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Surfaces with $p_g = q = 2$, $K^2 = 6$, and Albanese Map of Degree 2

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Abstract. We classify minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 6$ whose Albanese map is a generically finite double cover. We show that the corresponding moduli space is the disjoint union of three generically smooth irreducible components $\mathcal{M}_{Ia}$, $\mathcal{M}_{Ib}$, $\mathcal{M}_{II}$ of dimension 4, 4, 3, respectively.

Introduction

Minimal surfaces S of general type with $p_g = q = 2$ fall into two classes according to the behavior of their Albanese map $\alpha\colon S \to A$. Indeed, since $q = 2$, either $\alpha(S) = C$, where C is a smooth curve of genus 2, or α is surjective and S is of *Albanese general type*.

The surfaces that belong to the former case satisfy $K_S^2 = 8$ and are now completely classified; see [Z03, Pe11]. Those belonging to the latter case present a much richer and subtler geometry, and their full description is still missing; we refer the reader to the introduction of [PP10] and the references given there for a recent account on this topic.

So far, the only known example of a surface of general type with $p_g = q = 2$ and $K_S^2 = 6$ was the one given in [Pe11]; in that case, the Albanese map is a generically finite quadruple cover of an abelian surface with a polarization of type $(1, 3)$.

As the title suggests, in this paper we investigate surfaces with the above invariants and whose Albanese map is a generically finite *double* cover. The results that we obtain are quite satisfactory, indeed we are not only able to show the existence of such new surfaces, but we also provide their complete classification, together with a detailed description of their moduli space.

Before stating our results, let us introduce some notation. Let $(A, \mathcal{L})$ be a $(1, 2)$-polarized abelian surface and let us denote by $\phi_2\colon A[2] \to \widehat{A}[2]$ the restriction of the canonical homomorphism $\phi_{\mathcal{L}}\colon A \to \widehat{A}$ to the subgroup of 2-division points. Then $\operatorname{im}\phi_2$ consists of four line bundles $\{\mathcal{O}_A, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$. Let us denote by $\operatorname{im}\phi_2^{\times}$ the set $\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$.

Our first result is the following theorem.

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Theorem A (see Theorem 2.6) *Given an abelian surface A with a symmetric polarization $\mathcal{L}$ of type $(1,2)$, not of product type, for any $\mathcal{Q} \in \operatorname{im}\phi_2$ there exists a curve $D \in |\mathcal{L}^2 \otimes \mathcal{Q}|$ whose unique non-negligible singularity is an ordinary quadruple point at the origin $o \in A$. Let $\mathcal{Q}^{1/2}$ be a square root of $\mathcal{Q}$, and if $\mathcal{Q} = \mathcal{O}_A$, assume moreover that $\mathcal{Q}^{1/2} \neq \mathcal{O}_A$. Then the minimal desingularization S of the double cover of A branched over D and defined by $\mathcal{L} \otimes \mathcal{Q}^{1/2}$ is a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$, and Albanese map of degree 2.*

Conversely, every minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2 can be constructed in this way.

When $\mathcal{Q} = \mathcal{Q}^{1/2} = \mathcal{O}_A$ we obtain instead a minimal surface with $p_g = q = 3$; see Proposition 2.4 and Remark 2.5.

We use the following terminology:

- if $\mathcal{Q} = \mathcal{O}_A$, we say that S is a *surface of type I*. Furthermore, if $\mathcal{Q}^{1/2} \notin \operatorname{im}\phi_2^{\times}$, we say that S is of type *Ia*, whereas if $\mathcal{Q}^{1/2} \in \operatorname{im}\phi_2^{\times}$ we say that S is of type *Ib*;
- if $\mathcal{Q} \in \operatorname{im}\phi_2^{\times}$, we say that S is a *surface of type II*.

Since $q = 2$, the results in [Ca91] imply that the degree of the Albanese map is a topological invariant; see Proposition 3.1. Therefore we may consider the moduli space $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$, $K_S^2 = 6$, and Albanese map of degree 2. Let $\mathcal{M}_{Ia}, \mathcal{M}_{Ib}, \mathcal{M}_{II}$ be the algebraic subsets whose points parameterize isomorphism classes of surfaces of type *Ia*, *Ib*, *II*, respectively. Therefore $\mathcal{M}$ can be written as the disjoint union

$$\mathcal{M} = \mathcal{M}_{Ia} \sqcup \mathcal{M}_{Ib} \sqcup \mathcal{M}_{II}.$$

Our second result is the following theorem.

Theorem B (see Theorem 3.7) *The following hold:*

(i) *$\mathcal{M}_{Ia}, \mathcal{M}_{Ib}, \mathcal{M}_{II}$ are the connected components of $\mathcal{M}$;*

(ii) *these are also* irreducible *components of the moduli space of minimal surfaces of general type;*

(iii) *$\mathcal{M}_{Ia}, \mathcal{M}_{Ib}, \mathcal{M}_{II}$ are generically smooth, of dimension $4, 4, 3$, respectively;*

(iv) *the general surface in $\mathcal{M}_{Ia}$ and $\mathcal{M}_{Ib}$ has ample canonical class; all surfaces in $\mathcal{M}_{II}$ have ample canonical class.*

This work is organized as follows.

In Section 1 we fix notation and terminology, and we prove some technical results on abelian surfaces with $(1,2)$-polarization that are needed in the sequel of the paper.

In the Section 2 we give the proof of Theorem A. Moreover, we provide a description of the canonical system $|K_S|$ in each of the three cases *Ia*, *Ib*, *II*. It turns out that if S is either of type *Ia* or of type *II*, then the general curve in $|K_S|$ is irreducible, whereas if S is of type *Ib* then $|K_S| = Z + |\Phi|$, where $|\Phi|$ is a base-point free pencil of curves of genus 3.

Finally, Section 3 is devoted to the proof of Theorem B. Such a proof involves the calculation of the monodromy action of the paramodular group G_Δ on the set $\widehat{A}[2]$, with $\Delta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. This is probably well known to the experts, but, at least to our knowledge, it is nowhere explicitly written, so we dedicated an appendix to it.

Notation and Conventions

We work over the field $\mathbb{C}$ of complex numbers.

If A is an abelian variety and $\widehat{A} := \mathrm{Pic}^0(A)$ is its dual, we denote by o and $\widehat{o}$ the zero point of A and $\widehat{A}$, respectively. Moreover, $A[2]$ and $\widehat{A}[2]$ stand for the subgroups of 2-division points.

If $\mathcal{L}$ is a line bundle on A, we denote by $\phi_{\mathcal{L}}$ the morphism $\phi_{\mathcal{L}} \colon A \to \widehat{A}$ given by $x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. If $c_1(\mathcal{L})$ is non-degenerate, then $\phi_{\mathcal{L}}$ is an isogeny, and we denote by $K(\mathcal{L})$ its kernel.

A coherent sheaf $\mathcal{F}$ on A is called a *IT-sheaf of index i* if

$$H^j(A, \mathcal{F} \otimes \mathcal{Q}) = 0 \quad \text{for all } \mathcal{Q} \in \mathrm{Pic}^0(A) \quad \text{and } j \neq i.$$

If $\mathcal{F}$ is an IT-sheaf of index i and $\mathcal{P}$ it the normalized Poincaré bundle on $A \times \widehat{A}$, the coherent sheaf

$$\widehat{\mathcal{F}} := R^i \pi_{\widehat{A}*}(\mathcal{P} \otimes \pi_A^* \mathcal{F})$$

is a vector bundle of rank $h^i(A, \mathcal{F})$, called the *Fourier–Mukai transform* of $\mathcal{F}$.

By *surface* we mean a projective, non-singular surface S, and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler–Poincaré characteristic*. If $q(S) > 0$, we denote by $\alpha \colon S \to \mathrm{Alb}(S)$ the Albanese map of S.

Throughout the paper, we denote Cartier divisors on a variety by capital letters and the corresponding line bundles by italic letters, so we write for instance $\mathcal{L} = \mathcal{O}_S(L)$.

If $|L|$ is any complete linear system of curves on a surface, its base locus is denoted by $\mathrm{Bs}\,|L|$.

If X is any scheme, by a *first-order deformation* of X we mean a deformation over $\mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$.

In Section 1 we use the following special case of Eagon–Northcott complex. Let us consider a short exact sequence of sheaves on S of the form

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \otimes \mathcal{I}_p \longrightarrow 0,$$

where $\mathcal{L}, \mathcal{M}$ are line bundles, $\mathcal{F}$ is a rank 2 vector bundle, and p is a point. Then the symmetric powers $S^2\mathcal{F}$ and $S^3\mathcal{F}$ fit into short exact sequences

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{L} \longrightarrow S^2\mathcal{F} \longrightarrow \mathcal{M}^2 \otimes \mathcal{I}_p^2 \longrightarrow 0,$$
$$0 \longrightarrow S^2\mathcal{F} \otimes \mathcal{L} \longrightarrow S^3\mathcal{F} \longrightarrow \mathcal{M}^3 \otimes \mathcal{I}_p^3 \longrightarrow 0.$$

1 Abelian Surfaces with $(1,2)$-polarization

In this section we prove some technical facts about abelian surfaces with polarization of type $(1,2)$ that are needed in the sequel of the paper. The crucial results are Proposition 1.4, Proposition 1.5, Corollary 1.7, and Corollary 1.12. For the statements whose proof is omitted we refer the reader to [Ba87, HvM89, BPS09, PP10] and [BL04, Chapter 10].

Let A be an abelian surface and L an ample divisor on A with $L^2 = 4$. Then L defines a positive definite line bundle $\mathcal{L} := \mathcal{O}_A(L)$ on A, whose first Chern class is a polarization of type $(1, 2)$. By abuse of notation we consider the line bundle $\mathcal{L}$ itself as a polarization. Moreover, we have $h^0(A, \mathcal{L}) = 2$, so the linear system $|L|$ is a pencil.

Proposition 1.1 ([Ba87, p. 46]) *Let $(A, \mathcal{L})$ be a $(1, 2)$-polarized abelian surface and let $G \in |L|$. Then we have one of the following cases:*

(a) *G is a smooth, connected curve of genus* 3;

(b) *G is an irreducible curve of geometric genus* 2*, with an ordinary double point*;

(c) *$G = E + F$, where E and F are elliptic curves and $EF = 2$*;

(d) *$G = E+F_1+F_2$, with E, F_1, F_2 elliptic curves such that $EF_1 = 1$, $EF_2 = 1$, $F_1F_2 = 0$.*

Moreover, in case (c) the surface A admits an isogeny onto a product of two elliptic curves, and the polarization of A is the pull-back of the principal product polarization, whereas in case (d) the surface A itself is a product $E \times F$ and $\mathcal{L} = \mathcal{O}_A(E + 2F)$.

Let us denote by Δ the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, and by $\mathcal{A}_\Delta$ the moduli space of $(1, 2)$-polarized abelian surfaces; then there exists a Zariski dense set $\mathcal{U} \subset \mathcal{A}_\Delta$ such that, given any $(A, \mathcal{L}) \in \mathcal{U}$, all divisors in $|L|$ are irreducible, *i.e.,* of type (a) or (b); see [BPS09, Section 3].

Definition 1.2 If $(A, \mathcal{L}) \in \mathcal{U}$, we say that $\mathcal{L}$ is a *general* $(1, 2)$-polarization. If $|L|$ contains some divisor of type (c), we say that $\mathcal{L}$ is a *special* $(1, 2)$-polarization. Finally, if the divisors in $|L|$ are of type (d), we say that $\mathcal{L}$ is a *product* $(1, 2)$-polarization.

In the rest of this section we assume that $\mathcal{L}$ is not a product polarization. Then $|L|$ has four distinct base points $\{e_0, e_1, e_2, e_3\}$, which form an orbit for the action of $K(\mathcal{L}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ on A. Moreover all curves in $|L|$ are smooth at each of these base points; see [Ba87, Section 1].

Let us denote by $(-1)_A$ the involution $x \to -x$ on A. Then we say that a divisor C on A is *symmetric* if $(-1)_A^*C = C$. Analogously, we say that a vector bundle $\mathcal{F}$ on A is symmetric if $(-1)_A^*\mathcal{F} = \mathcal{F}$.

Since $\mathcal{L}$ is ample, [Ba87] implies that, up to translations, we may suppose that $\mathcal{L}$ is symmetric and that the base locus of $|L|$ coincides with $K(\mathcal{L})$. Moreover:

- for all sections $s \in H^0(A, \mathcal{L})$ we have $(-1)_A^*s = s$ (in particular, all divisors in $|L|$ are symmetric);
- we may assume that $e_0 = o$ and that e_1, e_2, e_3 are 2-division points, satisfying $e_1 + e_2 = e_3$.

There exist exactly three 2-torsion line bundles $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ on A, with $\mathcal{Q}_1 \otimes \mathcal{Q}_2 = \mathcal{Q}_3$, such that the linear system $|L + Q_i|$ contains an irreducible curve that is singular at o. More precisely, one shows that $h^0(A, \mathcal{L} \otimes \mathcal{Q}_i \otimes \mathcal{I}_o^2) = 1$ and that the unique curve $N_i \in |L + Q_i|$ that is singular at o actually has an ordinary double point there.

Denoting by $\phi_2 \colon A[2] \to \widehat{A}[2]$ the homomorphism induced by $\phi_{\mathcal{L}} \colon A \to \widehat{A}$ on the subgroups of 2-division points, both $\ker \phi_2$ and $\operatorname{im} \phi_2$ are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Indeed, we have

$$\ker \phi_2 = K(\mathcal{L}) \quad \text{and} \quad \operatorname{im} \phi_2 = \{\mathcal{O}_A, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}.$$

Let $\operatorname{im}\phi_2^\times$ be the set $\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$.

Proposition 1.3 *Let $\mathcal{Q}, \mathcal{Q}' \in \widehat{A}$ and $p \in \operatorname{Bs}|L+Q|$. Then*

$$h^0(A, \mathcal{L} \otimes \mathcal{Q}' \otimes \mathcal{I}_p^2) = \begin{cases} 0 & \textit{if } \mathcal{Q}' \otimes \mathcal{Q}^{-1} \notin \operatorname{im}\phi_2^\times, \\ 1 & \textit{if } \mathcal{Q}' \otimes \mathcal{Q}^{-1} \in \operatorname{im}\phi_2^\times. \end{cases}$$

Proof Since $p \in \operatorname{Bs}|L+Q|$, translating by p we see that $h^0(A, \mathcal{L} \otimes \mathcal{Q}' \otimes \mathcal{I}_p^2) \neq 0$ if and only if $h^0(A, \mathcal{L} \otimes \mathcal{Q}' \otimes \mathcal{Q}^{-1} \otimes \mathcal{I}_o^2) \neq 0$. Now the claim follows, because this holds precisely when $\mathcal{Q}' \otimes \mathcal{Q}^{-1} \in \operatorname{im}\phi_2^\times$. ∎

For any $\mathcal{Q} \in \widehat{A}$, let us consider the linear system $|\mathcal{L}^2 \otimes \mathcal{Q} \otimes \mathcal{I}_o^4| := \mathbb{P}H^0(A, \mathcal{L}^2 \otimes \mathcal{Q} \otimes \mathcal{I}_o^4)$ consisting of the curves in $|2L+Q|$ having a point of multiplicity at least 4 at o. We first analyze the case $\mathcal{Q} = \mathcal{O}_A$.

Proposition 1.4 *We have $h^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4) = 2$; that is, the linear system $|\mathcal{L}^2 \otimes \mathcal{I}_o^4| \subset |2L|$ is a pencil. Moreover, if $C \in |\mathcal{L}^2 \otimes \mathcal{I}_o^4|$, then we have one of the following cases:*

(a) *C is an irreducible curve of geometric genus* 3, *with an ordinary quadruple point* (*this corresponds to the general case*);

(b) *C is an irreducible curve of geometric genus* 2, *with an ordinary quadruple point and an ordinary double point*;

(c) *$C = 2C'$, where C' is an irreducible curve of geometric genus* 2 *with an ordinary double point*;

(d) *$\mathcal{L}$ is a special* $(1,2)$*-polarization and $C = 2C'$, where C' is the union of two elliptic curves intersecting transversally in two points.*

Proof Since the three curves $2N_i$ belong to $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ and each N_i is irreducible, by Bertini's theorem it follows that the general element $C \in |\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ is irreducible and smooth outside o. On the other hand we have $(2L)^2 = 16$, so C has an *ordinary* quadruple point at o. Blowing up this point, the strict transform of C has self-intersection 0, so $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ is a pencil.

Assume first that $\mathcal{L}$ is a general polarization. We have shown that the general curve in $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ belongs to case (a). In order to complete the proof, observe that $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ contains the following distinguished elements:

- three reduced, irreducible curves B_1, B_2, B_3 such that B_i has an ordinary quadruple point at o, an ordinary double point at e_i and no other singularities (see [BL04, Corollary 4.7.6]). These curves are as in case (b);
- three non-reduced elements, namely $2N_1$, $2N_2$, $2N_3$. These curves are as in case (c).

Moreover, all other elements of $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ are smooth outside o; this can be seen by blowing-up o and applying the Zeuthen–Segre formula to the fibration induced by the strict transform of the pencil, see [PP10, Section 1.2].

Finally, assume that $\mathcal{L}$ is a special polarization. Then there is just one more possibility, namely that $C = 2C'$, where C' is the translate of a reducible curve $E + F \in |L|$ by a suitable 2-division point. This yields case (d). ∎

Let us consider now the case where $\mathcal{Q}$ is non-trivial. In the sequel, $\{i, j, k\}$ always denotes a permutation of $\{1, 2, 3\}$.

Proposition 1.5 *Let $\mathcal{Q} \in \widehat{A}$ be non-trivial. Then $|\mathcal{L}^2 \otimes \mathcal{Q} \otimes \mathcal{I}_o^4|$ is empty, unless $\mathcal{Q} \in \operatorname{im} \phi_2^{\times}$. More precisely, for all $i \in \{1, 2, 3\}$ we have $h^0(A, \mathcal{L}^2 \otimes \mathcal{Q}_i \otimes \mathcal{I}_o^4) = 1$, so that $|\mathcal{L}^2 \otimes \mathcal{Q}_i \otimes \mathcal{I}_o^4|$ consists of a unique element, namely the curve $N_j + N_k$.*

Proof Assume that there exists an effective curve $C \in |\mathcal{L}^2 \otimes \mathcal{Q} \otimes \mathcal{I}_o^4|$. Blowing up the point o, the strict transform $\widetilde{C}$ of C is numerically equivalent to the strict transform of a general element of the pencil $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$. Since $\mathcal{Q}$ is non-trivial, by the description of the non-reduced elements of $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ given in Proposition 1.4 we must have $\widetilde{C} = \widetilde{N}_j + \widetilde{N}_k$, so $\mathcal{Q} = \mathcal{Q}_i$. ∎

Summing up, Propositions 1.4 and 1.5 imply the following corollary.

Corollary 1.6 *Let $\mathcal{Q} \in \widehat{A}$. Then $|\mathcal{L}^2 \otimes \mathcal{Q} \otimes \mathcal{I}_o^4|$ is empty, unless $\mathcal{Q} \in \operatorname{im} \phi_2$. In this case*

$$\dim |\mathcal{L}^2 \otimes \mathcal{Q} \otimes \mathcal{I}_o^4| = \begin{cases} 1 & \text{if } \mathcal{Q} = \mathcal{O}_A, \\ 0 & \text{if } \mathcal{Q} \in \operatorname{im} \phi_2^{\times}. \end{cases}$$

Corollary 1.7 *Let $(A, \mathcal{L})$ be a $(1, 2)$-polarized abelian surface, and let C be a* reduced *divisor numerically equivalent to $2L$ that has a quadruple point at $p \in A$. Then C belongs to one of the following cases, all of which occur:*

(i) *C is irreducible, with an ordinary quadruple point at p and no other singularities*;

(ii) *C is irreducible, with an ordinary quadruple point at p, an ordinary double point and no other singularities*;

(iii) *$C = C_1 + C_2$, where C_i is irreducible and numerically equivalent to L, with an ordinary double point at p and no other singularities. Since $C_1 C_2 = 4$, the singularity of C at p is again an ordinary quadruple point.*

Proposition 1.8 *There exists a rank 2 indecomposable vector bundle $\mathcal{F}$ on A, such that*

$$h^0(A, \mathcal{F}) = 1, \quad h^1(A, \mathcal{F}) = 0, \quad h^2(A, \mathcal{F}) = 0, \qquad (1.1)$$
$$c_1(\mathcal{F}) = \mathcal{L}, \quad c_2(\mathcal{F}) = 1.$$

Moreover, $\mathcal{F}$ is symmetric, and it is isomorphic to the unique locally free extension of the form

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \otimes \mathcal{I}_o \longrightarrow 0. \qquad (1.2)$$

Proof Let $\mathcal{L}^*$ be the $(1, 2)$-polarization on $\widehat{A}$ which is dual to $\mathcal{L}$. Then $\mathcal{L}^{*-1}$ is an IT-sheaf of index 2, and its Fourier–Mukai transform $\mathcal{F} := \widehat{\mathcal{L}^{*-1}}$ is a rank 2 vector bundle on A, which satisfies (1.1) by [BL04, Theorem 14.2.2] and [Mu81, Corollary 2.8]. In addition, [BL04, Proposition 14.4.3] implies $c_1(\mathcal{F}) = \mathcal{L}$. Finally, Hirzebruch–Riemann–Roch implies $c_2(\mathcal{F}) = 1$ and by [PP10, Proposition 2.2] and [PP10, Proposition 2.4], since $\mathcal{L}$ is not a product polarization, we infer that $\mathcal{F}$ is symmetric and isomorphic to the unique locally free extension (1.2). ∎

Proposition 1.9 *Let $\mathcal{Q} \in \widehat{A}$. The following holds:*

(i) *if $\mathcal{Q} \notin \operatorname{im} \phi_2^{\times}$, then*

$$h^0\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 0, \quad h^1\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 0,$$
$$h^2\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 0;$$

(ii) *if $\mathcal{Q} \in \operatorname{im} \phi_2^{\times}$, then*

$$h^0\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 1, \quad h^1\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 2,$$
$$h^2\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 1.$$

Proof Tensoring (1.2) with $\mathcal{Q}$ we obtain $h^0(A, \mathcal{F} \otimes \mathcal{Q}) = 1$; that is, $\mathcal{F} \otimes \mathcal{Q}$ has a non-trivial section. By [F98, Proposition 5 p. 33] there exists an effective divisor C and a zero-dimensional subscheme $W \subset A$ such that $\mathcal{F} \otimes \mathcal{Q}$ fits into a short exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{F} \otimes \mathcal{Q} \longrightarrow \mathcal{L} \otimes \mathcal{Q}^2 \otimes \mathcal{C}^{-1} \otimes \mathcal{I}_W \longrightarrow 0, \tag{1.3}$$

where $\mathcal{C} = \mathcal{O}_A(C)$. Then $h^0(A, \mathcal{C}) = 1$ and

$$1 = c_2(\mathcal{F} \otimes \mathcal{Q}) = C(L - C) + \ell(W). \tag{1.4}$$

Now there are three possibilities:

(i) C is an elliptic curve;
(ii) C is a principal polarization;
(iii) $C = 0$.

We want to show that (i) and (ii) cannot occur.

In case (i) we have $C^2 = 0$, then by (1.4) we obtain $CL = 1$ and $\ell(W) = 0$. Thus [BL04, Lemma 10.4.6] implies that $\mathcal{L}$ is a product polarization, a contradiction.

In case (ii), the Index Theorem yields $(CL)^2 \geq C^2L^2 = 8$, so using (1.4) we deduce $CL = 3$, $\ell(W) = 0$. Tensoring (1.3) by $\mathcal{Q}^{-1}$ and setting $\mathcal{C}' := \mathcal{C} \otimes \mathcal{Q}^{-1}$, we obtain

$$0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \otimes \mathcal{C}'^{-1} \longrightarrow 0.$$

Since $\mathcal{C}'$ is also a principal polarization, by applying the same argument used in the proof of [PP10, Proposition 2.2], we conclude again that $\mathcal{L}$ must be a product polarization.

Therefore the only possibility is (iii), namely $C = 0$. It follows $\ell(W) = 1$; that is, W consists of a unique point $p \in A$ and (1.3) becomes

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{F} \otimes \mathcal{Q} \longrightarrow \mathcal{L} \otimes \mathcal{Q}^2 \otimes \mathcal{I}_p \longrightarrow 0. \tag{1.5}$$

Moreover, since $\mathcal{F}\otimes\mathcal{Q}$ is locally free, we have $p \in \operatorname{Bs}|L+2Q|$, see [Ca90, Example 1.7] or [F98, Theorem 12 p. 39] . Applying the Eagon–Northcott complex to (1.5) and tensoring with $\bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}^{-1}$, we get

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q} \longrightarrow \mathcal{L} \otimes \mathcal{Q}^3 \otimes \mathcal{I}_p^2 \longrightarrow 0,$$

hence

$$h^0\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = h^0(A, \mathcal{L} \otimes \mathcal{Q}^3 \otimes \mathcal{I}_p^2).$$

On the other hand, since $p \in \operatorname{Bs}|L + 2Q|$, Proposition 1.3 yields

$$h^0(A, \mathcal{L} \otimes \mathcal{Q}^3 \otimes \mathcal{I}_p^2) = \begin{cases} 0 & \text{if } \mathcal{Q} \notin \operatorname{im} \phi_2^\times, \\ 1 & \text{if } \mathcal{Q} \in \operatorname{im} \phi_2^\times. \end{cases} \tag{1.6}$$

Using Serre duality, the isomorphism $\mathcal{F}^\vee \cong \mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee$, and (1.6), since $\mathcal{Q} \in \operatorname{im} \phi_2^\times$ if and only if $\mathcal{Q}^{-1} \in \operatorname{im} \phi_2^\times$, we obtain

$$h^2\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = h^0\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}^{-1}\big) = h^0\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big)$$

for all $\mathcal{Q} \in \widehat{A}$. Moreover Hirzebruch–Riemann–Roch gives $\chi(A, S^2\mathcal{F}\otimes\bigwedge^2 \mathcal{F}^\vee\otimes\mathcal{Q}) = 0$, hence we get

$$h^1\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 2 \cdot h^0\big(A, S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big).$$

This completes the proof. ■

Proposition 1.10 *For any $\mathcal{Q} \in \widehat{A}$, we have*

$$h^0\big(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 2, \quad h^1\big(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 0,$$

$$h^2\big(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) = 0.$$

Proof By Hirzebruch–Riemann–Roch we obtain $\chi(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}) = 2$, so it suffices to show that $h^i(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}) = 0$ for $i = 1, 2$. The sheaf $\mathcal{F} \otimes \mathcal{Q}^{-1}$ satisfies IT of index 0 and $h^0(A, \mathcal{F} \otimes \mathcal{Q}^{-1}) = 1$, so its Fourier–Mukai transform $\mathcal{L}_\delta^{-1} := \widehat{\mathcal{F} \otimes \mathcal{Q}^{-1}}$ is a line bundle on $\widehat{A}$, which satisfies IT of index 2 by [BL04, Theorem 14.2.2] and has $h^2(\widehat{A}, \mathcal{L}_\delta^{-1}) = 2$ by [Mu81, Corollary 2.8]. This means that $\mathcal{L}_\delta = (\mathcal{L}_\delta^{-1})^{-1}$ is a $(1, 2)$-polarization. Since $\mathcal{F}$ is a symmetric vector bundle, by using [Mu81, Corollary 2.4] we obtain

$$\widehat{\mathcal{L}_\delta} = (-1)_A^*(\mathcal{F} \otimes \mathcal{Q}^{-1}) = \mathcal{F} \otimes \mathcal{Q};$$

that is, the rank 2 vector bundle $\mathcal{F} \otimes \mathcal{Q}$ is the Fourier–Mukai transform of $\mathcal{L}_\delta$. Therefore, taking the isogeny $\phi = \phi_{\mathcal{L}_\delta^{-1}} \colon \widehat{A} \to A$ and using [Mu81, Proposition 3.11], we can write

$$\phi^*(\mathcal{F} \otimes \mathcal{Q}) = \mathcal{L}_\delta \oplus \mathcal{L}_\delta. \tag{1.7}$$

On the other hand, ϕ is a finite map, so we have

$$H^i\big(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) \cong \phi^* H^i\big(A, S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big) \subseteq H^i\Big(\widehat{A}, \phi^*\big(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big)\Big).$$

Since

$$S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q} = S^3(\mathcal{F} \otimes \mathcal{Q}) \otimes \bigwedge^2 (\mathcal{F} \otimes \mathcal{Q})^\vee,$$

by using (1.7) we deduce

$$H^i\Big(\widehat{A}, \phi^*\big(S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}\big)\Big) = H^i(\widehat{A}, \mathcal{L}_\delta)^{\oplus 4}.$$

The right-hand side vanishes for $i = 1, 2$, so we are done. ■

Let $\sigma\colon B \to A$ be the blow-up of A at o and let $E \subset B$ be the exceptional divisor. Since $\operatorname{Pic}^0(B) \cong \sigma^* \operatorname{Pic}^0(A)$, by abusing notation we will often identify degree 0 line bundles on B with degree 0 line bundles on A, and we will simply write $\mathcal{Q}$ instead of $\sigma^*\mathcal{Q}$.

The strict transform of the pencil $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$ gives the base-point free pencil $|\sigma^*(2L) - 4E|$ in B, whose general element is a smooth curve of genus 3.

Proposition 1.11 *Let $D \in |\sigma^*(2L) - 4E|$ be a smooth curve and let $\mathcal{Q} \in \widehat{A}$. Then $\mathcal{O}_D(Q) = \mathcal{O}_D$ if and only if $\mathcal{Q} \in \operatorname{im}\phi_2$.*

Proof If $\mathcal{Q} = \mathcal{O}_A$, the result is clear, so we assume that $\mathcal{Q} \in \widehat{A}$ is non-trivial. Since $h^1(B, \mathcal{Q}) = h^2(B, \mathcal{Q}) = 0$, by using the short exact sequence

$$0 \longrightarrow \mathcal{O}_B(Q - D) \longrightarrow \mathcal{O}_B(Q) \longrightarrow \mathcal{O}_D(Q) \longrightarrow 0$$

and Serre duality, we obtain

$$\begin{aligned}(1.8)\qquad h^1\big(D, \mathcal{O}_D(Q)\big) &= h^2\big(B, \mathcal{O}_B(Q - D)\big) = h^0\big(B, \mathcal{O}_B(D - Q + E)\big)\\ &= h^0\big(B, \sigma^*\mathcal{O}_A(2L - Q) - 3E\big) = h^0(A, \mathcal{L}^2 \otimes \mathcal{Q}^{-1} \otimes \mathcal{I}_o^3).\end{aligned}$$

In order to compute the last cohomology group, we will exploit the vector bundle $\mathcal{F}$. In fact, applying the Eagon–Northcott complex to (1.2) and tensoring with $\bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}^{-1}$, we get

$$(1.9)\quad 0 \longrightarrow S^2\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}^{-1} \longrightarrow S^3\mathcal{F} \otimes \bigwedge^2 \mathcal{F}^\vee \otimes \mathcal{Q}^{-1} \longrightarrow \mathcal{L}^2 \otimes \mathcal{Q}^{-1} \otimes \mathcal{I}_o^3 \longrightarrow 0.$$

By using (1.9) and Propositions 1.9 and 1.10, we obtain

$$h^0(A, \mathcal{L}^2 \otimes \mathcal{Q}^{-1} \otimes \mathcal{I}_o^3) = \begin{cases} 2 & \text{if } \mathcal{Q} \notin \operatorname{im} \phi_2^\times, \\ 3 & \text{if } \mathcal{Q} \in \operatorname{im} \phi_2^\times. \end{cases}$$

Since D is a smooth curve of genus 3, by using (1.8) and Riemann–Roch we deduce

$$h^0(D, \mathcal{O}_D(Q)) = h^1(D, \mathcal{O}_D(Q)) - 2 = \begin{cases} 0 & \text{if } \mathcal{Q} \notin \operatorname{im} \phi_2, \\ 1 & \text{if } \mathcal{Q} \in \operatorname{im} \phi_2. \end{cases}$$

This completes the proof. ■

Corollary 1.12 *Let $\mathcal{Q} \in \widehat{A}$ and let D be a smooth curve in the pencil $|\sigma^*(2L) - 4E|$. Then $\mathcal{O}_D(\sigma^*(L+Q) - 2E) = \mathcal{O}_D$ if and only if $\mathcal{Q} \in \operatorname{im}\phi_2$.*

Proof For all $i \in \{1, 2, 3\}$ the effective curve $\widetilde{N}_i \in |\sigma^*(L+Q_i)-2E|$ does not intersect D, so $\mathcal{O}_D(\sigma^*(L+Q_i) - 2E) = \mathcal{O}_D$. So we have $\mathcal{O}_D(\sigma^*(L+Q) - 2E) = \mathcal{O}_D$ if and only if $\mathcal{O}_D(Q - Q_i) \in \operatorname{im} \phi_2$, *i.e.*, if and only if $Q \in \operatorname{im} \phi_2$; see Proposition 1.11. ■

2 Surfaces with $p_g = q = 2$, $K^2 = 6$ and Albanese Map of Degree 2

In the sequel, S will be a smooth minimal surface of Albanese general type with $p_g = q = 2$, and $\alpha\colon S \to A$ will be its Albanese map, which we suppose is of degree 2. Let $D_A \subset A$ be the branch locus of α and let

$$\begin{array}{ccc} S & \longrightarrow & X \\ & \searrow^{\alpha} & \downarrow f \\ & & A \end{array}$$

be the Stein factorization of α. Then $f\colon X \to A$ is a finite double cover, and, since S is smooth, it follows that X is normal; see [BHPV03, Chapter I, Theorem 8.2]. In particular X has at most isolated singularities, hence the curve D_A is reduced.

Proposition 2.1 *Assume that $K_S^2 = 6$ and that the Albanese map $\alpha\colon S \to A$ is a generically finite double cover. Then there exists a polarization $\mathcal{L}_A = \mathcal{O}_A(L_A)$ of type $(1, 2)$ on A such that D_A is a curve in $|2L_A|$ whose unique non-negligible singularity is an ordinary quadruple point p.*

Proof D_A is linearly equivalent to $2L_A$ for some divisor L_A in A. There is a "canonical resolution" diagram

(2.1)
$$\begin{array}{ccc} \bar{S} & \longrightarrow & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\sigma} & A, \end{array}$$

where $\bar{S}$ is smooth and $\sigma\colon B \to A$ is composed of a series of blow-ups; see [BHPV03, Chapter III, Section 7]. Let $x_1, x_2, \dots, x_r$ be the centers of these blow-ups, and let E_i be the inverse image of x_i on B (with right multiplicities such that $E_iE_j = -\delta_{ij}$, $K_B =$

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$\sigma^* K_A + \sum_{i=1}^r E_i$). Then the branch locus D_B of $\beta\colon \bar{S} \to B$ is smooth and can be written as

$$D_B = \sigma^* D_A - \sum_{i=1}^{r} d_i E_i,$$

where the d_i are even positive integers, say $d_i = 2m_i$. Let us recall a couple of definitions:

• a *negligible singularity* of D_A is a point x_j such that $d_j = 2$, and $d_i \le 2$ for any point x_i infinitely near to x_j;
• a $[2d+1, 2d+1]$- *singularity* of D_A is a pair (x_i, x_j) such that x_i belongs to the first infinitesimal neighbourhood of x_j and $d_i = 2d+2$, $d_j = 2d$.

For example, a double point and an ordinary triple point are negligible singularities, whereas a $[3,3]$-point is not. By using the formulae in [BHPV03, p. 237] we obtain

$$2 = 2\chi(\mathcal{O}_{\bar{S}}) = L_A^2 - \sum_{i=1}^{r} m_i(m_i - 1), \quad K_{\bar{S}}^2 = 2L_A^2 - 2\sum_{i=1}^{r}(m_i - 1)^2, \tag{2.2}$$

which imply

$$6 = K_S^2 \ge K_{\bar{S}}^2 = 4 + 2\sum_{i=1}^{r}(m_i - 1).$$

If $m_i = 1$ for all i, then all the x_i are negligible singularities and (2.2) gives $K_S^2 = 4$, a contradiction. Then we can assume $m_1 = 2$, $m_2 = \cdots = m_r = 1$. Therefore (2.2) yields $L_A^2 = 4$, that is $\mathcal{L}_A := \mathcal{O}_A(L_A)$ is a polarization of type $(1,2)$ on A. Now we have two possibilities:

(i) x_1 is not infinitely near to x_2; then $D_A \in |2L_A|$ contains an ordinary quadruple point p and (possibly) some negligible singularities;

(ii) x_1 is infinitely near to x_2; then $D_A \in |2L_A|$ contains a point p of type $[3,3]$ and (possibly) some negligible singularities.

But in case (ii) the surface $\bar{S}$ contains a (-1)-curve, hence $K_S^2 = 7$, a contradiction. Therefore D_A must be a curve of type (i). The existence of such a curve was proven in Corollary 1.7, so we are done. ■

Remark 2.2 The argument used in the proof of Proposition 2.1 shows that if we were able to find a curve in $|2L_A|$ with a singular point of type $[3,3]$, then we could construct a surface S with $p_g = q = 2$ and $K_S^2 = 7$. Unfortunately, at present we do not know whether such a curve exists.

Proposition 2.3 *$\mathcal{L}_A$ is not a product polarization.*

Proof Assume by contradiction that $\mathcal{L}_A$ is a product polarization. Then $A = E \times F$, with natural projection maps $\pi_E\colon A \to E$ and $\pi_F\colon A \to F$, and $L \equiv E + 2F$. Let F_p be the fibre of π_E passing through p. Since D_A has a quadruple point at p and $D_A F_p = 2$,

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Bézout theorem implies that F_p is a component of D_A. Similarly, since $D_A - F_p$ has a triple point at p and $(D_A - F_p)F_p = 2$, it follows that F_p is a component of $D_A - F_p$. Therefore D_A contains the curve F_p with multiplicity at least 2, which is impossible, since D_A must be reduced. ■

Up to a translation we can now suppose $p = o$, and by using Corollary 1.6 we can write $\mathcal{L}_A^2 = \mathcal{L}^2 \otimes \mathcal{Q}$, where $\mathcal{Q} \in \operatorname{im} \phi_2$ and $\mathcal{L}$ is a symmetric polarization, not of product type, such that $h^0(A, \mathcal{L}^2 \otimes \mathcal{I}_o^4) = 2$.

In the rest of this section we assume for simplicity that D_A contains no negligible singularities besides the quadruple point o; this is an open condition, equivalent to the ampleness of K_S. Hence the map $\sigma\colon B \to A$ is just the blow-up at o, we have $\bar{S} = S$, and (2.1) induces the following commutative diagram

$$\begin{tikzcd} S \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & B \arrow[r, "\varphi"] \arrow[d, "\sigma"] & \mathbb{P}^1 \\ & A & \end{tikzcd}$$

where $\beta\colon S \to B$ is a *finite* double cover and $\varphi\colon B \to \mathbb{P}^1$ is the morphism induced by the base-point free pencil $|\sigma^*(2L) - 4E|$. The double cover β is branched along a *smooth* divisor

$$D_B \in |\sigma^*(2L + Q) - 4E|,$$

hence it is defined by a square root of $\mathcal{O}_B(D_B)$, namely $\mathcal{L}_B := \mathcal{O}_B(\sigma^*(L + Q^{1/2}) - 2E)$, where $\mathcal{Q}^{1/2}$ is a square root of $\mathcal{Q}$.

Proposition 2.4 *S is a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 6$, unless $\mathcal{Q} = \mathcal{Q}^{1/2} = \mathcal{O}_A$. In the last case we have instead $p_g = q = 3$ and $K_S^2 = 6$.*

Proof Standard formulae for double covers ([BHPV03, p. 237]) give $\chi(\mathcal{O}_S) = 1$ and $K_S^2 = 6$. Moreover we have $\beta_*\omega_S = \omega_B \oplus (\omega_B \otimes \mathcal{L}_B)$, hence we obtain

$$p_g(S) = h^0\big(B, \mathcal{O}_B(E)\big) + h^0\Big(B, \mathcal{O}_B\big(\sigma^*(L + Q^{1/2}) - E\big)\Big) = 1 + h^0(A, \mathcal{L} \otimes \mathcal{Q}^{1/2} \otimes \mathcal{I}_o).$$

If $\mathcal{Q}^{1/2}$ is not trivial, then $h^0(A, \mathcal{L} \otimes \mathcal{Q}^{1/2} \otimes \mathcal{I}_o) = 1$, otherwise $h^0(A, \mathcal{L} \otimes \mathcal{I}_o) = 2$. ■

Remark 2.5 If $\mathcal{Q} = \mathcal{Q}^{1/2} = \mathcal{O}_A$ then S is the symmetric product of a smooth curve of genus 3; see [HP02, Pi02]. Let us give an alternative construction of the double cover $f\colon S \to A$ in this particular case. Take a smooth curve C of genus 3, admitting a double cover $\varphi\colon C \to E$ onto an elliptic curve E. Let o be the identity in the group law of E, and for all $x \in C$ let us denote by x' the conjugate point of x with respect to the involution $C \to C$ induced by φ. Then $S := \mathrm{Sym}^2(C)$ contains the elliptic curve $Z := \{x + x' | x \in C\}$, which is isomorphic to E. Moreover, there is a morphism

$$\bar{\alpha}\colon S \longrightarrow \mathrm{Pic}^0(C) \quad \text{given by} \quad \bar{\alpha}(x + y) = \mathcal{O}_C\big(x + y - \varphi^*(o)\big).$$

Now take any point $x + x' \in Z$ and let $a := \varphi(x) = \varphi(x')$. We have

$$\bar{\alpha}(x + x') = \mathcal{O}_C\big(x + x' - \varphi^*(o)\big) = \varphi^*\mathcal{O}_E(a - o) \in \varphi^* \operatorname{Pic}^0(E);$$

that is, the induced map

$$\alpha\colon S \longrightarrow A := \operatorname{Pic}^0(C)/\varphi^* \operatorname{Pic}^0(E)$$

contracts Z to a point. Moreover, α has generic degree 2; in fact $\alpha(x+y) = \alpha(x'+y')$ for all $x, y \in C$.

Since we are interested in the case $p_g(S) = q(S) = 2$, in the sequel we always assume $\mathcal{Q}^{1/2} \neq \mathcal{O}_A$. Summing up, we have proven the following result.

Theorem 2.6 *Given an abelian surface A with a symmetric polarization $\mathcal{L}$ of type $(1,2)$, not of product type, for any $\mathcal{Q} \in \operatorname{im}\phi_2$ there exists a curve $D_A \in |\mathcal{L}^2 \otimes \mathcal{Q}|$ whose unique non-negligible singularity is an ordinary quadruple point at the origin $o \in A$. Let $\mathcal{Q}^{1/2}$ be a square root of $\mathcal{Q}$, and if $\mathcal{Q} = \mathcal{O}_A$, assume, moreover, that $\mathcal{Q}^{1/2} \neq \mathcal{O}_A$. Then the minimal desingularization S of the double cover of A branched over D_A and defined by $\mathcal{L} \otimes \mathcal{Q}^{1/2}$ is a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$, and Albanese map of degree 2.*

Conversely, every minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2 can be constructed in this way.

In order to proceed with the study of our surfaces, let us introduce the following definition.

Definition 2.7 Let S be a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$, and Albanese map of degree 2.

- If $\mathcal{Q} = \mathcal{O}_A$, we say that S is a *surface of type I*. Furthermore, if $\mathcal{Q}^{1/2} \notin \operatorname{im}\phi_2^{\times}$, we say that S is of type *Ia*, whereas if $\mathcal{Q}^{1/2} \in \operatorname{im}\phi_2^{\times}$ we say that S is of type *Ib*.
- If $\mathcal{Q} \in \operatorname{im}\phi_2^{\times}$ we say that S is a *surface of type II*.

Remark 2.8 If S is a surface of type I, then D_A is as in Corollary 1.7(i) or (ii). If S is a surface of type II, then D_A is as in Corollary 1.7(iii). See Figures 1 and 2.

We denote by $R \subset S$ the ramification divisor of $\beta\colon S \to B$ and by Z the divisor β^*E. Then Z is an elliptic curve and $Z^2 = -2$.

Proposition 2.9 *The pullback via $\beta\colon S \to B$ of the general curve D in the pencil $|D| = |\sigma^*(2L) - 4E|$ is reducible if and only if S is of type Ib.*

Proof The restriction of β to D is the trivial double cover if and only if $\mathcal{L}_B \otimes \mathcal{O}_D = \mathcal{O}_D$, *i.e.*, if and only if $\mathcal{O}_D(\sigma^*(L + Q^{1/2}) - 2E) = \mathcal{O}_D$. Thus the result follows from Corollary 1.12. ■

Now we want to describe the canonical system of our surfaces. Let us analyze first surfaces of type I. Then $\mathcal{Q}^{1/2}$ is a non-trivial 2-torsion line bundle, and, for the general surface S, the branch locus D_B of $\beta\colon S \to B$ is a smooth curve of genus 3 belonging to the pencil $|D| = |\sigma^*(2L) - 4E|$.

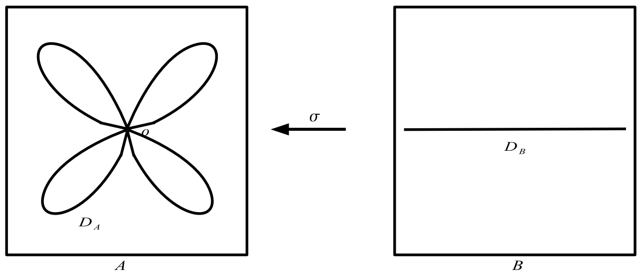

Figure 1: The branch curves D_A and D_B for a general surface of type *I*

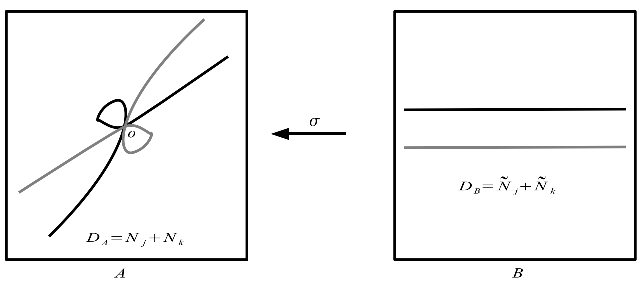

Figure 2: The branch curves D_A and D_B for a surface of type *II*

Proposition 2.10 *Let S be a surface of type I; then the following holds.*

(i) *If S is of type Ia, the pullback via $\beta\colon S \to B$ of the pencil $|D|$ on B is a base-point free pencil $|\Phi|$ on S, whose general element Φ is a smooth curve of genus 5 satisfying $\Phi Z = 8$. Moreover, the canonical system $|K_S|$ has no fixed part, hence the general canonical curve of S is irreducible. Finally, $2R \in |\Phi|$.*

(ii) *If S is of type Ib,* i.e., $\mathcal{Q}^{1/2} = \mathcal{Q}_i$ *for some* $i \in \{1, 2, 3\}$, *there is a commutative diagram*

(2.3)
$$\begin{tikzcd} S \arrow[r, "\beta"] \arrow[d, "\phi"'] & B \arrow[d, "\varphi"] \\ \mathbb{P}^1 \arrow[r, "b"] & \mathbb{P}^1, \end{tikzcd}$$

where $b\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a double cover branched in two points, namely the point corresponding to the branch locus D_B and the point corresponding to the curve

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$2\widetilde{N}_i$, where N_i is the unique curves in $|\mathcal{L} \otimes \mathcal{Q}_i|$ with an ordinary double point at o and "$\sim$" stands for the strict transform in B. The general fibre Φ of the map $\phi\colon S \to \mathbb{P}^1$ is a smooth curve of genus 3; moreover, Z is the fixed part of $|K_S|$ and $|K_S| = Z + |\Phi|$, i.e., *the canonical system is composed with the pencil $|\Phi|$. Finally, $R \in |\Phi|$.*

Proof (i) The fact that Φ is a smooth curve of genus 5 follows from Proposition 2.9; moreover $\Phi Z = (\beta^* D)(\beta^* E) = 2DE = 8$. We have $2R = \beta^* D_B \in |\Phi|$ and by Hurwitz's formula $K_S = \beta^* K_B + R = Z + R$. Since $\dim |K_S| = 1$ and neither Z nor R move in a pencil, we deduce that $|K_S|$ has no fixed part.

(ii) If S is of type *Ib*, then by Proposition 2.9 the pull-back via β of a general element of $|D|$ is the disjoint union of two smooth curves of genus 3. So there exists a base-point free genus 3 pencil $|\Phi|$ on S, and we obtain diagram (2.3). In this case $\mathcal{L}_B = \mathcal{O}_B(\widetilde{N}_i)$ is effective, and it is not difficult to see that $b\colon \mathbb{P}^1 \to \mathbb{P}^1$ is branched only at the two points corresponding to D_B and N_i. Moreover, $R = \beta^* \widetilde{N}_i \in |\Phi|$, so we can write

$$|K_S| = |\beta^* K_B + R| = Z + |\Phi|;$$

that is, the fixed part of the canonical pencil of S is $|Z|$ and its movable part is $|\Phi|$. ■

Let us consider now surfaces of type *II*. Then $\mathcal{Q} = \mathcal{Q}_i$ for some $i \in \{1, 2, 3\}$, so $\mathcal{Q}^{1/2}$ is a degree 0 line bundle whose order is exactly 4 and the curve D_B consists of two distinct half-fibres of $|D|$, namely $D_B = \widetilde{N}_j + \widetilde{N}_k$. Therefore S is of type *II* if and only if D_B is *disconnected*. Proposition 2.9 implies that the pullback via β of a general curve in $|D|$ is irreducible, so we obtain the following proposition.

Proposition 2.11 *If S is a surface of type II, then the pullback via $\beta\colon S \to B$ of the pencil $|D|$ on B is a base-point free pencil $|\Phi|$ on S, whose general element Φ is a smooth curve of genus 5 satisfying $\Phi Z = 8$. Moreover, the canonical system $|K_S|$ has no fixed part, hence the general canonical curve of S is irreducible. Finally, $R = R_1 + R_2$ with $4R_1, 4R_2 \in |\Phi|$.*

Proof The first two parts of the statement follow from Proposition 2.9 by the same argument used in the proof of Proposition 2.10(i). It remains only to prove the assertion about R. Let R_1, R_2 be the two effective curves in S such that $\beta^* \widetilde{N}_j = 2R_1$, $\beta^* \widetilde{N}_k = 2R_2$; then $R = R_1 + R_2$. Moreover, since $\widetilde{N}_j$ and $\widetilde{N}_k$ are both half-fibres of $|D|$, it follows that $4R_1, 4R_2 \in |\Phi|$, and we are done. ■

Remark 2.12 The general surface of type *I* has ample canonical divisor. In addition, all surfaces of type *II* have ample canonical divisors.

3 The Moduli Space

Let S be a minimal surface of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2; for a general choice of S we may assume that K_S is ample; see Remark 2.12. The following result can be found in [Ca11, Section 5].

Proposition 3.1 *Let S be a minimal surface of general type with $q(S) \geq 2$, and Albanese map $\alpha\colon S \to A$, and assume that $\alpha(S)$ is a surface. Then this is a topological property. If in addition $q(S) = 2$, then the degree of the α is a topological invariant.*

Proof By [Ca91] the Albanese map α induces a homomorphism of cohomology algebras

$$\alpha^*\colon H^*(\mathrm{Alb}(S), \mathbb{Z}) \longrightarrow H^*(S, \mathbb{Z}),$$

and $H^*(\mathrm{Alb}(S), \mathbb{Z})$ is isomorphic to the full exterior algebra $\bigwedge^* H^1(\mathrm{Alb}(S), \mathbb{Z})) \cong \bigwedge^* H^1(S, \mathbb{Z})$. In particular, if $q = 2$, the degree of the Albanese map equals the index of the image of $\bigwedge^4 H^1(S, \mathbb{Z})$ inside $H^4(S, \mathbb{Z})$, and it is therefore a topological invariant. ■

By Proposition 3.1 it follows that one may study the deformations of S by relating them to those of the flat double cover $\beta\colon S \to B$. By [Se06, p. 162] we have an exact sequence

$$0 \longrightarrow T_S \longrightarrow \beta^* T_B \longrightarrow \mathcal{N}_\beta \longrightarrow 0,$$

where $\mathcal{N}_\beta$ is a coherent sheaf supported on R called the *normal sheaf of* β.

Proposition 3.2 *Assume that K_S is ample. If S is a surface of type I, then $\mathcal{N}_\beta = \mathcal{O}_R$. If S is a surface of type II, then $\mathcal{N}_\beta$ is a non-trivial 2-torsion element of $\mathrm{Pic}^0(R)$.*

Proof Since K_S is ample, R is smooth and we have an isomorphism

$$\mathcal{N}_\beta = (N_{R/S})^{\otimes 2} = \mathcal{O}_R(2R);$$

see [Rol10, Lemma 3.2]. If S is of type I, then either $R \in |\Phi|$ or $2R \in |\Phi|$ (see Proposition 2.10), so $\mathcal{N}_\beta$ is trivial. If S is of type II, then $4R \in |\Phi|$ (see Proposition 2.11), so $\mathcal{N}_\beta$ is a non-trivial 2-torsion line bundle. ■

Proposition 3.3 *Assume that K_S is ample. Then the sheaf $\beta^* T_B$ satisfies*

$$h^0(S, \beta^* T_B) = 0, \quad h^1(S, \beta^* T_B) = 4, \quad h^2(S, \beta^* T_B) = 4.$$

Proof Since $\beta\colon S \to B$ is a finite map, by using projection formula and the Leray spectral sequence we deduce

$$h^i(S, \beta^* T_B) = h^i(B, \beta_* \beta^* T_B) = h^i(B, T_B) + h^i(B, T_B \otimes \mathcal{L}_B^{-1}), \quad i = 0, 1, 2. \tag{3.1}$$

There is a short exact sequence

$$0 \longrightarrow T_B \longrightarrow \sigma^* T_A \longrightarrow \mathcal{O}_E(-E) \longrightarrow 0; \tag{3.2}$$

see [Se06, p. 73]. Then a straightforward computation yields

$$h^0(B, T_B) = 0, \quad h^1(B, T_B) = 4, \quad h^2(B, T_B) = 2. \tag{3.3}$$

Now let us tensor (3.2) with $\mathcal{L}_B^{-1}$. Since $\sigma^* T_A = \mathcal{O}_B \oplus \mathcal{O}_B$ and $\mathcal{L}_B^{-1} \otimes \mathcal{O}_E(-E) = \mathcal{O}_E(E)$, by taking cohomology we obtain

$$h^i(B, T_B \otimes \mathcal{L}_B^{-1}) = 2 \cdot h^i(B, \mathcal{L}_B^{-1}), \quad i = 0, 1, 2. \tag{3.4}$$

Moreover we have

$$h^0(B, \mathcal{L}_B^{-1}) = 0, \quad h^1(B, \mathcal{L}_B^{-1}) = 0, \quad h^2(B, \mathcal{L}_B^{-1}) = 1, \tag{3.5}$$

where the first equality comes from the fact that $D_A = 2L_B$ is an effective divisor, the third equality follows from Serre duality and $h^0(B, \mathcal{L}_B \otimes \mathcal{O}_B(E)) = 1$, since $\mathcal{Q}^{1/2} \neq \mathcal{O}_A$, and the second one is a consequence of Riemann–Roch.

Therefore the claim follows using (3.1), (3.3), (3.4), and (3.5). ∎

We have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_S \arrow[r] \arrow[d] & \beta^* T_B \arrow[r] \arrow[d] & \mathcal{N}_\beta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_S \arrow[r] \arrow[d] & \alpha^* T_A \arrow[r] \arrow[d] & \mathcal{N}_\alpha \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{O}_Z(-Z) \arrow[r] \arrow[d] & \mathcal{O}_Z(-Z) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{3.6}$$

whose central column is the pullback of (3.2) via $\beta\colon S \to B$.

Proposition 3.4 *Let S be a minimal surface with $p_g = q = 2$, $K_S^2 = 6$, and Albanese map of degree 2, and assume that K_S is ample. Then*

$$h^1(S, T_S) = \begin{cases} 4 & \textit{if } S \textit{ is of type I}, \\ 3 & \textit{if } S \textit{ is of type II}. \end{cases}$$

Proof Proposition 3.3 yields $H^0(S, \beta^* T_B) = 0$, so looking at the central column of diagram (3.6) we obtain the long exact sequence in cohomology

$$0 \longrightarrow H^0(S, \alpha^* T_A) \longrightarrow H^0(Z, \mathcal{O}_Z(-Z)) \longrightarrow H^1(S, \beta^* T_B) \xrightarrow{\delta} H^1(S, \alpha^* T_A) \longrightarrow 0.$$

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Since $h^0(S, \alpha^* T_A) = h^0(Z, \mathcal{O}_Z(-Z)) = 2$, it follows that the map δ is an isomorphism. Therefore the commutativity of (3.6) implies that the image of $H^1(S, T_S)$ in $H^1(S, \beta^* T_B)$ coincides with the image of $H^1(S, T_S)$ in $H^1(S, \alpha^* T_A) \cong H^1(A, T_A)$. So we obtain the exact sequence

$$0 \longrightarrow H^0(R, \mathcal{N}_\beta) \longrightarrow H^1(S, T_S) \xrightarrow{\gamma} H^1(A, T_A). \tag{3.7}$$

We claim that the image of γ has dimension 3. In order to prove this, we borrow an argument from [PP10, Section 6]. Take a positive integer $m \geq 2$ such that there exists a smooth pluricanonical divisor $\Gamma \in |mK_S|$ and let Γ' be the image of Γ in A. By [Se06, Section 3.4.4 p. 177], the first order deformations of a pair (X, Y), where $X \subset Y$ is a closed subscheme and Y is nonsingular, are parameterized by the vector space $H^1(Y, T_Y\langle X\rangle)$, where $T_Y\langle X\rangle$ is the sheaf of germs of tangent vectors to Y that are tangent to X. Notice that $T_Y\langle X\rangle$ is usually denoted by $T_Y(-\log X)$ when X is a normal crossing divisor with smooth components. In our situation, a first-order deformation of the pair (Γ, S) induces a first-order deformation of the pair (Γ', A), because the the differential map $d\alpha\colon T_S \to T_A$ sends vectors tangent to Γ into vectors tangent to Γ'. Hence we have a commutative diagram

$$\begin{tikzcd} H^1(S, T_S\langle\Gamma\rangle) \arrow[r, "\gamma'"] \arrow[d, "\epsilon"'] & H^1(A, T_A\langle\Gamma'\rangle) \arrow[d, "\epsilon'"] \\ H^1(S, T_S) \arrow[r, "\gamma"] & H^1(A, T_A). \end{tikzcd}$$

Let us now observe the following facts.

- Since S is smooth, the line bundle ω_S^m extends along any first-order deformation of S, because the relative dualizing sheaf is locally free for any smooth morphism of schemes; see [Man08, p. 182]. Moreover, since S is minimal of general type, we have $h^1(S, \omega_S^m) = 0$, so every section of ω_S^m extends as well; see [Se06, Section 3.3.4]. This means that no first-order deformation of S makes Γ disappear; in other words, ϵ is surjective. Therefore $\operatorname{im}\gamma \subseteq \operatorname{im}\epsilon'$.
- Since $(\Gamma')^2 > 0$, the line bundle $\mathcal{O}_A(\Gamma')$ is ample on A; therefore it deforms along a subspace of $H^1(A, T_A)$ of dimension 3; see [Se06, p. 152]. Since every first-order deformation of the pair (A, Γ') induces a first-order deformation of the pair $(A, \mathcal{O}_A(\Gamma'))$, it follows that the image of ϵ' is at most 3-dimensional.

According to the above remarks, we obtain

$$\dim(\operatorname{im}\gamma) \leq \dim(\operatorname{im}\epsilon') \leq 3.$$

On the other hand, given any abelian surface A with a $(1, 2)$-polarization, not of product type, by the results of Section 2 we can construct a surface S of type I or II

such that $\mathrm{Alb}(S) = A$. Then the dimension of $\operatorname{im}\gamma$ equals the dimension of the moduli space of $(1,2)$-polarized abelian surfaces, which is precisely 3. So (3.7) implies

$$h^1(S, T_S) = 3 + h^0(R, \mathcal{N}_\beta),$$

and by using Proposition 3.2 we are done. ■

By Proposition 3.1 we may consider the moduli space $\mathcal{M}$ of minimal surfaces S of general type with $p_g = q = 2$, $K_S^2 = 6$, and Albanese map of degree 2. Let $\mathcal{M}_{Ia}$, $\mathcal{M}_{Ib}$, $\mathcal{M}_{II}$ be the subsets whose points parameterize isomorphism classes of surfaces of type Ia, Ib, II, respectively. Therefore $\mathcal{M}$ can be written as the disjoint union

$$\mathcal{M} = \mathcal{M}_{Ia} \sqcup \mathcal{M}_{Ib} \sqcup \mathcal{M}_{II}.$$

Moreover, set $\mathcal{M}_I := \mathcal{M}_{Ia} \sqcup \mathcal{M}_{Ib}$.

Proposition 3.5 *The following holds:*

(i) *$\mathcal{M}_{Ia}$ and $\mathcal{M}_{Ib}$ are irreducible, generically smooth of dimension* 4;
(ii) *$\mathcal{M}_{II}$ is irreducible, generically smooth of dimension* 3.

Proof (i) The construction of a surface of type I depends on the following data:

- the choice of a $(1,2)$-polarized abelian surface $(A, \mathcal{L})$, not of product type ;
- the choice of a general divisor D_A in the pencil $|\mathcal{L}^2 \otimes \mathcal{I}_o^4|$;
- the choice of a non-trivial line bundle $\mathcal{Q}$ such that $\mathcal{Q}^2 = \mathcal{O}_A$.

Let $\mathcal{A}_\Delta[2]$ be the space of pairs $(A, \mathcal{Q})$, where A is a $(1,2)$-polarized abelian surface and $\mathcal{Q} \in \widehat{A}$ is the isomorphism class of a non-trivial, 2-torsion line bundle. In the appendix (see Proposition A.2) we show that $\mathcal{A}_\Delta[2]$ is a quasi-projective variety, disjoint union of two connected, irreducible components of dimension 3

$$\mathcal{A}_\Delta^{(a)}[2] \quad \text{and} \quad \mathcal{A}_\Delta^{(b)}[2],$$

which correspond to $\mathcal{Q} \notin \operatorname{im}\phi_2^\times$ and $\mathcal{Q} \in \operatorname{im}\phi_2^\times$, respectively. Therefore there are two generically finite dominant maps

$$\mathcal{P}^{(a)} \longrightarrow \mathcal{M}_{Ia}, \quad \mathcal{P}^{(b)} \longrightarrow \mathcal{M}_{Ib},$$

where $\mathcal{P}^{(a)}$ and $\mathcal{P}^{(b)}$ are suitable projective bundles on $\mathcal{A}_\Delta^{(a)}[2]$ and $\mathcal{A}_\Delta^{(b)}[2]$. It follows that $\mathcal{M}_{Ia}$ and $\mathcal{M}_{Ib}$ are irreducible of dimension 4. On the other hand, Proposition 3.4 implies that for a general $[S] \in \mathcal{M}_I$ we have

$$\dim T_{[S]}\mathcal{M}_I = h^1(S, T_S) = 4.$$

This shows that both $\mathcal{M}_{Ia}$ and $\mathcal{M}_{Ib}$ are generically smooth.

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(ii) The construction of a surface of type II depends on the following data:

- the choice of a $(1,2)$-polarized abelian surface $(A, \mathcal{L})$, not of product type ;
- the choice of $\mathcal{Q} \in \operatorname{im} \phi_2^{\times}$, which yields the unique curve $D_A \in |\mathcal{L}^2 \otimes \mathcal{Q} \otimes \mathcal{I}_o^4|$;
- the choice of a square root $\mathcal{Q}^{1/2}$ of $\mathcal{Q}$.

Let $\mathcal{A}_{\Delta}[2,4]$ be space of triplets $(A, \mathcal{Q}, \mathcal{Q}^{1/2})$, where A is the isomorphism class of a $(1,2)$-polarized abelian surface, $\mathcal{Q} \in \operatorname{im} \phi_2^{\times}$, and $\mathcal{Q}^{1/2}$ is a square root of $\mathcal{Q}$. In the appendix (see Proposition A.3) we show that $\mathcal{A}_{\Delta}[2,4]$ is a 3-dimensional, irreducible, quasi-projective variety. We have a generically finite, dominant map

$$\mathcal{A}_{\Delta}[2,4] \longrightarrow \mathcal{M}_{II},$$

so $\mathcal{M}_{II}$ is irreducible of dimension 3. On the other hand, Proposition 3.4 implies that for a general $[S] \in \mathcal{M}_{II}$ we have

$$\dim T_{[S]}\mathcal{M}_{II} = h^1(S, T_S) = 3,$$

hence $\mathcal{M}_{II}$ is generically smooth. ■

Proposition 3.6 *$\mathcal{M}_{Ia}$, $\mathcal{M}_{Ib}$, and $\mathcal{M}_{II}$ are connected components of $\mathcal{M}$.*

Proof We proved that $\mathcal{M}$ is the disjoint union of three irreducible constructible sets

$$\mathcal{M} = \mathcal{M}_{Ia} \sqcup \mathcal{M}_{Ib} \sqcup \mathcal{M}_{II},$$

so it is sufficient to show that $\mathcal{M}_{Ia}$, $\mathcal{M}_{Ib}$, $\mathcal{M}_{II}$ are all open in $\mathcal{M}$. In other words, given a flat family $\mathscr{S} \to \mathscr{D}$ over a small disk $\mathscr{D}$, such that $S_0 \in \mathcal{M}_{Ia}$ (resp. $S_0 \in \mathcal{M}_{Ib}$, $\mathcal{M}_{II}$), we must show that $S_t \in \mathcal{M}_{Ia}$ (resp. $S_t \in \mathcal{M}_{Ib}$, $\mathcal{M}_{II}$) for $t \neq 0$. We may associate with the family $\mathscr{S} \to \mathscr{D}$ the family $\mathscr{X} \to \mathscr{D}$, whose fibre over $t \in \mathscr{D}$ is the Stein factorization X_t of S_t, that is the contraction of the elliptic curve $Z_t \subset S_t$. By the previous results it follows that, up to a base change, the family $\mathscr{S} \to \mathscr{D}$ is the double cover of a family $\mathscr{B} \to \mathscr{D}$ of blow-ups B_t of $(1,2)$-polarized abelian surfaces and the family $\mathscr{X} \to \mathscr{D}$ is the double cover of the family $\mathscr{A} \to \mathscr{D}$, where A_t is the minimal model of B_t. Globalizing the results of Section 2 we see that the polarizations $\mathcal{L}_t$ on the abelian surfaces A_t glue together in order to give an ample line bundle $\mathscr{L}$ on $\mathscr{A}$ and that there exists a divisor $\mathscr{D}_{\mathscr{B}}$ on $\mathscr{B}$ whose restriction to the fibre B_t is the branch locus D_{B_t} of $\beta_t \colon S_t \to B_t$. Moreover, we find a commutative diagram

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & \mathscr{B} \arrow[d, "\sigma"] \\ & \mathscr{A} \end{tikzcd}$$

and a line bundle $\mathscr{Q} \in \operatorname{Pic}^0(\mathscr{A})$ of order 2 such that $\mathscr{D}_{\mathscr{B}} \cong \sigma^*(2\mathscr{L} + \mathscr{Q}) - 4\mathscr{E}$, where $\sigma \colon \mathscr{B} \to \mathscr{A}$ is the relative blow-down and $\mathscr{E}$ is the exceptional divisor of σ. We denote by $\mathcal{Q}_t$ the restriction of $\mathscr{Q}$ to A_t.

Now let us consider the three cases separately.

- *$\mathcal{M}_{II}$ is open in $\mathcal{M}$.*

It is equivalent to prove that $\mathcal{M}_I$ is closed in $\mathcal{M}$, namely that $S_t \in \mathcal{M}_I$ for $t \neq 0$ implies $S_0 \in \mathcal{M}_I$. The condition $S_t \in \mathcal{M}_I$ for $t \neq 0$ implies that D_{B_t} is connected for any $t \neq 0$; it follows that D_{B_0} is also connected, hence S_0 is again a surface of type I.

- *$\mathcal{M}_{Ia}$ is open in $\mathcal{M}$.*

Assume that $S_0 \in \mathcal{M}_{Ia}$. By Proposition 2.10, this is equivalent to say that the branch locus D_{B_0} of $\beta_0 \colon S_0 \to B_0$ is connected and that $|K_{S_0}|$ is base-point free. Clearly these are both open conditions, so $\mathcal{M}_{Ia}$ is open in $\mathcal{M}$.

- *$\mathcal{M}_{Ib}$ is open in $\mathcal{M}$.*

Assume that $S_0 \in \mathcal{M}_{Ib}$. Then we have $(A_0, \mathcal{Q}_0) \in \mathcal{A}_{\Delta}^{(b)}[2]$. By Proposition A.2 in the appendix it follows that $\mathcal{A}_{\Delta}^{(b)}[2]$ is a connected component of $\mathcal{A}_{\Delta}[2]$, in particular it is open therein. Hence $(A_t, \mathcal{Q}_t) \in \mathcal{A}_{\Delta}^{(b)}[2]$ for $t \neq 0$, proving that $\mathcal{M}_{Ib}$ is open in $\mathcal{M}$. Notice that the same argument gives an alternative proof of the fact that $\mathcal{M}_{Ia}$ is open in $\mathcal{M}$, since $\mathcal{A}_{\Delta}^{(a)}[2]$ is the other connected component of $\mathcal{A}_{\Delta}[2]$.

This completes the proof of Proposition 3.6. ■

Summing up, Propositions 3.5 and 3.6 and Remark 2.12 imply the following result.

Theorem 3.7 *Let $\mathcal{M}$ be the moduli space of minimal surfaces S of general type with $p_g = q = 2$, $K_S^2 = 6$, and Albanese map of degree 2. Then the following holds:*

(i) *$\mathcal{M}$ is the disjoint union of three connected components, namely*

$$\mathcal{M} = \mathcal{M}_{Ia} \sqcup \mathcal{M}_{Ib} \sqcup \mathcal{M}_{II};$$

(ii) *these are also irreducible components of the moduli space of minimal surfaces of general type*;

(iii) *$\mathcal{M}_{Ia}, \mathcal{M}_{Ib}, \mathcal{M}_{II}$ are generically smooth of dimension $4, 4, 3$, respectively*;

(iv) *the general surface in $\mathcal{M}_{Ia}$ and $\mathcal{M}_{Ib}$ has ample canonical class*; *all surfaces in $\mathcal{M}_{II}$ have ample canonical class.*

A Appendix: The Spaces $\mathcal{A}_{\Delta}[2]$ and $\mathcal{A}_{\Delta}[2,4]$ and their Connected Components

First let us recall some well-known facts about the moduli space of polarized abelian surfaces that can be found, for instance, in [BL04, Chapter 8].

Let us denote by Δ the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and let

$$\mathfrak{H}_2 := \{Z \in M_2(\mathbb{C}) \mid {}^tZ = Z, \operatorname{Im} Z > 0\}$$

be the Siegel upper half-space. We define a polarized abelian surface of type Δ with symplectic basis to be a triplet $(A, H, \{\lambda_1, \lambda_2, \mu_1, \mu_2\})$ with $A = \mathbb{C}^2/\Lambda$ an abelian surface, H a polarization of type Δ on A, and $\{\lambda_1, \lambda_2, \mu_1, \mu_2\}$ a basis of the lattice

Λ, symplectic with respect to H. Then any $Z \in \mathfrak{H}_2$ determines a polarized abelian surface of type Δ with symplectic basis $(A_Z, H_Z, \{\lambda_1, \lambda_2, \mu_1, \mu_2\})$ as follows: just set

$$\lambda_Z := (Z, D)\mathbb{Z}^{2g}, \quad H_Z = (\operatorname{Im} Z)^{-1}$$

and let $\{\lambda_1, \lambda_2, \mu_1, \mu_2\}$ be the columns of the matrix (Z, D). Moreover, there exists a universal family, that is, a holomorphic family $\mathcal{X}_\Delta \to \mathfrak{H}_2$ parameterizing these objects; see [BL04, Section 8.7].

If $Z, Z' \in \mathfrak{H}_2$, the polarized abelian surfaces (A_Z, H_Z) and $(A_{Z'}, H_{Z'})$ are isomorphic if and only if $Z' = M \cdot Z$, where

$$G_\Delta := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \frac{1}{2}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q})$$

is the full paramodular group (see [BL04, Chapter 8], [Mu99]), and the action is defined as follows: for any $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G_\Delta$ and $Z \in \mathfrak{H}_2$, we set

$$M \cdot Z := (\alpha Z + \beta)(\gamma Z + \delta)^{-1}. \tag{A.1}$$

Notice that the following special matrices lie in G_Δ:

$$M_b := \begin{pmatrix} 1 & 0 & b_{11} & 2b_{12} \\ 0 & 1 & 2b_{12} & 2b_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_d := \begin{pmatrix} d_{22} & -d_{21} & 0 & 0 \\ -2d_{12} & d_{11} & 0 & 0 \\ 0 & 0 & d_{11} & 2d_{12} \\ 0 & 0 & d_{21} & d_{22} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} d_{11} & 2d_{12} \\ d_{21} & d_{22} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

$$M_{1,2} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix}.$$

The action (A.1) is properly discontinuous, so the moduli space $\mathcal{A}_\Delta$ of $(1,2)$-polarized abelian surfaces is a quasi-projective variety of dimension 3, obtained as the quotient $\mathfrak{H}_g / G_\Delta$. Then G_Δ is the orbifold fundamental group of $\mathcal{A}_\Delta$, and there is an induced monodromy action of G_Δ on both $A[2]$ and $\widehat{A}[2]$; see [Har79].

Proposition A.1 *The monodromy action of G_Δ on $\widehat{A}[2]$ has precisely three orbits, namely*

$$\{\mathcal{O}_A\}, \quad \mathrm{im}\phi_2^\times, \quad \textit{and} \quad \widehat{A}[2] \setminus \mathrm{im}\phi_2.$$

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Proof Let us start by making a couple of observations. First, the trivial line bundle $\mathcal{O}_A$ is obviously invariant for the monodromy action. Second, for the computation of the monodromy we may assume that $\mathrm{NS}(A)$ is 1-dimensional, generated by the numerical class of $\mathcal{L}$. Then for any $M \in G_\Delta$ the monodromy transformation associated with M sends $\mathcal{L}$ to $\mathcal{L} \otimes \mathcal{Q}$, with $\mathcal{Q} \in \widehat{A}$. Since $K(\mathcal{L}) = K(\mathcal{L} \otimes \mathcal{Q})$, it follows that $\ker \phi_2$ is invariant under the monodromy action; hence $\operatorname{im} \phi_2^\times$ is invariant too. It remains to show that $\widehat{A}[2] \setminus \operatorname{im} \phi_2$ forms a single orbit.

Set $A = \mathbb{C}^2/\Lambda$ and write the period matrix for A as

$$\begin{pmatrix} z_{11} & z_{12} & 1 & 0 \\ z_{21} & z_{22} & 0 & 2 \end{pmatrix},$$

with $Z := \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right) \in \mathfrak{H}_2$. Then the lattice Λ is spanned by the four column vectors

$$\lambda_1 := \begin{pmatrix} z_{11} \\ z_{21} \end{pmatrix}, \quad \lambda_2 := \begin{pmatrix} z_{12} \\ z_{22} \end{pmatrix}, \quad \mu_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2 := \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and the matrix of the alternating form $E \colon \Lambda \times \Lambda \to \mathbb{Z}$ with respect to this basis is $\left(\begin{smallmatrix} 0 & \Delta \\ -\Delta & 0 \end{smallmatrix}\right)$. Therefore,

$$E(\lambda_1, \mu_1) = 1, \quad E(\mu_1, \lambda_1) = -1, \quad E(\lambda_2, \mu_2) = 2, \quad E(\mu_2, \lambda_2) = -2,$$

and all the other values are 0.

The finite subgroup $\widehat{A}[2]$ of $\widehat{A}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, and, by the Appell–Humbert theorem, its elements can be canonically identified with the 16 characters $\Lambda \to \mathbb{C}^*$ with values in $\{\pm 1\}$, see [BL04, Chapter 2]. Since

$$K(L) = \{x \in A \mid E(x, \Lambda) \subseteq \mathbb{Z}\},$$

it follows that $K(L) = \langle \frac{\lambda_2}{2}, \frac{\mu_2}{2} \rangle$ and $\operatorname{im} \phi_2 = \langle \phi_2(\frac{\lambda_1}{2}), \phi_2(\frac{\mu_1}{2}) \rangle$. In other words, $\operatorname{im} \phi_2$ corresponds to the four characters

$$e^{2\pi i(\cdot, x)} \colon \Lambda \longrightarrow \{\pm 1\}$$

with $x = 0, \frac{\lambda_1}{2}, \frac{\mu_1}{2}, \frac{\lambda_1 + \mu_1}{2}$. We will denote a character $\chi \colon \Lambda \to \{\pm 1\}$ by the vector $(\chi(\lambda_1), \chi(\lambda_2), \chi(\mu_1), \chi(\mu_2))$. Therefore, $\operatorname{im} \phi_2$ consists of

$$\chi_0 := (1,1,1,1), \ \chi_1 := (1,1,-1,1), \ \chi_2 := (-1,1,1,1), \ \chi_3 := (-1,1,-1,1),$$

whereas the 12 elements of $\widehat{A}[2] \setminus \operatorname{im} \phi_2$ correspond to

$$\begin{array}{lll} \psi_1 := (1,1,1,-1), & \psi_2 := (1,1,-1,-1), & \psi_3 := (1,-1,1,1), \\ \psi_4 := (1,-1,1,-1), & \psi_5 := (1,-1,-1,1), & \psi_6 := (1,-1,-1,-1), \\ \psi_7 := (-1,1,1,-1), & \psi_8 := (-1,1,-1,-1), & \psi_9 := (-1,-1,1,1), \\ \psi_{10} := (-1,-1,1,-1), & \psi_{11} := (-1,-1,-1,1), & \psi_{12} := (-1,-1,-1,-1). \end{array}$$

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Now take $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G_\Delta$, where

$$\alpha = \begin{pmatrix} a_{11} & a_{12} \\ 2a_{21} & a_{22} \end{pmatrix}, \ \beta = \begin{pmatrix} b_{11} & 2b_{12} \\ 2b_{21} & 2b_{22} \end{pmatrix}, \ \gamma = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & \frac{c_{22}}{2} \end{pmatrix}, \ \delta = \begin{pmatrix} d_{11} & 2d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

and $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{Z}$. By [BL04, proof of Proposition 8.1.3], the monodromy action of M on Λ is given by the matrix $\left(\begin{smallmatrix} \mathbb{I}_2 & 0 \\ 0 & \Delta \end{smallmatrix}\right)^{-1} {}^t M \left(\begin{smallmatrix} \mathbb{I}_2 & 0 \\ 0 & \Delta \end{smallmatrix}\right)$, so the induced action over a character χ is as follows:

$$\begin{aligned}
(M\cdot\chi)(\lambda_1) &= \chi(\lambda_1)^{a_{11}}\chi(\lambda_2)^{a_{12}}\chi(\mu_1)^{b_{11}}\chi(\mu_2)^{b_{12}},\\
(M\cdot\chi)(\lambda_2) &= \chi(\lambda_1)^{2a_{21}}\chi(\lambda_2)^{a_{22}}\chi(\mu_1)^{2b_{21}}\chi(\mu_2)^{b_{22}},\\
(M\cdot\chi)(\mu_1) &= \chi(\lambda_1)^{c_{11}}\chi(\lambda_2)^{c_{12}}\chi(\mu_1)^{d_{11}}\chi(\mu_2)^{d_{12}},\\
(M\cdot\chi)(\mu_2) &= \chi(\lambda_1)^{2c_{21}}\chi(\lambda_2)^{c_{22}}\chi(\mu_1)^{2d_{21}}\chi(\mu_2)^{d_{22}}.
\end{aligned} \tag{A.2}$$

For instance, we have

$$M\cdot\chi_1 = \big((-1)^{b_{11}}, 1, (-1)^{d_{11}}, 1\big), \quad M\cdot\chi_2 = \big((-1)^{a_{11}}, 1, (-1)^{c_{11}}, 1\big),$$

hence the set $\operatorname{im}\phi_2^\times$ is G_Δ-invariant (and by using the matrices of type M_b one checks that it is a single G_Δ-orbit, as expected).

Now we are ready to compute the monodromy action of G_Δ on $\widehat{A}[2] \setminus \operatorname{im}\phi_2$ or, equivalently, on the set $\{\psi_1, \dots, \psi_{12}\}$. By using (A.2), one shows that

- the monodromy permutation associated with a matrix of type M_b is
  - $(\psi_2\psi_8)(\psi_5\psi_{11})(\psi_6\psi_{12})$ if b_{11} is odd and b_{12}, b_{22} are even;
  - $(\psi_1\psi_7)(\psi_2\psi_8)(\psi_4\psi_{10})(\psi_6\psi_{12})$ if b_{12} is odd and b_{11}, b_{22} are even;
  - $(\psi_1\psi_4)(\psi_2\psi_6)(\psi_7\psi_{10})(\psi_8\psi_{12})$ if b_{22} is odd and b_{11}, b_{12} are even;
- the monodromy permutation associated with a matrix of type M_d is
  - $(\psi_3\psi_9)(\psi_4\psi_{10})(\psi_5\psi_{11})(\psi_6\psi_{12})$ if d_{21} is odd and d_{12} is even;
  - $(\psi_1\psi_2)(\psi_4\psi_6)(\psi_7\psi_8)(\psi_{10}\psi_{12})$ if d_{12} is odd and d_{21} is even;
- the monodromy permutation associated with the matrix $M_{1,2}$ is $(\psi_1\psi_3)(\psi_2\psi_9)(\psi_5\psi_7)(\psi_6\psi_{10})(\psi_8\psi_{11})$.

Therefore, the subgroup of the symmetric group $\mathcal{S}_{12}$ corresponding to the monodromy action of G_Δ on $\{\psi_1, \dots, \psi_{12}\}$ contains

$$\begin{aligned}
T := \big\langle &(2\ 8)(5\ 11)(6\ 12),\ (1\ 7)(2\ 8)(4\ 10)(6\ 12),\ (1\ 4)(2\ 6)(7\ 10)(8\ 12),\\
&(3\ 9)(4\ 10)(5\ 11)(6\ 12),\ (1\ 2)(4\ 6)(7\ 8)(10\ 12),\ (1\ 3)(2\ 9)(5\ 7)(6\ 10)(8\ 11)\big\rangle.
\end{aligned}$$

A straightforward computation, for instance by using the Computer Algebra System `GAP4` (see [GAP4]), shows that T is a *transitive* subgroup of $\mathcal{S}_{12}$; therefore, $\{\psi_1, \dots, \psi_{12}\}$ form a single orbit for the G_Δ-action. This completes the proof. ■

Now let $(A = \mathbb{C}^2/\Lambda, H)$ be a polarized abelian surface of type Δ. A symplectic basis $\{\lambda_1, \lambda_2, \mu_1, \mu_2\}$ of Λ for H determines the 15 non-trivial characters $\chi_1, \dots, \chi_3$, $\psi_1, \dots, \psi_{12}$. Therefore we can consider the set of pairs

$$(Z, \rho), \quad Z \in \mathfrak{H}_2, \rho \in \{\chi_1, \dots, \chi_3, \psi_1, \dots, \psi_{12}\} \subset \widehat{A_Z}[2],$$

which can be seen as a subscheme of the relative Picard scheme $\mathrm{Pic}^0(\mathcal{X}_\Delta/\mathfrak{H}_2)$.

The group G_Δ acts on this set of pairs, the action being defined by (A.1) on the first component and by the monodromy on the second one. The corresponding quotient $\mathcal{A}_\Delta[2]$ is a quasi-projective variety, and by construction we have a degree 15 cover $\pi\colon \mathcal{A}_\Delta[2] \to \mathcal{A}_\Delta$. We can identify $\mathcal{A}_\Delta[2]$ with the set of pairs $(A, \mathcal{Q})$, where A is the isomorphism class of a $(1, 2)$-polarized abelian variety and $\mathcal{Q}$ is a non-trivial, 2-torsion line bundle on A; then the map π is just the forgetful map $(A, \mathcal{Q}) \to A$.

Proposition A.2 *$\mathcal{A}_\Delta[2]$ is the disjoint union of two connected components:*

$$\mathcal{A}_\Delta^{(a)}[2] \quad \textit{and} \quad \mathcal{A}_\Delta^{(b)}[2],$$

corresponding to $\mathcal{Q} \notin \mathrm{im}\phi_2^\times$ and $\mathcal{Q} \in \mathrm{im}\phi_2^\times$, respectively. The forgetful maps

$$\pi_1\colon \mathcal{A}_\Delta^{(a)}[2] \longrightarrow \mathcal{A}_\Delta, \quad \pi_2\colon \mathcal{A}_\Delta^{(b)}[2] \longrightarrow \mathcal{A}_\Delta$$

are finite covers of degree 12 *and* 3*. Finally, both $\mathcal{A}_\Delta^{(a)}[2]$ and $\mathcal{A}_\Delta^{(b)}[2]$ are irreducible and generically smooth.*

Proof The first part of the statement follows immediately, since the action of G_Δ on the set of non-trivial characters $\Lambda \to \{\pm 1\}$ has precisely two orbits, namely $\{\chi_1, \dots \chi_3\}$ and $\{\psi_1, \dots, \psi_{12}\}$ (Proposition A.1). Moreover, π_1 and π_2 are étale covers on a smooth Zariski open set $\mathcal{A}_\Delta^0 \subset \mathcal{A}_\Delta$; then they are generically smooth. Finally, by construction $\mathcal{A}_\Delta^{(a)}[2]$ and $\mathcal{A}_\Delta^{(b)}[2]$ are normal varieties, because they only have quotient singularities. Then, since they are connected, they must be also irreducible. ■

Similarly, there is an action of G_Δ on the set of triplets $(Z, \chi, \chi^{1/2})$, where $Z \in \mathfrak{H}_2$, $\chi \in \{\chi_1, \chi_2, \chi_3\} \subset \widehat{A_Z}[2]$ and $\chi^{1/2}\colon \Lambda_Z \to \mathbb{C}^*$ is a character whose square is χ. The corresponding quotient is a quasi-projective variety that can be identified with the space $\mathcal{A}_\Delta[2, 4]$ of triples $(A, \mathcal{Q}, \mathcal{Q}^{1/2})$, where A is the isomorphism class of a $(1, 2)$-polarized abelian surface, $\mathcal{Q} \in \mathrm{im}\, \phi_2$, and $\mathcal{Q}^{1/2}$ is a square root of $\mathcal{Q}$. There is forgetful map $\pi\colon \mathcal{A}_\Delta[2, 4] \to \mathcal{A}_\Delta$, sending $(A, \mathcal{Q}, \mathcal{Q}^{1/2})$ to A; it is a finite cover of degree 48.

Proposition A.3 *$\mathcal{A}_\Delta[2, 4]$ is irreducible and generically smooth.*

Proof It is sufficient to check that the monodromy action of G_Δ is transitive on the set

$$\{(\mathcal{Q}, \mathcal{Q}^{1/2}) \mid \mathcal{Q} \in \operatorname{im} \phi_2^{\times}, (\mathcal{Q}^{1/2})^2 = \mathcal{Q}\}.$$

This is a straightforward computation that can be carried out as the one in the proof of Proposition A.1, so it is left to the reader. ■

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