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L^p–L^r ESTIMATES FOR THE POISSON SEMIGROUP ON HOMOGENEOUS TREES

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Abstract

Let $\mathfrak{X}$ be a homogeneous tree of degree at least three. In this paper we investigate for which values of p and r the (σ, θ)-Poisson semigroup is $L^p - L^r$-bounded, and we give sharp estimates for the corresponding operator norms.

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Let $\mathfrak{X}$ be a homogeneous tree of degree $q+1$, that is, a connected graph with no loops in which every vertex is adjacent to $q+1$ other vertices. We will write $x \sim y$ if x and y are adjacent. On $\mathfrak{X}$ there is a natural Laplace operator defined by the formula

$$\mathscr{L} f(x) = \frac{1}{q+1} \sum_{x \sim y} [f(x) - f(y)] .$$

If $L^p(\mathfrak{X})$ denotes the Lebesgue space on $\mathfrak{X}$ with respect to counting measure, $\mathscr{L}$ is bounded from L^p to L^r for every $1 \le p \le r \le +\infty$, and is self-adjoint on L^2.

Let $\sigma_p(\mathscr{L})$ be the L^p spectrum of $\mathscr{L}$, and $b_p = \inf \operatorname{Re}\left(\sigma_p(\mathscr{L})\right)$. For θ in $[0, 1]$ and σ in $(0, 1)$, the θ-heat and the (σ, θ)-Poisson semigroups $\left(\mathscr{H}_{t,\theta}\right)_{t>0}$ and $\left(\mathscr{P}^{\sigma}_{\theta,t}\right)_{t>0}$ are spectrally defined, for all t in $(0, +\infty)$ and f in $L^2(\mathfrak{X})$, by

$$\mathscr{H}_{t,\theta} f = \int_{\sigma_2(\mathscr{L})} \exp\left(-t[\lambda - \theta b_2]\right) dP_\lambda f,$$
$$\mathscr{P}^{\sigma}_{\theta,t} f = \int_{\sigma_2(\mathscr{L})} \exp\left(-t[\lambda - \theta b_2]^{\sigma}\right) dP_\lambda f,$$

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where P_λ denotes the spectral resolution of $\mathscr{L}$.

If $1 \leq p, r \leq +\infty$ and an operator $\mathscr{A} : L^2(\mathfrak{X}) \to L^2(\mathfrak{X})$ satisfies a norm inequality of the form

$$\|\mathscr{A} f\|_r \leq C\|f\|_p \quad \forall f \in L^p(\mathfrak{X}) \cap L^2(\mathfrak{X}),$$

then the operator $\mathscr{A}$ is said to be *$L^p - L^r$-bounded.*

The case of the heat semigroup was considered in [CMS2], where a detailed study of the $L^p - L^r$ operator norms of $\mathscr{H}_{t,\theta}$ was carried out.

This paper is concerned with the study of Poisson semigroup. By spectral theory, $\mathscr{P}^\sigma_{\theta,t}$ is clearly bounded on L^2, but while the heat semigroup is $L^p - L^r$-bounded whenever $1 \leq p \leq r \leq +\infty$, this turns out not to be the case for the (σ, θ)-Poisson semigroup. In this paper we investigate the pairs (p, r) for which $\mathscr{P}^\sigma_{\theta,t}$ is $L^p - L^r$-bounded, and we determine exactly the corresponding operator norms.

Our study parallels that of [CGM], where the Poisson semigroup on Riemannian symmetric spaces of non-compact type is considered, and our results correspond exactly to what they obtain in the rank one case. The most notable difference is that in the present setting the infinitesimal generator $(\mathscr{L} - \theta b_2)^\sigma$ of the semigroup is bounded on L^p whenever $\mathscr{P}^\sigma_{\theta,t}$ is, and this allows a considerable simplification of the analysis.

In order to state our main result we need to introduce some notation: For every p in $[1, +\infty]$, we write $\delta(p)$ for $1/p - 1/2$ and p' for the conjugate index $p/(p-1)$. Given a non-negative real number β, we denote by $\mathbf{S}_\beta$ and $\bar{\mathbf{S}}_\beta$ the strips $\{z \in \mathbb{C} : |\mathrm{Im}\,(z)| < \beta\}$ and $\{z \in \mathbb{C} : |\mathrm{Im}\,(z)| \leq \beta\}$, respectively, and let $\gamma : \mathbb{C} \to \mathbb{C}$ be the function defined by

$$\gamma(z) = \frac{q^{1/2}}{q+1}\left(q^{iz} + q^{-iz}\right).$$

Then

$$\gamma(z) = \frac{2q^{1/2}}{q+1}\cos(z \log q) = \gamma(0)\cos(z \log q).$$

It is known (see, for example, [P1, Theorem 3.1], or [FTP, Chapter 3]) that $\sigma_p(\mathscr{L})$ is the image under the map $1 - \gamma$ of the strip $\bar{\mathbf{S}}_{|\delta(p)|}$. A simple computation then shows that $\sigma_p(\mathscr{L})$ is the region of all w such that

$$\left(\frac{1 - \mathrm{Re}\,(w)}{\gamma(0)\cosh(\delta(p)\log q)}\right)^2 + \left(\frac{\mathrm{Im}\,(w)}{\gamma(0)\,(\sinh(\delta(p)\log q))}\right)^2 \leq 1.$$

In particular $\sigma_2(\mathscr{L})$ degenerates to the segment $[1 - \gamma(0), 1 + \gamma(0)]$ on the real axis, and, if we denote by b_p the infimum of $\mathrm{Re}\,(\sigma_p(\mathscr{L}))$, from the expression above we deduce that

$$b_p = 1 - \gamma(0)\cosh(\delta(p)\log q) = 1 - \gamma(i\delta(p)).$$

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For θ in $[0, 1]$ let $p_\theta \in [1, 2]$ be the threshold index for which the L^p spectrum of $\mathscr{L} - \theta b_2$ is contained in the right half plane, and tangent to the vertical axis, or equivalently (cf. [CMS2, Theorem 2.2 (i)]) the threshold index for which $\mathscr{H}_{t,\theta}$ is contractive on $L^p(\mathfrak{X})$. Thus, p_θ is the unique solution of the equation $\gamma(i\delta(p_\theta)) = 1 - b_2\theta$ in the interval $[1, 2]$.

Given two functions $A(t)$, $B(t)$, both defined on a set $\mathbf{D}$, we say that $A(t) \sim B(t)$ in $\mathbf{D}$ if there exist positive constants C, C' such that $CA(t) \leq B(t) \leq C'A(t)$ for all t in $\mathbf{D}$.

With this notation, our main result is the following:

THEOREM 1. *Assume that $\theta \in [0, 1]$, $\sigma \in (0, 1)$, and let $1 \leq p, r \leq +\infty$. Then the following hold:*

(i) *For every $t > 0$, $\mathscr{P}^\sigma_{\theta,t}$ is $L^p - L^r$-bounded if and only if $p \leq r$, $p \leq p_\theta'$, and $r \geq p_\theta$;*

(ii) *If $p_\theta \leq p \leq p_\theta'$, then*

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,p} = \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(i\delta(p))]^\sigma\right) \quad \forall t \in [0, +\infty);$$

(iii) *If $p < r = 2$, or if $2 = p < r$, then*

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,r} \sim \min\left\{1, t^{-3/4}\right\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(0)]^\sigma\right) \quad \forall t \in [0, +\infty);$$

(iv) *If $p < 2 < r$, then*

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,r} \sim \min\left\{1, t^{-3/2}\right\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(0)]^\sigma\right) \quad \forall t \in [0, +\infty);$$

(v) *If $p < r < 2$, and $r > p_\theta$, then*

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,r} \sim \min\left\{1, t^{-1/2r'}\right\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(i\delta(r))]^\sigma\right) \quad \forall t \in [0, +\infty);$$

(vi) *If $p < r = p_\theta$, then*

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,r} \sim \min\left\{1, t^{-1/\sigma r'}\right\} \quad \forall t \in [0, +\infty);$$

(vii) *If $2 < p < r$, and $p < p_\theta'$, then*

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,r} \sim \min\left\{1, t^{-1/2p}\right\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(i\delta(p))]^\sigma\right) \quad \forall t \in [0, +\infty);$$

(viii) *If $p_\theta' = p < r$, then*

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,r} \sim \min\left\{1, t^{-1/\sigma p}\right\} \quad \forall t \in [0, +\infty).$$

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We note that since $\mathscr{P}^\sigma_{\theta,t}$ is subordinated to the θ-heat semigroup, the upper bounds in Theorem 1 could be derived by subordination from the estimates for $|||\mathscr{H}_{t,\theta}|||_{p,r}$ that follows from [CMS2, Theorem 2.2]. The lower bounds however cannot, and we prove both upper and lower bounds using techniques of spherical analysis on $\mathfrak{X}$, which are briefly summarised in Section 1 below.

In Section 2 we obtain sharp estimates for the L^p norm of the convolution kernel associated to $\mathscr{P}^\sigma_{\theta,t}$, which are then used to prove Theorem 1.

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1. Preliminaries on spherical analysis on $\mathfrak{X}$

Let o be a fixed reference point on $\mathfrak{X}$. We say that a function f on $\mathfrak{X}$ is radial if $f(x)$ depends only on the distance $d(o,x)$ also denoted by $|x|$, between x and o, where, for x y in $\mathfrak{X}$, $d(x,y)$ is defined as the number of edges between the vertices x and y. If $E(\mathfrak{X})$ is a function space on $\mathfrak{X}$, we will denote by $E(\mathfrak{X})^\sharp$ the subspace of radial elements in $E(\mathfrak{X})$.

Let G be the group of automorphisms of the tree, that is, of isometries of d, and let K be the isotropy subgroup of o. Then $\mathfrak{X}$ can be naturally identified with the coset space G/K, and functions and radial functions on $\mathfrak{X}$ with K−right-invariant and K−bi-invariant functions on G, respectively. By means of this identification we can define the convolution of two functions on $\mathfrak{X}$ as

$$f_1 * f_2(g\cdot o) = \int_G f_1(h\cdot o) f_2(h^{-1}g\cdot o)dh \quad \forall g \in G,$$

whenever the integral makes sense. We observe that in case f_2 is radial we can write

$$f_1 * f_2(x) = \sum_{n=0}^{+\infty} f_2(x_n) \sum_{d(x,y)=n} f_1(y),$$

where, for every n, x_n is chosen in such a way that $|x_n| = n$. It follows that $\mathscr{L}f = f*(\delta_o - \nu)$, where δ_o is the Dirac measure at o, and ν is the normalised radial measure concentrated on the set $\mathfrak{S}_1 = \{x \in \mathfrak{X} : |x| = 1\}$. Moreover every G−invariant (in the sense that $\mathscr{A}(f\circ g) = (\mathscr{A}f)\circ g$ for every g in G) continuous operator from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$ (weak-star continuous if $r = +\infty$) is given by right convolution with a K−bi-invariant kernel k: $\mathscr{A}f(x) = f*k(x)$.

We recall now the main features of spherical analysis on $\mathfrak{X}$. The spherical functions are defined as the radial eigenfunctions of the Laplace operator $\mathscr{L}$ satisfying the

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normalisation condition $\phi(o) = 1$, and are given by

$$\phi_z(x) = \begin{cases} \left(1 + \frac{q-1}{q+1}|x|\right) q^{-|x|/2} & \forall z \in \tau\mathbb{Z}, \\ \left(1 + \frac{q-1}{q+1}|x|\right) q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2 + \tau\mathbb{Z}, \\ \mathbf{c}(z) q^{(iz-1/2)|x|} + \mathbf{c}(-z) q^{(-iz-1/2)|x|} & \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}, \end{cases}$$

where $\tau = 2\pi/\log q$ and $\mathbf{c}$ is the meromorphic function defined by the rule

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1} \frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}} \quad \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}.$$

The spherical Fourier transform $\widetilde{f}$ of a function $f \in L^1(\mathfrak{X})^\sharp$ is then defined by the formula

$$\widetilde{f}(z) = \sum_{x \in \mathfrak{X}} f(x)\phi_z(x) \quad \forall z \in \bar{\mathbf{S}}_{1/2}.$$

Let $\mathbb{T}$ be the torus $\mathbb{T} = \mathbb{R}/\tau\mathbb{Z}$, usually identified with $[-\tau/2, \tau/2)$, and let μ be the Plancherel measure on $\mathbb{T}$ defined by the formula

$$d\mu(s) = c_G |\mathbf{c}(s)|^{-2}\, ds,$$

where $c_G = q \log q / 4\pi(q+1)$. Then the spherical Fourier transformation extends to an isometry of $L^2(\mathfrak{X})^\sharp$ onto $L^2(\mathbb{T}, d\mu(s))$, and corresponding Plancherel and inversion formulae hold:

$$\|f\|_2 = \left(\int_{-\tau/2}^{\tau/2} |\widetilde{f}(s)|^2 d\mu(s) \right)^{1/2} \quad \forall f \in L^2(\mathfrak{X})^\sharp,$$

and

$$f(x) = \int_{-\tau/2}^{\tau/2} \widetilde{f}(s)\phi_s(x)\, d\mu(s) \qquad \forall x \in \mathfrak{X} \quad \forall f \in L^2(\mathfrak{X})^\sharp.$$

See, for instance, [FTN, Ch. 2].

If f is in $L^p(\mathfrak{X})^\sharp$ with $1 \le p < 2$, then $\widetilde{f}$ extends to a holomorphic function in the strip $\mathbf{S}_{|\delta(p)|}$, with boundary values $\widetilde{f}(\cdot \pm i\delta(p))$ belonging to $L^{p'}(\mathbb{T})$ and the following version of the classical Hausdorff–Young inequality holds

$$\left[\int_{-\tau/2}^{\tau/2} |\widetilde{f}(s \pm i\delta(p)|^{p'}\, ds \right]^{1/p'} \le C\|f\|_p. \tag{1}$$

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On the other hand, if $1 \leq p < 2$, and a radial function f is such that $f\phi_{i\delta(p)}$ is in $L^1(\mathfrak{X})^\sharp$ then f is in L^p and

$$\|f\|_p \leq C\left\|f\phi_{i\delta(p)}\right\|_1^{2\delta(p)}\left(\int_{-\tau/2}^{\tau/2}|f(s+i\delta(p))|^2\,ds\right)^{1/2-\delta(p)}. \tag{2}$$

Referring to [CMS1, Theorems 1.1, 1.2] for the proofs, we remark that the assumption that $f\phi_{i\delta(p)}$ is in L^1 is equivalent to the requirement that f is in the Lorentz space $L^{p,1}(\mathfrak{X})^\sharp$. Also, $\left\|f\phi_{i\delta(p)}\right\|_1 = |f|^\sim(i\delta(p))$.

We conclude this section by recalling that if the operator $f \mapsto f * k$ of right convolution with the radial kernel k is bounded from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$, then, by the extension to $\mathfrak{X}$ of a well-known theorem of Hörmander ([Hö, Theorem 1.1]), $p \leq r$, and k is in $L^r(\mathfrak{X})^\sharp$ (because the Dirac measure at o is in every L^p, and $\delta_o * k = k$). In particular, if $r < 2$, $\widetilde{k}$ extends to a holomorphic function in $\mathbf{S}_{i\delta(r)}$, and, in the special case where $p = r$, by the Clerc–Stein ([CS]) condition $\widetilde{k}$ is actually holomorphic and bounded in $\mathbf{S}_{i\delta(p)}$.

2. Estimates for the (σ, θ)-Poisson semigroup

Denote by $p^\sigma_{\theta,t}$ and l^σ_θ respectively the convolution kernels of $\mathscr{P}^\sigma_{\theta,t}$ and of its infinitesimal generator $(\mathscr{L}-\theta b_2)^\sigma$, so that $(\mathscr{L}-\theta b_2)^\sigma f = f*l^\sigma_\theta$ and $\mathscr{P}^\sigma_{\theta,t} f = f*p^\sigma_{\theta,t}$ for every f in $L^2(\mathfrak{X})$. Since $(\delta_o-\nu)\widetilde{\ } = 1-\gamma$, and $1-\theta b_2 = \gamma(1/p_\theta)$ by the definition of p_θ, it follows that

$$\widetilde{l}^\sigma_\theta(z) = [1-\gamma(z)-\theta b_2]^\sigma = [\gamma(1/p_\theta)-\gamma(z)]^\sigma$$

and

$$\widetilde{p}^\sigma_{\theta,t}(z) = e^{-t[\gamma(1/p_\theta)-\gamma(z)]^\sigma},$$

the powers being defined using the principal branch of the logarithm. Note that $\widetilde{l}^\sigma_\theta$ is holomorphic in the strip $\mathbf{S}_{\delta(p)t}$, continuous on the closure, and its restriction to the boundary of the strip is the Euclidean Fourier transform of a function in $L^1(\mathbb{Z})$. Indeed the function $s \mapsto [\gamma(i\delta(p_\theta)) - \gamma(s+ip_\theta)]^\sigma$ is continuous on $\mathbb{T}$ and its derivative is in $L^{1+\epsilon}(\mathbb{T})$ for every $\epsilon < 1-\sigma$, so that the claim follows from [Z, vol. 1 p. 241]. By [CMS1, Theorem 1.1], l^σ_θ in the Lorentz space $L^{p_\theta,1}(\mathfrak{X})^\sharp$, and then a theorem by Pytlik ([P2]), and duality, imply that $(\mathscr{L}-\theta b_2)^\sigma$ is bounded on L^p for every $p_\theta \leq p \leq p_\theta{}'$. Therefore

$$\mathscr{P}^\sigma_{\theta,t} = \sum_0^{+\infty}\frac{(-t)^k}{k!}\,[(\mathscr{L}-\theta b_2)^\sigma]^k$$

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where the series converges in the L^p uniform operator norm for every p in $[p_\theta, p_\theta']$, and defines a uniformly bounded semigroup of operators.

Now we come to $p^\sigma_{\theta,t}$. Arguing as above one shows that $p^\sigma_{\theta,t}$ belongs to $L^{p_\theta,1}(\mathfrak{X})^\sharp$. On the other hand $p^\sigma_{\theta,t}$ is not in $L^p(\mathfrak{X})$ for every $p < p_\theta$ because $\widetilde{p}^\sigma_{\theta,t}$ does not extend holomorphically to any strip $\mathbf{S}_\beta$ strictly containing $\mathbf{S}_{\delta(p_\theta)}$.

By spherical Fourier inversion we can write

$$p^\sigma_{\theta,t}(x) = c_G \int_{-\tau/2}^{\tau/2} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(s)]^\sigma\right) \phi_s(x) \, |\mathbf{c}(s)|^{-2} \, ds.$$

We will also need the fact that $p^\sigma_{\theta,t}$ is non-negative on $\mathfrak{X}$, for every $t > 0$. This is most easily seen using semigroup subordination. Indeed $\mathscr{P}^\sigma_{\theta,t}$ can be expressed in terms of the θ-heat semigroup via the formula

$$\mathscr{P}^\sigma_{\theta,t} = \int_0^\infty f_{\sigma,t}(s) \mathscr{H}_{s,\theta} \, ds, \tag{3}$$

where $f_{t,\sigma}$ is the function defined by

$$f_{\sigma,t}(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zs - tz^\sigma} \, dz \quad (a > 0,\ t > 0,\ s \geq 0,\ 0 < \sigma < 1)$$

(see [Y, Chapter IX.11]), and therefore

$$p^\sigma_{\theta,t}(x) = \int_0^{+\infty} f_{\sigma,t}(s) e^{s\theta b_2} h_s(x) \, ds,$$

where $h_s(x)$ is the heat kernel of $\mathfrak{X}$. The claim now follows from the positivity of h_s and of $f_{\sigma,t}$.

As remarked in the introduction, one could use formula (3) above and the results of [CMS2, Theorem 2.2] to derive upper bounds for the operator norms $|||\mathscr{P}^\sigma_{\theta,t}|||_{p,r}$. Techniques of spherical Fourier analysis however are computationally easier, and allow us to obtain lower bounds that cannot be obtained by semigroup subordination.

As a first step towards the proof of Theorem 1, we begin by proving sharp L^p norm estimates for $p^\sigma_{\theta,t}$. To obtain our estimates we will make extensive use of the following version of Laplace method, whose proof can be found in [E, Section 2.4].

LEMMA 2. *Let g and h be functions defined on the interval (α, β) and assume that for every sufficiently large positive t the integral*

$$I(t) = \int_\alpha^\beta g(s) e^{th(s)} \, ds$$

exists. Let h be real valued, continuous at $s = \alpha$, continuously differentiable with $h'(s) < 0$ for $\alpha < s \leq \alpha + \eta$, $\eta > 0$, and such that $h(s) \leq h(\alpha) - \epsilon$, $\epsilon > 0$, for

$\alpha+\eta \leq s \leq \beta$. *Assume furthermore that, as* $s \to \alpha$, $h'(s) = -a(s-\alpha)^{\rho-1}(1+o(1))$, *and* $g(s) = b(s-\alpha)^{\lambda-1}(1+o(1))$, $\lambda, \rho > 0$. *Then, as* $t \to +\infty$,

$$I(t) = (b/\rho)\Gamma(\lambda/\rho)(\rho/\alpha t)^{\lambda/\rho} e^{th(\alpha)}(1+o(1)).$$

For future use we also note that a straightforward computation shows that

$$|\mathbf{c}(s)|^{-2} = \frac{4(q+1)^2\sin^2(s\log q)}{(q+1)^2\sin^2(s\log q)+(q-1)^2\cos^2(s\log q)}$$

and since

$$(q-1)^2 \leq (q+1)^2\sin^2(s\log q)+(q-1)^2\cos^2(s\log q) \leq (q+1)^2 \quad \forall s \in \mathbb{T},$$

we have

$$|\mathbf{c}(s)|^{-2} \sim \sin^2(s\log q) \quad \forall s \in \mathbb{T}. \tag{4}$$

LEMMA 3. *For every* t *in* $[0,+\infty)$ *the following norm estimates hold:*

(i) *If* $p=2$, *then*

$$\left\|p^{\sigma}_{\theta,t}\right\|_p \sim \min\left\{1, t^{-3/4}\right\}\exp\left(-t[\gamma(i\delta(p_\theta))-\gamma(0)]^{\sigma}\right);$$

(ii) *If* $p=+\infty$, *then*

$$\left\|p^{\sigma}_{\theta,t}\right\|_p \sim \min\left\{1, t^{-3/2}\right\}\exp\left(-t[\gamma(i\delta(p_\theta))-\gamma(0)]^{\sigma}\right);$$

(iii) *If* $p_\theta < p < 2$, *then*

$$\left\|p^{\sigma}_{\theta,t}\right\|_p \sim \min\left\{1, t^{-1/2p'}\right\}\exp\left(-t[\gamma(i\delta(p_\theta))-\gamma(i\delta(p))]^{\sigma}\right);$$

(iv) *If* $p = p_\theta < 2$, *then* $\left\|p^{\sigma}_{\theta,t}\right\|_p \sim \min\left\{1, t^{-1/\sigma p_\theta'}\right\}$.

PROOF. The proof follows the lines of that of [CGM, Lemma 3] (cf. also the proof of [CMS2, Lemma 2.1]). By the Plancherel formula

$$\left\|p^{\sigma}_{\theta,t}\right\|_2^2 = c_G\int_{-\tau/2}^{\tau/2} e^{-2t[\gamma(i\delta(p_\theta))-\gamma(s)]^{\sigma}}\,|\mathbf{c}(s)|^{-2}\,ds.$$

Since the right-hand side is decreasing in t and bounded above by 1 for all t, we immediately conclude that

$$\left\|p^{\sigma}_{\theta,1}\right\|_2 \leq \left\|p^{\sigma}_{\theta,t}\right\|_2 \leq \left\|p^{\sigma}_{\theta,0}\right\|_2 = 1$$

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when $0 \le t \le 1$. On the other hand, using $\gamma(s) = \gamma(0)\cos(s\log q)$ and (4) above, we can write

$$\left\| p^{\sigma}_{\theta,t} \right\|_2^2 \sim \int_{-\tau/2}^{\tau/2} e^{-2t[\gamma(i\delta(p_\theta))-\gamma(0)\cos(s\log q)]^{\sigma}} \sin^2(s\log q)\, ds.$$

The function $-2[\gamma(i\delta(p_\theta)) - \gamma(1/2)\cos(s\log q)]^{\sigma}$ attains its maximum at $s = 0$, where its derivative vanishes of order one, while $\sin^2(s\log q)$ vanishes there of order two, so that an application of Lemma 2 yields

$$\left\| p^{\sigma}_{\theta,t} \right\|_2^2 = Ct^{-3/2} \exp\left(-2t[\gamma(i\delta(p_\theta)) - \gamma(0)]\right)(1 + o(1)), \quad \text{as } t \to +\infty,$$

and (i) follows.

To prove (ii) we note that using the inequality $|\phi_z(x)| \le \phi_z(o) = 1$ in the inversion formula shows that $\left|p^{\sigma}_{\theta,t}(x)\right| \le p^{\sigma}_{\theta,t}(o)$ for every x in $\mathfrak{X}$, and that $p^{\sigma}_{\theta,t}(o)$ is decreasing in t and bounded above by one. Therefore

$$\left\| p^{\sigma}_{\theta,t} \right\|_\infty = p^{\sigma}_{\theta,t}(o) \quad \forall t \in [0, +\infty),$$

and

$$1 = p^{\sigma}_{\theta,0}(o) \le \left\| p^{\sigma}_{\theta,t} \right\|_\infty \le p^{\sigma}_{\theta,1}(o),$$

for t in $[0, 1]$. Using (4) in the inversion formula we obtain

$$p^{\sigma}_{\theta,t}(o) \sim \int_{-\tau/2}^{\tau/2} e^{-t[\gamma(i\delta(p_\theta))-\gamma(0)\cos(s\log q)]^{\sigma}} \sin^2(s\log q)\, ds.$$

Proceeding as in (i) to estimate the integral yields (ii).

We next prove (iii). By formula (1) above we have

$$\begin{aligned}
\left\| p^{\sigma}_{\theta,t} \right\|_p &\ge C\left(\int_{-\tau/2}^{\tau/2} \left|\widetilde{p}^{\sigma}_{\theta,t}(s + i\delta(p))\right|^{p'} ds\right)^{1/p'} \\
&= C\left(\int_{-\tau/2}^{\tau/2} e^{-tp'\,\mathrm{Re}([\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p))]^{\sigma})}\, ds\right)^{1/p'} \\
&\ge C\left(\int_{-\tau/2}^{\tau/2} e^{-tp'|\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p))|^{\sigma}}\, ds\right)^{1/p'}.
\end{aligned}$$

The integral on the right-hand side defines a decreasing function of t, so that

$$\left\| p^{\sigma}_{\theta,t} \right\|_p \ge C\left(\int_{-\tau/2}^{\tau/2} e^{-p'|\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p))|^{\sigma}}\, ds\right)^{1/p'} > 0 \quad \forall t \in [0, 1].$$

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Moreover, using the fact that

$$\text{(5)}\qquad \gamma(s+i\delta(p)) = \gamma(i\delta(p))\cos(s\log q) - i\sinh(\delta(p)\log q)\sin(s\log q),$$

it is easy to verify that the function $s \mapsto -tp'|\gamma(1/p_\theta) - \gamma(1/p+is)|^\sigma$ attains its maximum at $s=0$, and there its derivative vanishes of order one. Therefore by Lemma 2,

$$\int_{-\tau/2}^{\tau/2} e^{-tp'|[\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p))]|^\sigma}\,ds = Ct^{-1/2}e^{-tp'[\gamma(i\delta(p_\theta))-\gamma(i\delta(p))]^\sigma}(1+o(1)),$$

as $t \to +\infty$, and from this we may conclude that

$$\left\|p^\sigma_{\theta,t}\right\|_p \geq Ct^{-1/2p'}e^{-t[\gamma(i\delta(p_\theta))-\gamma(i\delta(p))]^\sigma} \quad \forall t \in [1,+\infty).$$

To prove a comparable upper bound, we use (2) above, and the positivity of $p^\sigma_{\theta,t}$ to write

$$\left\|p^\sigma_{\theta,t}\right\|_p \leq C\widetilde{p}^\sigma_{\theta,t}(i\delta(p))^{2\delta(p)}\left(\int_{-\tau/2}^{\tau/2} e^{-2t\,\mathrm{Re}([\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p))]^\sigma)}\,ds\right)^{1/2-\delta(p)}.$$

The right-hand side decreasing in t and finite for $t=0$. Therefore

$$\left\|p^\sigma_{\theta,t}\right\|_p \leq C \quad \forall t \in [0,+\infty).$$

Since the inequality $\mathrm{Re}\,(z^\sigma) \geq (\mathrm{Re}\,(z))^\sigma$ holds for $\mathrm{Re}\,(z) \geq 0$, and $0<\sigma<1$, using (5) we can estimate

$$\int_{-\tau/2}^{\tau/2} e^{-2t\,\mathrm{Re}([\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p))]^\sigma)}\,ds \leq \int_{-\tau/2}^{\tau/2} e^{-2t[\gamma(i\delta(p_\theta))-\gamma(i\delta(p))\cos(s\log q)]^\sigma}\,ds,$$

and, again by Lemma 2, the integral on the right-hand side is equal to

$$Ct^{-1/2}e^{-2t[\gamma(i\delta(p_\theta))-\gamma(i\delta(p))]^\sigma}(1+o(1)) \quad \text{as } t \to +\infty.$$

Therefore

$$\left\|p^\sigma_{\theta,t}\right\|_p \leq Ct^{-1/2p'}e^{-t[\gamma(i\delta(p_\theta))-\gamma(i\delta(p))]^\sigma} \quad \forall t \in [1,+\infty),$$

as required to finish the proof of (iii).

The proof of (iv) proceeds in much the same way. To obtain a lower bound we write

$$\begin{aligned}\left\|p^\sigma_{\theta,t}\right\|_{p_\theta} &\geq C\left(\int_{-\tau/2}^{\tau/2} \left|\widetilde{p}^\sigma_{\theta,t}(s+i\delta(p_\theta))\right|^{p_\theta'}ds\right)^{1/p_\theta'} \\ &\geq C\left(\int_{-\tau/2}^{\tau/2} e^{-tp_\theta'|\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p_\theta))|^\sigma}\,ds\right)^{1/p_\theta'}.\end{aligned}$$

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The right-hand side is decreasing in t and therefore bounded away from zero for t in $[0, 1]$. Moreover the maximum of the function $s \mapsto -|\gamma(i\delta(p_\theta)) - \gamma(s + i\delta(p_\theta))|^\sigma$ is attained at $s = 0$ and, as $s \to 0$, its derivative is asymptotic to $-Cs^{\sigma-1}$. By Lemma 2 the integral is equal to

$$Ct^{-1/\sigma}(1 + o(1)) \quad \text{as } t \to +\infty,$$

and therefore

$$\left\| p^\sigma_{\theta,t} \right\|_{p_\theta} \geq Ct^{-1/\sigma p_\theta'} \quad \forall t \in [1, +\infty).$$

As for the upper bound, we estimate

$$\left\| p^\sigma_{\theta,t} \right\|_{p_\theta} \leq C \widetilde{p}^\sigma_{\theta,t}(i\delta(p_\theta))^{2\delta(p_\theta)} \left(\int_{-\tau/2}^{\tau/2} e^{-2t \operatorname{Re}([\gamma(i\delta(p_\theta)) - \gamma(s+i\delta(p_\theta))]^\sigma)} \, ds \right)^{1/2-\delta(p_\theta)}.$$

As in (iii), the right-hand side is bounded above by a constant for every $t \geq 0$ and using (5) it is not hard to check that the function $s \mapsto -2\operatorname{Re}([\gamma(i\delta(p_\theta)) - \gamma(s + i\delta(p_\theta))]^\sigma)$ attains its absolute maximum in $[-\tau/2, \tau/2]$ at $s = 0$, where it vanishes of order σ. Thus Lemma 2 shows that

$$\int_{-\tau/2}^{\tau/2} e^{-2t \operatorname{Re}([\gamma(i\delta(p_\theta)) - \gamma(s+i\delta(p_\theta))]^\sigma)} \, ds = Ct^{-1/\sigma}(1 + o(1)) \quad \text{as } t \to +\infty,$$

and therefore

$$\left\| p^\sigma_{\theta,t} \right\|_{p_\theta} \leq Ct^{-(1/2-\delta(p_\theta))/\sigma} = Ct^{-1/\sigma p_\theta'} \quad \forall t \in [1, +\infty).$$

PROOF (THEOREM 1). We have already noted that $\mathscr{P}^\sigma_{\theta,t}$ is bounded on $L^p(\mathfrak{X})$ for every p in $[p_\theta, p_\theta']$. Duality and the inclusion properties of the $L^p(\mathfrak{X})$ spaces immediately imply that $\mathscr{P}^\sigma_{\theta,t}$ is $L^p - L^r$-bounded for p and r in the range specified in the statement. On the other hand, if $\mathscr{P}^\sigma_{\theta,t}$ is $L^p - L^r$-bounded then $p \leq r$ by what was remarked at end of Section 1, and then, by duality, $\mathscr{P}^\sigma_{\theta,t}$ is also bounded from $L^{r'}(\mathfrak{X})$ to $L^{p'}(\mathfrak{X})$. Thus $p^\sigma_{\theta,t}$ is in $L^s(\mathfrak{X})$ with $s = \min\{r, p'\}$, and consequently $\widetilde{p}^\sigma_{\theta,t}$ extends to a holomorphic function in $\mathbf{S}_{\delta(s)}$. This forces $s \geq p_\theta$, that is, $r \geq p_\theta$, and $p \leq p_\theta'$, as required to complete the proof of (i).

To prove (ii) we observe that, since $p^\sigma_{\theta,t}$ is non-negative, when $p_\theta \leq p \leq 2$, by the Herz principle de majoration [H] (cf. [CMS1, Proposition 2.3]) we have

$$|||\mathscr{P}^\sigma_{\theta,t}|||_{p,p} = \widetilde{p}^\sigma_{\theta,t}(i\delta(p)) = e^{-t[\gamma(i\delta(p_\theta)) - \gamma(i\delta(p))]^\sigma}$$

for all t in $[0, +\infty)$. Duality, and the identity $\gamma(i\delta(p)) = \gamma(i\delta(p'))$, imply that this holds for every p in $[p_\theta, p_\theta']$.

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We consider next (iii). It suffices to examine the case $p < r = 2$, for then the case $2 = p < r$ follows by duality. By the radial form of the Kunze–Stein phenomenon (cf. [N], and [CMS1, Section 2]), and Lemma 3 (i) we have

$$|||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,2} \leq C \left\| p^{\sigma}_{\theta,t} \right\|_2 \leq C \min\left\{1, t^{-3/4}\right\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(0)]^{\sigma}\right)$$

for all t in $[0, +\infty)$. On the other hand,

$$\left\| p^{\sigma}_{\theta,t} \right\|_2 = \left\| \delta_o * p^{\sigma}_{\theta,t} \right\|_2 \leq |||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,2} \|\delta_o\|_p = |||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,2},$$

and, again by Lemma 3 (i),

$$|||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,2} \geq C \min\left\{1, t^{-3/4}\right\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(0)]^{\sigma}\right),$$

for all t in $[0, +\infty)$, and (iii) follows.

Assume now that $p < 2 < r$. The semigroup property and (iii) give

$$\begin{aligned} |||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,r} &\leq |||\mathscr{P}^{\sigma}_{\theta,t/2}|||_{p,2} |||\mathscr{P}^{\sigma}_{\theta,t/2}|||_{2,r} \\ &\leq C \left(\min\left\{1, (t/2)^{-3/4}\right\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(0)]^{\sigma}/2\right)\right)^2, \end{aligned}$$

for every t in $[0, +\infty)$. By the inclusion properties of the $L^p(\mathfrak{X})$ spaces, and Lemma 3 (ii) we also have

$$|||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,r} \geq |||\mathscr{P}^{\sigma}_{\theta,t}|||_{1,\infty} = \left\| p^{\sigma}_{\theta,t} \right\|_\infty \geq C \min\{1, t^{-3/2}\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(0)]^{\sigma}\right),$$

and (iv) is proved.

Now let $p < r < 2$ and $r > p_\theta$. We use again the radial Kunze–Stein phenomenon and Lemma 3 (iii) to get

$$|||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,r} \leq C \left\| p^{\sigma}_{\theta,t} \right\|_r \leq C \min\{1, t^{-1/2r'}\} \exp\left(-t[\gamma(i\delta(p_\theta)) - \gamma(i\delta(r))]^{\sigma}\right)$$

for all t in $[0, +\infty)$, and since the reverse inequality is a consequence of Lemma 3 (iii) and of $|||\mathscr{P}^{\sigma}_{\theta,t}|||_{p,r} \geq |||\mathscr{P}^{\sigma}_{\theta,t}|||_{1,r} = \left\| p^{\sigma}_{\theta,t} \right\|_r$, (v) follows.

The case $p < r = p_\theta$ is treated as in (vi), using Lemma 3 (iv) instead of Lemma 3 (iii). Since (vii) and (viii) follow from duality respectively from (v) and (vi), the proof of Theorem 1 is complete.

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