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GENERIC DIFFERENTIABILITY OF LOCALLY LIPSCHITZ FUNCTIONS ON PRODUCT SPACES

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Although it is known that locally Lipschitz functions are densely differentiable on certain classes of Banach spaces, it is a minimality condition on the subdifferential mapping of the function which enables us to guarantee that the set of points of differentiability is a residual set. We characterise such minimality by a quasi continuity property of the Dini derivatives of the function and derive sufficiency conditions for the generic differentiability of locally Lipschitz functions on a product space.

1. INTRODUCTION

A real valued function ψ on an open subset A of a normed linear space X is *locally Lipschitz* if for each $x_0 \in A$ there exists a $K_0 > 0$ and $\delta_0 > 0$ such that

$$|\psi(x) - \psi(y)| \leqslant K_0 \|x - y\| \text{ for all } x, y \in B(x_0; \delta_0).$$

The function ψ is *Gâteaux differentiable at $x \in A$ in the direction $y \in X$* if

$$\psi'(x)(y) \equiv \lim_{\lambda \to 0} \frac{\psi(x + \lambda y) - \psi(x)}{\lambda}$$

exists and is *Gâteaux differentiable at $x \in A$* if it is Gâteaux differentiable at x in all directions $y \in X$ and $\psi'(x)$ is a continuous linear functional on X. The function ψ is *Fréchet differentiable* at $x \in A$ if it is Gâteaux differentiable at x and the limit is approached uniformly for all $y \in S(X)$. A Banach space X is said to be *smoothable* if there exists an equivalent norm on X which is Gâteaux differentiable everywhere except at the origin. A Banach space X is an *Asplund* space if every continuous convex function on an open convex subset of X is Fréchet differentiable on a residual subset of its domain.

The determination of differentiability properties of locally Lipschitz functions is particularly important for applications in optimisation. The differentiability of a locally

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Lipschitz function ψ on an open subset A of a normed linear space X is studied using the *Clarke directional derivative*

$$\psi^0(x)(y) \equiv \limsup_{\substack{z \to x \\ \lambda \to 0+}} \frac{\psi(z+\lambda y) - \psi(z)}{\lambda}$$

at each $x \in A$ in the direction $y \in X$ and $\psi^0(x)(y)$ is a continuous sublinear functional in y. The *Clarke subdifferential*

$$\partial\psi^0(x) \equiv \{f \in X^* : f(y) \leqslant \psi^0(x)(y) \text{ for all } y \in X\}$$

at each $x \in A$, is a non-empty weak* compact convex set.

The key result generalising the classical Rademacher Theorem from Euclidean to Banach spaces was given by David Preiss, [**14**].

PREISS' THEOREM. *A locally Lipschitz function ψ on an open subset A of a smoothable (Asplund) space is Gâteaux (Fréchet) differentiable on a dense subset D of A and the Clarke subdifferential is generated by the Gâteaux (Fréchet) derivatives; that is, given $x \in A$*

$$\partial\psi^0(x) = \bigcap_{r>0} \overline{co}^{w^*} \{\psi'(z) : z \in B(x;r) \cap D\}.$$

However, the set of points of differentiability need not be a residual subset of the domain and this can inhibit our analysis.

A set-valued mapping Φ from a topological space A into subsets of a linear topological space X is *upper semi-continuous* at $a \in A$ if given an open subset W of X such that $\Phi(a) \subseteq W$ there exists an open neighbourhood U of a such that $\Phi(U) \subseteq W$. When Φ is upper semi-continuous on A and $\Phi(a)$ is convex and compact for each $a \in A$ we call Φ a *cusco* on A. We say that Φ is a *minimal cusco* on A if its graph does not contain the graph of any other cusco with the same domain.

For a locally Lipschitz function ψ on an open subset A of a normed linear space X, the *Clarke subdifferential mapping* $x \mapsto \partial\psi^0(x)$ is a weak* cusco on A but is not in general a minimal weak* cusco.

A locally Lipschitz function ψ on an open subset A of a normed linear space X is said to be *strictly differentiable at $x \in A$ in the direction $y \in X$* if

$$\lim_{\substack{z \to x \\ \lambda \to 0+}} \frac{\psi(z+\lambda y) - \psi(z)}{\lambda}$$

exists and is said to be *strictly differentiable at x* if it is strictly differentiable at x in all directions $y \in X$. Further, ψ is said to be *uniformly strictly differentiable* at x if this limit is approached uniformly for all $y \in S(X)$. Obviously, if ψ is strictly differentiable at $x \in A$ then ψ is Gâteaux differentiable at x. Further, if ψ is uniformly strictly differentiable at $x \in A$ then ψ is Fréchet differentiable at x.

Clearly, ψ is strictly differentiable at $x \in A$ if and only if $\partial\psi^0(x)$ is singleton. But also, ψ is uniformly strictly differentiable at $x \in A$ if and only if $\partial\psi^0(x)$ is singleton and the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is norm upper semi-continuous at x, [**5**, p.374]. With certain minimal weak* cuscos we can associate significant residual subsets of the domain.

Proposition 1.1. *Consider a minimal weak* cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X.*

(i) *If X is smoothable then Φ is single-valued on a residual subset of A,* [**15**].
(ii) *If X is Asplund then Φ is single-valued and norm upper semi-continuous at the points of a residual subset of A,* [**12**, p.106].

The implications for differentiability of locally Lipschitz functions are immediate.

Corollary 1.2. *A locally Lipschitz function ψ on an open subset A of a smoothable (Asplund) space X is strictly (uniformly strictly) differentiable on a residual subset of A if the subdifferential mapping $x \mapsto \partial\psi^0(x)$ on A is minimal.*

To establish this minimality for the subdifferential mapping can be a problem so there is considerable value in determining properties sufficient to guarantee it. Some work has already been done in this area, [**1, 2, 3**] and more recently [**11**].

Here we give a characterisation of minimality for the subdifferential mapping using quasi continuity and provide two sufficiency conditions for minimality on a product space. This in turn enables us to deduce sufficiency conditions for the generic differentiability of locally Lipschitz functions on a product space.

2. A characterisation of minimal subdifferential mappings

The minimality of a cusco has the following useful characterisation, [**8**, p.252].

Lemma 2.1. *A cusco Φ from a topological space A into subsets of a separated locally convex X is a minimal cusco if and only if for any open set U in A and open half-space W in X where $\Phi(U) \cap W \neq \emptyset$, there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$.*

Proof: Suppose that Φ is a minimal cusco on A and for an open set $U \subseteq A$ and open half-space W we have $\Phi(U) \cap W \neq \emptyset$. If there exists an $a \in U$ such that $\Phi(a) \subseteq W$ then by the upper semi-continuity of Φ there exists a non-empty open neighbourhood V of a such that $\Phi(V) \subseteq W$. If not, then $\Phi(a) \cap C(W) \neq \emptyset$ for every

$a \in U$. Consider the set-valued mapping Ψ from A into subsets of X where

$$\Psi(a) = \begin{cases} \Phi(a) \cap C(W) & \text{for } a \in U \\ \Phi(a) & \text{for } a \notin U. \end{cases}$$

Then Ψ is a cusco on A whose graph is contained in that of Φ. But this contradicts the minimality of Φ

Conversely, suppose that Φ is a cusco which is not minimal. Then there exists a cusco Ψ whose graph is contained in that of Φ but for some $a_0 \in A$ there exists an $x_0 \in \Phi(a_0) \setminus \Psi(a_0)$. Since $\Psi(a_0)$ is convex and compact there exist disjoint open half spaces W_1 and W_2 such that $\Psi(a_0) \subseteq W_1$, and $x_0 \in W_2$. Since Ψ is upper semi-continuous at a_0 there exists an open neighbourhood U of a_0 such that $\Psi(U) \subseteq W_1$. But then $\Phi(U) \cap W_2 \neq \emptyset$ and $\Phi(a) \cap C(W_2) \neq \emptyset$ for all $a \in U$. □

The minimality of a weak* cusco can be characterised by the minimality of associated cuscos into subsets of the real numbers, [**11**, Proposition 1.4].

Lemma 2.2. *Consider a weak* cusco Φ from a topological space A into subsets of X^* the dual of a Banach space X. Then Φ is a minimal weak* cusco if and only if for each $x \in S(X)$ the set-valued mapping T_x from A into subsets of $\mathbb{R}$ where $a \mapsto T_x(a) = \widehat{x}(\Phi(a))$ is a minimal cusco.*

Proof: Suppose that Φ is a minimal weak* cusco on A. Given $x \in S(X)$ it is easy to see that T_x is a cusco; we show that T_x is minimal. Given $\alpha \in \mathbb{R}$ and an open set U in A such that $T_x(U) \cap (\alpha, \infty) \neq \emptyset$ then for some $a \in U$ and $f \in \Phi(a)$ we have $\widehat{x}(f) > \alpha$. Consider W the open half-space, $W \equiv \{f \in X^* : f(x) > \alpha\}$. Now $\Phi(U) \cap W \neq \emptyset$. But since Φ is a minimal weak* cusco, from Lemma 2.1 there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. That is, $\widehat{x}(\Phi(a)) > \alpha$ for all $a \in V$ which implies that $T_x(V) \subseteq (\alpha, \infty)$. A similar argument applies for subsets of $\mathbb{R}$ of the form $(-\infty, \alpha)$ and we conclude from Lemma 2.1 that T_x is a minimal cusco on A.

Conversely, suppose that Φ is not a minimal weak* cusco on A. Then there exists a weak* cusco Ψ on A whose graph is strictly contained in that of Φ. So there exists an $a_0 \in A$ such that $\Psi(a_0) \subsetneq \Phi(a_0)$ and an $x_0 \in S(X)$ such that max $\widehat{x}_0(\Phi(a_0)) > \max \widehat{x}_0(\Psi(a_0))$. Now consider the two set-valued mappings T_{x_0} and S_{x_0} from A into subsets of $\mathbb{R}$ where $a \mapsto T_{x_0}(a) = \widehat{x}_0(\Phi(a))$ and $a \mapsto S_{x_0}(a) = \widehat{x}_0(\Psi(a))$. Clearly, $S_{x_0}(a) \subseteq T_{x_0}(a)$ for all $a \in A$. However, max $S_{x_0}(a_0) = \max \widehat{x}_0(\Psi(a_0)) < \max \widehat{x}_0(\Phi(a_0)) = \max T_{x_0}(a_0)$ so $S_{x_0}(a_0) \neq T_{x_0}(a_0)$ and we conclude that T_{x_0} is not a minimal cusco on A. □

For a locally Lipschitz function ψ on an open subset A of a normed linear space X, the *upper Dini derivative* of ψ at $x \in A$ in the direction $y \in X$ is

$$\psi^+(x)(y) \equiv \limsup_{\lambda \to 0+} \frac{\psi(x + \lambda y) - \psi(x)}{\lambda}$$

and the *lower Dini derivative* of ψ at $x \in A$ in the direction $y \in X$ is

$$\psi^-(x)(y) \equiv \liminf_{\lambda \to 0+} \frac{\psi(x+\lambda y) - \psi(x)}{\lambda}.$$

An equivalent formulation for the subdifferential of ψ at $x \in A$ is

$$\partial\psi^0(x) \equiv \left\{ f \in X^* : -(-\psi)^0(x)(y) \leqslant f(y) \leqslant \psi^0(x)(y) \text{ for all } y \in X \right\}$$

and we note that

$$-(-\psi)^0(x)(y) = \liminf_{\substack{z \to x \\ \lambda \to 0+}} \frac{\psi(z+\lambda y) - \psi(z)}{\lambda}.$$

It is convenient to express the Clarke directional derivatives in terms of the directional and Dini derivatives, [7, p.837].

Lemma 2.3. *Consider a locally Lipschitz function ψ on an open subset A of a normed linear space X. Given $y \in X$ and $x \in A$,*

$$\psi^0(x)(y) = \limsup_{\substack{z \to x \\ z \in D_y}} \psi'(z)(y) = \limsup_{z \to x} \psi^+(z)(y)$$

$$-(-\psi)^0(x)(y) = \liminf_{\substack{z \to x \\ z \in D_y}} \psi'(z)(y) = \liminf_{z \to x} \psi^-(z)(y)$$

where D_y is the set of points in A where ψ is Gâteaux differentiable in the direction y.

Proof: Clearly, $\psi^0(x)(y) \geqslant \limsup_{z \to x} \psi^+(z)(y) \geqslant \limsup_{\substack{z \to x \\ z \in D_y}} \psi'(z)(y)$. But also, given $\varepsilon > 0$, in any neighbourhood of x there exists a $z_0 \in A$ and $\lambda_0 > 0$ such that $z_0 + \lambda_0 y \in A$ and

$$\frac{\psi(z_0+\lambda_0 y) - \psi(z_0)}{\lambda_0} > \psi^0(x)(y) - \varepsilon.$$

Consider ψ restricted to the interval $[z_0, z_0+\lambda_0 y]$. Since ψ is locally Lipschitz it follows from Lebesgue's Differentiation Theorem that there exists a $0 \leqslant \lambda_1 \leqslant \lambda_0$ such that

$$\psi'(z_0+\lambda_1 y)(y) \geqslant \frac{\psi(z_0+\lambda_0 y) - \psi(z_0)}{\lambda_0}.$$

So $\limsup_{z \to x} \psi^+(z)(y) \geqslant \limsup_{\substack{z \to x \\ z \in D_y}} \psi'(z)(y) \geqslant \psi^0(x)(y)$ and our first result follows. Now for all $x \in A$ and $y \in X$, $-(-\psi)^0(x)(y) = -\psi^0(x)(-y)$ and $\psi^-(x)(y) = -(-\psi)^+(x)(y)$. So $-(-\psi)^0(x)(y) = -\limsup_{\substack{z \to x \\ z \in D_y}} \psi'(z)(-y) = \liminf_{\substack{z \to x \\ z \in D_y}} \psi'(z)(y)$. But also

$$-(-\psi)^0(x)(y) = -\limsup_{\substack{z \to x \\ z \in D_y}} (-\psi)'(z)(y) = -\limsup_{z \to x} (-\psi)^+(z)(y) = \liminf_{z \to x} \psi^-(z)(y). \quad \square$$

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From Preiss' Theorem we see that for a locally Lipschitz function on an open subset of a smoothable (Asplund) space the subdifferential is generated by the dense set of derivatives of the function and so in this case we have a tighter result.

LEMMA 2.4. *Consider a locally Lipschitz function ψ on an open subset A of a smoothable (Asplund) space X. Given $y \in X$ and $x \in A$,*

$$\psi^0(x)(y) = \limsup_{\substack{z \to x \\ z \in D}} \psi'(z)(y) = \limsup_{z \to x} \psi^+(z)(y)$$

$$-(-\psi)^0(x)(y) = \liminf_{\substack{z \to x \\ z \in D}} \psi'(z)(y) = \liminf_{z \to x} \psi^-(z)(y)$$

where D is the set of points where ψ is Gâteaux (Fréchet) differentiable on A.

Consider a real valued function ϕ on a topological space A. Now ϕ is said to be *quasi upper semi-continuous* at $a_0 \in A$ if given $\varepsilon > 0$ and an open neighbourhood U of a_0, there exists a non-empty open set $V \subseteq U$ such that $\phi(a) < \phi(a_0) + \varepsilon$ for all $a \in V$, and is said to be *quasi lower semi-continuous* at $a_0 \in A$ if $-\phi$ is quasi upper semi-continuous at a_0. The function ϕ is said to be *quasi continuous* at $a_0 \in A$ if given $\varepsilon > 0$ and an open neighbourhood U of a_0, there exists a non-empty open set $V \subseteq U$ such that

$$\phi(a_0) - \varepsilon < \phi(a) < \phi(a_0) + \varepsilon \text{ for all } a \in V,$$

If ϕ is quasi upper semi continuous on a Baire space A then ϕ is continuous on a residual subset of A, [**4**, p.369].

We now present our characterisation for minimality of the Clarke subdifferential mapping in terms of quasi-continuity. The result is similar to that given in [**11**, Theorem 2.14].

THEOREM 2.5. *For a locally Lipschitz function ψ on an open subset A of a normed linear space X, the following are equivalent.*

(i) *the Clarke subdifferential mapping $x \mapsto \partial\psi^0(x)$ is a minimal weak* cusco on A,*

(ii) *for each $y \in X$, $\psi^+(x)(y)$ is quasi upper semi-continuous on A,*

(iii) *for each $y \in X$, $\psi^-(x)(y)$ is quasi lower semi-continuous on A,*

(iv) *for each $y \in X$, $\psi'(x)(y)$ is quasi upper semi-continuous on D_y,*

(v) *for each $y \in X$, $\psi'(x)(y)$ is quasi lower semi-continuous on D_y,*

where D_y is the set of points in A where ψ is Gâteaux differentiable in the direction y.

PROOF: (i) $\Rightarrow$ (ii) Given $x \in A$ and $\varepsilon > 0$ and any neighbourhood U of x there exists a non-empty open set $V \subseteq U$ such that

$$\left[-(-\psi)^0(z)(y), \psi^0(z)(y)\right] \subseteq \left(-\infty, -(-\psi)^0(x)(y) + \varepsilon\right) \text{ for all } z \in V.$$

Then for each $z' \in V$ there exists an open neighbourhood V' of z' where $V' \subseteq V$ such that

$$\psi^+(z)(y) < -(-\psi)^0(x)(y) + \varepsilon < \psi^+(x)(y) + \varepsilon \text{ for all } z \in V';$$

that is, $\psi^+(x)(y)$ is quasi upper semi-continuous on A.

(i) $\Rightarrow$ (iii) Given $x \in A$ and $\varepsilon > 0$ and any neighbourhood U of x there exists a non-empty open set $V \subseteq U$ such that

$$\left[-(-\psi)^0(z)(y), \psi^0(z)(y)\right] \subseteq \left(\psi^0(x)(y) - \varepsilon, \infty\right) \text{ for all } z \in V.$$

Then as in (i) $\Rightarrow$ (ii) we deduce that $\psi^-(x)(y)$ is quasi lower semi-continuous on A.

(ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v). It follows from Lebesgue's Differentiation Theorem that D_y is dense in A and so we have these results.

(iv) $\Leftrightarrow$ (v) Given $x \in D_y$, $\psi'(x)(y) = -\psi'(x)(-y)$. So $\psi'(x)(-y)$ is quasi upper semi-continuous on D_y if and only if $\psi'(x)(y)$ is quasi lower semi-continuous on D_y.

(iv) $\Rightarrow$ (i) Given $x \in A$ and $\varepsilon > 0$ and any neighbourhood U of x, by Lemma 2.3 there exists an $x' \in U \cap D_y$ such that

$$\psi'(x')(y) < -(-\psi)^0(x)(y) + \frac{\varepsilon}{2}.$$

Since $\psi'(x)(y)$ is quasi upper semi-continuous at x', there exists a non-empty open set $V \subseteq U$ such that

$$\psi'(z)(y) < \psi'(x')(y) + \frac{\varepsilon}{2} \text{ for all } z \in V \cap D_y.$$

But then

$$\psi^0(z)(y) \leqslant \psi'(x')(y) + \frac{\varepsilon}{2} < -(-\psi)^0(x)(y) + \varepsilon \text{ for all } z \in V.$$

So

$$\left[-(-\psi)^0(z)(y), \psi^0(z)(y)\right] \subseteq \left(-\infty, -(-\psi)^0(x)(y) + \varepsilon\right) \text{ for all } z \in V.$$

Now $\psi^0(x)(-y) = (-\psi)^0(x)(y)$ and $-(-\psi)^0(x)(-y) = -\psi^0(x)(y)$. So applying our results to $-y \in X$ and $x \in A$ and neighbourhood U of x there exists a non-empty open set $V \subseteq U$ such that

$$\left[-(-\psi)^0(z)(-y), \psi^0(z)(-y)\right] \subseteq \left(-\infty, -(-\psi)^0(x)(-y) + \varepsilon\right);$$

that is,

$$\left[-\psi^0(z)(y), (-\psi)^0(z)(y)\right] \subseteq \left(-\infty, -\psi^0(x)(y) + \varepsilon\right) \text{ for all } z \in V.$$

So

$$\left[-(-\psi)^0(z)(y), \psi^0(z)(y)\right] \subseteq \left(\psi^0(x)(y) - \varepsilon, \infty\right) \text{ for all } z \in V.$$

We conclude that the Clarke subdifferential mapping $x \mapsto \partial\psi^0(x)$ is a minimal weak* cusco on A. ∎

Using Lemma 2.4 we have a tighter result for a locally Lipschitz function on an open subset of a smoothable (Asplund) space.

THEOREM 2.6. *For a locally Lipschitz function ψ on an open subset A of a smoothable (Asplund) space X, the Clarke subdifferential mapping $x \mapsto \partial\psi^0(x)$ is a minimal weak* cusco on A if and only if for each $y \in X$, $\psi'(x)(y)$ is quasi upper semi-continuous on D, the set of points in A where ψ is Gâteaux (Fréchet) differentiable.*

The proof in one direction follows from Theorem 2.5 (i) $\Rightarrow$ (iv). In the other direction it is similar to Theorem 2.5 (iv) $\Rightarrow$ (i) but using Lemma 2.4.

A locally Lipschitz function ψ on an open subset A of a normed linear space X is strictly differentiable at $x \in A$ in the direction $y \in X$ if and only if $\psi^+(x)(y)$ is continuous at x [**7,** p.837]. Using the fact that, given $x \in A$, $\psi^+(x)(y)$ is continuous in y, [**6,** p.207] and the generic continuity of quasi upper semi-continuous functions, we can make the following deduction.

COROLLARY 2.7. *For a locally Lipschitz function ψ on an open subset A of a separable Banach space X, if the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is minimal then ψ is strictly differentiable on a residual subset of A.*

We should note that such a result is not true for non-separable spaces. On ℓ_∞ the semi-norm p defined for $x \equiv \{x_1, x_2, \dots, x_n, \dots\}$ by $p(x) = \limsup |x_n|$, has a minimal subdifferential mapping $x \mapsto \partial p(x)$, but p is nowhere Gâteaux differentiable, [**12,** p.13]. Further, the converse of Corollary 2.7 does not hold in general. Pompeiu [**13**], has given an example of a real valued differentiable function ψ with a bounded non-negative derivative on an interval (a, b) where the sets $\{x \in (a,b) : \psi'(x) = 0\}$ and $\{x \in (a,b) : \psi'(x) > 0\}$ are both dense in (a, b). Clearly at each point of $\{x \in (a,b) : \psi'(x) > 0\}$, ψ' is not quasi lower semi-continuous and so the subdifferential mapping $x \mapsto \partial\psi^0(x)$ is not minimal. However, since ψ is differentiable on (a,b), ψ is strictly differentiable on a residual subset of (a,b), [**6,** p.210].

At this stage it is worth noting that a real valued differentiable function ψ on an interval (a,b) with derivative ψ' continuous almost everywhere, has ψ' quasi continuous on (a,b), [**9,** p.974], and so has a minimal subdifferential mapping $x \mapsto \partial\psi^0(x)$. On the other hand there exists a real-valued function ψ on (a,b) with bounded derivative which is quasi continuous on (a,b) but where the derivative is discontinuous on a set of positive measure, [**9,** p.975].

A locally Lipschitz function ψ on an open subset A of a normed linear space X is said to be *pseudo-regular* at $x \in A$ *in the direction* $y \in X$ if $\psi^+(x)(y) = \psi^0(x)(y)$ and *pseudo-regular at* x if it is pseudo-regular at x in all directions $y \in X$. Since

$\psi^0(x)(y) = \limsup_{z\to x} \psi^+(z)(y)$, it follows that ψ is pseudo-regular at $x \in A$ in the direction $y \in X$ if and only if $\psi^+(x)(y)$ is upper semi-continuous at x, [**7**, p.836]. So we can make the following deduction, [**11**, Theorem 2.5].

Corollary 2.8. *A locally Lipschitz function ψ which is pseudo-regular on an open subset A of a normed linear space X, has a minimal subdifferential mapping $x \mapsto \partial\psi^0(x)$ on A.*

3 Minimal subdifferential mappings on product spaces

Given topological spaces X, Y and Z and a function θ from $X \times Y$ into Z, we define for $p \in X$, the function θ_p from Y into Z where

$$\theta_p(y) = \theta(p, y)$$

and for $q \in Y$, the function θ_q from X into Z where

$$\theta_q(x) = \theta(x, q).$$

The following lemma relates separate and joint quasi continuity modelled on the proof of a similar result, [**10**, p.39].

Lemma 3.1. *Consider a real valued function θ on $X \times Y$ where X is a Baire space and Y is second countable. If θ_x is quasi upper semi-continuous on Y for all $x \in X$ and θ_y is both quasi upper and quasi lower semi-continuous on X for all $y \in Y$ then θ is quasi upper semi-continuous on $X \times Y$.*

Proof: Suppose that θ is not quasi upper semi-continuous at $(p,q) \in X \times Y$. Then there is an $r > 0$ and a neighbourhood $U \times V$ of (p,q) such that in every non-empty open subset of $U \times V$ there exists an (x,y) such that

$$\theta(x,y) \geqslant \theta(p,q) + r.$$

Since θ_q is quasi upper semi-continuous at p, there exists a non-empty open set $E \subseteq U$ such that

$$\theta(x,q) < \theta(p,q) + \frac{r}{3} \text{ for all } x \in E.$$

Consider $\mathcal{V}$ a countable base for Y and $\{V_n : n \in \mathbb{N}\}$ those elements from $\mathcal{V}$ contained in V. For each $n \in \mathbb{N}$, write

$$A_n \equiv \left\{x \in E : \theta(x,y) < \theta(x,q) + \frac{r}{3} \text{ for all } y \in V_n\right\}.$$

Consider $x \in E$. Since θ_x is quasi upper semi–continuous at q there exists a non–empty open set $F \subseteq V$ such that $\theta(x,y) < \theta(x,q) + r/3$ for all $y \in F$. But there exists $k \in \mathbb{N}$ such that $V_k \subseteq F$. So $x \in A_k$ and $E = \bigcup_1^\infty A_n$.

Consider E' a non–empty open subset of E and $n \in \mathbb{N}$. Then $E' \times V_n \subseteq U \times V_n$ and there is a $(x', y') \in E' \times V_n$ such that $\theta(x', y') \geqslant \theta(p,q) + r$. Since $\theta_{y'}$ is quasi lower semi–continuous at x', there exists a non–empty open set $E'' \subseteq E'$ such that

$$\theta(x, y') > \theta(x', y') - \frac{r}{3} \text{ for all } x \in E''.$$

For $x \in E''$,

$$\theta(x, y') > \theta(x', y') - \frac{r}{3} \geqslant \theta(p,q) + \frac{2r}{3} > \theta(x,q) + \frac{r}{3}.$$

But since $y' \in V_n$ then $x \notin A_n$ and so $E'' \cap A_n = \emptyset$. Therefore, A_n is nowhere dense and E is of first Baire category. This contradicts the fact that X is a Baire space. ▯

This Lemma with Theorem 2.5 gives an improved sufficiency theorem for minimal subdifferential mappings of locally Lipschitz functions on certain product spaces.

Theorem 3.2. *Consider a locally Lipschitz function ψ on a product space $X \times Y$ where X and Y are Banach spaces and Y is separable. The subdifferential mapping $(x,y) \mapsto \partial\psi^0(x,y)$ is minimal on $X \times Y$ if given $(u,v) \in X \times Y$, for each $p \in X$, $\psi^+(p,y)(u,v)$ is quasi upper semi–continuous on Y and for each $q \in Y$, $\psi^+(x,q)(u,v)$ is both quasi upper and quasi lower semi–continuous on X.*

From Theorem 3.2 and Proposition 1.1 we can deduce the following generic differentiability properties of locally Lipschitz functions on a product space.

Corollary 3.3. *Consider a locally Lipschitz function ψ a product space $X \times Y$ where X and Y are Banach spaces and Y is separable and ψ satisfies the hypothesis of Theorem 3.2.*

(i) *If X is smoothable, then ψ is strictly differentiable on a residual subset of $X \times Y$.*
(ii) *If X is Asplund and Y has separable dual, then ψ is uniformly strictly differentiable on a residual subset of $X \times Y$.*

Proof:

(i) If X is smoothable and Y is separable, then Y is smoothable and so $X \times Y$ is smoothable.
(ii) If X is Asplund and Y has separable dual, then closed separable subspaces of $X \times Y$ have separable duals and $X \times Y$ is Asplund, [**12**, p.32].

Our result now follows from Proposition 1.1 and Corollary 1.2. ▯

Theorem 3.2 provides a test for minimality for locally Lipschitz functions on a product space using the behaviour of associated functions on each of the component spaces. Our other theorem gives a similar result using the behaviour of derivatives in component directions.

THEOREM 3.4. *Consider a locally Lipschitz function ψ on an open subset A of a smoothable (Asplund) product space $X \times Y$ where X and Y are Banach spaces. The subdifferential mapping $(x,y) \mapsto \partial\psi^0(x,y)$ is minimal on A if one of $\psi'(x,y)(u,0)$ and $\psi'(x,y)(0,v)$ is upper semi-continuous on D and the other is quasi upper semi-continuous on D where D is the set of points in A where ψ is Gâteaux (Fréchet) differentiable.*

PROOF: Given $(u,v) \in X \times Y$ and $(x,y) \in D$ then

$$\psi'(x,y)(u,v) = \psi'(x,y)(u,0) + \psi'(x,y)(0,v).$$

It follows that $\psi'(x,y)(u,v)$ is quasi upper semi-continuous on D and Theorem 2.6 gives our result. ∎

In particular, ψ satisfies the hypothesis of this theorem when ψ is pseudo-regular on $X \times Y$, in directions $(u,0)$ and $(0,v)$, [**6**, p.837]. So Theorem 3.4 can be considered to be a generalisation of Corollary 2.8.

From Theorem 3.4 and Proposition 1.1 we can deduce generic differentiability properties.

COROLLARY 3.5. *A locally Lipschitz function ψ on an open subset A of a smoothable (Asplund) product space $X \times Y$ where X and Y are Banach spaces and ψ satisfies the hypothesis of Theorem 3.4, has ψ strictly (uniformly strictly) differentiable on a residual subset of A.*

It is a classical result that a real valued locally Lipschitz function on Euclidean space with continuous partial derivatives at a point is strictly differentiable at the point. A proof of this follows from the more general local result.

THEOREM 3.6. *Consider a locally Lipschitz function ψ on an open subset A of a a product space $X \times Y$ where X and Y are normed linear spaces. If ψ is strictly differentiable at (x_0, y_0) in both directions $(u,0)$ and $(0,v)$ then ψ is strictly differentiable at (x_0, y_0).*

PROOF: Consider $f \in \partial\psi^0(x_0,y_0)$. Since ψ is strictly differentiable at (x_0,y_0) in directions $(u,0)$ and $(0,v)$ then

$$f(u,0) = \psi^0(x_0,y_0)(u,0) \text{ and } f(0,v) = \psi^0(x_0,y_0)(0,v).$$

So $f(u,v) = \psi^0(x_0,y_0)(u,v)$ and we conclude that $\partial\psi^0(x_0,y_0)$ is singleton. ∎

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