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UNIFORM APPROXIMATION BY ELEMENTARY OPERATORS

BOJAN MAGAJNA

Department of Mathematics, University of Ljubljana, Jadranska 21, Ljubljana 1000, Slovenia (bojan.magajna@fmf.uni-lj.si)

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Abstract On a separable C^*-algebra A every (completely) bounded map which preserves closed two-sided ideals can be approximated uniformly by elementary operators if and only if A is a finite direct sum of C^*-algebras of continuous sections vanishing at ∞ of locally trivial C^*-bundles of finite type.

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1. Introduction and the main result

Throughout this paper A will denote a C^*-algebra, A_+ will denote the positive and A_h will denote the self-adjoint part of A. An *elementary operator on* A is a map of the form

$$\psi(x) = \sum_{i=1}^{m} a_i x b_i, \quad x \in A, \tag{1.1}$$

where a_i and b_i are fixed elements of the multiplier algebra $M(A)$ of A. The smallest m for which ψ can be expressed in the form (1.1) is called the *length* of ψ. The set of all elementary operators on A is denoted by $\mathrm{E}(A)$ and its norm closure (in the set of all bounded operators on A) by $\overline{\overline{\mathrm{E}(A)}}$. By $\mathcal{I}_{\mathrm{c}}(A)$ we will denote the set of all closed two-sided ideals in A and by $\mathrm{IB}(A)$ (respectively, $\mathrm{ICB}(A)$) the set of all bounded (respectively, all completely bounded [**21**]) maps that preserve all ideals in $\mathcal{I}_{\mathrm{c}}(A)$. By an *ideal* we shall always mean a closed two-sided ideal. Clearly,

$$\mathrm{E}(A) \subseteq \mathrm{ICB}(A) \subseteq \mathrm{IB}(A).$$

In this note we characterize C^*-algebras for which the equalities $\mathrm{ICB}(A) = \overline{\overline{\mathrm{E}(A)}}$ or $\mathrm{IB}(A) = \overline{\overline{\mathrm{E}(A)}}$ hold.

Theorem 1.1. *For a separable C^*-algebra A the inclusion* $\mathrm{ICB}(A) \subseteq \overline{\overline{\mathrm{E}(A)}}$ *holds if and only if A is a finite direct sum of homogeneous C^*-algebras of finite type; in this case* $\mathrm{IB}(A) = \mathrm{E}(A) = \mathrm{ICB}(A)$.

We recall that a C^*-algebra A is called *n-homogeneous* if all its irreducible representations are of the same finite dimension n. (By the dimension of a representation π we mean the dimension of the Hilbert space of π.) Then, by [**11**, **26**], A is isomorphic to the C^*-algebra $\Gamma_0(E)$ of all continuous sections vanishing at ∞ of a locally trivial C^*-bundle E with fibres isomorphic to $\mathrm{M}_n(\mathbb{C})$. (E is just a usual vector bundle such that the local trivializations, restricted to fibres, are isomorphisms of C^*-algebras.) If the base space Δ of this bundle admits a finite open covering (Δ_i) such that $E|\Delta_i$ is trivial for each i (as a C^*-bundle), then E is said to be of *finite type* [**14**] and we shall say that in this case A is of *finite type*.

We note that a weaker form of approximation is always possible: namely, for every C^*-algebra A the set $\mathrm{E}(A)$ is dense in $\mathrm{ICB}(A)$ (and in $\mathrm{IB}(A)$) in the point norm-topology (see [**18**, 2.3] and [**5**, 5.3.4]). However, there is in general no control on the norms in this approximation: not every complete contraction $\phi \in \mathrm{ICB}(A)$ can be approximated by a net of complete contractions in $\mathrm{E}(A)$. If A is a von Neumann algebra, then each $\phi \in \mathrm{CB}(A)$ preserving all weak* closed ideals can be approximated by complete contractions in $\mathrm{E}(A)$ in the point-weak* topology if and only if A is injective [**8**] (at least if the predual of A is separable). For a general C^*-algebra A, the question of when every complete contraction $\phi \in \mathrm{ICB}(A)$ can be approximated pointwise by complete contractions in $\mathrm{E}(A)$ is connected to the theory of tensor products of C^*-algebras and the complete answer is still not clear to the author. Concerning elementary operators, we mention that in recent years interest has shifted from spectral and structural theory [**9**, **13**] to questions related to the natural map μ from the central Haagerup tensor product $M(A) \overset{h}{\otimes}_Z M(A)$ into $\mathrm{CB}(A)$ (see [**4**,**5**,**8**,**19**,**24**] and references therein). In particular, the problem of when μ is isometric has been much studied by several authors for special cases of C^*-algebras (see [**5**, Chapter 5]) and was finally solved for general C^*-algebras in [**6**, **25**]. Clearly, the range of μ is contained in $\overline{\overline{\mathrm{E}(A)}}$ and Theorem 1.1 characterizes C^*-algebras in which the range of μ is as large as possible.

In one direction the proof of Theorem 1.1 is easy. Namely, if $A = \Gamma_0(E)$ with E of finite type, the usual (finite) partition of unity argument reduces the proof to the case when $E \cong \Delta \times \mathrm{M}_n(\mathbb{C})$ is trivial, so that $A \cong C_0(\Delta, \mathrm{M}_n(\mathbb{C}))$ (continuous matrix-valued functions vanishing at ∞). In this special case a bounded linear map ϕ, which preserves all ideals of the form $J_t = \{f \in A : f(t) = 0\}$, $t \in \Delta$, decomposes into a bounded continuous collection of maps on fibres $A/J_t \cong \mathrm{M}_n(\mathbb{C})$; in other words, ϕ is in the set $\mathrm{B}_{Z_0}(A) = \mathrm{B}_{Z_0}(\mathrm{M}_n(Z_0))$ of bimodule maps over the centre $Z_0 = C_0(\Delta)$ of A. With e_{ij} the standard matrix units in $\mathrm{M}_n(\mathbb{C})$ and $\eta_{kl} : \mathrm{M}_n(Z_0) \to Z_0$ the maps $\eta_{kl}([z_{ij}]) = z_{kl}$, we have that

$$\phi_{ki}^{jl} : Z_0 \to Z_0, \quad \phi_{ki}^{jl}(z) := \eta_{kl}(\phi(ze_{ij})), \quad z \in Z_0,$$

are bimodule maps over Z_0 (thus, double centralizers since Z_0 is commutative), and hence given by multiplications with certain elements c_{ki}^{jl} of the multiplier algebra $Z = C_b(\Delta)$ of Z_0. Then for $[z_{ij}] = \sum_{i,j=1}^{n} z_{ij} e_{ij}$ we have that

$$\phi([z_{ij}]) = \sum_{i,j=1}^{n} \phi(z_{ij}e_{ij}) = \sum_{i,j,k,l=1}^{n} c_{ki}^{jl} z_{ij} e_{kl} = \sum_{k,l,r,s=1}^{n} c_{ks}^{rl} e_{ks} \bigg(\sum_{i,j=1}^{n} z_{ij} e_{ij} \bigg) e_{rl},$$

so that ϕ is an elementary operator with coefficients in $\mathrm{M}_n(Z)$, the multiplier C^*-algebra of $A = \mathrm{M}_n(Z_0)$.

In certain special cases (say, if A is prime) one can use the Akemann–Pedersen characterization of C^*-algebras having only inner derivations [**2**] together with some additional work to give a relatively short proof of a part of Theorem 1.1. But in general the proof of Theorem 1.1 requires construction of new classes of maps preserving ideals, which cannot be uniformly approximated by elementary operators. It is perhaps not very surprising that such maps exist if the dimensions of irreducible representations of A are not bounded. They can be taken to be of the form $x \mapsto \sum_{k=1}^{\infty} e_k x f_k$, where the sum is norm convergent for all $x \in A$, but not uniformly convergent. It will be shown in § 2 that an appropriate choice of the coefficients e_k and f_k is possible so that such a map cannot be approximated uniformly by elementary operators.

But even if A is subhomogeneous we do not always have the inclusion $\mathrm{ICB}(A) \subseteq \overline{\overline{\mathrm{E}(A)}}$. Consider, for example, the C^*-subalgebra A_0 of $C([0,1], \mathrm{M}_2(\mathbb{C}))$ consisting of all x such that $x(0)$ is a diagonal matrix with 0 on the $(2,2)$ position (or, alternatively, a general diagonal matrix) and the map $x \mapsto \phi(x) = e_{12} x e_{12}$, where e_{12} has 1 on the position $(1,2)$ and 0 elsewhere. It can be shown that ϕ preserves ideals but $\phi \notin \overline{\overline{\mathrm{E}(A_0)}}$ (the details are just a special case of those in the proof of Lemma 4.1; see the paragraphs containing (4.4)–(4.8)). Such examples suggest the way to a part of the proof of Theorem 1.1. Namely, for an n-subhomogeneous C^*-algebra A which is not a direct sum of homogeneous C^*-algebras, it will be shown in § 4 that the multiplier algebra $M(J)$ of the n-homogeneous ideal J of A contains an element b such that the two-sided multiplication $\phi : x \mapsto bxb$ maps A into A and $\phi \in \mathrm{ICB}(A) \setminus \overline{\overline{\mathrm{E}(A)}}$ (provided that J is essential in A, the general case will be reduced to this situation). As a preparation for this, we shall show in § 3 that, if J is not unital, $M(J)$ is the C^*-algebra of continuous sections of a (not necessarily locally trivial) C^*-bundle over the Stone–Čech compactification $\beta(U)$ of the spectrum U of J and the n-homogeneous ideal of $M(J)$ properly contains J (Lemma 3.4). This will enable us to show in § 4 (as the first step towards the proof of Theorem 1.1 in the case of subhomogeneous C^*-algebras) that there exists a point in $\beta(U)$ at which A ($A \subseteq M(J)$) looks in a certain respect essentially like A_0 of the example mentioned above.

On the other hand, the explanation that the homogeneous summands in Theorem 1.1 must be of finite type is simple and can be given immediately.

Proof that A must be of finite type. Assume that a locally trivial C^*-bundle E over a locally compact space Δ with fibres $\mathrm{M}_n(\mathbb{C})$ is not of finite type. Then E is not of finite type as a vector bundle by [**23**, 2.9] and it follows that for any finite set $\{a_1, \dots, a_m\}$ of bounded continuous sections of E there exists a point $t_0 \in \Delta$ such that

$$\dim \operatorname{span}\{a_1(t_0), \dots, a_m(t_0)\} < n^2.$$

Indeed, if this were not the case, then the map

$$f : \Delta \times \mathbb{C}^m \to E, \quad f(t; \lambda_1, \dots, \lambda_m) = \sum_{j=1}^{m} \lambda_j a_j(t)$$

would be a surjective morphism of vector bundles and E would be isomorphic to the subbundle $(\ker f)^{\perp}$ of $\Delta \times \mathbb{C}^m$, and hence of finite type by [**14**, 3.5.8]. It follows that for each elementary operator ψ on $A = \Gamma_0(E)$ there is a point $t_0 \in \Delta$ such that the induced elementary operator ψ_{t_0} on $A/J_{t_0} \cong \mathrm{M}_n(\mathbb{C})$ has length at most $n^2 - 1$. On the other hand, the (normalized) central trace τ on A (defined by $\tau(x)(t) = (1/n)\operatorname{tr} x(t)$, $t \in \Delta$) preserves all (primitive ideals J_t hence all) ideals of A; hence, $\tau \in \mathrm{ICB}(A)$. But, denoting by $e_{i,j}$, $i, j = 1, \dots, n$, the usual matrix units in $\mathrm{M}_n(\mathbb{C})$, we have that

$$\tau(x)(t_0) = (1/n) \sum_{i,j=1}^{n} e_{i,j} x(t) e_{j,i},$$

so that τ_{t_0} on $\mathrm{M}_n(\mathbb{C})$ has length n^2. Therefore, τ_{t_0} has a positive distance d to the closed set of all elementary operators of length less than or equal to $n^2 - 1$ on $\mathrm{M}_n(\mathbb{C})$. This implies that the distance of τ to $\mathrm{E}(A)$ is at least d, so $\tau \notin \overline{\overline{\mathrm{E}(A)}}$. $\square$

Throughout this paper we shall denote by $\hat{A}$ the spectrum of A (equal to the set of all equivalence classes of irreducible representations) and by $\check{A}$ the primitive spectrum of A (equal to the set of all primitive ideals) equipped with the Jacobson topology. The norm and the weak* closure of a set S will be denoted by $\bar{\bar{S}}$ and $\bar{S}$, respectively.

2. A reduction to subhomogeneous C^*-algebras

Lemma 2.1. *Let A be an irreducible C^*-subalgebra in $\mathrm{B}(\mathcal{H})$, let $x_1, \dots, x_n$ be arbitrary elements of A, let $g \in A_+ \setminus \{0\}$, let $B = \overline{\overline{gAg}}$ be the hereditary C^*-subalgebra generated by g and let $\varepsilon > 0$. If $\operatorname{rank} g > n$, then there exist $e, f \in B_+$ such that $\|e\| = 1 = \|f\|$ and*

$$\|e x_j f\| < \varepsilon, \quad j = 1, \dots, n.$$

Proof. Choose a unit vector $\eta \in \mathcal{K} := [B\mathcal{H}]$. Note that $b = (b|\mathcal{K}) \oplus (0|\mathcal{K}^{\perp})$ for each $b \in B$ (since $B\mathcal{K}^{\perp} \subseteq \mathcal{K} \cap \mathcal{K}^{\perp} = 0$), that $B|\mathcal{K}$ is irreducible [**20**, 5.5.2] and that $\dim \mathcal{K} > n$ since $\operatorname{rank} g > n$. Hence, by the Kadison transitivity theorem there exists $e \in B$ with $\|e\| = 1$ such that e annihilates the projections of all vectors $x_j\eta$ to $\mathcal{K}$. Thus, (since $e\mathcal{K}^{\perp} = 0$),

$$e x_j \eta = 0, \quad j = 1, \dots, n.$$

Moreover, replacing e by e^*e, we may assume that $e \in B_+$. By the algebraic irreducibility [**20**, 5.2.3] there exists $b \in B_+$ with $\|b\| = 1$ and $b\eta = \eta$. Then the vector state $\omega_\eta(x) := \langle x\eta, \eta \rangle$ annihilates the element

$$x_0 := \sum_{j=1}^{n} b x_j^* e^2 x_j b$$

of B. Since ω_η is a pure state on B (by irreducibility of B on $\mathcal{K}$), by [**1**] there exists a positive element h in the unitization of B such that $\|h\| = 1$,

$$\|h x_0 h\| < \varepsilon^2 \quad \text{and} \quad \omega_\eta(h) = 1.$$

This implies that $\|ex_jbh\| < \varepsilon$ for all $j = 1, \ldots, n$, and (since $\|h\| = 1$ and $\langle h\eta, \eta\rangle = 1$) $h\eta = \eta$. Set $f := |hb| = |(bh)^*|$. Then $f \in B_+$, $\|f\| = 1$ (since $hb\eta = \eta$) and (using the polar decomposition $(bh)^* = uf$ of $(bh)^*$) we deduce that $\|ex_jf\| = \|ex_jbh\| < \varepsilon$ for all $j = 1, \ldots, n$. □

Lemma 2.2. *If A is separable and has an infinite-dimensional irreducible representation $\pi : A \to \mathrm{B}(\mathcal{H})$, then there exist two bounded sequences (e_i) and (f_i) in A_+ such that $\|\pi(e_i)\| = 1 = \|\pi(f_i)\|$, $e_ie_j = 0 = f_if_j$ if $i \neq j$ and the sum*

$$\phi(x) = \sum_{n=1}^{\infty} e_n x f_n \tag{2.1}$$

is norm convergent for each $x \in A$.

Proof. Since $\pi(A)$ is irreducible and $\mathcal{H}$ infinite dimensional, $\pi(A)$ must be infinite dimensional and the same for any of its maximal abelian self-adjoint subalgebras [**15**, 4.6.12]. Thus, by functional calculus we may find a sequence $(\dot{g}_i)$ in $\pi(A)_+$ with $\|\dot{g}_i\| = 1$, $\dot{g}_i\dot{g}_j = 0$ if $i \neq j$ and $\operatorname{rank} \dot{g}_i > i$. Set $P = \ker \pi$ and identify A/P with $\pi(A)$. Let (x_j) be a bounded sequence with dense span in A. By Lemma 2.1 for each n there exist elements $\dot{e}_n$ and $\dot{f}_n$ in $\overline{(\dot{g}_n\pi(A)\dot{g}_n)}_+$ such that $\|\dot{e}_n\| = 1 = \|\dot{f}_n\|$ and

$$\|\dot{e}_n\pi(x_j)\dot{f}_n\| < \frac{1}{2^n}, \quad j = 1, \ldots, n. \tag{2.2}$$

Since the $\dot{g}_n$ are orthogonal (that is, $\dot{g}_i\dot{g}_j = 0$ if $i \neq j$), the same holds for $\dot{e}_n$ and for $\dot{f}_n$. By [**15**, 4.6.20] we may lift $(\dot{e}_n)$ (and similarly $(\dot{f}_n)$) from $\pi(A)$ to orthogonal sequences $(\tilde{e}_n)$ (and $(\tilde{f}_n)$) of norm 1 elements in A_+. Recall that, with (u_k) an approximate unit in P, we have $\|\pi(x)\| = \lim \|(1 - u_k)x(1 - u_k)\|$ for all $x \in A$; hence, from (2.2) for each n there exists $u_n \in P$, $0 \leqslant u_n \leqslant 1$, such that

$$\|(1 - u_n)\tilde{e}_n x_j \tilde{f}_n(1 - u_n)\| < \frac{1}{2^n}, \quad j = 1, \ldots, n. \tag{2.3}$$

Set $e_n = \tilde{e}_n(1 - u_n)\tilde{e}_n$ and $f_n = \tilde{f}_n(1 - u_n)\tilde{f}_n$. Then $e_ie_j = 0 = f_if_j$ if $i \neq j$, $\|e_n\| = 1 = \|f_n\|$ (since $\|\pi(e_n)\| = 1$, $\|\pi(f_n)\| = 1$ and $\|e_n\|, \|f_n\| \leqslant 1$), and (2.3) implies that

$$\|e_n x_j f_n\| < \frac{1}{2^n}, \quad j = 1, \ldots, n. \tag{2.4}$$

Since $\|\sum_{n=1}^{\infty} e_n^2\| = \max_n \|e_n^2\| = 1$ (by orthogonality) and $\|\sum_{n=1}^{\infty} f_n^2\| = 1$, it follows that (2.1) defines a (complete) contraction ϕ from A into the von Neumann envelope $\bar{A}$ of A. We have

$$\phi(x_j) = \sum_{n=1}^{j-1} e_n x_j f_n + \sum_{n=j}^{\infty} e_n x_j f_n,$$

where the sum on the right side is norm convergent by (2.4). Since the sequence (x_j) has dense span in A it follows that the sum (2.1) is convergent for each $x \in A$. □

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If $p_i \in \mathrm{B}(\mathcal{H})$, $i = 1, \dots, n$, are non-zero orthogonal projections and $\phi \in \mathrm{E}(\mathrm{B}(\mathcal{H}))$ is defined by $\phi(x) = \sum_{i=1}^n p_i x p_i$, the distance of ϕ to the set E_{n-1} of elementary operators of length at most $n-1$ turns out to be (not 1, but) at most $1/n$. (For a proof, let $\psi \in \mathrm{E}(\mathrm{B}(\mathcal{H}))$ be defined by

$$\psi(x) = \sum_{i,j=1}^{n} \left(\delta_{i,j} - \frac{1}{n}\right) p_i x p_j$$

and note that $\phi(x) - \psi(x) = pxp/n$, where $p = \sum_{i=1}^n p_i$, so that $\|\phi - \psi\| = 1/n$. To show that the length of ψ is at most $n-1$, observe that the $n \times n$ matrix $[\delta_{i,j} - 1/n]$ is (a projection) of rank $n-1$; therefore, there exist $\alpha_{i,j}, \beta_{i,j} \in \mathbb{C}$ such that

$$\delta_{i,j} - \frac{1}{n} = \sum_{k=1}^{n-1} \alpha_{k,i} \beta_{k,j} \quad \text{for all } i, j.$$

Now, with $a_k := \sum_{i=1}^n \alpha_{k,i} p_i$ and $b_k := \sum_{i=1}^n \beta_{k,i} p_i$ we have that

$$\psi(x) = \sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \alpha_{k,i} \beta_{k,j} p_i x p_j = \sum_{k=1}^{n-1} a_k x b_k$$

for all $x \in \mathrm{B}(\mathcal{H})$.) However, we shall only need an asymptotic estimate stated in the following lemma.

Lemma 2.3. *For each $m \in \mathbb{N}$ there exists $n(m) \in \mathbb{N}$ such that for every $\theta \in \mathrm{E}(\mathrm{B}(\mathcal{H}))$ of the form*

$$\theta(x) = \sum_{i=1}^{n} e_i x f_i, \quad x \in \mathrm{B}(\mathcal{H}),$$

where $n \geqslant n(m)$ and $e_i, f_i \in \mathrm{B}(\mathcal{H})_+$ are norm-1 elements satisfying $e_i e_j = 0 = f_i f_j$ if $i \neq j$, the distance $d(\theta, E_m)$ of θ to the set E_m of all elementary operators of length at most m is at least $\frac{1}{5}$.

Proof. Denote by $\mathrm{B}(\mathcal{H})^\sharp$ the dual of $\mathrm{B}(\mathcal{H})$ and note that the map

$$\kappa : \mathrm{E}(\mathrm{B}(\mathcal{H})) \to \mathrm{B}(\mathrm{B}(\mathcal{H})^\sharp, \mathrm{B}(\mathcal{H})), \quad \kappa\left(\sum_{i=1}^{n} a_i \otimes b_i\right)(\rho) = \sum_{i=1}^{n} \rho(a_i) b_i (\rho \in \mathrm{B}(\mathcal{H})^\sharp)$$

is contractive, where the elements $\psi = \sum a_i \otimes b_i \in \mathrm{E}(\mathrm{B}(\mathcal{H}))$ have the usual operator norm $\|\psi\| = \sup\{\|\sum a_i x b_i\| : x \in \mathrm{B}(\mathcal{H}), \|x\| \leqslant 1\}$. This follows from

$$\|\kappa(\psi)\| = \sup\left\{\left|\sum \rho(a_i)\omega(b_i)\right| : \omega, \rho \in \mathrm{B}(\mathcal{H})^\sharp,\ \|\omega\| \leqslant 1,\ \|\rho\| \leqslant 1\right\},$$

by noting first that the supremum does not change if we restrict ω and ρ to be of rank 1 (since the unit ball of $\mathrm{B}(\mathcal{H})^\sharp$ is the weak* closure of the convex hull of rank-1 functionals of the form $x \mapsto \langle x\xi, \eta\rangle$, where $\xi, \eta \in \mathcal{H}$ have norm at most 1) and then noting that the supremum is equal to

$$\sup\Big\{\Big\|\sum a_i x b_i\Big\| : x \in \mathrm{B}(\mathcal{H}),\ \|x\| \leqslant 1,\ \operatorname{rank} x \leqslant 1\Big\},$$

and hence dominated by $\|\psi\|$.

Let θ be as in the lemma (but with n arbitrary). Given $\psi \in E_m$ of the form

$$\psi(x) = \sum_{j=1}^{m} a_j x b_j,$$

let U be the closed unit ball of $V := \operatorname{span}\{b_1, \dots, b_m\}$. By the orthogonality of the e_i we may choose ρ_i in the unit ball of $\mathrm{B}(\mathcal{H})^\sharp$ so that $\rho_i(e_j) = \delta_{i,j}$; hence,

$$\varepsilon := \|\theta - \psi\| \geqslant \|\kappa(\theta) - \kappa(\psi)\| \geqslant \Big\|f_i - \sum_{j=1}^{m} \rho_i(a_j) b_j\Big\|.$$

This shows that the distance of f_i to V is at most ε and, since $\|f_i\| = 1$, it follows that $\operatorname{dist}(f_i, U) \leqslant 2\varepsilon$. Thus, we may choose $h_i \in U$ with $\|f_i - h_i\| \leqslant 2\varepsilon$, from which we have (since $\|f_i - f_j\| = 1$ if $i \neq j$)

$$\|h_i - h_j\| \geqslant \|f_i - f_j\| - \|f_i - h_i\| - \|f_j - h_j\| \geqslant 1 - 4\varepsilon.$$

Suppose that $\varepsilon < \frac{1}{5}$, so that $\|h_i - h_j\| > \frac{1}{5}$ for all $i \neq j$. If we equip V with a suitable Euclidean norm $\|\cdot\|_2$ (by proclaiming an Auerbach basis of V to be orthonormal), then $\|\xi\|/\sqrt{m} \leqslant \|\xi\|_2 \leqslant \|\xi\|\sqrt{m}$ for all $\xi \in V$. Thus, $\|h_i - h_j\|_2 > 1/(5\sqrt{m})$ if $i \neq j$, while all the vectors h_i, $i = 1, \dots, n$, are contained in the same at most m-dimensional Euclidean ball of radius $\sqrt{m}$. This is clearly impossible if n is large enough. □

Lemma 2.4. *Suppose that A is separable. If* $\mathrm{ICB}(A) \subseteq \overline{\overline{\mathrm{E}(A)}}$, *then A is subhomogeneous, that is,* $\sup_{[\pi]\in\hat{A}} \dim \pi < \infty$.

Proof. First we will show that all irreducible representations of A must be finite dimensional. Suppose to the contrary that $\pi : A \to \mathrm{B}(\mathcal{H})$ is an infinite-dimensional irreducible representation and consider the map ϕ defined in Lemma 2.2. Clearly, $\phi \in \mathrm{ICB}(A)$. Denote by $\dot{\phi}$ the map on $\dot{A} := A/\ker\pi$ induced by ϕ. From the norm convergent series (2.1) we have that $\dot{\phi}(x) = \sum_{n=1}^{\infty} \pi(e_n) x \pi(f_n)$, $x \in \pi(A)$, and by the same formula $\dot{\phi}$ can be extended uniquely to a weak* continuous (complete) contraction $\bar{\phi}$ on $\mathrm{B}(\mathcal{H})$ (the weak* closure of $\pi(A)$). If $\phi \in \overline{\overline{\mathrm{E}(A)}}$, then

$$\dot{\phi} \in \overline{\overline{\mathrm{E}(\dot{A})}}$$

and (since the norm of any weak* continuous operator on $\pi(A)$ agrees with the norm of its weak* continuous extension to $\overline{\pi(A)}$, a consequence of the Kaplansky density theorem)

$$\bar{\phi} \in \overline{\overline{\mathrm{E}(\mathrm{B}(\mathcal{H}))}}.$$

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Thus, there exists $\psi \in \mathrm{E}(\mathrm{B}(\mathcal{H}))$, say $\psi(x) = \sum_{j=1}^{m} a_j x b_j$, such that

$$\|\bar{\phi} - \psi\| < \tfrac{1}{5}. \tag{2.5}$$

Now, for each $N \in \mathbb{N}$ denote by P_N and Q_N the projections onto $\sum_{n=1}^{N} \pi(e_n)\mathcal{H}$ and $\sum_{n=1}^{N} \pi(f_n)\mathcal{H}$, respectively. From orthogonality of each of the sequences (e_n) and (f_n), the operator $P_N\bar{\phi}Q_N$ has the form $P_N\bar{\phi}Q_N(x) = \sum_{n=1}^{N} \pi(e_n)x\pi(f_n)$. But (2.5) implies that $\|P_N\bar{\phi}Q_N - P_N\psi Q_N\| < \frac{1}{5}$ for all N and, since $P_N\psi Q_N$ is an elementary operator of length at most m, this contradicts Lemma 2.3.

Thus, for each irreducible representation π the C^*-algebra $\pi(A)$ is isomorphic to $\mathrm{M}_r(\mathbb{C})$ for some $r \in \mathbb{N}$, we may identify $\hat{A}$ with $\check{A}$ and each primitive ideal P of A is maximal. A point $P \in \check{A}$ is called *Hausdorff* (or separated) if for each $Q \in \check{A}$, $Q \neq P$, there exist disjoint open neighbourhoods of P and Q in $\check{A}$. (Note that in our situation singletons are automatically closed sets since primitive ideals are maximal.) By [**10**, 3.9.4] the set S of Hausdorff points is dense in $\check{A}$. If S is finite, then $S = \check{A}$, A is finite dimensional and the proof is finished in this case. So we may assume that S is infinite. Since for each $g \in A_+$ the trace function $[\pi] \mapsto \operatorname{tr}\pi(g)$ is lower semicontinuous on $\hat{A}$ [**22**], the same holds for the rank function (for $\operatorname{rank}\pi(g) = \sup_n \operatorname{tr} \sqrt[n]{\pi(g)}$ if $\|g\| \leqslant 1$). Thus, if we assume that $\sup_{[\pi]\in\hat{A}} \dim\pi = \infty$, then there exists a sequence (σ_k) in S with $\dim\sigma_k$ tending to ∞ as $k \to \infty$. Suppose first that there exists a limit point σ of (σ_k) in $\hat{A}$. Since σ_1 is a Hausdorff point, there exist disjoint open neighbourhoods U_1 of σ_1 and V_1 of σ. Put $[\pi_1] = \sigma_1$ and choose any $[\pi_2] \in V_1 \cap (\sigma_k)$ such that $\dim\pi_2 > 2\cdot 3$. Since $[\pi_2]$ is a Hausdorff point, there exist disjoint open neighbourhoods $U_2 \subseteq V_1$ of $[\pi_2]$ and $V_2 \subseteq V_1$ of σ. Continuing in this way, we find a sequence $([\pi_k]) \subseteq \hat{A}$ such that $\dim\pi_k > k(k+1)$, and open neighbourhoods U_k of $[\pi_k]$ and V_k of σ such that $U_k \cap V_k = \emptyset$ and $U_{k+1}, V_{k+1} \subseteq V_k$. In particular, $U_n \cap \bigcup_{k\neq n} U_k = \emptyset$; hence, $[\pi_k] \notin U_n$ if $k \neq n$, which implies that the kernel P_n of π_n is not contained in the closure of the set $\{P_k : k \neq n\}$. If the sequence (σ_k) has no limit points, then we simply let $([\pi_k])$ be a subsequence with $\dim\pi_k > k(k+1)$ and then again $P_n = \ker\pi_n$ is not in the closure of $\{P_k : k \neq n\}$.

Setting $R_n = \bigcap_{k\neq n} P_k$, this means that P_n does not contain R_n; hence, $P_n + R_n = A$ since P_n is maximal. Since $\pi_n(A)$ is of the form $\mathrm{M}_r(\mathbb{C})$ for some $r > n(n+1)$, there exist mutually orthogonal projections $\pi_n(g_{ni})$, $i = 1,\dots,n$, in $\pi_n(A)$ such that $\operatorname{rank}\pi(g_{ni}) > n$ and $\sum_{i=1}^{n} \pi_n(g_{ni}) = 1$. These may be lifted to mutually orthogonal positive contractions g_{ni} in A [**15**, 4.6.20]. Moreover, since $R_{n+} + P_{n+} = A_+$ and $P_n = \ker\pi_n$, we may see that $g_{ni} \in R_n$. Set $\tilde{g}_n = \sum_{i=1}^{n} g_{ni}$ and define recursively $g_1 = \tilde{g}_1$, $g_n = (1 - g_1 - \cdots - g_{n-1})\tilde{g}_n(1 - g_1 - \cdots - g_{n-1})$. Then $\sum_{n=1}^{m} g_n \leqslant 1$ for all m (by an induction, using the fact that $h^2 \leqslant h$ if $0 \leqslant h \leqslant 1$); hence, $\sum_{n=1}^{\infty} g_n \leqslant 1$ (in the von Neumann envelope of A) and $\pi_n(g_n) = 1$ since $\pi_n(\tilde{g}_n) = 1$ and $g_m \in R_m \subseteq P_n = \ker\pi_n$ if $m \neq n$.

Let (x_j) be a sequence with a dense span in A and let $\|x_k\| \leqslant 1$. By Lemma 2.1 there exist positive norm 1 elements $\dot{e}_{ni}$ and $\dot{f}_{ni}$ in $\pi_n(g_{ni}Ag_{ni})$ such that

$$\|\dot{e}_{ni}\pi_n(x_j)\dot{f}_{ni}\| < \frac{1}{n2^n}, \quad i,j = 1,\dots,n. \tag{2.6}$$

Note that $\sum_{i=1}^{n} \dot{e}_{ni} \leqslant 1$ (and similarly for $\dot{f}_{ni}$) by mutual orthogonality of the projections $\pi_n(g_{ni})$ for a fixed n. For each n we may lift $\dot{e}_{n1}$ to a positive element e_{n1} in A such that $e_{n1} \leqslant g_n$ since $\dot{e}_{n1} \leqslant \pi_n(g_n) = 1$ (see [**15**, 4.6.21]). Assuming inductively that for some $i < n$ we already have elements e_{nj}, $j = 1, \dots, i$, in A_+ such that $\pi_n(e_{nj}) = \dot{e}_{nj}$ and $e_{n1} + \dots + e_{ni} \leqslant g_n$, then by [**15**, 4.6.21] we may find $e_{n,i+1}$ in A_+ such that $\pi_n(e_{n,i+1}) = \dot{e}_{n,i+1}$ and $e_{n,i+1} \leqslant g_n - (e_{n1} + \dots + e_{n,i})$ since $\dot{e}_{n,i+1} \leqslant 1 - \dot{e}_{ni} - \dots - \dot{e}_{ni} = \pi_n(g_n - e_{n1} - \dots - e_{ni})$. Thus, we may find $e_{n,i}$ so that $\sum_{i=1}^{n} e_{ni} \leqslant g_n$ and it follows that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} e_{ni} \leqslant 1. \tag{2.7}$$

Similarly, there exist elements $f_{ni} \in A_+$ such that $\pi_n(f_{ni}) = \dot{f}_{ni}$ and

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} f_{ni} \leqslant 1.$$

Given $u \in P_n$ with $0 \leqslant u \leqslant 1$, we may replace the elements e_{ni} ($i = 1, \dots, n$, n fixed) by $|(1-u)e_{ni}|^2$ without violating (2.7) (since $e_{ni}(1-u)^2 e_{ni} \leqslant e_{ni}^2 \leqslant e_{ni}$). Choosing u from an (increasing) approximate unit of P_n, we have from (2.6) that

$$\begin{aligned} \inf_u \|e_{ni}(1-u)^2 e_{ni} x_j f_{ni}\| &\leqslant \lim_u \|(1-u)e_{ni} x_j f_{ni}\| \\ &= \|\dot{e}_{ni} \pi_n(x_j) \dot{f}_{ni}\| \\ &< \frac{1}{n2^n}, \quad i, j = 1, \dots, n. \end{aligned}$$

Thus, we may assume that e_{ni} and f_{ni} have been chosen so that (note that $e^2 \leqslant e$ if $0 \leqslant e \leqslant 1$)

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} e_{ni}^2 \leqslant 1, \qquad \sum_{n=1}^{\infty} \sum_{i=1}^{n} f_{ni}^2 \leqslant 1, \tag{2.8}$$

$$\pi_n(e_{ni})\pi_n(e_{nj}) = 0 = \pi_n(f_{ni})\pi_n(f_{nj}) \quad \text{if } i \neq j, \qquad \|\pi_n(e_{ni})\| = 1 = \|\pi_n(f_{ni})\| \tag{2.9}$$

and

$$\|e_{ni} x_j f_{ni}\| < \frac{1}{n2^n}, \quad i, j = 1, \dots, n. \tag{2.10}$$

By (2.8) we may define a (complete) contraction $\phi : A \to \bar{A}$ by

$$\phi(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} e_{ni} x f_{ni}, \quad x \in A. \tag{2.11}$$

Since the sequence (x_j) has dense span in A, (2.10) implies that the series (2.11) is norm convergent for each $x \in A$ and, consequently, $\phi \in \mathrm{ICB}(A)$.

If $\|\phi - \psi\| < \frac{1}{5}$ for some $\psi \in \mathrm{E}(A)$ of length (say) m, then also

$$\|\phi_n - \psi_n\| < \tfrac{1}{5}, \tag{2.12}$$

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where ϕ_n and ψ_n are the maps on $A_n := \pi_n(A) \cong A/P_n \cong \mathrm{M}_{r(n)}(\mathbb{C})$ induced by ϕ and ψ (respectively). Since $\pi_n(e_{mi}) = 0$ if $m \neq n$ (for $e_{mi} \leqslant g_m$),

$$\phi_n(\dot{x}) = \sum_{i=1}^{n} \dot{e}_{ni}\dot{x}\dot{f}_{ni} \quad \text{for all } \dot{x} \in A/P_n.$$

Since the length of ψ_n is at most m for each n, by Lemma 2.3 the inequality (2.12) cannot hold for all n; hence, $\|\phi - \psi\| \geqslant \frac{1}{5}$ and

$$\phi \notin \overline{\overline{\mathrm{E}(A)}}.$$

□

3. The multiplier algebra of a homogeneous C^*-algebra

Recall that a C^*-algebra A is called n-subhomogeneous ($n \in \mathbb{N}$) if n is the maximal dimension of irreducible representations of A. Then the intersection of the kernels of all irreducible representations of dimension at most $n-1$ is an ideal J of A such that all irreducible representations of J are n dimensional. J is called the *n-homogeneous ideal of A*; it is the largest ideal of A which is n-homogeneous as a C^*-algebra.

For an ideal J in A we shall denote by $J^{\perp}$ the annihilator of J in A. Note that the left annihilator is equal to the right annihilator, that is, $aJ = 0$ if and only if $Ja = 0$, $a \in A$.

Lemma 3.1. *Suppose that A is n-subhomogeneous, J is the n-homogeneous ideal of A, $B = A/J^{\perp}$, K is the n-homogeneous ideal of B and $q : A \to B$ is the quotient map. Then $q(J) = K$ and K is an essential ideal in B.*

Proof. Since $J \cap J^{\perp} = 0$, $q|J$ is injective, so $q(J)$ is isomorphic to J and hence n-homogeneous. Since $q(J)$ is an ideal in B, it follows that $q(J) \subseteq K$. Thus, $J \subseteq q^{-1}(K)$ and then $J + J^{\perp} \subseteq q^{-1}(K)$. If $J + J^{\perp} \neq q^{-1}(K)$, then there exists an irreducible representation π of A such that $\pi(J + J^{\perp}) = 0$ and $\pi(q^{-1}(K)) \neq 0$. Since the set $S := \{[\sigma] \in \hat{A} : \dim \sigma \leqslant n-1\}$ is closed in $\hat{A}$ [**22**, 4.4.10] and J is just the intersection of kernels of representations (the equivalence classes of which are) in S, (the class of) every irreducible representation that annihilates J must be in S. Thus, $[\pi]$ is in S, so $\dim \pi < n$. Furthermore, $\pi(J^{\perp}) = 0$ implies that π descends to an irreducible representation σ of B (so that $\pi = \sigma q$) and $\sigma(K) \neq 0$, since $\pi(q^{-1}(K)) \neq 0$. But $\dim \sigma = \dim \pi < n$, which contradicts the definition of K as the intersection of kernels of all irreducible representations of B of dimension less than n.

The ideal $q(J)$ in $B = A/J^{\perp}$ is essential, since $aJ \subseteq J^{\perp}$, $a \in A$, means that in fact $aJ \subseteq J \cap J^{\perp} = 0$; hence, $a \in J^{\perp}$. □

If Z is the centre of a unital C^*-algebra A (or more generally, a C^*-subalgebra of the centre of the multiplier algebra of a not necessarily unital A such that ZA is dense in A), Δ is the maximal ideal space of Z and for each $t \in \Delta$ we denote by $A(t)$ the quotient algebra $A(t) = A/(At)$, then for every $x \in A$ the function $t \mapsto \|x(t)\|$ is upper semicontinuous (see [**27**, C.10] and [**17**]) on Δ (and vanishes at ∞). If these functions are

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continuous, then the set $E = \{(t, x(t)) : t \in \Delta,\ x \in A\}$ can be equipped with a topology such that E becomes a C^*-bundle with fibres $A(t)$ in the sense of [**27**, Appendix C] or [**12**] and A is (isomorphic to) the C^*-algebra $\Gamma_0(E)$ of all continuous sections of E vanishing at ∞. Since we do not need this topology here, we only recall that a section of E is a map $s : \Delta \to E$ such that $s(t) \in A(t)$ for all $t \in \Delta$.

The following lemma can be deduced as a special case from a more general result in [**3**], but we shall sketch a short direct proof. For a C^*-bundle E let $\Gamma_{\mathrm{b}}(E)$ be the C^*-algebra of all continuous bounded sections of E and let $\Gamma_0(E)$ be the ideal in $\Gamma_{\mathrm{b}}(E)$ consisting of all sections vanishing at ∞.

Lemma 3.2. *If the fibres of a C^*-bundle E over a locally compact space Δ are finite dimensional, then $M := \Gamma_{\mathrm{b}}(E)$ is just the multiplier algebra of $J := \Gamma_0(E)$.*

Proof. For each point $e \in E$ there is a section in J passing through e and it follows that J is an essential ideal in M. It suffices to prove that for each C^*-algebra A, which contains J as an essential ideal, the inclusion $J \to A$ can be extended to a $*$-homomorphism $L : A \to M$. For each $t \in \Delta$ and $a \in A$ define a map $L_{t,a}$ on the fibre E_t of E by

$$L_{t,a}(s(t)) = (as)(t), \quad s \in J.$$

Here we have used the fact that each element of E_t is of the form $s(t)$ for some $s \in J$, but since s is not unique, we need to check that $s(t) = 0$ implies $(as)(t) = 0$. This follows from

$$(as)(t)^*(as)(t) = ((as)^*(as))(t) \leqslant \|a\|^2(s^*s)(t) = \|a\|^2 s(t)^* s(t),$$

which shows also that $\|L_{t,a}\| \leqslant \|a\|$. Clearly, $L_{t,a}$ is linear and, to check that $L_{t,a}$ is a left multiplication by an element of E_t, it suffices to verify that $L_{t,a}$ commutes with all right multiplications $R_{z(t)}$, $z \in J$. For each $s \in J$ we indeed have

$$L_{t,a}(s(t)z(t)) = L_{t,a}((sz)(t)) = (asz)(t) = (as)(t)z(t) = L_{t,a}(s(t))z(t).$$

Thus, the function $L(a)$ which sends $t \in \Delta$ to $L_{t,a}$ is a bounded section of E. To show that it is continuous, choose an approximate unit (e_k) in J and observe that $L(a)$ is the uniform limit on compact subsets of Δ of continuous sections $L(a)e_k = ae_k \in J$. Indeed, for each $t \in \Delta$ and $s \in J$ we have

$$\|(L(a)(t) - (L(a)e_k)(t))s(t)\| = \|(a(1 - e_k)s)(t)\| \xrightarrow{k} 0,$$

which implies, since E_t is finite dimensional (with all elements of the form $s(t)$), that $\|(L(a)(1 - e_k))(t)\| \xrightarrow{k} 0$. To show that the convergence is uniform on compact sets, note that

$$\|(L(a)(1 - e_k))(t)\|^2 = \|(L(a)(1 - e_k)^2 L(a)^*)(t)\| \leqslant \|(L(a)(1 - e_k)L(a)^*)(t)\|$$

and that the net of functions $t \mapsto \|(L(a)(1 - e_k)L(a)^*)(t)\|$ is decreasing (since the approximate unit (e_k) is increasing), so Dini's theorem applies. This shows that $L(a) \in M$ and it can be verified that the map $a \to L(a)$ is a contractive homomorphism from A to M. □

If J is an n-homogeneous C^*-algebra, then J is (isomorphic to) $\Gamma_0(E)$ for some locally trivial C^*-bundle E over $U := \hat{J}$ by [**11**, **26**]. The multiplier algebra $M(J) = \Gamma_{\mathrm{b}}(E)$ is n-subhomogeneous by [**7**, IV.1.4.6], but in general not n-homogeneous as we shall now explain.

If E is of finite type (that is, if U admits a finite covering by open subsets U_i with $E|U_i$ trivial), then E can be extended to a locally trivial C^*-bundle F over the Stone–Čech compactification $\beta(U)$ [**23**, 2.9] and it follows (since such a bundle is a direct summand of a trivial bundle and bounded continuous functions on U have unique continuous extensions to $\beta(U)$) that $M(J) = \Gamma_{\mathrm{b}}(E)$ is isomorphic to the C^*-algebra $\Gamma(F)$ of all continuous sections of F; hence, $M(J)$ is n-homogeneous in this case.

Conversely, if $M := M(J)$ is n-homogeneous, then by [**11**] $M = \Gamma(F)$ for a locally trivial C^*-bundle F over the compact Hausdorff space $\hat{M} \sim \hat{Z}_M$, where Z_M is the centre of M, and (by the Dauns–Hofmann theorem) $\hat{Z}_M$ can be identified with $\beta(\hat{J}) \cong \beta(\hat{Z}_J)$. Since J is an ideal in $M = \Gamma(F)$, it follows that J is of the form $J = \{s \in \Gamma(F) : s|\Lambda = 0\}$ for a closed set $\Lambda \subseteq \beta(\hat{Z}_J)$ and, considering the characters of the centre, Λ must be $\beta(\hat{Z}_J) \setminus \hat{Z}_J$. We conclude that $J = \Gamma_0(F|\hat{Z}_J)$, and the C^*-bundle $F|\hat{Z}_J$ has an extension to a locally trivial C^*-bundle F over a compact space, hence is of finite type by [**23**, 2.9]. Thus, we can state the following remark.

Remark 3.3. The multiplier algebra of an n-homogeneous C^*-algebra J, $n \in \mathbb{N}$, is n-homogeneous if and only if J is of finite type.

We shall need the fact that for a non-unital n-homogeneous C^*-algebra J the n-homogeneous ideal of $M(J)$ is strictly larger than J.

Lemma 3.4. *Let E be a locally trivial C^*-bundle with fibres $\mathrm{M}_n(\mathbb{C})$, $n \in \mathbb{N}$, over a non-compact, locally compact space U, $J := \Gamma_0(E)$, M the multiplier C^*-algebra of J and K the n-homogeneous ideal of M. Regard each point $t \in \beta(U)$ (the Stone–Čech compactification of U) as a maximal ideal of the centre Z_M of M. Then M is the C^*-algebra of continuous sections of a (not necessarily locally trivial) C^*-bundle E_0, with fibres $M(t) := M/(Mt)$, over $\beta(U)$, extending E, such that $F := E_0|\hat{K}$ is locally trivial. Moreover, at least if U is metrizable, $\hat{K}$ properly contains U (that is, K properly contains J).*

Proof. For each $x \in M$ denote by $x(t)$ the coset of x in $M(t)$. The function $\check{x}(t) := \|x(t)\|$ is upper semicontinuous on $\check{Z}_M = \beta(U)$ [**27**, C10]. Moreover, $\check{x}$ must be lower semicontinuous on U as the supremum $\sup\{(xy)\check{} : y \in J,\ \|y\| \leqslant 1\}$ of continuous functions (note that $xy \in J = \Gamma_0(E)$ if $y \in J$). To show that $\check{x}$ is continuous on all $\beta(U)$, we may assume that $x \geqslant 0$ (otherwise just replace x by $|x|$). It suffices now to prove that $\check{x}$ coincides with the unique continuous extension $\tilde{x}$ of the bounded continuous function $\check{x}|U$. In other words, we have to show, for each $t' \in \beta(U) \setminus U$ and each net $(t_\nu) \subseteq U$ converging to t', the equality

$$\check{x}(t') = \lim \check{x}(t_\nu).$$

The inequality $\tilde{x}(t') \leqslant \check{x}(t')$ follows from the continuity of $\tilde{x}$ and the upper semicontinuity of $\check{x}$ since the two functions coincide on the dense set U. Suppose that $\tilde{x}(t') < \check{x}(t')$. Then,

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by continuity of $\tilde{x}$, for a small $\varepsilon > 0$ we have the inequality $\tilde{x}(t) \leqslant \check{x}(t') - \varepsilon$ for all t in an open neighbourhood V of t' in $\beta(U)$. Choose a continuous function $f : [0, \infty) \to [0, 1]$ such that $f([0, \check{x}(t') - \varepsilon]) = 0$ and $f(\check{x}(t')) = 1$. Note that for $t \in U \cap V$ the spectrum of $x(t)$ is contained in $[0, \check{x}(t') - \varepsilon]$; hence, $f(x)(t) = f(x(t)) = 0$ and therefore by continuity $\widetilde{f(x)}(t) = 0$ for all $t \in V$. Furthermore, $(f(x))\check{}(t') = \|f(x)(t')\| = \|f(x(t'))\| = 1$, since $\check{x}(t')$ is in the spectrum of $x(t)$ and $f(\check{x}(t')) = 1$. Thus, replacing x by $f(x)$, we have that $\tilde{x}(t) = 0$ if $t \in V$ and $\check{x}(t') = 1$. Choosing a continuous function χ on $\beta(U)$ with values in $[0, 1]$, supported in V and with $\chi(t') = 1$, and replacing x by χx (where χ is regarded as an element of Z_M by the Dauns–Hofmann theorem), we find an element $x \in M$ such that $\tilde{x}(t) = 0$ for all $t \in \beta(U)$ (hence $x = 0$) and $\check{x}(t') = 1$, which is a contradiction. The continuity of $\check{x}$ proved above means that M is the C^*-algebra of continuous sections of a C^*-bundle E_0 over $\beta(U)$ with fibres $M(t)$ [**27**, Appendix C].

In general the map $\zeta : \hat{M} \to \check{Z}_M = \beta(U)$, $\zeta([\pi]) = \ker(\pi|Z_M)$, is continuous, but since the functions $\check{x}$ $(x \in M)$ are continuous this map is also open [**27**, C.10]. Since J and K $(J \subseteq K)$ are essential ideals in M, one can verify the inclusion of the centres $Z_J \subseteq Z_K \subseteq Z_M$ as ideals in Z_M. Furthermore, $\zeta(\hat{K}) = \check{Z}_K$. (More precisely, denoting for each $[\pi] \in \hat{K}$ by $\tilde{\pi}$ the unique extension of π to the irreducible representation of M, $\tilde{\pi}|Z_K = \pi|Z_K$.) Since K is n-homogeneous, we may identify $\hat{K}$ with $\check{Z}_K$, that is, ζ maps $\hat{K}$ onto $\check{Z}_K \subseteq \beta(U)$ homeomorphically, and we may regard $\hat{K}$ as an open subset in $\beta(U)$. Since K is n-homogeneous, for each $t \in \check{Z}_K$ there is (up to a unitary equivalence) a unique irreducible representation π_t of K such that $\ker(\pi|Z_K) = t \cap Z_K$. Then the extension $\tilde{\pi}_t$ of π_t to M is the unique irreducible representation σ of M with $\ker(\sigma|Z_M) = t$. (Namely, $\ker(\sigma|Z_M) = t$ implies that $\ker(\sigma|Z_K) = t \cap Z_K$; hence, $\sigma|K$ must coincide, up to a unitary equivalence, with π_t, since irreducible representations of a homogeneous C^*-algebra K are determined by their restrictions to the centre. This implies that $\sigma = \tilde{\pi}_t$, since extensions of non-degenerate representations from ideals are unique.) Since each Mt is an intersection of primitive ideals, it follows that Mt must be a primitive ideal in M (for by the above there is only one primitive ideal containing t) and $M/(Mt) \cong \mathrm{M}_n(\mathbb{C})$ for all $t \in \check{Z}_K$. Furthermore, if $t \in \beta(U) \setminus \check{Z}_K$, then Mt must be the intersection of kernels of certain irreducible representations π with $[\pi] \in \hat{M} \setminus \hat{K}$ only. It follows that for a section $x \in M$ we have that $x(t) = 0$ for all $t \in \beta(U) \setminus \check{Z}_K$ if and only if $\pi(x) = 0$ for all $[\pi] \in \hat{M} \setminus \hat{K}$. This means that the ideal $\varGamma_0(E_0|\hat{K})$ in $\varGamma(E_0) = M$ must be K. Since K is n-homogeneous, it follows (using [**11**]) that $F := E_0|\hat{K}$ must be locally trivial. Finally, since K contains J as an ideal, $J = \{s \in K : s|(\hat{K} \setminus U) = 0\} = \varGamma_0(F|U)$ [**12**, II.14.8]; hence $F|U \cong E$.

To show that $\hat{K}$ properly contains U, choose a sequence (t_k) in U with no limit points in U (recall that U is assumed metrizable) and sections $s_{ij} \in M = \varGamma_{\mathrm{b}}(E)$ such that $s_{ij}(t_k)$ $(i, j = 1, \dots, n)$ are the matrix units in the fibres $E_{t_k} \cong \mathrm{M}_n(\mathbb{C})$. For each section $s \in M$ we expand $s(t_k) = \sum_{i,j=1}^{n} \alpha_{ij}(t_k) s_{ij}(t_k)$, $\alpha_{ij}(t_k) \in \mathbb{C}$, extend each (bounded) sequence $(\alpha_{ij}(t_k))_k$ to a continuous function α_{ij} on $\beta(U)$, choose a limit point $t_0 \in \beta(U) \setminus U$ of (t_k) and set

$$\pi_{t_0}(s) := \sum_{i,j=1}^{n} \alpha_{i,j}(t_0) e_{ij} = [\alpha_{ij}(t_0)] \in \mathrm{M}_n(\mathbb{C}),$$

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where e_{ij} are the standard matrix units in $\mathrm{M}_n(\mathbb{C})$. This defines a representation π_{t_0} of M into $\mathrm{M}_n(\mathbb{C})$ ($\pi_{t_0}(s)$ is a kind of a limit point of $(s(t_k))$), which is surjective (hence irreducible) since $\pi_{t_0}(s_{ij}) = e_{ij}$. If $[\pi_{t_0}]$ was not in $\hat{K}$, then $\pi_{t_0}(K) = 0$, which would imply (by the definition of K) that $\ker \pi_{t_0}$ is in the closure of the set of kernels of all irreducible representations of M of dimension less than n. But this is impossible since the set is closed. □

4. A reduction to locally homogeneous C^*-algebras

Lemma 4.1. *If a separable n-subhomogeneous C^*-algebra A is not a direct sum of homogeneous C^*-algebras, then* $\mathrm{ICB}(A) \not\subseteq \overline{\overline{\mathrm{E}(A)}}$.

Since the proof of the lemma occupies the entire section, it will be divided into several steps. Let J be the n-homogeneous ideal of A, let U be the primitive spectrum of the centre Z_J of J, let E be the locally trivial C^*-bundle over U such that $J = \Gamma_0(E)$ and let $M = M(J) = \Gamma_{\mathrm{b}}(E)$ be the multiplier C^*-algebra of J. If J is unital, then A is isomorphic to $J \oplus (A/J)$, where A/J is m-subhomogeneous for some $m < n$, and the proof reduces to a smaller degree of subhomogeneity. So by an induction we may assume that J is not unital; hence, U is not compact. We shall show that in this case $\mathrm{ICB}(A) \not\subseteq \overline{\overline{\mathrm{E}(A)}}$. By Lemma 3.4 the n-homogeneous ideal K of M properly contains J and the corresponding locally trivial C^*-bundle F over the open subset $\hat{K}$ of $\beta(U)$ (so that $K = \Gamma_0(F)$) extends E, while $M = \Gamma(E_0)$ for a (not necessarily locally trivial) C^*-bundle E_0 over $\beta(U)$ extending F. We denote by Z_K and Z_M the centres of K and M, identify $\hat{K}$ and $\hat{J}$ with $\check{Z}_K$ and $\check{Z}_J$, respectively, and regard them as open subsets of $\check{Z}_M = \beta(U)$. Choose $t_0 \in \hat{K} \setminus \hat{J}$ and an open neighbourhood V of t_0 in $\beta(U)$ such that $\bar{V} \subseteq \hat{K}$ and $F|\bar{V}$ is trivial. Using a fixed isomorphism $E_0|\bar{V} = F|\bar{V} \cong \bar{V} \times \mathrm{M}_n(\mathbb{C})$, we shall identify the two bundles over V.

Step 1 (suppose that the ideal J in A is essential). We may thus regard A as a C^*-subalgebra of M. Since all n-dimensional irreducible representations of A are (up to a unitary equivalence) evaluations at points of U, for each $t \in \bar{V} \setminus U$ the evaluation π_t of sections of E_0 at t must be reducible as a representation of A. Let m be the maximal dimension of irreducible subrepresentations of $\pi_t|A$ as t ranges over $\bar{V} \setminus U$ and let $t_1 \in \bar{V} \setminus U$ be a point where this maximum is attained. Then (up to a unitary equivalence) $\pi_{t_1}|A$ has the form

$$\pi_{t_1}(a) = \begin{bmatrix} \sigma_{t_1}^{(k)}(a) & 0 \\ 0 & \rho_{t_1}(a) \end{bmatrix}, \quad a \in A, \tag{4.1}$$

where $\sigma_{t_1} : A \to \mathrm{M}_m(\mathbb{C})$ is an irreducible representation, $k \in \mathbb{N}$ and $\rho_{t_1} : A \to \mathrm{M}_{n-km}$ is a representation disjoint from σ_{t_1}. Denote by e_{ij}, $i, j = 1, \dots, m$, the standard matrix units in $\mathrm{M}_m(\mathbb{C})$. By [**10**, 4.2.5] there exist $a_{ij} \in A$ such that $\pi_{t_1}(a_{ij}) = e_{ij}^{(k)} \oplus 0$ (relative to the decomposition (4.1)). By continuity, if t is close to t_1, $\pi_t(a_{ij})$ will be approximate matrix units in $\mathrm{M}_m(\mathbb{C})$ and well-known arguments (using functional calculus and polar decomposition, as in [**16**, § 12.1]) show that there exist $b_{ij} \in A$ such that $\pi_t(b_{ij})$, $i, j = 1, \dots, m$, are $m \times m$ matrix units in $\mathrm{M}_n(\mathbb{C})$; in other words, $\pi_t(A)$ contains a copy of

$\mathrm{M}_m(\mathbb{C})$ for all t in a neighbourhood $W \subseteq \bar{W} \subseteq V$ of t_1. It follows now by maximality of m that (up to a conjugation with a unitary $u \in C(W, \mathrm{M}_n(\mathbb{C}))$) $\pi_t|A$ has the form

$$a(t) := \pi_t(a) = \begin{bmatrix} \sigma_t(a) & 0 \\ 0 & \theta_t(a) \end{bmatrix}, \quad a \in A,\ t \in W \setminus U, \tag{4.2}$$

where $\sigma_t : A \to \mathrm{M}_m(\mathbb{C})$ is an irreducible representation and $\theta : A \to \mathrm{M}_{n-m}(\mathbb{C})$ is a (possibly degenerate) representation.

Choose a continuous function χ on $\beta(U) \setminus U$, supported in $W \setminus U$, with values in $[0, 1]$ and $\chi(t_1) = 1$. Let $v \in \mathrm{M}_{m,n-m}(\mathbb{C})$ be any matrix with $\|v\| = 1$. Since

$$M = \Gamma(E_0) \quad \text{and} \quad J = \Gamma_0(E) = \{s \in \Gamma(E_0) : s|(\beta(U) \setminus U) = 0\},$$

$M/J = \Gamma(E_0|(\beta(U) \setminus U))$ (using the Tietze extension theorem for sections of Banach bundles [**12**, II.14.8]). Define a section $s \in M/J$ on $\beta(U) \setminus U$ by

$$s(t) = \begin{cases} \begin{bmatrix} 0 & \chi(t)v \\ 0 & 0 \end{bmatrix} & \text{if } t \in W \setminus U, \\ 0 & \text{if } t \in (\beta(U) \setminus U) \setminus W \end{cases} \tag{4.3}$$

and let $b \in M$ be any lift of s (that is, a continuous extension of s to a section of E_0). Finally, let $\phi : A \to M$ be the two-sided multiplication $x \mapsto bxb$.

Step 2 (proof that $\phi(A) \subseteq A$ and that ϕ preserves ideals). Given $a \in A$, the value $\phi(a)(t)$ of $\phi(a) \in M$ at each $t \in \beta(U) \setminus U$ is 0. Indeed, $b(t) = s(t) = 0$ if $t \in (\beta(U) \setminus U) \setminus W$, while for $t \in W \setminus U$ we have that $\phi(a)(t) = b(t)a(t)b(t) = s(t)\pi_t(a)s(t) = 0$, as can be verified by performing the matrix multiplication with $\pi_t(a)$ and $s(t)$ of the form (4.2) and (4.3). This implies that $\phi(a) \in J$; in particular, ϕ maps A into A. To show that ϕ preserves all ideals in $\mathcal{I}_{\mathrm{c}}(A)$, let (e_k) be an approximate unit in the n-homogeneous ideal J. Note that (since $\phi(a) \in J$)

$$\phi(a) = \lim e_k \phi(a) e_k = \lim (e_k b) a (b e_k),$$

where the two-sided multiplications $a \mapsto (e_k b)a(be_k)$ preserve the ideals, since $e_k b$ and be_k are in $J \subseteq A$. Thus, ϕ is a pointwise limit of maps preserving ideals, so ϕ must preserve (closed) ideals.

Step 3 (proof that $\phi \notin \overline{\overline{\mathrm{E}(A)}}$). First we shall 'localize' the proof to W (to work with matrix-valued functions instead of bundles), then we shall show by an explicit computation that $\phi \notin \overline{\overline{\mathrm{E}(A)}}$.

Let $J_W = \{a \in M : a(t) = 0\ \forall t \in \bar{W}\}$ and let ϕ_W be the map on $A_W := A/(J_W \cap A)$ induced by ϕ. Note that A_W is (naturally isomorphic to) a C^*-subalgebra of $M/J_W = \Gamma(E_0|\bar{W}) = \Gamma_0(F|\bar{W}) = C(\bar{W}, \mathrm{M}_n(\mathbb{C}))$, and ϕ_W is just the two-sided multiplication

$$\phi_W(x) = dxd, \quad x \in A_W \subseteq C(\bar{W}, \mathrm{M}_n(\mathbb{C})),$$

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where d is the coset of b in M/J_W. As an element of $C(\bar{W}, \mathrm{M}_n(\mathbb{C}))$, decomposing $\mathrm{M}_n(\mathbb{C})$ into blocks according to (4.2), d can be represented by a block matrix of continuous functions

$$d = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix},$$

where (by the definitions of b and s) $d_{11}(t_1) = 0$, $d_{21}(t_1) = 0$, $d_{22}(t_1) = 0$ and $d_{12}(t_1) = v$. It follows now from $\phi_W(x) = dxd$ that

$$\phi_W(x)(t_1) = \begin{bmatrix} 0 & vx_{21}(t_1)v \\ 0 & 0 \end{bmatrix} \quad \text{for all } x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \text{ in } A_W \subseteq C(\bar{W}, \mathrm{M}_n(\mathbb{C})).$$

Given $\varepsilon > 0$, by continuity of functions d_{ij} (that is, since $\|d(t) - d(t_1)\|$ is small if $t \in W$ is close to t_1) there exists a neighbourhood $W_1 \subseteq W$ of t_1 such that we have, uniformly for all $x = [x_{ij}] \in A_W$ with $\|x\| \leqslant 1$, the estimate

$$\left\| \phi_W(x)(t) - \begin{bmatrix} 0 & vx_{21}(t)v \\ 0 & 0 \end{bmatrix} \right\| < \varepsilon \quad \text{for all } t \in W_1. \tag{4.4}$$

The evaluation π_{t_1} maps A into block-diagonal matrices according to (4.2), but we shall need the same for $M(A)$ (π_{t_1} can be degenerate). Since J is essential in A, and hence also in $M(A)$, we have that $M(A) \subseteq M(J) = M$; hence, each $f \in M(A)$ can be represented over W by a 2×2 block matrix $f|W = [f_{ij}]$ in accordance with the decomposition (4.2). Let $p \in \mathrm{M}_n(\mathbb{C})$ be the projection onto $[\sigma_{t_1}^{(k)}(A)\mathbb{C}^n]$ (where σ_{t_1} is as in (4.1)). Then $p \in \pi_{t_1}(A)$ since ρ_{t_1} and $\sigma_{t_1}^{(k)}$ are disjoint. With respect to the decomposition (4.2), p has the form $p = 1 \oplus q$, where 1 is the $m \times m$ identity matrix and q is a projection. Since $f(t_1)p \in \pi_{t_1}(A)$ and $pf(t_1) \in \pi_{t_1}(A)$ and matrices in $\pi_{t_1}(A)$ are block diagonal, a matrix multiplication shows that $f_{21}(t_1) = 0$ and $f_{12}(t_1) = 0$. Thus, $\pi_{t_1}(M(A))$ consists of block-diagonal matrices only.

Suppose that there exists $\psi \in \mathrm{E}(A)$ with $\|\psi - \phi\| < \varepsilon$; hence,

$$\|\psi_W - \phi_W\| < \varepsilon, \tag{4.5}$$

where ψ_W is the map induced on A_W by ψ. Then ψ is of the form

$$\psi(x) = \sum_{k=1}^{\ell} a^k x b^k, \quad x \in A,$$

where $a^k, b^k \in M(A) \subseteq M$. By the previous paragraph, $a^k(t_1) = a_{11}^k(t_1) \oplus a_{22}^k(t_1)$ and $b^k(t_1) = b_{11}^k(t_1) \oplus b_{22}^k(t_1)$ are block diagonal. Now, for matrices of the form

$$x = \begin{bmatrix} 0 & 0 \\ x_{21} & 0 \end{bmatrix} \tag{4.6}$$

we have that $\sum_{k=1}^{\ell} a^k(t_1)xb^k(t_1)$ is of the form

$$\begin{bmatrix} 0 & 0 \\ \displaystyle\sum_{k=1}^{\ell} a_{22}^k(t_1)x_{21}b_{11}^k(t_1) & 0 \end{bmatrix};$$

hence, by continuity of the coefficients a^k and b^k (on W) there exists a neighbourhood $W_2 \subseteq W$ of t_1 such that

$$\left\| \psi_W(x)(t) - \begin{bmatrix} 0 & 0 \\ \displaystyle\sum_{k=1}^{\ell} a_{22}^k(t)x_{21}(t)b_{11}^k(t) & 0 \end{bmatrix} \right\| < \varepsilon \quad \text{for all } t \in W_2 \tag{4.7}$$

uniformly for all $x \in A_W$ of the form (4.6) with $\|x\| \leqslant 1$. From (4.4), (4.5) and (4.7) we conclude that

$$\left\| \begin{bmatrix} 0 & vx_{21}(t)v \\ -\displaystyle\sum_{k=1}^{\ell} a_{22}^k(t)x_{21}(t)b_{11}^k(t) & 0 \end{bmatrix} \right\| < 3\varepsilon \tag{4.8}$$

for all $t \in W_1 \cap W_2$ and $x \in A_W$ of the form (4.6) with $\|x\| \leqslant 1$. But, for each $t \in W_1 \cap W_2 \cap U$, we have that $A_W(t) = \pi_t(A) = \mathrm{M}_n(\mathbb{C})$ (since we already have $J(t) = \mathrm{M}_n(\mathbb{C})$); hence, we may choose $x \in A_W$ of the form (4.6) so that $\|x_{21}(t)\| = 1$ and $\|vx_{21}(t)v\| = 1$ (for a fixed t), which contradicts (4.8) if $\varepsilon < \frac{1}{3}$. Thus,

$$\phi \notin \overline{\overline{\mathrm{E}(A)}}.$$

This proves the lemma in the case where J is essential in A.

Step 4 (a reduction to the case when J is essential). Let $B = A/J^{\perp}$, let $q : A \to B$ be the quotient map and let $K = q(J)$. By Lemma 3.1 K is the n-homogeneous ideal of B and is an essential ideal in B. By what we have proved above, there exists $b \in M(K)$ such that the two-sided multiplication $\phi(x) = bxb$ maps B into K and $\phi \in \mathrm{ICB}(B) \setminus \overline{\overline{\mathrm{E}(B)}}$. Define $\phi_0 : A \to A$ as the composition

$$\phi_0 = (q|J)^{-1}\phi q.$$

Then $\phi_0(A) \subseteq J$. To show that ϕ_0 preserves ideals of A, let (e_k) be an approximate unit in J and choose $a \in M(J)$ so that $\tilde{q}(a) = b$, where $\tilde{q}$ is the extension to $M(J) \to M(K)$ of the isomorphism $q|J : J \to K$. Then, for $x \in A$,

$$\begin{aligned} \phi_0(x) &= \lim e_k\phi_0(x)e_k \\ &= \lim e_k(q|J)^{-1}(bq(x)b)e_k \\ &= \lim (q|J)^{-1}(q(e_k)bq(x)bq(e_k)) \\ &= \lim (q|J)^{-1}q(e_kaxae_k) \\ &= \lim (e_ka)x(ae_k); \end{aligned}$$

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hence, $\phi_0(x)$ is in (the closed two-sided) ideal generated by x since $e_k a \in J \subseteq A$. To show that

$$\phi_0 \notin \overline{\overline{\mathrm{E}(A)}},$$

assume the contrary: for each $\varepsilon > 0$ there exists $\psi \in \mathrm{E}(A)$ with $\|\phi_0 - \psi\| \leqslant \varepsilon$. Denote by $\dot{\psi}$ the elementary operator on B induced by ψ, so that $\dot{\psi} q = q\psi$. Then for each x in the unit ball of A we have that $\|\phi_0(x) - \psi(x)\| \leqslant \varepsilon$, which implies that $\|\phi q(x) - \dot{\psi} q(x)\| = \|q(\phi_0(x) - \psi(x))\| \leqslant \varepsilon$. Since q maps the closed unit ball of A onto that of B, it follows that $\|\phi - \dot{\psi}\| \leqslant \varepsilon$. But this would imply that $\phi \in \overline{\overline{\mathrm{E}(B)}}$: a contradiction. This completes the proof of Lemma 4.1.

Finally, Theorem 1.1 follows from Lemmas 2.4 and 4.1 from the part already proved in § 1.

The author does not know if Theorem 1.1 also holds for non-separable C^*-algebras.

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