Can. J. Math., Vol. XXX, No. 4, 1978, pp. 1087–1091

# CURVATURE BOUNDS FOR THE SPECTRUM OF CLOSED EINSTEIN SPACES

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The following is our main result.

(A) THEOREM. *Let* $(M, g)$ *be a closed connected Einstein space,* $n = \dim M \geqq 2$ *(with constant scalar curvature* $R$*). Let* $\kappa_0$ *be the lower bound of the sectional curvature. Then either* $(M, g)$ *is isometrically diffeomorphic to a sphere and the first nonzero eigenvalue* $\lambda_1$ *of the Laplacian fulfils*

$$\lambda_1 = n\,R = n\kappa_0$$

*or each eigenvalue* $\lambda$ *of the Laplacian satisfies the inequality*

$$\lambda > 2n\kappa_0.$$

(B) *Remark.* As for a sphere of constant sectional curvature $\kappa$ the first nonzero eigenvalue is given by $\lambda_1 = n\kappa$, the second by $\lambda_2 = (2n + 1)\kappa$. The second eigenvalue $\lambda_2$ of the Laplacian on closed Einstein spaces of dimension $n \geqq 3$ generally satisfies

$$\lambda_2 > 2n\kappa_0.$$

So on closed Einstein spaces, $n \geqq 3$, there is no eigenvalue $\lambda$ such that

$$n\kappa_0 < \lambda \leqq 2n\kappa_0.$$

Examples in [**1**] (cf. pp. 43 and 47; choose $s = 2$ for $\Psi_{n,s}$) and the value of $\lambda_2$ on spheres lead to the following.

CONJECTURE. *On closed Einstein spaces,* $n \geqq 3$, *there is no eigenvalue* $\lambda$ *such that*

$$n\kappa_0 < \lambda < 2(n + 1)\kappa_0.$$

*Both bounds are the best possible.*

A result related to Theorem A was proved by S. Tanno [**8**]. The author thanks S. Tanno and the referee for valuable hints.

**1. Notations and auxiliary results.** Let $(M, g)$ be a connected Riemannian manifold of class $C^\infty$, $n = \dim M \geqq 2$, denote by $\nabla$ the corresponding covariant differentiation and by $g_{ij}$(respectively $g^{ij}$) the components of the metric tensor $g$ (respectively $g^{-1}$) in local coordinates $(u^i)$; denote by *do* the

Received June 16, 1977 and in revised form, September 19, 1977.

volume element on $M$ and by $R^h{}_{ijk}$ (respectively $R_{ij}$) the components of the curvature tensor (respectively the Ricci tensor) (with the sign of [**3**, I, p. 201]); let $R$ denote the scalar curvature (such that $R = 1$ on the unit sphere). As usual, raising and lowering of indices are defined.

Let $f : M \to \mathbf{R}$ be a $C^\infty$-function, let $f_{ij} := \nabla_j \nabla_i f$ be the Hessian and

(1.1) $\Delta f := g^{ij} f_{ij}$ the Laplacian and

(1.2) $\nabla(f, f) := g^{ij} \nabla_i f \nabla_j f$ the first Beltrami operator.

1.3. LEMMA ([**6**, (7*a*-*b*]). *Let $f$: $M \to \mathbf{R}$ be a $C^\infty$-function. Then $f$ fulfils the equation*

$$\tfrac{1}{2}\Delta(f_{ij}f^{ij}) = 2\sum_{i<j} \kappa_{ij}(\sigma_i - \sigma_j)^2 + f^{ij}\nabla_j\nabla_i(\Delta f) + \nabla_k f_{ij}\nabla^k f^{ij} + f^{ij}f^k[2\nabla_i R_{jk} - \nabla_k R_{ij}],$$

*where $\sigma_1, \ldots, \sigma_n$ are the eigenvalues of the Hessian, $E_1, \ldots, E_n$ are corresponding orthonormal eigenvectors and $\kappa_{ij}$ is the sectional curvature of the plane $\{E_i, E_j\}_{i \neq j}$.*

1.4. LEMMA. *Let $M$ be a closed Einstein manifold,* dim $M = n > 2$. *There exists a nontrivial function $f$: $M \to \mathbf{R}, f \in C^\infty$, which fulfils*

(1.4.1) $n \cdot f_{ij} = \Delta f \cdot g_{ij}$

*if and only if $M$ is isometrically diffeomorphic to a sphere.*

*Proof.* Cf. [**9**].

For the following two lemmata cf. [**5**, Lemma 2.6 and Lemma 2.8].

1.5. LEMMA. *Let* $(M, g)$ *be closed* (*compact without boundary*), dim $M \geqq 2$. *Let $f, h : M \to \mathbf{R}$ be $C^\infty$-functions. Then*

$$\int f_{ij}h^{ij}do - \int \Delta f \Delta h do + \int R^{ij}f_i h_j do = 0.$$

1.6. LEMMA. *If $M$ is closed and $f : M \to \mathbf{R}$ is a $C^\infty$-function, then*

$$\int \sum_{i<j} (\sigma_i - \sigma_j)^2 do = (n-1)\int (\Delta f)^2 do - n\int R^{ij}f_i f_j do.$$

**2. Proof of the main theorem.** Let $(M, g)$ be a closed connected Einstein space, $n = \dim M > 2$. Then $R$ is a constant. We assume $(M, g)$ not to be isometrically diffeomorphic to a sphere. Then each eigenvalue fulfils $\lambda > nR$ [**4**]. We make the following calculations.

(a) From (1.5) we get for $h = \Delta f = -\lambda f$

$$(2.1) \quad \int f^{ij}\nabla_j\nabla_i(\Delta f)do = -\lambda\int (\Delta f)^2 do + (n-1)R\cdot\lambda\int \nabla(f, f)do$$
$$= \lambda[(n-1)\cdot R - \lambda]\int \nabla(f, f)do$$

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by Green's theorem. $\triangle f + \lambda f = 0$ and (1.6) imply

$$(2.2)\quad \int \sum_{i<j} (\sigma_i - \sigma_j)^2 do = (n-1)(\lambda - nR) \int \nabla(f,f) do;$$

therefore from (2.1) we get

$$(2.3)\quad \int f^{ij}\nabla_j \nabla_i(\triangle f) do = \frac{\lambda[(n-1)R - \lambda]}{(n-1)(\lambda - nR)} \int \sum_{i<j} (\sigma_i - \sigma_j)^2 do.$$

(b) As

$$0 \leqq \nabla_k\left(f_{ij} - \frac{1}{n}\triangle f \cdot g_{ij}\right)\nabla^k\left(f^{ij} - \frac{1}{n}\triangle f \cdot g^{ij}\right) = \nabla_k f_{ij}\nabla^k f^{ij} - \frac{1}{n}\nabla(\triangle f, \triangle f)$$

we get from $\triangle f + \lambda f = 0$ and (2.1)

$$\begin{aligned}(2.4)\quad 0 &\leqq \int \nabla_k f_{ij}\nabla^k f^{ij} do - \frac{1}{n}\int \nabla(\triangle f, \triangle f) do \\ &= \int \nabla_k f_{ij}\nabla^k f^{ij} do - \frac{1}{n}\lambda^2 \int \nabla(f,f) do \\ &= \int \nabla_k f_{ij}\nabla^k f^{ij} do - \frac{\lambda^2}{n(n-1)(\lambda - nR)} \int \sum_{i<j} (\sigma_i - \sigma_j)^2 do.\end{aligned}$$

(c) Applying (1.3) to a closed Einstein space, using (2.1)–(2.4), we get

$$0 = \tfrac{1}{2}\int \triangle(f_{ij}f^{ij}) do = \int \left\{\nabla_k f_{ij}\nabla^k f^{ij} - \frac{1}{n}\nabla(\triangle f, \triangle f)\right\} do + \int \sum_{i<j} (\sigma_i - \sigma_j)^2\left[2\kappa_{ij} - \frac{\lambda}{n}\right] do.$$

Assume $\lambda \leqq 2n\kappa_0$; then $\nabla_k f_{ij}\nabla^k f^{ij} - (1/n)\nabla(\triangle f, \triangle f) = 0$. $M$ ~~is~~ irreducible (as the sectional curvature is positive from $0 < \lambda \leqq 2n\kappa_0$), therefore

$$\nabla_k(f_{ij} - (1/n)(\triangle f) \cdot g_{ij}) = 0$$

implies $f_{ij} - (1/n)(\triangle f) g_{ij} = \mu \cdot g_{ij}$, $\mu \in \mathbf{R}$, which again together with $\triangle f + \lambda f = 0$ implies $\mu = 0$. But then $(M, g)$ is isometrically diffeomorphic to a sphere [**9**] which contradicts our assumption at the beginning of the proof. Therefore $\lambda > 2n\kappa_0$.

**3. Two dimensional Riemannian manifolds.** Let $M$ be closed, dim $M = 2$, and let $\kappa$ denote the curvature of $(M, g)$. (1.5) gives

$$(3.1)\quad \lambda^2 \int f^2 do = \int (\triangle f)^2 do = \int f_{ij} f^{ij} do + \int R^{ij} f_i f_j do.$$

Now $2f_{ij}f^{ij} = (\sigma_1 - \sigma_2)^2 + (\triangle f)^2$; therefore

$$(3.2)\quad 2\int f_{ij}f^{ij} do = \int (\sigma_1 - \sigma_2)^2 do + \lambda \int \nabla(f,f) do.$$

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Furthermore

$$\int \nabla(f,f)do = -\int f\triangle f do = \lambda \int f^2 do.$$

The integral formulas above give:

3.3. LEMMA. *Let* $(M, g)$ *be a closed, connected Riemannian manifold,* dim $M = 2$. *Then each eigenvalue* $\lambda$ *of the Laplacian fulfils*

$$\frac{\int (\sigma_1 - \sigma_2)^2 do}{\int \nabla(f,f)do} + 2\min \kappa \leqq \lambda \leqq \frac{\int (\sigma_1 - \sigma_2)^2 do}{\int \nabla(f,f)do} + 2\max \kappa,$$

*where* $f$ *is an eigenfunction corresponding to the eigenvalue* $\lambda$ *and* $\sigma_1$, $\sigma_2$ *are the eigenvalues of the Hessian of* $f$.

3.4. THEOREM. *Let* $(M, g)$ *be a closed, connected two dimensional Riemannian manifold of genus zero. Then the first nonzero eigenvalue* $\lambda_1$ *of the Laplacian fulfils*

(3.4.1) $\quad 2 \min \kappa \leqq \lambda_1 \leqq 2 \max \kappa$

*and the equality on the left or on the right implies* $(M, g)$ *to be isometrically diffeomorphic to a sphere.*

*Proof.* J. Hersch [**2**] proved

$$\lambda_1 = 8\pi\left\{\int do\right\}^{-1}$$

where equality holds if and only if $(M, g)$ is isometrically diffeomorphic to a sphere. Using the theorem of Gauss-Bonnet we get the assertion for the right hand side of (3.4.1). (3.3) implies $2 \min \kappa \leqq \lambda_1$ and equality if and only if $\sigma_1 = \sigma_2 = :\sigma$ on $M$. But this gives $f_{ij} = \sigma g_{ij}$ and $-\lambda f = \triangle f = 2\sigma$; so finally we have

$$f_{ij} + \frac{\lambda}{2} f g_{ij} = 0$$

and $(M, g)$ is isometrically diffeomorphic to a sphere [**4**].

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