SUBGROUPS OF FINITE INDEX IN FREE GROUPS

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1. Introduction. This paper[1] has as its chief aim the establishment of two formulae associated with subgroups of finite index in free groups. The first of these (Theorem 3.1) gives an expression for the total length of the free generators of a subgroup U of the free group F_r with r generators. The second (Theorem 5.2) gives a recursion formula for calculating the number of distinct subgroups of index n in F_r.

Of some independent interest are two theorems used which do not involve any finiteness conditions. These are concerned with ways of determining a subgroup U of F. The first (Theorem 4.1) gives a criterion for recognizing different representations of the same group, and the second (Theorem 5.1) yields a determination of a subgroup U by a set of permutations.

2. The standard representation. We shall be considering a free group F_r with a finite number of generators $s_1, s_2, \ldots, s_r$. A *word* f in F_r is any finite string $f = a_1 \ldots a_t$, each $a_i =$ some s_j or s_j^{-1} and the length of f, $l^*(f) = t$ the number of terms in the string. Every *element* f of F_r may be written in various ways as a word but in only one way as a reduced word. We shall write $l(f)$ for the length of the reduced form of the element f, whence

$$l^*(f) = l(f)$$

if and only if f is in reduced form.

A *Schreier system* is a set S of reduced words in F_r such that if any $f = a_1 \ldots a_t$ belongs to F_r then also $a_1 \ldots a_{t-1}$ belongs to F_r. If U is any subgroup of F_r then in the left cosets of U in F_r

$$F_r = Ug_1 + Ug_2 + \ldots + Ug_n \tag{2.1}$$

the representatives $g_1 = 1, g_2, \ldots, g_n$ may be chosen as a Schreier system. In this paper we shall be considering primarily subgroups U of finite index in F_r, $[F_r : U] = n$. In a recent paper [1] a standard representation for U was given. If G is a generic term for the g_i and S for the s_i, let $\phi(GS^\epsilon)$, $\epsilon = \pm 1$ be the G for the coset to which GS^ϵ belongs, H being a generic term for elements GS^ϵ, then ϕ is a function defined for arguments H. It was shown that U is completely determined by the Schreier system $S = \{G\}$ and the function $\phi(H)$. $\phi(H)$ satisfies:

(2.2.1) $\phi(H)$ is a G;

(2.2.2) $\phi(H) = G$ if $H = G$;

(2.2.3) $\phi[\phi(GS^\epsilon)S^{-\epsilon}] = G$.

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Hence we may consider $\{G\}$ and $\phi(H)$ as giving a representation of U

(2.3) $$U = U[\{G\}, \phi(H)]$$

which we shall call the standard representation for U.

3. Coset representatives and generators. Suppose the subgroup U of F_r to be given by its standard representation (2.3). Then an element u

(3.1) $$u = u(i, \alpha) = g_i s_\alpha \phi(g_i s_\alpha)^{-1}$$

is either a reduced word as it stands or reduces to the identity [1, §3]. Moreover, those u's different from the identity are a free set of generators of U. Since both g_i and $\phi(g_i s_\alpha)^{-1} = g_j^{-1}$ are reduced as they stand, $u(i, \alpha)$ reduces to the identity if and only if one of two things happens: (1) g_i ends in s_α^{-1}, (2) g_j ends in s_α.

Let $n(s_\alpha)$ be the number of G's ending in s_α, $n(s_\alpha^{-1})$ the number ending in s_α^{-1}. Then

$$\sum_\alpha [n(s_\alpha) + n(s_\alpha^{-1})] = n - 1,$$

since every $G \neq 1$ is counted exactly once on the left. Hence, as Schreier proved [2] the number of generators of U is $nr - (n - 1) = n(r - 1) + 1 = N$. But there is also a relation connecting the lengths of the generators. Let us write

(3.2.1) $$L = \sum_{i=1}^{n} l(g_i),$$

(3.2.2) $$K = \sum_{k=1}^{N} l(u_k),$$

where in (3.2.2) $u_1, \ldots, u_N$ are the generators $u(i, \alpha) \neq 1$ of U. From 3.1 it follows that

(3.3) $$\sum_{i=1}^{n} l^*[u(i,\alpha)] = \sum_{i=1}^{n} l^*[g_i s_\alpha \phi(g_i s_\alpha)^{-1}] = 2L + n,$$

since the $g_j = \phi(g_i s_\alpha)$ are a permutation of the G's. In (3.3) let us separate the summation into three classes: (A) terms for which g_i ends in s_α^{-1}; (B) terms for which $g_j = \phi(g_i s_\alpha)$ ends in s_α and (C) the remaining terms. Then

(3.4) $$2L + n = \sum_A l^*[u(i, \alpha)] + \sum_B l^*[u(i, \alpha)] + \sum_C l^*[u(i, \alpha)].$$

If we now write

(3.5) $$\lambda(s_\alpha) = \sum_j l(g_j),\ g_j \text{ ends in } s_\alpha;$$

$$\lambda(s_\alpha^{-1}) = \sum_i l(g_i),\ g_i \text{ ends in } s_\alpha^{-1};$$

we have

(3.6) $$2L + n = 2\,\lambda(s_\alpha^{-1}) + 2\lambda(s_\alpha) + \sum_C l[u(i,\alpha)],$$

since for $\sum_A$ if g_i ends in s_α^{-1}, $l[\phi(g_i s_\alpha)] = l(g_i) - 1$ and $l^*[g_i s_\alpha \phi(g_i s_\alpha)^{-1}] = 2l(g_i)$. Similarly for $\sum_B$ if $g_j = \phi(g_i s_\alpha)$ ends in s_α, then $l[\phi(g_j)] = l(g_i) + 1$ and $l^*[g_i s_\alpha \phi(g_i s_\alpha)^{-1}] = 2l(g_j)$. Summing (3.6) over $\alpha = 1, 2, \ldots, r$ we have

$$(3.7) \qquad (2L + n)r = 2L + \sum_{k=1}^{N} l(u_k),$$

since $\sum_{\alpha}[\lambda(s_\alpha^{-1}) + \lambda(s_\alpha)] = L$ and the u's of class C for $\alpha = 1, 2, \ldots, r$ are precisely the $N = (r - 1)n + 1$ free generators of U. Hence,

THEOREM 3.1. *Let U be a subgroup of index n in a free group F_r with r generators, given by its standard representation. Thus, if L is the total length of the coset representatives and K the total length of the free generators of U given by the standard representation, then $K = (2L + n)r - 2L$.*

4. Different representations of the same subgroup. Given a subgroup U of F, the choice of coset representatives, even as a Schreier system, is not in general unique. The choice of representatives, given U, determines the function ϕ. We seek a criterion for recognizing different representations of the same subgroup. The following theorem does not assume either that the number of generators of F is finite or that U is of finite index.

THEOREM 4.1. *Let $U_1 = U_1[\{G^1\}, \phi^1(H)]$, $U_2 = U_2[\{G^2\}, \phi^2(H)]$ be standard representations of subgroups U_1 and U_2 of the free group F. Then $U_1 = U_2$ if and only if we may find a one to one correspondence between the coset representatives mapping the identity onto itself such that if $g_i^1 \rightleftarrows g_j^2$, including $1 = g_1^1 \rightleftarrows g_1^2 = 1$ then for any s_α^ϵ, $\phi^1(g_i^1 s_\alpha^\epsilon) \rightleftarrows \phi^2(g_j^2 s_\alpha^\epsilon)$.*

Proof. First suppose $U_1 = U_2 = U$. Then if $Ug_i^1 = Ug_j^2$ is the same left coset of U as given in the two representations this establishes a $1 - 1$ correspondence $g_i^1 \rightleftarrows g_j^2$ which includes $1 = g_1^1 \rightleftarrows g_1^2 = 1$ such that if $g_i^1 \rightleftarrows g_j^2$ then $\phi^1(g_i^1 s_\alpha^\epsilon) \rightleftarrows \phi^2(g_j^2 s_\alpha^\epsilon)$. Conversely, suppose the $1 - 1$ correspondence given with $1 = g_1^1 \rightleftarrows g_1^2 = 1$ such that if $g_i^1 \rightleftarrows g_j^2$ then $\phi^1(g_i^1 s_\alpha^\epsilon) \rightleftarrows \phi^2(g_j^2 s_\alpha^\epsilon)$. By induction on length we may show that an element belonging to a coset of U_1 belongs to the corresponding coset of U_2. This is true for $l(f) = 0$, since $f = 1$, and $1 \rightleftarrows 1$. And if f is in the cosets $U_1 g_i^1 \rightleftarrows U_2 g_j^2$, then fs_α^ϵ is in the corresponding cosets $U_1\phi^1(g_i^1 s_\alpha^\epsilon) \rightleftarrows U_2\phi^2(g_j^2 s_\alpha^\epsilon)$. Hence corresponding cosets contain the same elements and in particular $U_1 = U_2$. Examples exist (subgroups of index 3 in F_2) showing that the requirement $1 = g_1^1 \rightleftarrows g_1^2 = 1$ is not redundant. As an illustration of the content of Theorem 3.1, it may be observed that there are 18 Schreier systems for subgroups of index 3 in F_2 and each of these yields four or six subgroups of index 3, and yet of these only 13 are different.

5. The number of subgroups of index n in F_r. If U is a subgroup of any group F (free or not) generated by $s_1, \ldots, s_i, \ldots$ then by multiplication on the right each s_i induces a permutation P_i on the left cosets of U in F. Since the s_i's generate F, the P_i's will generate a group transitive on the cosets. The following theorem is a kind of converse for free groups. No finiteness assumptions are needed here.

THEOREM 5.1. *Given a free group F with generators $s_1, \ldots s_i, \ldots$ and a set of indices $I = \{1, \ldots, t, \ldots\}$. With each generator s_i associate a permutation P_i of the indices. Suppose $J = \{1, \ldots, j, \ldots\}$ to be the subset of I which is*

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the transitive constituent of I including the index 1. *Then in F there is a Schreier system* $\{G\} : g_1 = 1, \ldots, g_j, \ldots$, *which may be indexed by J and a* ϕ *function such that* $\phi(g_j s_i^\epsilon) = g_k$ *if and only if* P_i^ϵ *takes j into k.*

Proof. The permutations P_i generate a group E. In E let D_1 be the subgroup which fixes the symbol 1. The mapping $s_i \to P_i$ determines a homomorphism of F onto E, $F \to E$. In this homomorphism let U be the subgroup of F mapped onto D_1, $U \to D_1$. We may choose coset representatives $\{G\}$ of U in F as a Schreier system. If $G \to A$, then $UG \to D_1A$. Let A take the index 1 into j, which we write $(1)A = j$. Here let us assign the index j to G, putting $G = g_j$. Hence the Schreier system $\{G\}$ has been indexed by those indices into which elements of E take the index 1, which form the set J. Here if $g_j \to A$, $s_i^\epsilon \to P_i^\epsilon$, then $g_j s_i^\epsilon \to AP_i^\epsilon$. Now if $(j)P_i^\epsilon = k$, then $(1)AP_i^\epsilon = k$ and AP_i^ϵ belongs to the coset D_1B of D_1, consisting of those elements X of E such that $(1)X = k$. Here $Ug_k \to D_1B$. Hence $g_j s_i^\epsilon$ belongs to Ug_k or $\phi(g_j s_i^\epsilon) = g_k$.

THEOREM 5.2. *The number* $N_{n,r}$ *of subgroups of index n in* F_r *is given recursively by* $N_{1,r} = 1$,

$$N_{n,r} = n(n!)^{r-1} - \sum_{i=1}^{n-1} [(n-i)!]^{r-1} N_{i,r}.$$

Proof. $N_{1,r} = 1$ states merely that F_r is its own unique subgroup of index 1. From Theorem 5.1 each subgroup U of index α is determined by r permutations $P_1, \ldots, P_r$, generating a group transitive on numbers $1, b_2, \ldots, b_\alpha$. From Theorem 4.1 replacement of $1, b_2, \ldots, b_\alpha$ by $1, c_2, \ldots, c_\alpha$ in $P_1, \ldots, P_r$ yields the same group U.

Consider r permutations $P_1, P_2, \ldots, P_r$ on n letters. The number of possible choices for $P_1, P_2, \ldots, P_r$ is $(n!)^r$. In general $P_1, \ldots, P_r$ will not generate a group transitive on all n letters. Let $1, b_2, b_3, \ldots, b_\alpha$ be the transitive constituent which includes the identity. Disregarding the rest of the numbers, we may associate $P_1, \ldots, P_r$ with a unique subgroup of index α. The other $n - \alpha$ letters may occur in $[(n-\alpha)!]^r$ ways. Also $b_2, \ldots, b_\alpha$ could be replaced by numbers $c_2, \ldots, c_\alpha$ from $2, \ldots, n$ in $(n-1)(n-2) \ldots (n-\alpha+1)$ ways. Hence, of the permutations $P_1, \ldots, P_r$, on $1, 2, \ldots, n$ a total of $(n-1)(n-2) \ldots (n-\alpha+1)$ $[(n-\alpha)!]^r = (n-1)![(n-\alpha)!]^{r-1}$ are associated with the same subgroup of index α in F_r. Hence $(n-1)![(n-\alpha)!]^{r-1} N_{\alpha,r}$ permutations are associated with subgroups of index α. Hence

$$(n!)^r = \sum_{\alpha=1}^{n} (n-1)![(n-\alpha)!]^{r-1} N_{\alpha,r}.$$

If here we divide by $(n-1)!$ and transpose the sum for $\alpha = 1$ to $n - 1$ we have the formula of the Theorem.

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