/2} + x^{1/2} q^{1/2} \right) (\log 3x)^{k^2/2}.$$

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Proof. By the Cauchy–Schwarz inequality

$$\begin{aligned}&\bigg(\sum_{M\leqslant m<2M}\tau_k(m)\min\Big\{\frac{x}{M},\frac{1}{\|\alpha m\|}\Big\}\bigg)^2\\&\leqslant\bigg(\sum_{M\leqslant m<2M}\tau_k(m)^2\frac{x}{M}\bigg)\cdot\bigg(\sum_{M\leqslant m<2M}\min\Big\{\frac{x}{M},\frac{1}{\|\alpha m\|}\Big\}\bigg)\\&\ll_k x(\log 3x)^{k^2-1}\cdot\bigg(\frac{x}{q}+M+q\bigg)(\log 3x),\end{aligned}$$

by a standard ingredient in type I estimates (see, e.g., [IK04, Formula before Lemma 13.7]). □

Proof of Lemma 8.3. We can clearly assume that $M\leqslant x^{1/2}/Q$. Write S for the left-hand side of the claim, and write $\overline{m}$ for the inverse of $m \pmod{rQ}$. Then

$$S\leqslant\sum_{r\leqslant x^{1/2}}\max_{(c,rQ)=1}\sum_{\substack{M\leqslant m<2M\\(m,rQ)=1}}\bigg|\sum_{\substack{x/m\leqslant n<2x/m\\n\equiv c\overline{m}\pmod{rQ}}}e(\alpha mn)\bigg|.$$

Writing $n=c\overline{m}+krQ$, with k running over an interval with elements of size $x/(mrQ)\gg 1$, and summing the geometric series, we see that

$$S\ll\sum_{r\leqslant x^{1/2}}\sum_{M\leqslant m<2M}\min\Big\{\frac{x}{mrQ},\frac{1}{\|\alpha mrQ\|}\Big\}\leqslant\sum_{d\leqslant 2Mx^{1/2}}\tau(d)\min\Big\{\frac{x/Q}{d},\frac{1}{\|(\alpha Q)d\|}\Big\}.$$

By our assumption on α, we have $|\alpha Q-(Qa/h)/(q/h)|<1/q^2\leqslant 1/(q/h)^2$. Hence, after a dyadic division on d, Lemma B.3 gives

$$S\ll\bigg(\frac{x}{Q}\cdot\frac{1}{(q/h)^{1/2}}+\Big(\frac{x}{Q}\Big)^{1/2}(Mx^{1/2})^{1/2}+\Big(\frac{x}{Q}\Big)^{1/2}(q/h)^{1/2}\bigg)(\log x)^3.\qquad\square$$

B.3 Minor arc estimates for type II sums

In the proof of Lemma 8.4 we use the following auxiliary exponential sum estimate due to Mikawa [Mik00], in the proof of which one Fourier expands the min-function on the left-hand side and uses Weyl differencing.

Lemma B.4. *Let $|\alpha-a/q|<1/q^2$ for some $(a,q)=1$. For $0<M,J\leqslant x$, one has*

$$M\sum_{M\leqslant m<2M}\sum_{J\leqslant j<2J}\tau_3(j)\min\Big\{\frac{x}{m^2j},\frac{1}{\|\alpha m^2j\|}\Big\}\ll\bigg(M^2J+x^{3/4}\Big(\frac{x}{q}+\frac{x}{M}+q\Big)^{1/4}\bigg)(\log x)^8.$$

Proof of Lemma 8.4. Let us first note that in case $D\leqslant(\log x)^C$, we can combine $dr=d'\in[D',4D']$ with $2D'=2DR\leqslant 2(\log x)^CM/x^{1/2}\leqslant x/(MQ(\log x)^C)$ which is still at most the upper bound for D in Lemma 8.4. This allows us to assume that $R=1$ in case $D\leqslant(\log x)^C$; combining $dr=d'$ introduces at worst a divisor function $\tau(d')$, but the claim follows in any case if we can show the claimed upper bound for

$$I=\sum_{D\leqslant d<2D}\tau(d)\sum_{\substack{R\leqslant r<2R\\(r,c'dQ)=1}}\theta(d,r)\sum_{\substack{x\leqslant mn<2x\\mn\equiv c'\pmod r\\mn\equiv c_d\pmod{dQ}\\M\leqslant m<2M}}a_mb_ne(\alpha mn),$$

for any choice of residue class $c_d \pmod{dQ}$ with $(c_d, dQ) = 1$ and any choice of $\theta(d,r) \in \mathbb{C}$ with $|\theta(d,r)| = 1$. By the Cauchy–Schwarz inequality, we have

$$|I|^2 \ll DM(\log x)^6 \sum_{D \leqslant d < 2D} \sum_{M \leqslant m < 2M} \Bigg| \sum_{\substack{R \leqslant r < 2R \\ (r, c'dQ)=1}} \theta(d,r) \sum_{\substack{x/m \leqslant n < 2x/m \\ mn \equiv c' \pmod r \\ mn \equiv c_d \pmod{dQ}}} b_n e(\alpha mn) \Bigg|^2.$$

Expanding out the square, moving the sum over m inside, and noting that $|b_{n_1} b_{n_2}| \leqslant \tau(n_1)^2 + \tau(n_2)^2$, we obtain

$$|I|^2 \ll DM(\log x)^6 \sum_{D \leqslant d < 2D} \sum_{\substack{R \leqslant r_1, r_2 < 2R \\ (r_1, c'dQ)=(r_2, c'dQ)=1}} \sum_{x/2M \leqslant n_1, n_2 \leqslant 2x/M} \tau(n_1)^2$$
$$\times \Bigg| \sum_{\substack{M \leqslant m < 2M \\ x/n_j \leqslant m < 2x/n_j \\ mn_1 \equiv c' \pmod{r_1} \\ mn_2 \equiv c' \pmod{r_2} \\ mn_1 \equiv mn_2 \equiv c_d \pmod{dQ}}} e(\alpha m(n_1 - n_2)) \Bigg|.$$

The simultaneous congruences above are soluble if and only if $(n_1, r_1 dQ) = (n_2, r_2 dQ) = 1$ and $n_1 \equiv n_2 \pmod{dQ(r_1, r_2)}$, in which case they reduce to the single equation $m \equiv b \pmod{dQ[r_1, r_2]}$ for some b. Thus, substituting $m = b + kdQ[r_1, r_2]$ (and noticing $DQR^2 \leqslant M$), we see that the inner sum over m is

$$\ll \min\left(\frac{M}{dQ[r_1, r_2]}, \frac{1}{\|\alpha(n_1 - n_2) dQ[r_1, r_2]\|} \right).$$

Writing $n_1 = n_2 + \ell \cdot dQ(r_1, r_2)$, we have

$$\alpha(n_1 - n_2) dQ[r_1, r_2] = \alpha \ell (dQ)^2 (r_1, r_2)[r_1, r_2] = \alpha \ell (dQ)^2 r_1 r_2,$$

so that

$$|I|^2 \ll DM(\log x)^6 \sum_{D \leqslant d < 2D} \sum_{R \leqslant r_1, r_2 < 2R} \sum_{x/2M \leqslant n_1 \leqslant 2x/M} \tau(n_1)^2 \sum_{|\ell| \leqslant 2x/MdQ(r_1, r_2)}$$
$$\times \min\left(\frac{M}{dQ[r_1, r_2]}, \frac{1}{\|\alpha \ell (dQ)^2 r_1 r_2\|} \right)$$
$$\ll Dx(\log x)^9 \sum_{D \leqslant d < 2D} \sum_{R \leqslant r_1, r_2 < 2R} \sum_{|\ell| \leqslant 2x/MdQ(r_1, r_2)} \min\left(\frac{M}{dQ[r_1, r_2]}, \frac{1}{\|\alpha \ell (dQ)^2 r_1 r_2\|} \right).$$

The terms with $\ell = 0$ contribute to the right-hand side

$$\ll D^2 x (\log x)^9 \sum_{R \leqslant r_1, r_2 < 2R} \frac{M}{DQ[r_1, r_2]} \ll D^2 x (\log x)^{10} \frac{M}{DQ} \ll \frac{x^2}{Q^2} (\log x)^{-C+10}$$

by the assumption on D, which is acceptable. To treat the terms with $\ell \neq 0$, write $j = \ell r_1 r_2$ so that

$$0 < |j| < 4R^2 \cdot \frac{2x}{MDQ(r_1, r_2)} \leqslant \frac{8R^2 x}{MDQ}$$

and that

$$\frac{M}{dQ[r_1, r_2]} = \frac{M(r_1, r_2)}{dQ r_1 r_2} \ll \frac{M}{DQR^2} \cdot \frac{R^2 x}{MDQ|j|} \ll \frac{x}{(dQ)^2 |j|}.$$

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It follows that

$$|I|^2 \ll \frac{x^2}{Q^2}(\log x)^{-C+10} + Dx(\log x)^9 \sum_{D\leqslant d<2D} \sum_{0<|j|\leqslant 8R^2x/(MDQ)} \tau_3(j)\min\left(\frac{x}{(dQ)^2|j|}, \frac{1}{\|\alpha(dQ)^2 j\|}\right).$$

By a dyadic division, it suffices to show that

$$D\sum_{D\leqslant d<2D}\sum_{J\leqslant j<2J}\tau_3(j)\min\left(\frac{x/Q^2}{d^2 j}, \frac{1}{\|(\alpha Q)^2 d^2 j\|}\right) \ll \frac{x}{Q^2}\left(\frac{(\log x)^C}{(q/h)^{1/4}} + (\log x)^C Q\left(\frac{q}{x}\right)^{1/4} + \frac{1}{(\log x)^{C/4}}\right)(\log x)^8 \tag{B.1}$$

for $1 \leqslant J \leqslant 4R^2x/(DMQ)$. To prove this we divide into two cases depending on whether D is large or small.

Case 1. First assume that $D > (\log x)^C$. Note that, by assumption,

$$\left|\alpha Q^2 - \frac{aQ^2/h}{q/h}\right| \leqslant \frac{1}{4q^2(\log x)^{2C}} \leqslant \frac{1}{(q/h)^2}.$$

It is also easy to see that $D, J \leqslant x/Q^2$. We may thus apply Lemma B.4 to bound the left-hand side of (B.1) by

$$\ll \left(D^2 J + \left(\frac{x}{Q^2}\right)^{3/4}\left(\frac{x/Q^2}{q/h} + \frac{x/Q^2}{D} + q\right)^{1/4}\right)(\log x)^8$$
$$\ll \frac{x}{Q^2}\left(\frac{1}{(\log x)^C} + \frac{1}{(q/h)^{1/4}} + \frac{1}{(\log x)^{C/4}} + Q^{1/2}\left(\frac{q}{x}\right)^{1/4}\right)(\log x)^8$$

by the upper bound on J and the assumptions on D and R.

Case 2. Now assume that $D \leqslant (\log x)^C$. Recall that in this case we can assume that $R = 1$. In this case, for each fixed $D \leqslant d < 2D$ we have, by assumption,

$$\left|\alpha(dQ)^2 - \frac{a(dQ)^2}{q}\right| \leqslant \frac{1}{q^2}.$$

Moreover, the denominator of the fraction $a(dQ)^2/q$ is at least $q/hD^2 \geqslant q/(h(\log x)^{2C})$ after reducing it to the reduced form. Applying Lemma B.3 to the inner sum over j in (B.1) (noting that $J \leqslant x/(D^2Q^2)$), we may bound the left-hand side of (B.1) by

$$\ll D^2\left(\frac{x/(Q^2D^2)}{(q/(h(\log x)^{2C}))^{1/2}} + \left(\frac{x}{Q^2D^2}\right)^{1/2}\cdot\left(\frac{x}{DMQ}\right)^{1/2} + \left(\frac{x}{Q^2D^2}\right)^{1/2} q^{1/2}\right)(\log x)^{9/2}$$
$$\ll \frac{x}{Q^2}\left(\frac{(\log x)^C}{(q/h)^{1/2}} + \frac{1}{(\log x)^{C/2}} + Q\left(\frac{q}{x}\right)^{1/2}(\log x)^C\right)(\log x)^{9/2}$$

by our assumptions on D and M. $\square$

B.4 Minor arc estimates for sums over primes

Proof of Lemma 8.6. By partial summation it is enough to consider the claim of Lemma 8.6 with

$$\sum_{\substack{x\leqslant p<2x\\ p\equiv c' \pmod r\\ p\equiv c \pmod Q}} e(\alpha p) \quad \text{replaced by} \quad \sum_{\substack{x\leqslant n<2x\\ n\equiv c' \pmod r\\ n\equiv c \pmod Q}} \Lambda(n)e(\alpha n).$$

Then, by a dyadic splitting on r, Vaughan's identity (see [IK04, Proposition 13.4]) with $y=z=x^{2/3}$, and further partial summation, it is then enough to show that, for any $M\leqslant x^{1/3}$, $R\leqslant x^{1/2-\varepsilon/2}$ and any $|a_m|\leqslant 1$, one has the type I estimate

$$\begin{aligned}\Bigg|\sum_{\substack{R\leqslant r<2R\\ (r,c'Q)=1}} \mu(r)^2\lambda_r \sum_{\substack{x\leqslant mn<2x\\ mn\equiv c' \pmod r\\ mn\equiv c \pmod Q\\ M\leqslant m<2M}} a_m e(\alpha mn)\Bigg| \\ \leqslant \frac{x}{Q}\cdot\left(\frac{(\log x)^{C/2}}{(q/h)^{1/8}} + (\log x)^{C/2}Q^{1/2}\frac{q^{1/8}}{x^{1/8}} + \frac{1}{(\log x)^{C/8}}\right)(\log x)^{12}.\end{aligned} \tag{B.2}$$

and that, for any $x^{1/3}\leqslant M\leqslant x^{2/3}$, $R\leqslant x^{1/2-\varepsilon/2}$ and any $|a_k|,|b_k|\leqslant\tau(k)$, one has the type II estimate

$$\begin{aligned}\Bigg|\sum_{\substack{R\leqslant r<2R\\ (r,c'Q)=1}} \mu(r)^2\lambda_r \sum_{\substack{x\leqslant mn<2x\\ mn\equiv c' \pmod r\\ mn\equiv c \pmod Q\\ M\leqslant m<2M}} a_m b_n e(\alpha mn)\Bigg| \\ \leqslant \frac{x}{Q}\cdot\left(\frac{(\log x)^{C/2}}{(q/h)^{1/8}} + (\log x)^{C/2}Q^{1/2}\frac{q^{1/8}}{x^{1/8}} + \frac{1}{(\log x)^{C/8}}\right)(\log x)^{12}.\end{aligned} \tag{B.3}$$

The estimate (B.2) follows directly from Lemma 8.3. On the other hand, to estimate (B.3), by symmetry we may assume that $M\geqslant x^{1/2}$, and we take $D=\min\{R, x/(MQ(\log x)^C)\}$ and $R'=R/D$. Note that $D\geqslant\min\{R,x^{1/3-\varepsilon}\}$ and, thus, for either possibility of λ_r from Hypothesis 6.3(i), by the well-factorability property we always get for the left-hand side of (B.3) the upper bound

$$\begin{aligned}&\sum_{\substack{d\leqslant D\\ (d,c'Q)=1}}\sum_{\substack{r'\leqslant R'\\ (r',c'dQ)=1}}\Bigg|\sum_{\substack{x\leqslant mn<2x\\ mn\equiv c' \pmod{dr'}\\ mn\equiv c \pmod Q\\ M\leqslant m<2M}} a_m b_n e(\alpha mn)\Bigg| \\ &\leqslant \sum_{\substack{d\leqslant D\\ (d,Q)=1}}\max_{(c_0,dQ)=1}\sum_{\substack{r'\leqslant R'\\ (r',c'dQ)=1}}\Bigg|\sum_{\substack{x\leqslant mn<2x\\ mn\equiv c' \pmod{r'}\\ mn\equiv c_0 \pmod{dQ}\\ M\leqslant m<2M}} a_m b_n e(\alpha mn)\Bigg|,\end{aligned}$$

and the claim follows from Lemma 8.4 after dividing the variables d and r' dyadically. □

Let us note that, in the previous proof, in order to apply Lemma 8.4 when M is close to $x^{2/3}$, we needed to take D to be slightly smaller than $x^{1/3}$. This is in contrast to what was claimed in [Mat09, Remark 10], but the caused mistake in the proof of [Mat09, Theorem 2] could be easily fixed by using a slight modification of Chen's weights used here in Appendix A.

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Proof of Lemma 8.5. The proof is analogous to the proof of Lemma 8.6 but, since $r \leqslant x^{1/8}$, after a dyadic division to $R \leqslant r < 2R$, we can always take $D = R$ when we apply Lemma 8.4. $\square$

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Vinogradov's three primes theorem with almost twin primes

Kaisa Matomäki ksmato@utu.fi
Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland

Xuancheng Shao Xuancheng.Shao@maths.ox.ac.uk
Mathematical Institute, Radcliffe Observatory Quarter, Woodstock Road,
Oxford OX2 6GG, UK