ON EXISTENCE, UNIQUENESS AND L^r-EXPONENTIAL STABILITY FOR STATIONARY SOLUTIONS TO THE MHD EQUATIONS IN THREE-DIMENSIONAL DOMAINS

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Abstract

The existence of stationary solutions to the MHD equations in three-dimensional bounded domains will be proved. At the same time if the assumption of smallness is made on external forces, uniqueness of the stationary solutions can be guaranteed and it can be shown that any L^r ($r > 3$) global bounded non-stationary solution to the MHD equations approaches the stationary solution under both L^2 and L^r norms exponentially as time goes to infinity.

1. Introduction

Let Ω be a three-dimensional bounded domain with smooth boundary $\partial\Omega$ and $Q = \Omega \times (0, \infty)$. Consider the MHD equations [3] in Q as follows:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u - \dfrac{1}{\rho\mu}(B \cdot \nabla)B \\
\quad + \dfrac{1}{2\rho\mu}\nabla(|B|^2) + \dfrac{1}{\rho}\nabla\Pi_1 = f(x), \quad (x, t) \in Q, \\
\dfrac{\partial B}{\partial t} - \lambda\Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u = 0, \quad (x, t) \in Q, \\
\nabla \cdot u = 0, \quad \nabla \cdot B = 0, \quad (x, t) \in Q, \\
u|_{\partial\Omega} = 0, \quad B|_{\partial\Omega} = 0, \quad t \in (0, \infty), \\
u(x, 0) = u_0(x), \quad B(x, 0) = B_0(x), \quad x \in \Omega.
\end{cases}
\tag{$E.S$}
$$

In $(E.S)$, $u = (u^1(x, t), u^2(x, t), u^3(x, t))$ and $B = (B^1(x, t), B^2(x, t), B^3(x, t))$ are an unknown velocity vector and magnetic field respectively, $f(x)$ is a known external

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force and Π_1 is pressure and can be uniquely determined by (u, B, f) up to a constant. We note that ν, μ, ρ are constants of kinematic viscosity, magnetic permeability and density of Eulerian flow respectively and that $\lambda = \eta/\mu$ with electrical resistivity η.

Clearly, the stationary solutions of $(E.S)$ satisfy the following equations:

$$
\begin{cases}
-\nu \Delta v + (v \cdot \nabla)v - \dfrac{1}{\rho\mu}(b \cdot \nabla)b + \dfrac{1}{2\rho\mu}\nabla(|b|^2) + \dfrac{1}{\rho}\nabla\pi_1 = f(x), & x \in \Omega, \\
-\lambda \Delta b + (v \cdot \nabla)b - (b \cdot \nabla)v = 0, & x \in \Omega, \\
\nabla \cdot v = 0, \quad \nabla \cdot b = 0, \quad x \in \Omega, \\
v|_{\partial\Omega} = 0, \quad b|_{\partial\Omega} = 0.
\end{cases}
\tag{S.S}
$$

In a similar manner, π_1 can also be uniquely determined by (v, b, f) up to a constant.

In this paper, we mainly study the existence and uniqueness of stationary solutions, and stability relations between the global L^r $(r > 3)$ bounded non-stationary solutions and stationary solutions. First, let us recollect some related methods and results about stability for stationary solutions to the well-known Navier-Stokes equations $(N.S)$. Temam [12] studied the existence and uniqueness of stationary solutions to $(N.S)$ and obtained L^2-exponential stability for the stationary solutions under the assumption that external forces are sufficiently small or the viscosity constant of fluids is sufficiently large. Recently, Schonbek [9] and Guo and Zhang [5] presented some decay properties of solutions to the MHD equations and showed that under some assumptions on the decay properties of external forces, any non-stationary solutions to the MHD equations on the whole three-dimensional or two-dimensional space decay to zero algebraically as time goes to infinity. Qu and Wang studied L^p stability for stationary solutions of $(N.S)$ on three-dimensional bounded domains in [7]. Motivated and inspired by the above mentioned work, we studied existence, uniqueness and L^r $(r > 3)$-exponential stability for stationary solutions to the MHD equations on three-dimensional bounded domains. We prove the existence of at least one stationary solution to the MHD equations in three-dimensional bounded domains and uniqueness of the stationary solutions if external forces are sufficiently small or ν and λ are sufficiently large. The main purpose of this paper is to prove that under the assumption of smallness of external forces there exist positive constants c, $\bar{k}$, $\tilde{\beta}$ independent of t, u, B such that

$$
\|u(t) - v\|_r^r + \|B(t) - b\|_r^r \le c\big(\|u_0(x) - v\|_r^r + \|B_0(x) - b\|_r^r\big)^{\bar{k}} e^{-\tilde{\beta}t}
$$

for all $t > 0$, where $\|\cdot\|_r$ denotes the usual L^r-norm.

This paper is arranged as follows. We present some mathematical preliminaries in Section 2. In Section 3, we prove the existence of stationary solutions to $(E.S)$. Moreover, if external forces are sufficiently small or μ and λ are sufficiently large, uniqueness of the stationary solution can be guaranteed. Additionally in this section, we also prove our main results on L^2-exponential stability of stationary solutions to

$(E.S)$. The main results on L^r $(r > 3)$-exponential stability are presented and proved in Section 4.

2. Mathematical preliminaries

In this section we present some mathematical preliminaries. First, let us introduce some vector-valued functional spaces and notation as follows: $L^r(\Omega)$ denotes the usual vector-valued functional space consisting of r-times integrable functions on Ω. Here H_r = closure of $\{\phi \in C_0^\infty(\Omega), \nabla \cdot \phi = 0\}$ in $L^r(\Omega)$ and $H_{\pi,r}$ = closure of $\{\nabla \phi, \phi \in C^1(\Omega)\}$ in $L^r(\Omega)$. As described by Temam [11],

$$L^r(\Omega) = H_r \oplus H_{\pi,r}.$$

Here $W^{m,r}(\Omega)$ $(m \geq 0)$ denotes the well-known vector-valued Sobolev spaces. In particular, if $r = 2$, $H^m = W^{m,2}(\Omega)$.

Let $P_r : L^r(\Omega) \to H_r$ be a Helmholtz projection operator [11] and let $A_r = -P_r\Delta$ be the well-known Stokes operator with domain $D(A_r) = H_r \cap W_0^{1,r}(\Omega) \cap W^{2,r}(\Omega)$. Let V = closure of $\{\phi \in C_0^\infty(\Omega), \nabla \cdot \phi = 0\}$ in $H^1(\Omega)$ and let V' be the dual space of V. For simplicity, we omit subscripts in the notation H_2, A_2, P_2 and $\| \cdot \|_2$. We denote by $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ the dual product between V and V' and the inner product of H respectively.

Applying P to both sides of the first two equations in $(E.S)$ and in $(S.S)$ yields

$$\frac{\partial u}{\partial t} + \nu A u + P(u \cdot \nabla)u - \frac{1}{\rho\mu}P(B \cdot \nabla)B = Pf, \tag{2.1}$$

$$\frac{\partial B}{\partial t} + \lambda A B + P(u \cdot \nabla)B - P(B \cdot \nabla)u = 0 \tag{2.2}$$

and

$$\nu A v + P(v \cdot \nabla)v - \frac{1}{\rho\mu}P(b \cdot \nabla)b = Pf, \tag{2.3}$$

$$\lambda A b + P(v \cdot \nabla)b - P(b \cdot \nabla)v = 0. \tag{2.4}$$

Since $A^{-1} : H \to D(A)$ is a positive compact operator, A has eigenvalues $\{\lambda_j\}$ $(j = 1, 2, \dots)$ and corresponding eigenvectors $\{\omega_j\}$ $(j = 1, 2, \dots)$ satisfying

$$A\omega_j = \lambda_j \omega_j, \quad 0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots .$$

Moreover, $\lambda_j \to \infty$ as $j \to \infty$.

DEFINITION 1. For any $u_0(x)$, $B_0(x) \in H \cap L^r(\Omega)$ $(r > 3)$ and $f(x) \in L^r(\Omega)$, $(u(x, t), B(x, t))$ is called an L^r-bounded solution of $(E.S)$ if

(i) both $\|u(x, t)\|_r$ and $\|B(x, t)\|_r$ are uniformly bounded with respect to $t \in [0, \infty)$;

(ii) $u(x, t), B(x, t) \in W^{1,r}(\Omega)$, for any $t \in (0, \infty)$;

(iii) $(u(x, t), B(x, t))$ satisfies $(E.S)$ in a weak or strong sense.

For existence and uniqueness of local or global strong (weak) solutions of $(E.S)$, see references [2, 6, 8].

3. Existence, uniqueness and L^2-exponential stability for stationary solutions

In this section, we study the existence, uniqueness and L^2-exponential stability for stationary solutions to the MHD equations. We use a Fadeo-Galerkin approximation combined with Schaefer's fixed point theorem to show the existence of stationary solutions, and the energy estimates method to show uniqueness and L^2-exponential stability for stationary solutions.

THEOREM 3.1. *If $f(x) \in V'$, then there exists at least one solution $(v, b) \in V \times V$ to (2.3)–(2.4). Moreover, if $f(x) \in H$, then $v, b \in D(A)$. Additionally, if $\|f(x)\|$ is sufficiently small or both v and λ are sufficiently large, then the solution (v, b) to (2.3)–(2.4) is unique.*

PROOF. We implement the well-known Fadeo-Galerkin method [11] to prove the existence of solutions (v, b) to (2.3)–(2.4). For $m \in N$, we look for an approximate solution (v_m, b_m) such that $v_m = \sum_{i=1}^m \xi_{im}\omega_i$, $b_m = \sum_{i=1}^m \tilde{\xi}_{im}\omega_i$, $\xi_{im}, \tilde{\xi}_{im} \in R$, and

$$v(\nabla v_m, \nabla \hat{v}) + b^*(v_m, v_m, \hat{v}) - \frac{1}{\rho\mu}b^*(b_m, b_m, \hat{v}) = \langle f, \hat{v} \rangle, \qquad (3.1)$$

$$\lambda(\nabla b_m, \nabla \hat{b}) + b^*(v_m, b_m, \hat{b}) - b^*(b_m, v_m, \hat{b}) = 0, \qquad (3.2)$$

for every $\hat{v}, \hat{b} \in W_m = \text{span}\{\omega_1, \ldots, \omega_m\}$, where

$$b^*(u, v, w) = \sum_{i,j=1}^3 \int_\Omega u^i \frac{\partial v^i}{\partial x_j} w^j \, dx.$$

Equations (3.1)-(3.2) are also equivalent to

$$vAv_m + \hat{P}_m(v_m \cdot \nabla)v_m - \frac{1}{\rho\mu}\hat{P}_m(b_m \cdot \nabla)b_m = \hat{P}_mf, \qquad (3.3)$$

$$\lambda Ab_m + \hat{P}_m(v_m \cdot \nabla)b_m - \hat{P}_m(b_m \cdot \nabla)v_m = 0 \qquad (3.4)$$

in V', where $\hat{P}_m : H \to W_m$ is a projection operator.

Next we prove the existence of a solution (v_m, b_m) to (3.3)–(3.4) by the well-known Schaefer fixed point theorem [4]. Since W_m is a finite-dimensional subspace of H, to apply the fixed point theorem, it is sufficient to show the uniform boundedness of $\|\nabla v_m\|$ and $\|\nabla b_m\|$ with respect to $0 \le \alpha \le 1$ for all possible solutions of the following equations:

$$\nu A v_m + \alpha \hat{P}_m (v_m \cdot \nabla) v_m - \frac{\alpha}{\rho \mu} \hat{P}_m (b_m \cdot \nabla) b_m = \hat{P}_m f, \tag{3.5}$$

$$\lambda A b_m + \alpha \hat{P}_m (v_m \cdot \nabla) b_m - \alpha \hat{P}_m (b_m \cdot \nabla) v_m = 0. \tag{3.6}$$

In fact, taking the dual product of V and V' with v_m on both sides of (3.5) and with $b_m / \rho \mu$ on both sides of (3.6), then summing them together, yields

$$\nu \|\nabla v_m\|^2 + \frac{\lambda}{\rho \mu} \|\nabla b_m\|^2 = \langle \hat{P}_m f, v_m \rangle \le \frac{\nu}{2} \|\nabla v_m\|^2 + \frac{1}{2\nu} \|f\|_{V'}^2.$$

Therefore

$$\nu \|\nabla v_m\|^2 + \frac{2\lambda}{\rho \mu} \|\nabla b_m\|^2 \le \frac{1}{\nu} \|f\|_{V'}^2.$$

Thus due to the Schaefer fixed point theorem acting on $W_m \times W_m$, the existence of (v_m, b_m) is guaranteed.

Next we give some *a priori* estimates of v_m and b_m.

First, taking $\hat{v} = v_m$, $\hat{b} = b_m / \rho \mu$ in (3.1)–(3.2) and summing them together, we get

$$\nu \|\nabla v_m\|^2 + \frac{\lambda}{\rho \nu} \|\nabla b_m\|^2 = \langle f, v_m \rangle \le \frac{1}{2\nu} \|f\|_{V'}^2 + \frac{\nu}{2} \|\nabla v_m\|^2.$$

Therefore

$$\nu \|\nabla v_m\|^2 + \frac{2\lambda}{\rho \mu} \|\nabla b_m\|^2 \le \frac{1}{\nu} \|f\|_{V'}^2. \tag{3.7}$$

So we can extract from (v_m, b_m) a subsequence $(v_{m'}, b_{m'})$ which converges weakly in V to some limit (v, b), and since the injection of V in H is compact, $(v_{m'}, b_{m'})$ converges to (v, b) strongly in H, that is, $b_{m'} \to b$ and $v_{m'} \to v$ weakly in V and strongly in H. Passing to the limit in (3.1)–(3.2), we find that (v, b) is a solution of (3.3)–(3.4) in V'. We note that if $v, b \in V$, then according to [12, (2.26)] it follows that $P(v \cdot \nabla)v$, $P(b \cdot \nabla)b$, $P(v \cdot \nabla)b$ and $P(b \cdot \nabla)v$ belong to $D(A^{-1/2})$. Hence

$$v = \nu^{-1} A^{-1} \left(Pf - P(v \cdot \nabla)v + \frac{1}{\rho \mu} P(b \cdot \nabla)b \right) \in D(A^{3/2}), \tag{3.8}$$

$$b = \lambda^{-1} A^{-1} \left(P(b \cdot \nabla)v - P(v \cdot \nabla)b \right) \in D(A^{3/2}). \tag{3.9}$$

If $f(x) \in L^2(\Omega)$, applying [12, Lemma 2.1] to (3.8)–(3.9) yields $P(v \cdot \nabla)v$, $P(b \cdot \nabla)b$, $P(v \cdot \nabla)b$, $P(b \cdot \nabla)v \in H$. Therefore v and $b \in D(A)$. Since $v, b \in D(A) \subset H^2(\Omega)$ we can define $\|v\|_r$, $\|b\|_r$ for any $r \geq 2$ according to the Sobolev embedding theorem [4].

Next we give some *a priori* estimates of (v, b) in V and $D(A)$ respectively. If $f(x) \in L^2(\Omega)$, from (3.7) and lower-semicontinuity of the norm $\| \cdot \|$, it follows that

$$\nu \|\nabla v\|^2 + \frac{2\lambda}{\rho\mu} \|\nabla b\|^2 \leq \frac{1}{\nu\lambda_1} \|f\|^2.$$

For the norm in $D(A)$, we infer from (2.3)–(2.4) for $r = 2$ that

$$\nu\|Av\| \leq \|f\| + \|P(v \cdot \nabla)v\| + \frac{1}{\rho\mu}\|P(b \cdot \nabla)b\|$$

$$\leq \|f\| + c_1 \left(\|\nabla v\|^{3/2}\|Av\|^{1/2} + \frac{1}{\rho\mu}\|\nabla b\|^{3/2}\|Ab\|^{1/2} \right)$$

$$\leq \|f\| + \frac{\nu}{4}\|Av\| + \frac{c_1^2}{\nu}\|\nabla v\|^3 + \frac{\lambda}{4}\|Ab\| + \frac{c_1^2}{\rho^2\mu^2\lambda}\|\nabla b\|^3,$$

where we have used Young's inequality. Similarly,

$$\lambda\|Ab\| \leq \frac{\nu}{4}\|Av\| + \frac{c_1^2}{\nu}\|\nabla b\|^3 + \frac{\lambda}{4}\|Ab\| + \frac{c_1^2}{\lambda}\|\nabla v\|^3.$$

Finally,

$$\nu\|Av\| + \lambda\|Ab\|$$

$$\leq 2\|f\| + \frac{2c_1^2}{\nu}\big(\|\nabla v\|^3 + \|\nabla b\|^3\big) + \frac{2c_1^2}{\rho^2\mu^2\lambda}\|\nabla b\|^3 + \frac{2c_1^2}{\lambda}\|\nabla v\|^3$$

$$\leq 2\|f\| + 2c_1^2 \left[\frac{1}{\nu^3\lambda_1^{3/2}} \left(\frac{1}{\nu} + \frac{1}{\lambda} \right) + \frac{(\rho\mu)^{3/2}}{(\lambda\nu\lambda_1)^{3/2}} \left(\frac{1}{\rho^2\mu^2\lambda} + \frac{1}{\nu} \right) \right] \|f\|^3. \quad (3.10)$$

To prove the uniqueness result, let us assume that (v_i, b_i), $i = 1, 2$, are two solutions of (2.3)–(2.4) for $r = 2$, that is,

$$\nu(\nabla v_i, \nabla \hat{v}) + b^*(v_i, v_i, \hat{v}) - \frac{1}{\rho\mu}b^*(b_i, b_i, \hat{v}) = \langle f, \hat{v} \rangle,$$

$$\lambda(\nabla b_i, \nabla \hat{b}) + b^*(v_i, b_i, \hat{b}) - b^*(b_i, v_i, \hat{b}) = 0,$$

for $i = 1, 2$ and $\hat{v}, \hat{b} \in V$.

Let $w = v_1 - v_2$, $\tilde{b} = b_1 - b_2$. We get

$$\nu(\nabla w, \nabla \hat{v}) + b^*(v_1, w, \hat{v}) + b^*(w, v_2, \hat{v}) - \frac{1}{\rho\mu}\big(b^*(b_1, \tilde{b}, \hat{v}) + b^*(\tilde{b}, b_2, \hat{v})\big) = 0,$$

$$\lambda(\nabla \tilde{b}, \nabla \hat{b}) + b^*(v_1, \tilde{b}, \hat{b}) + b^*(w, b_2, \tilde{b}) - \big(b^*(b_1, w, \hat{b}) + b^*(\tilde{b}, v_2, \hat{b})\big) = 0.$$

Taking $\hat{v} = w$ and $\hat{b} = \tilde{b}$ in the above equalities and adding them together yields

$$\nu\|\nabla w\|^2 + \frac{\lambda}{\rho\mu}\|\nabla\tilde{b}\|^2$$

$$= -b^*(w, v_2, w) + \frac{1}{\rho\mu}\big(b^*(\tilde{b}, b_2, w) + b^*(\tilde{b}, v_2, \tilde{b}) - b^*(w, b_2, \tilde{b})\big)$$

$$\leq c_1\|\nabla w\|^2\|\nabla v_2\| + \frac{c_1}{\rho\mu}\big(2\|\nabla\tilde{b}\|\|\nabla w\|\|\nabla b_2\| + \|\nabla\tilde{b}\|^2\|\nabla v_2\|\big)$$

$$\leq c_1\|\nabla w\|^2\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{1}{\sqrt{\rho\mu\lambda\nu\lambda_1}}\right)\|f\|$$

$$+ \frac{c_1}{\rho\mu}\|\nabla\tilde{b}\|^2\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{1}{\sqrt{\rho\mu\lambda\nu\lambda_1}}\right)\|f\|.$$

Therefore

$$\left(\nu - c_1\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{1}{\sqrt{\rho\mu\lambda\nu\lambda_1}}\right)\|f\|\right)\|\nabla w\|^2$$

$$+ \frac{1}{\rho\mu}\left(\lambda - c_1\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{1}{\sqrt{\rho\mu\lambda\nu\lambda_1}}\right)\|f\|\right)\|\nabla\tilde{b}\|^2 \leq 0.$$

If $\|f(x)\|$ is sufficiently small or both ν and λ are sufficiently large so that

$$\nu - c_1\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{1}{\sqrt{\rho\mu\lambda\nu\lambda_1}}\right)\|f\| > 0$$

and

$$\lambda - c_1\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{1}{\sqrt{\rho\mu\lambda\nu\lambda_1}}\right)\|f\| > 0,$$

it follows that $v_1 = v_2$ and $b_1 = b_2$. Hence uniqueness of the solution is proved and the proof of Theorem 3.1 is completed.

THEOREM 3.2. *Let $(u(t), B(t))$ be any solution of (2.1)–(2.2) with initial data $u_0, B_0 \in H$ and $f(x) \in H$ be sufficiently small or ν and λ be sufficiently large. Then there exist constants $c, \beta > 0$ independent of t, u and B such that, for all $t > 0$,*

$$\|u(t) - v\|^2 + \frac{1}{\rho\mu}\|B(t) - b\|^2 \leq c\left(\|u_0 - v\|^2 + \frac{1}{\rho\mu}\|B_0 - b\|^2\right)e^{-\beta t}.$$

PROOF. Let $w(t) = u(t) - v$ and $\tilde{B}(t) = B(t) - b$. We have by difference,

$$\frac{\partial w}{\partial t} + \nu Aw + P(u\cdot\nabla)w + P(w\cdot\nabla)v - \frac{1}{\rho\mu}\big(P(B\cdot\nabla)\tilde{B} + P(\tilde{B}\cdot\nabla)b\big) = 0, \quad (3.11)$$

$$\frac{\partial\tilde{B}}{\partial t} + \lambda A\tilde{B} + P(u\cdot\nabla)\tilde{B} + P(w\cdot\nabla)b - \big(P(B\cdot\nabla)w + P(\tilde{B}\cdot\nabla)v\big) = 0, \quad (3.12)$$

$$w(x, 0) = u_0 - v, \quad \tilde{B}(x, 0) = B_0 - b.$$

Taking the scalar product of (3.11) with $w(t)$, and of (3.12) with $\tilde{B}(t)/\rho\mu$ in H and summing them together, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w(t)\|^2 + \frac{1}{\rho\mu}\|\tilde{B}(t)\|^2\right) + \nu\|\nabla w\|^2 + \frac{\lambda}{\rho\mu}\|\nabla\tilde{B}\|^2$$
$$= -b^*(w, v, w) + \frac{1}{\rho\mu}\left(b^*(\tilde{B}, b, w) + b^*(\tilde{B}, v, \tilde{B}) - b^*(w, b, \tilde{B})\right). \quad (3.13)$$

Next we give estimates of each term on the right-hand side of (3.13)

$$|b^*(w, v, w)| \leq c_1\|w\|^{\frac{3}{2}}\|\nabla w\|^{1/2}\|Av\| \leq \frac{\nu}{6}\|\nabla w\|^2 + \frac{c_2}{\nu^{1/3}}\|w\|^2\|Av\|^{4/3},$$
$$|b^*(\tilde{B}, b, w)| \leq c_1\|\tilde{B}\|^{3/4}\|\nabla\tilde{B}\|^{1/4}\|w\|^{3/4}\|\nabla w\|^{1/4}\|Ab\|$$
$$\leq \frac{\nu}{6}\|\nabla w\|^2 + \frac{\lambda}{6\rho\mu}\|\nabla\tilde{B}\|^2 + c_2\frac{(\rho\mu)^{1/3}}{\lambda^{1/3}}\|\tilde{B}\|^2\|Ab\|^{4/3}$$
$$+ \frac{c_2}{\nu^{1/3}}\|w\|^2\|Ab\|^{4/3},$$

where we have used Young's inequality.

Similarly,

$$|b^*(\tilde{B}, v, \tilde{B})| \leq \frac{\lambda}{6\rho\mu}\|\nabla\tilde{B}\|^2 + \frac{c_2(\rho\mu)^{1/3}}{\lambda^{1/3}}\|\tilde{B}\|^2\|Av\|^{4/3},$$
$$|b^*(w, b, \tilde{B})| \leq c_1\|\tilde{B}\|^{3/4}\|\nabla\tilde{B}\|^{1/4}\|w\|^{3/4}\|\nabla w\|^{1/4}\|Ab\|$$
$$\leq \frac{\nu}{6}\|\nabla w\|^2 + \frac{\lambda}{6\rho\mu}\|\nabla\tilde{B}\|^2 + c_2\frac{(\rho\mu)^{1/3}}{\lambda^{1/3}}\|\tilde{B}\|^2\|Ab\|^{4/3}$$
$$+ \frac{c_2}{\nu^{1/3}}\|w\|^2\|Ab\|^{4/3},$$

also using Young's inequality here.

Substituting the above estimates into (3.13), we get

$$\frac{d}{dt}\left(\|w\|^2 + \frac{1}{\rho\mu}\|\tilde{B}\|^2\right) + \nu\|\nabla w\|^2 + \frac{\lambda}{\rho\mu}\|\nabla\tilde{B}\|^2$$
$$\leq 2c_2\|w\|^2\left(\frac{1}{\nu^{1/3}}\|Av\|^{4/3} + \frac{2}{\nu^{2/3}}\|Ab\|^{4/3}\right)$$
$$+ 2c_2\|\tilde{B}\|^2\left(\frac{(\rho\mu)^{1/3}}{\lambda^{1/3}}\|Av\|^{4/3} + \frac{2(\rho\mu)^{1/3}}{\lambda^{1/3}}\|Ab\|^{4/3}\right).$$

Therefore

$$\frac{d}{dt}\left(\|w\|^2 + \frac{1}{\rho\mu}\|\tilde{B}\|^2\right) + \beta\left(\|w\|^2 + \frac{1}{\rho\mu}\|\tilde{B}\|^2\right) \leq 0, \quad (3.14)$$

where

$$\beta = \min(\nu, \lambda)\lambda_1$$

$$- 2\max\left\{ \frac{c_2\|Av\|^{4/3}}{\nu^{1/3}} + \frac{2c_2\|Ab\|^{4/3}}{\nu^{1/3}}, \frac{c_2(\rho\mu)^{4/3}\|Av\|^{4/3}}{\lambda^{1/3}} + \frac{2c_2(\rho\mu)^{4/3}\|Ab\|^{4/3}}{\lambda^{1/3}} \right\}.$$

From (3.10), we know

$$\|Av\| \le \frac{2}{\nu}\|f\| + 2c_1^2\left[\frac{1}{\nu^4\lambda_1^{3/2}}\left(\frac{1}{\nu} + \frac{1}{\lambda}\right) + \frac{(\rho\mu)^{3/2}}{\nu^{5/2}(\lambda\lambda_1)^{3/2}}\left(\frac{1}{\rho^2\mu^2\lambda} + \frac{1}{\nu}\right) \right]\|f\|^3$$

and

$$\|Ab\| \le \frac{2}{\lambda}\|f\| + 2c_1^2\left[\frac{1}{\lambda\nu^3\lambda_1^{3/2}}\left(\frac{1}{\nu} + \frac{1}{\lambda}\right) + \frac{(\rho\mu)^{3/2}}{\lambda^{5/2}(\nu\lambda_1)^{3/2}}\left(\frac{1}{\rho^2\mu^2\lambda} + \frac{1}{\nu}\right) \right]\|f\|^3.$$

Thus it follows that if ν and λ are sufficiently large or $\|f\|$ is sufficiently small, then $\beta > 0$.

Taking (3.14) into account, we get

$$\|u(t) - v\|^2 + \frac{1}{\rho\mu}\|B(t) - b\|^2 \le c\left(\|u_0 - v\|^2 + \frac{1}{\rho\mu}\|B_0 - b\|^2 \right)e^{-\beta t}$$

for all $t > 0$. The proof of Theorem 3.2 is completed.

4. Main results of L^r $(r > 3)$-exponential stability

In this section we will study the stability of the solutions to $(E.S)$ with initial values $u_0(x)$, $B_0(x) \in H \cap L^r(\Omega)$. The main tools we use in this section are energy estimates.

From $(E.S)$ and $(S.S)$, by difference, we get

$$\begin{cases} \dfrac{\partial w}{\partial t} - \nu\Delta w + (u \cdot \nabla)w + (w \cdot \nabla)v - \dfrac{1}{\rho\mu}\big((B \cdot \nabla)\tilde{B} + (\tilde{B} \cdot \nabla)b\big) \\[2mm] \quad + \dfrac{1}{2\rho\mu}\nabla(|B|^2 - |b|^2) + \dfrac{1}{\rho}\nabla(\Pi_1 - \pi_1) = 0, \quad (x, t) \in Q, \\[3mm] \dfrac{\partial \tilde{B}}{\partial t} - \lambda\Delta\tilde{B} + (u \cdot \nabla)\tilde{B} + (w \cdot \nabla)b \\[2mm] \quad - (B \cdot \nabla)w - (\tilde{B} \cdot \nabla)v = 0, \quad (x, t) \in Q, \\[2mm] \nabla \cdot w = 0, \quad \nabla \cdot \tilde{B} = 0, \quad (x, t) \in Q, \\[2mm] w|_{\partial\Omega} = 0, \quad \tilde{B}|_{\partial\Omega} = 0, \quad t \in (0, \infty), \\[2mm] w(x, 0) = u_0(x) - v, \quad \tilde{B}(x, 0) = B_0(x) - b, \quad x \in \Omega. \end{cases} \quad (D.E.S)$$

Let us denote $\Pi = (|B|^2 - |b|^2)/(2\rho\mu) + (\Pi_1 - \pi_1)/\rho$. From $(D.E.S)$ we see

$$\frac{\partial w}{\partial t} - \nu\Delta w + (u \cdot \nabla)w + (w \cdot \nabla)v - \frac{1}{\rho\mu}\big((B \cdot \nabla)\tilde{B} + (\tilde{B} \cdot \nabla)b\big) + \Pi = 0, \quad (4.1)$$

$$\frac{\partial \tilde{B}}{\partial t} - \lambda\Delta\tilde{B} + (u \cdot \nabla)\tilde{B} + (w \cdot \nabla)b - (B \cdot \nabla)w - (\tilde{B} \cdot \nabla)v = 0. \quad (4.2)$$

LEMMA 4.1. *The following estimate of Π holds:*

$$\|\Pi\|^2_{(r+2)/2} \le c\|w\|^2_{r+2}\big(\|v\|^2_{r+2} + \|w\|^2_{r+2}\big) + \frac{c}{\rho\mu}\|\tilde{B}\|^2_{r+2}\big(\|b\|^2_{r+2} + \|\tilde{B}\|^2_{r+2}\big). \quad (4.3)$$

PROOF. To estimate $\|\Pi\|_{(r+2)/2}$, we take the divergence of (4.1) and obtain

$$-\Delta\Pi = \sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}\big(w^i(u^j + v^j)\big) - \frac{1}{\rho\mu}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}\big(\tilde{B}^i(B^j + b^j)\big).$$

From the Calderon-Zygmund inequality [10] it follows that

$$\|\Pi\|_{(r+2)/2} \le c\sum_{i,j=1}^{n}\big\|w^i(v^j + u^j)\big\|_{(r+2)/2} + \frac{c}{\rho\mu}\sum_{i,j=1}^{n}\big\|\tilde{B}^i(b^j + B^j)\big\|_{(r+2)/2}.$$

Noticing that $u = w + v$ and $B = \tilde{B} + b$ we obtain

$$\|\Pi\|_{(r+2)/2} \le c\sum_{i,j=1}^{n}\big\|w^i(2v^j + w^j)\big\|_{(r+2)/2} + \frac{c}{\rho\mu}\sum_{i,j=1}^{n}\big\|\tilde{B}^i(2b^j + \tilde{B}^j)\big\|_{(r+2)/2}.$$

Then by Holder's inequality, we get

$$\|\Pi\|^2_{(r+2)/2} \le c\|w\|^2_{r+2}\big(\|v\|^2_{r+2} + \|w\|^2_{r+2}\big) + \frac{c}{\rho\mu}\|\tilde{B}\|^2_{r+2}\big(\|b\|^2_{r+2} + \|\tilde{B}\|^2_{r+2}\big).$$

Lemma 4.1 is proved.

Next we present two lemmas which were proved in [1].

LEMMA 4.2. *Let $r > 2$ and $N_r(w) = \int_\Omega |\nabla w|^2 |w|^{r-2}\,dx$. Then*

$$N_r(w) \ge c\|w\|^{-4r/3(r-2)}\|w\|_r^{r+(4r/3(r-2))},$$

for any $w \in W^{1,r}(\Omega)$ and $w \ne 0$.

LEMMA 4.3. *Let $r > 3$ and $w \in W^{1,r}(\Omega)$. Then $\|w\|_{r+2}^{r+2} \le c\|w\|_r^{r-1}N_r(w)^{3/r}$.*

Multiplying both sides of (4.1) by $|w|^{r-2}w$ and both sides of (4.2) by $|\tilde{B}|^{r-2}\tilde{B}$ and integrating over Ω, after suitable integrations by parts, we obtain

$$
\frac{1}{r}\frac{d}{dt}\|w\|_r^r + \nu N_r(w) + \frac{4\nu(r-2)}{r^2}\int_\Omega \big|\nabla|w|^{r/2}\big|^2\, dx
$$

$$
= -\int_\Omega (u\cdot\nabla)w\cdot|w|^{r-2}w\, dx - \int_\Omega (w\cdot\nabla)v\cdot|w|^{r-2}w\, dx
$$

$$
+ \frac{1}{\rho\mu}\left(\int_\Omega (B\cdot\nabla)\tilde{B}\cdot|w|^{r-2}w\, dx + \int_\Omega (\tilde{B}\cdot\nabla)b\cdot|w|^{r-2}w\, dx\right)
$$

$$
- \int_\Omega \nabla\Pi\cdot|w|^{r-2}w\, dx \tag{4.4}
$$

and

$$
\frac{1}{r}\frac{d}{dt}\|\tilde{B}\|_r^r + \lambda N_r(\tilde{B}) + \frac{4\lambda(r-2)}{r^2}\int_\Omega \big|\nabla|\tilde{B}|^{r/2}\big|^2\, dx
$$

$$
= -\int_\Omega (u\cdot\nabla)\tilde{B}\cdot|\tilde{B}|^{r-2}\tilde{B}\, dx - \int_\Omega (w\cdot\nabla)b\cdot|\tilde{B}|^{r-2}\tilde{B}\, dx
$$

$$
+ \int_\Omega (B\cdot\nabla)w\cdot|\tilde{B}|^{r-2}\tilde{B}\, dx + \int_\Omega (\tilde{B}\cdot\nabla)v\cdot|\tilde{B}|^{r-2}\tilde{B}\, dx. \tag{4.5}
$$

Noticing that the first integral on the right-hand side of (4.4)–(4.5) vanishes since u, w and $\tilde{B}$ are divergence free, we need only give estimates of other terms on the right-hand sides of (4.4)–(4.5), that is,

$$
\left|\int_\Omega (w\cdot\nabla)v\cdot|w|^{r-2}w\, dx\right| \le (r-1)\int_\Omega |w|^{r-1}|\nabla w||v|\, dx
$$

$$
\le (r-1)N_r(w)^{1/2}\left(\int_\Omega |w|^r|v|^2\, dx\right)^{1/2}
$$

$$
\le \frac{\nu}{32}N_r(w) + \frac{8(r-1)^2}{\nu}\|w\|_{r+2}^r\|v\|_{r+2}^2.
$$

On the other hand, $\|w\|_{r+2}^r \le cN_r(w)^{3/(r+2)}\|w\|_r^{r(r-1)/(r+2)}$. Thus

$$
\|w\|_{r+2}^r\|v\|_{r+2}^2 \le \frac{\nu}{32}N_r(w) + c\|w\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)}.
$$

Therefore

$$
\left|\int_\Omega (w\cdot\nabla)v\cdot|w|^{r-2}w\, dx\right| \le \frac{\nu}{16}N_r(w) + c\|w\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)}. \tag{4.6}
$$

Similarly,

$$
\left|\int_\Omega (\tilde{B}\cdot\nabla)b\cdot|w|^{r-2}w\, dx\right| \le (r-1)N_r(w)^{1/2}\left(\int_\Omega |w|^{r-2}|\tilde{B}|^2|v|^2\, dx\right)^{1/2}
$$

$$
\le \frac{\nu}{32}N_r(w) + \frac{8(r-1)^2}{\nu}\|w\|_{r+2}^{r-2}\|\tilde{B}\|_{r+2}^2\|v\|_{r+2}^2.
$$

Noticing that

$$\|w\|_{r+2}^{r-2}\|\tilde{B}\|_{r+2}^2 \le cN_r(w)^{3(r-2)/r(r+2)}\|w\|_r^{(r-1)(r-2)/(r+2)}N_r(\tilde{B})^{6/r(r+2)}\|\tilde{B}\|_r^{2(r-1)/(r+2)},$$

we get

$$\|w\|_{r+2}^{r-2}\|\tilde{B}\|_{r+2}^2\|v\|_{r+2}^2 \le \frac{\nu}{32}N_r(w) + \frac{\lambda}{32}N_r(\tilde{B})$$
$$+ c\|w\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)} + c\|\tilde{B}\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)}.$$

Hence

$$\left|\int_\Omega (\tilde{B}\cdot\nabla)b\cdot|w|^{r-2}w\,dx\right| \le \frac{\nu}{16}N_r(w) + \frac{\lambda}{32}N_r(\tilde{B})$$
$$+ c\|w\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)} + c\|\tilde{B}\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)}. \quad (4.7)$$

By a similar manner, we get

$$\left|\int_\Omega (w\cdot\nabla)b\cdot|\tilde{B}|^{r-2}\tilde{B}\,dx\right| \le \frac{\nu}{32}N_r(w) + \frac{\lambda}{16}N_r(\tilde{B})$$
$$+ c\|w\|_r^r\|b\|_{r+2}^{2(r+2)/(r-1)} + c\|\tilde{B}\|_r^r\|b\|_{r+2}^{2(r+2)/(r-1)}, \quad (4.8)$$
$$\left|\int_\Omega (\tilde{B}\cdot\nabla)v\cdot|\tilde{B}|^{r-2}\tilde{B}\,dx\right| \le \frac{\lambda}{16}N_r(\tilde{B}) + c\|\tilde{B}\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)} \quad (4.9)$$

and

$$\left|\int_\Omega \nabla\Pi\cdot|w|^{r-2}w\,dx\right| \le (r-1)\int_\Omega |\Pi||w|^{r-2}|\nabla w|\,dx$$
$$\le (r-1)N_r(w)^{1/2}\|w\|_{r+2}^{(r-2)/2}\|\Pi\|_{(r+2)/2}$$
$$\le \frac{\nu}{32}N_r(w) + c\|w\|_{r+2}^{r-2}\|\Pi\|_{(r+2)/2}^2.$$

Applying Lemma 4.1, we have

$$\|w\|_{r+2}^{r-2}\|\Pi\|_{(r+2)/2}^2$$
$$\le \|w\|_{r+2}^r\left(\|v\|_{r+2}^2 + \|w\|_{r+2}^2\right) + \frac{c}{\rho\mu}\|w\|_{r+2}^{r-2}\|\tilde{B}\|_{r+2}^2\left(\|b\|_{r+2}^2 + \|\tilde{B}\|_{r+2}^2\right)$$
$$\le cN_r(w)^{3/(r+2)}\|w\|_r^{r(r-1)/(r+2)}\|v\|_{r+2}^2 + cN_r(w)^{3/r}\|w\|_r^{r-1}$$
$$+ cN_r(w)^{3(r-2)/r(r+2)}\|w\|_r^{(r-1)(r-2)/(r+2)}N_r(\tilde{B})^{6/r(r+2)}\|\tilde{B}\|_r^{2(r-1)/(r+2)}\|b\|_{r+2}^2$$
$$+ cN_r(w)^{3(r-2)/r(r+2)}\|w\|_r^{(r-1)(r-2)/(r+2)}N_r(\tilde{B})^{12/r(r+2)}\|\tilde{B}\|_r^{4(r-1)/(r+2)}$$
$$\le \frac{\nu}{32}N_r(w) + \frac{\lambda}{32}N_r(\tilde{B}) + c\|w\|_r^r\|v\|_{r+2}^{2(r+2)/(r-1)} + c\|w\|_r^{r(r-1)/(r-3)}$$
$$+ c\|w\|_r^r\|b\|_{r+2}^{2(r+2)/(r-1)} + c\|\tilde{B}\|_r^r\|b\|_{r+2}^{2(r+2)/(r-1)} + c\|\tilde{B}\|_r^{r(r-1)/(r-3)}.$$

Hence

$$\left| \int_\Omega \nabla \Pi \cdot |w|^{r-2} w \, dx \right| \leq \frac{\nu}{16} N_r(w) + \frac{\lambda}{16} N_r(\tilde{B}) + c\|w\|_r^r \|v\|_{r+2}^{2(r+2)/(r-1)}$$
$$+ c\|w\|_r^{r(r-1)/(r-3)} + c\|w\|_r^r \|b\|_{r+2}^{2(r+2)/(r-1)}$$
$$+ c\|\tilde{B}\|_r^r \|b\|_{r+2}^{2(r+2)/(r-1)} + c\|\tilde{B}\|_r^{r(r-1)/(r-3)}. \qquad (4.10)$$

Since

$$\left| \int_\Omega (B \cdot \nabla)\tilde{B} \cdot |w|^{r-2} w \, dx \right| \leq \left| \int_\Omega (b \cdot \nabla)\tilde{B} \cdot |w|^{r-2} w \, dx \right|$$
$$+ \left| \int_\Omega (\tilde{B} \cdot \nabla)\tilde{B} \cdot |w|^{r-2} w \, dx \right|,$$

$$\left| \int_\Omega (b \cdot \nabla)\tilde{B} \cdot |w|^{r-2} w \, dx \right| \leq (r-1) \int_\Omega |w|^{r-2} |\nabla w| |b| |\tilde{B}| \, dx$$
$$\leq \frac{\nu}{32} N_r(w) + \frac{\lambda}{32} N_r(\tilde{B}) + c\|w\|_r^r \|b\|_{r+2}^{2(r+2)/(r-1)}$$
$$+ c\|\tilde{B}\|_r^r \|b\|_{r+2}^{2(r+2)/(r-1)},$$

$$\left| \int_\Omega (\tilde{B} \cdot \nabla)\tilde{B} \cdot |w|^{r-2} w \, dx \right| \leq (r-1) \int_\Omega |w|^{r-2} |\nabla w| |\tilde{B}|^2 \, dx$$
$$\leq \frac{\nu}{32} N_r(w) + c\|w\|_{r+2}^{r-2} \|\tilde{B}\|_{r+2}^4$$

and Lemma 4.3 tells us

$$\|w\|_{r+2}^{r-2} \|\tilde{B}\|_{r+2}^4 \leq c N_r(w)^{3(r-2)/r(r+2)} \|w\|_r^{(r-1)(r-2)/(r+2)} N_r(\tilde{B})^{12/r(r+2)} \|\tilde{B}\|_r^{4(r-1)/(r+2)}$$
$$\leq \frac{\nu}{32} N_r(w) + \frac{\lambda}{32} N_r(\tilde{B}) + c\|w\|_r^{r(r-1)(r-2)/(r+2)(r-3)} \|\tilde{B}\|_r^{4r(r-1)/(r+2)(r-3)}$$
$$\leq \frac{\nu}{32} N_r(w) + \frac{\lambda}{32} N_r(\tilde{B}) + c\|w\|_r^{r(r-1)/(r-3)} + c|\tilde{B}|_r^{r(r-1)/(r-3)},$$

we get

$$\left| \int_\Omega (B \cdot \nabla)\tilde{B} \cdot |w|^{r-2} w \, dx \right| \leq \frac{\nu}{16} N_r(w) + \frac{\lambda}{16} N_r(\tilde{B}) + c\|w\|_r^r \|b\|_{r+2}^{2(r+2)/(r-1)}$$
$$+ c\|\tilde{B}\|_r^r \|b\|_{r+2}^{2(r+2)/(r-1)} + c\|w\|_r^{r(r-1)/(r-3)}$$
$$+ c\|\tilde{B}\|_r^{r(r-1)/(r-3)}. \qquad (4.11)$$

Similarly,

$$\left| \int_\Omega (B \cdot \nabla)w \cdot |\tilde{B}|^{r-2} \tilde{B} \, dx \right| \leq \frac{\nu}{16} N_r(w) + \frac{\lambda}{16} N_r(\tilde{B}) + c\|w\|_r^r \|b\|_{r+2}^{2(r+2)/(r-1)}$$
$$+ c\|\tilde{B}\| \|b\|_{r+2}^{2(r+2)/(r-1)} + c\|w\|_r^{r(r-1)/(r-3)}$$
$$+ c\|\tilde{B}\|_r^{r(r-1)/(r-3)}. \qquad (4.12)$$

Substituting estimates (4.6)-(4.12) into (4.4)-(4.5) and summing them together, we get

$$
\frac{1}{r}\frac{d}{dt}\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big) + \frac{1}{2}\min(\nu,\lambda)(N_r(w) + N_r(\tilde{B}))
$$
$$
\leq c\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big)\big(\|v\|_{r+2}^{2(r+2)/(r-1)} + \|b\|_{r+2}^{2(r+2)/(r-1)}\big)
$$
$$
+ c\big(\|w\|^{r(r-1)/(r-3)} + \|\tilde{B}\|^{r(r-1)/(r-3)}\big).
$$

By interpolation, we have $\|w\|_r \leq \|w\|^{4/(3r-2)}\|w\|_{3r}^{3(r-2)/(3r-2)}$. From Lemma 4.2 it follows that $N_r(w) \geq c\|w\|^{-4r/(3(r-2))}\|w\|_r^{r+(4r/3(r-2))}$. Using these estimates, we get

$$
\frac{1}{r}\frac{d}{dt}\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big) + \frac{1}{2}\min(\nu,\lambda)\big(\|w\|^{-4r/3(r-2)}\|w\|_r^{r+(4r/3(r-2))}
$$
$$
+ \|\tilde{B}\|^{-4r/3(r-2)}\|\tilde{B}\|_r^{r+(4r/3(r-2))}\big)
$$
$$
\leq c\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big)\big(\|v\|_{r+2}^{2(r+2)/(r-1)}\|b\|_{r+2}^{2(r+2)/(r-1)}\big)
$$
$$
+ c\big(\|\tilde{B}\|_r^{r(r-1)/(r-3)} + \|w\|_r^{r(r-1)/(r-3)}\big).
$$

From (3.10), Theorem 3.1 and $H^2(\Omega) \hookrightarrow L^{r+2}(\Omega)$, we get

$$
\frac{1}{r}\frac{d}{dt}\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big) + c\big(\|w_0\|^2 + \|\tilde{B}_0\|^2\big)^{-k}e^{\beta kt}\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big)^{1+(4/3(r-2))}
$$
$$
\leq c\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big) + c\big(\|w\|_r^r + \|\tilde{B}\|_r^r\big)^{(r-1)/(r-3)}, \tag{4.13}
$$

where $k = 2r/3(r-2)$.

Let $y(t) = \|w(t)\|_r^r + \|\tilde{B}(t)\|_r^r$. From (4.13) we get

$$
\frac{1}{r}y'(t) + c\big(\|w_0\|_r^r + \|\tilde{B}_0\|_r^r\big)^{-k_*}e^{\beta kt}y(t)^{1+(4/3(r-2))} \leq cy(t) + cy(t)^{(r-1)/(r-3)},
$$

where $k_* = 4/3(r-2)$.

Applying Lemma 4.5, we get the following theorem.

THEOREM 4.4. *Suppose (u, B) is any L^r-bounded solution of $(E.S)$, $f(x) \in L^r(\Omega)$ and $u_0(x)$, $B_0(x) \in L^r(\Omega) \cap H$, where (v, b) is the solution of $(S.S)$. If the L^2-norm of $f(x)$ is sufficiently small or ν and λ are sufficiently large, then there exist constants c, β, $k_* > 0$ independent of t, u and B such that, for all $t > 0$,*

$$
\|u(t) - v\|_r^r + \|B(t) - b\|_r^r \leq c\big(\|u_0 - v\|_r^r + \|B_0 - b\|_r^r\big)^{k_*}e^{-\beta kt}.
$$

LEMMA 4.5 ([7]). *Let us make the following assumptions:*

(i) *There exists a constant $M > 0$ such that for all $t > 0$, the function $y(t) < M$.*

(ii) *There exists a differentiable function $h(t) > 0$ and continuous functions $f_1(t)$,*
$\ldots, f_m(t)$ *for $t > 0$ such that*

$$\lim_{t \to \infty} \frac{h'(t)}{h(t)} = l > 0, \quad \lim_{t \to \infty} \frac{f_i(t)}{h(t)} = 0, \quad (i = 1, \ldots, m).$$

(iii) *There are constants $a_1 > a_2 > \cdots > a_m > a_0 > 1$.*
(iv) *The function $y(t)$ satisfies the differential inequality*

$$y'(t) + ch(t)y(t)^{a_0} \le b_0 y(t) + \sum_{i=1}^{m} f_i(t)y(t)^{a_i},$$

where $c, b_0 > 0$. Then the estimate $y(t)^{a_0-1} \le (b_0/cl)h(t)^{-1}$ holds for all $t > 0$.

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