BULL. AUSTRAL. MATH. SOC. 41A28, 41A30
VOL. 43 (1991) [295–302]

APPROXIMATION IN FUNCTION MODULES

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We investigate the existence of best approximation of an element α in a function module from a subfunction module whose fibers satisfy the intersection property of balls. Also we investigate the lower semicontinuity of the metric projection associated with such a subfunction module.

1. INTRODUCTION

Let E be a normed linear space and G a closed subspace of E. The set

$$\mathsf{P}_G(x) = \{g_0 \in G : \|x - g_0\| = \inf \|x - g\|,\ g \in G\} \tag{1.1}$$

is called the set of all best approximations to x from G. This defines a set valued mapping P_G which is called the metric projection onto G. A mapping $s\colon E \to G$ is called a selection for P_G if $s(x) \in \mathsf{P}_G(x)$ for all $x \in E$. A subspace G of a normed linear space E is called proximinal (respectively Chebychev) if $\mathsf{P}_G(x)$ contains at least (exactly) one element for all $x \in E$.

The set valued mapping P_G is called lower semicontinuous (l.s.c.) if the set

$$\{x \in E : \mathsf{P}_G(x) \cap U \neq \emptyset\} \tag{1.2}$$

is open for each open subset U of G or, what is equivalent, for each sequence $\{x_n\}$ in E converging to x in E and for each g in $\mathsf{P}_G(x)$, there is a sequence $\{g_n\}$ in G such that for each $n \in \mathbb{N}$, $g_n \in \mathsf{P}_G(x_n)$ and $g_n \to g$, see [**2**, p.365].

DEFINITION 1.1: A subspace G of the Banach space E is said to have the two-ball property for open balls if for any pair $B(x_1, r_1)$ and $B(x_2, r_2)$ of open balls such that $B(x_1, r_1) \cap B(x_2, r_2) \neq \emptyset$ and $B(x_i, r_i) \cap G \neq \emptyset$ for $i = 1, 2$, the intersection $(G \cap B(x_1, r_1) \cap B(x_2, r_2))$ is nonvoid; see [**1**, Definition 2.16].

Let T be a nonvoid compact Hausdorff space and (E_t) a family of Banach spaces over T. Consider the Banach space $\prod\limits_{t\in T}^{\infty} E_t = \{\alpha \in \prod\limits_{t\in T} E_t : \|\alpha\|_\infty = \sup\limits_{t\in T} \|\alpha(t)\|_t < \infty\}$ (where $\|.\|_t$ is the norm on the Banach space E_t). Closed subspaces of $\prod\limits_{t\in T}^{\infty} E_t$ will be called Banach spaces of a vector valued function on T.

Received 30 April 1990

DEFINITION 1.2: A function module is a triple $\left(T, (E_t)_{t\in T}, E_\infty\right)$, where T is a nonvoid compact Hausdorff space (called base space), $(E_t)_{t\in T}$ a family of Banach spaces (the component spaces) and E_∞ a closed subspace of the space $\prod\limits_{t\in T}^{\infty} E_t$ such that:

(1) E_∞ is a $C(T)$-module (where $C(T)$ is the Banach algebra of all continuous scalar valued functions on T), $(f, \alpha)(t) = f(t)\alpha(t)$, $f \in C(T)$, $\alpha \in E_\infty$.

(2) For every $\alpha \in E_\infty$, the map $t \mapsto \|\alpha(t)\|_t$ is upper semicontinuous.

(3) $E_t = \{\alpha(t) : \alpha \in E_\infty\}$ for every $t \in T$.

(4) $\{t : t \in T, E_t \neq \{0\}\} = T$.

REMARK. Instead of "$\left(T, (E_t)_{t\in T}, E_\infty\right)$ is a function module" we will often say that E_∞ is a function module in $\prod\limits_{t\in T}^{\infty} E_t$ or (if T and $(E_t)_{t\in T}$ are understood) that E_∞ itself is a function module, see [1, Definition 4.1].

DEFINITION 1.3: Let G_∞ be a sub-function module in the function module E_∞ and α be an element of E_∞. The element γ_0 in G_∞ is called:

(1) global best approximation of α from G_∞ if

$$\|\alpha - \gamma_0\|_\infty = \inf\{\|\alpha - \gamma\|_\infty : \gamma \in G_\infty\};$$

(2) local best approximation if for each $t \in T$

$$\|\alpha(t) - \gamma_0(t)\|_t = \inf\{\|\alpha(t) - g\|_t : g \in G_t\};$$

that is, $\gamma(t)$ in $\mathsf{P}_{G_t}(\alpha(t))$ for each $t \in T$.

Note that local best approximations are always global but the converse is not always true.

2. RESULTS

THEOREM 2.1. *Let G be a closed subspace of the Banach space E. If G satisfies the two-ball property for open balls, then G is proximinal and P_G is lower semicontinuous.*

PROOF: Let x be an arbitrary element in $E \setminus G$ and define $r = d(x, G) = \inf\{\|x - g\| : g \in G\}$. For any positive real number ε and any $g \in G$ such that $\|x - g\| < r + \varepsilon$ we have $B(x, r + \varepsilon/2) \cap B(g, \varepsilon/2) \neq \emptyset$, $B(x, r + \varepsilon/2) \cap G \neq \emptyset$ and $B(g, \varepsilon/2) \cap G \neq \emptyset$. Therefore $B(x, r + \varepsilon/2) \cap B(g, \varepsilon/2) \cap G \neq \emptyset$, and then there exists an element g_0 in G satisfying the following:

$$\|x - g_0\| \leqslant r + \frac{\varepsilon}{2} \text{ and } \|g - g_0\| \leqslant \frac{\varepsilon}{2}. \tag{2.1}$$

By applying (2.1) inductively, we can construct a sequence $\{g_n\}$ in G satisfying the following:

$$\|x - g_n\| \leqslant r + 2^{-n} \text{ and } \|g_n - g_{n+1}\| \leqslant 2^{-n}. \tag{2.2}$$

The sequence $\{g_n\}$ is Cauchy, and hence it has a limit g in G. Moreover, we have $\|x - g\| = r$; that is, $g \in \mathrm{P}_G(x)$.

For the lower semi-continuity of P_G, let U be an arbitrary open subset of G and $V = \{x \in E : \mathrm{P}_G(x) \cap U \neq \emptyset\}$. We may assume without loss of generality that $V \neq \emptyset$ and show that V^c (the complement of V in E) is closed. For, let $\{x_n\}$ be a sequence in V^c converging to x in V, $g \in \mathrm{P}_G(x) \cap U$ and $\varepsilon > 0$ such that $B(g, \varepsilon) \subseteq U$. Define $r_n = d(x_n, G)$ and $r = d(x, G)$. Let N be the positive integer such that $\|x_n - x\| < \varepsilon/2$ and $\|r_n - r\| < \varepsilon/2$ for each $n \geqslant N$. Now, for each $n \geqslant N$, the two balls $B(x_n, r_n)$ and $B(g, \varepsilon)$ satisfy the following:

$$B(x_n, r_n) \cap B(g, \varepsilon) \neq \emptyset$$

(since $\|x_n - g\| \leqslant \|x_n - x\| + \|x - g\| < \varepsilon/2 + r < \varepsilon/2 + (r_n + \varepsilon/2) = r_n + \varepsilon$,) $B(x_n, r_n) \cap G \neq \emptyset$, and $B(g, \varepsilon) \cap G \neq \emptyset$. Hence $B(x_n, r_n) \cap B(g, \varepsilon) \cap G \neq \emptyset$, or, what is equivalent, there is a $g_n \in \mathrm{P}_G(x_n)$ such that $g_n \in B(g, \varepsilon) \subseteq U$. This contradicts the assumption. Thus V must be open. ∎

Theorem 2.2. *Let E_∞ be a function module in $\prod\limits_{t\in T}^{\infty} E_t$, such that for each α in E_∞ the mapping $t \mapsto \|\alpha(t)\|_t$ is continuous. If $\alpha_1, \ldots, \alpha_n$ are elements of E_∞ such that for each t in T, span$\langle\alpha_1(t), \ldots, \alpha_n(t)\rangle$ has dimension n and satisfies the two-ball property for open balls, then $\underset{C(T)}{\text{span}}\langle\alpha_1, \ldots, \alpha_n\rangle$ contains a local best approximation for each $\alpha \in E_\infty$.*

In order to prove the above theorem, we need the following lemma, which perhaps is interesting in itself.

Lemma 2.3. *With the assumption of Theorem 2.2, for each α in E_∞ the function $\rho: T \to \mathrm{R}$ defined by $\rho(t) = d(\alpha(t), G_t)$ is continuous.*

Proof: Let $H: T \times \ell_1^n \to \mathrm{R}$ be the mapping defined by $H(t, a) = \left\|\alpha(t) - \sum\limits_{i=1}^{n} a_i\alpha_i(t)\right\|_t$ (where $a_i = h_i(a) and \{h_i\}$ is the sequence of coefficient functionals associated with the unit vector basis of ℓ_1^n). Let (t_0, r) be a fixed point in $T \times \ell_1^n$.

https://doi.org/10.1017/S0004972700029087 Published online by Cambridge University Press

Then

$$\begin{aligned}|H(t,a)-H(t_0,r)| &\leqslant |H(t,a)-H(t,r)|+|H(t,r)-H(t_0,r)|\\ &\leqslant \sum_{i=1}^{n}|a_i-r_i|\,\|\alpha_i(t)\|_t\\ &\qquad + \left|\left\|\alpha(t)-\sum_{i=1}^{n}r_i.\alpha_i(t)\right\|_t - \left\|\alpha(t_0)-\sum_{i=1}^{n}r_i.\alpha_i(t_0)\right\|_{t_0}\right|.\end{aligned}$$

This inequality and the continuity of the map $t \mapsto \left\|\alpha(t)-\sum_{i=1}^{n}r_i.\alpha_i(t)\right\|_t$ imply that H is continuous on T. For each $t \in T$, define $\Lambda_t\colon \ell_1^n \to G_t$ by $a \mapsto \sum_{i=1}^{n}a_i.\alpha_i(t)$. Here Λ_t is a one to one onto linear mapping. Moreover for each $t \in T$, $a \in \ell_1^n$ we have

$$\begin{aligned}\|\Lambda_t a\|_t = \left\|\sum_{i=1}^{n}a_i.\alpha_i(t)\right\|_t &\leqslant \sum_{i=1}^{n}\|a_i\|\,\|\alpha_i(t)\|_t\\ &\leqslant n\max_i\|h_i\|\,.\max_i\|\alpha_i\|_\infty.\end{aligned}$$

Hence the open mapping theorem and the uniform boundedness principle give positive reals m and k such that

$$m\|a\| \leqslant \|\Lambda_t a\|_t \leqslant k\|a\| \qquad \forall t\in T,\quad \forall a \in \ell_1^n. \tag{2.3}$$

Now, let t_0 be a fixed point in T, and $\{t_b\}_{b\in B}$ be any net in T converging to t_0. Pick $g \in \mathrm{P}_{G_{t_0}}(\alpha(t_0))$ and write $g = \sum_{i=1}^{n}g_i.\alpha_i(t_0)$, $\beta = \sum_{i=1}^{n}g_i.\alpha_i$ ($\beta \in G_\infty =$ span of $\langle\alpha_1,\ldots,\alpha_n\rangle$ over $C(T)$). Let ε be any positive real number, and U_{t_0} the neighbourhood of t_0 such that $\|\alpha(t)-\beta(t)\|_t < \|\alpha(t_0)-\beta(t_0)\|_{t_0}+\varepsilon = \rho(t_0)+\varepsilon$. But

$$\rho(t) \leqslant \|\alpha(t)-\beta(t)\|_t < \rho(t_0)+\varepsilon \qquad \forall t \in U_{t_0} \tag{2.4}$$

(since $\beta(t) = \sum_{i=1}^{n}g_i.\alpha_i(t) \in G_t$). The net $\{a(t_b)\}_{b\in B}$ (where $a(t_b) = (a_1(t_b),\ldots,a_n(t_b))$ and $\left\|\alpha(t_b)-\sum_{i=1}^{n}a_i(t_b).\alpha_i(t_b)\right\|_{t_b} = \rho(t_b)$) is eventually bounded since

$$\begin{aligned}\|a(t_b)\| &\leqslant \frac{1}{m}\left\|\sum_{i=1}^{n}a_i(t_b)\alpha_i(t_b)\right\|_{t_b}\\ &\leqslant \frac{1}{m}\left(\left\|\alpha(t_b)-\sum_{i=1}^{n}a_i(t_b).\alpha_i(t_b)\right\|_{t_b}+\|\alpha(t_b)\|_{t_b}\right).\end{aligned}$$

By (2.4) there is a $c \in B$ such that $\rho(t_b) < \rho(t_0) + 1$ for each $b \geqslant c$. Thus $\|a(t_b)\| \leqslant (1/m)(\rho(t_0) + 1 + \|\alpha\|_\infty)$. We may assume without loss of generality that $a(t_b) \to a$.

$$\rho(t) - \rho(t_0) \leqslant H(t,\, a(t_0)) + H(t_0,\, a(t_0)). \tag{2.5}$$

$$\rho(t_0) - \rho(t) \leqslant H(t_0,\, a(t)) + H(t,\, a(t)). \tag{2.6}$$

$$\begin{aligned}|H(t_0,\, a(t_b)) - H(t_b,\, a(t_b))| \leqslant{}& |H(t_0,\, a) - H(t_0,\, a(t_b))| \\ &+ |H(t_0,\, a) - H(t_b,\, a(t_b))|.\end{aligned}$$

The continuity of the map H and $a(t_b) \to a$ imply that $|H(t_0,\, a(t_b)) - H(t_b,\, a(t_b))| \to 0$ as $t_b \to t$. Consequently, (2.5) and (2.6) imply that ρ is continuous. ∎

PROOF OF THE THEOREM: It suffices to show that for each $\alpha \in E_\infty$, $t_0 \in T$, $g \in \mathrm{P}_{G_{t_0}}(\alpha(t_0))$ and $\varepsilon > 0$ there exists $\gamma \in G_\infty$ such that $\gamma(t_0) = g$, and $d(\gamma(t), \mathrm{P}_{G_t}(\alpha(t))) < \varepsilon$ for each $t \in T$. Write $g = \sum_{i=1}^{n} g_i.\alpha_i(t_0)$, $\alpha_0 = \sum_{i=1}^{n} g_i.\alpha_i \in G_\infty$. Let U_{t_0} be the neighbourhood of t_0 such that, for $s \in U_{t_0}$,

$$\|\alpha_0(s) - \alpha(s)\|_s < \|\alpha_0(t_0) - \alpha(t_0)\|_{t_0} + \varepsilon/4$$

and

$$\begin{aligned}\rho(t_0) &< \rho(s) + \varepsilon/4.\ \|\alpha_0(s) - \alpha(s)\|_s \\ &< \|\alpha_0(t_0) - \alpha(t_0)\|_{t_0} + \varepsilon/4 \\ &= \rho(t_0) + \varepsilon/4 < \rho(s) + \varepsilon/2.\end{aligned}$$

Thus for each $s \in U_{t_0}$, we have the following:

$$B(\alpha(s),\, \rho(s)) \cap B(\alpha_0(s),\, \varepsilon/2) \neq \emptyset,$$
$$B(\alpha(s),\, \rho(s)) \cap G_s \neq \emptyset$$

and

$$B(\alpha_0(s),\, \varepsilon/2) \cap G_s \neq \emptyset.$$

Therefore, there is g_s in $B(\alpha(s), \rho(s)) \cap B(\alpha_0(s), \varepsilon/2) \cap G_s$, and then $d(\alpha_0(s), \mathrm{P}_{G_s}(\alpha(s)))$: $\|g_s - \alpha_0(s)\|_s \leqslant \varepsilon/2 < \varepsilon$.

Now, for each $t \in T$, $t \neq t_0$, select a g_t from $\mathrm{P}_{G_t}(\alpha(t))$. By the above there is $\alpha_t \in G_\infty$ and a neighbourhood U_t of t (we may assume that $U_{t_0} \cap U_t = \emptyset$, since T is Hausdorff) such that $\alpha_t(t) = g_t$ and $d(\alpha_t(p), \mathrm{P}_{G_p}(\alpha(p))) < \varepsilon$ for each p in U_t. Let $f_t : T \to [0, 1]$ be the continuous function such that $f_{|u_{t_0}} = 1$ and $f_{|u_t} = 0$. For $\beta_t = f_t.\alpha_0 + (1 - f_t).\alpha_t$, we have $\beta_{t|u_{t_0}} = \alpha_0$, $\beta_{t|u_t} = \alpha_t$ and $d(\beta_t(s), \mathrm{P}_{G_s}(\beta(s))) < \varepsilon$ for each s in $V_t = U_{t_0} \cup U_t$. The collection $\{V_t : t \in T\}$ forms an open covering of T; then there are $t_1, \ldots, t_n$ in T such that $T = \bigcup_{i=1}^{n} V_{t_i}$. Let $\{h_i\}_{i=1}^{n}$ be the partition of

https://doi.org/10.1017/S0004972700029087 Published online by Cambridge University Press

unity subordinate to $\{V_{t_i}\}$. A simple calculation will show that $\gamma = \sum_{i=1}^{n} h_i.\beta_{t_i}$ is the desired element of G_∞. Since ε was arbitrary, the result follows from the closeness of $\mathrm{P}_{G_t}(\alpha(t))$ and the fact that $d(\alpha(t), G_t) \leqslant d(\alpha, G_\infty)$ for each t in T. □

Theorem 2.4. *Let E_∞ be a function module in $\prod_{t\in T}^{\infty} E_t$. If G_∞ is a sub-$C(T)$-module of E_∞ such that for each t in T the fiber $G_t = \{\gamma(t) : \gamma \in G_\infty\}$ has the two-ball property for open balls, then G_∞ is proximinal (global best approximation exists).*

Proof: Let α be any fixed element of E_∞. For each t in T, let $g(t) \in \mathrm{P}_{G_t}(\alpha(t))$. Define

$$r = \inf_{\gamma\in G_\infty} \|\alpha - \gamma\|_\infty \geqslant \sup_{t\in T} \|\alpha(t) - g(t)\|_t. \tag{2.7}$$

We shall show that there is a Cauchy sequence $\{\gamma_n\}$ in G_∞ such that $\|\alpha - \gamma_n\|_\infty \to r$. For, let $\varepsilon > 0$; then by definition of r there is $\beta \in G_\infty$ such that $\|\beta - \alpha\|_\infty < r + \varepsilon$. We will show that there is another element $\gamma \in G_\infty$. such that

$$\|\alpha - \gamma\|_\infty < r + \frac{\varepsilon}{2} \quad \text{and} \quad \|\beta - \gamma\|_\infty \leqslant \frac{\varepsilon}{2}. \tag{2.8}$$

For each $t \in T$, the two balls $B(\alpha(t), r)$ and $B(\beta(t), \varepsilon)$ satisfy the conditions of the two-ball property (since $\|\alpha(t) - g(t)\|_t \leqslant r$ and $\|\beta(t) - \alpha(t)\|_t \leqslant \|\beta - \alpha\|_\infty < r + \varepsilon$). Let $x(t) \in G_t$ be such that $\|\alpha(t) - x(t)\|_t \leqslant r$ and $\|\beta(t) - x(t)\|_t < \varepsilon$. Put $y_t = (x(t) + \beta(t))/2$; then

$$\begin{aligned} &\|\alpha(t) - y(t)\|_t \leqslant \|\alpha(t) - x(t)\|_t + \|x(t) - y(t)\|_t < r + \varepsilon/2 \quad \text{and} \\ &\qquad\qquad \|\beta(t) - y(t)\|_t < \varepsilon/2. \end{aligned} \tag{2.9}$$

Now, let $\gamma_t \in G_\infty$ be such that $\gamma_t(t) = y_t$ and V_t the neighbourhood of t such that for each s in U_t

$$\|\alpha(s) - \gamma_t(s)\|_s < r + \varepsilon/2 \quad \text{and} \quad \|\beta(s) - \gamma_t(s)\|_s < \frac{\varepsilon}{2}; \tag{2.10}$$

(such U_t exists by (u.s.c.) of the norm functions). The collection $\{U_t : t \in T\}$ forms an open covering of T. Let $t_1, \dots, t_n$ be in T such that $T = \bigcup_{i=1}^{n} U_{t_i}$ and $\{f_i\}_{i=1}^{n}$ the partition of unity subordinate to $\{U_{t_i}\}_{i=1}^{n}$. A simple calculation will show that $\gamma = \sum_{i=1}^{n} f_i.\gamma_{t_i}$ is the desired element.

By applying (2.10) inductively, we can construct a sequence $\{\gamma_n\}$ in G_∞ such that

$$\|\alpha - \gamma_n\|_\infty \leqslant r + 2^{-n} \quad \text{and} \quad \|\gamma_n - \gamma_{n+1}\|_\infty \leqslant 2^{-n}. \tag{2.11}$$

The second inequality of (2.11) implies that $\{\gamma_n\}$ is Cauchy; hence it has a limit γ_0 in G_∞ and the first inequality of (2.11) implies that $\|\alpha - \gamma\|_\infty = r$; that is, γ in $\mathsf{P}_{G_\infty}(\alpha)$. ∎

THEOREM 2.5. *Let E_∞ be a function module in $\prod\limits_{t\in T}^{\infty} E_t$. If G_∞ is a sub-$C(T)$-module of E_∞ such that for each t in T the fiber $G_t = \{\gamma(t) : \gamma \in G_\infty\}$ has the two-ball property for open balls, then P_{G_∞} is (l.s.c.).*

In order to prove the above theorem, we need the following lemma, which maybe is interesting in itself.

LEMMA 2.6. *With the assumption of Theorem 2.5, for each α in E_∞, t in T and x_t in G_t such that $\|\alpha(t) - x_t\|_t \leqslant r = d(\alpha, G_\infty)$, there is γ in $\mathsf{P}_{G_\infty}(\alpha)$ such that $\gamma(t) = x_t$.*

PROOF: We shall show that for each positive ε there are two elements β_ε and γ_ε in G_∞ such that

$$\beta_\varepsilon(t) = \gamma_\varepsilon(t) = x_t; \tag{2.12}$$

$$\|\alpha - \beta_\varepsilon\|_\infty < r + \varepsilon; \tag{2.13}$$

$$\|\alpha - \gamma_\varepsilon\|_\infty < r + \frac{\varepsilon}{2} \quad \text{and} \quad \|\beta_\varepsilon - \gamma_\varepsilon\|_\infty < \frac{\varepsilon}{2}. \tag{2.14}$$

To see this, let $s \in T$, $x_s \in G_s$ such that $\|x_s - \alpha(s)\|_s \leqslant r$ (if $s = t$ take $x_x = x_t$). Let φ, φ_s be the elements of G_∞ such that $\varphi(t) = x_t$ and $\varphi_s(s) = x_s$ and $h : T \longrightarrow [0, 1]$ the continuous function such that $h(t) = 0$ and $h(s) = 1$. Take $\beta_s = (1 - h).\varphi + h.\varphi_s$ and let U_s be the neighbourhood of s such that $\|\alpha(p) - \beta_s(p)\|_p < r + \varepsilon$ for each p in U_s. The collection $\{U_s : s \in T\}$ forms an open covering of T. Let $s_1, \ldots, s_n$ in T be such that $T = \bigcup\limits_{i=1}^{n} U_{s_i}$ and $\{f_i\}_{i=1}^{n}$ the partition of unity subordinate to $\{U_{s_i}\}_{i=1}^{n}$. Take $\beta_\varepsilon = \sum\limits_{i=1}^{n} f_i.\beta_{s_i}$. A simple calculation will show that β_ε satisfies (2.12) and (2.13). Now, for $s \neq t$, let $y_s \in G_s$ be such that $\|\alpha(s) - y_s\|_s \leqslant r$ and $\|y_s - \beta(s)\|_s < \varepsilon$ ($y_s \in B(\alpha(s), r) \cap B(\beta_\varepsilon(s), \varepsilon) \cap G_s$). Let $a_s = (y_s + \beta_\varepsilon(s))/2$ and $\Omega, \Omega_s \in G_\infty$ such that $\Omega(t) = x_t$ and $\Omega_s(s) = a_s$. Put $\gamma_s = (1 - f)\Omega + f\Omega_s$ (where $f\colon T \to [1, 0]$ such that f is continuous, $f(t) = 0$ and $f(s) = 1$). Let U_s be the neighbourhood of s such that $\|\alpha(p) - \gamma_s(p)\|_p < r + \varepsilon/2$ for each p in U_s. Again $\{U_s \colon s \in T\}$ forms an open covering of T. Let $s_1, \ldots, s_n$ in T be such that $T = \bigcup\limits_{i=1}^{n} U_{s_i}$ and $\{h_i\}_{i=1}^{n}$ the partition of unity subordinate to $\{U_{s_i}\}_{i=1}^{n}$. It can easily be checked that $\gamma_\varepsilon = \sum\limits_{i=1}^{n} h_i.\gamma_{s_i}$ satisfies (2.12) and (2.14). Now, apply (2.12)–(2.14) inductively to construct a sequence $\{\gamma_n\}$ in G_∞ with the following:

$$\alpha_n(t) = x_t,\ \|\alpha - \gamma_n\|_\infty \leqslant r + 2^{-n} \text{ and } \|\gamma_n - \gamma_{n+1}\|_\infty \leqslant 2^{-n}. \tag{2.15}$$

The third inequality on the right in (2.15) implies that $\{\gamma_n\}$ is Cauchy, and then it has a limit γ in G_∞. Clearly γ is the desired element. ∎

PROOF OF THE THEOREM: Let $\{\alpha_n\}$ be a sequence in E_∞ converging to α, and β an element in $\mathrm{P}_{G_\infty}(\alpha)$. Define $r = d(\alpha, G_\infty)$ and $r_n = d(\alpha_n, G_\infty)$. For each $t \in T$ and $n \in \mathbb{N}$, define $\mathcal{A}_t^n = B(\beta(t), d_n) \cap B(\alpha_n(t), r_n) \cap G$ (where $d_n = \|\alpha_n - \alpha\|_\infty + |r_n - r| + 1/n$).

$$(2.16)\qquad \begin{aligned} \|\beta(t) - \alpha_n(t)\|_t &\leqslant \|\beta(t) - \alpha(t)\|_t + \|\alpha(t) - \alpha_n(t)\|_t \\ &\leqslant r + \|\alpha_n - \alpha\|_\infty \\ &< r_n + |r_n - r| + \|\alpha_n - \alpha\|_\infty + \frac{1}{n}. \end{aligned}$$

By (2.16) $\mathcal{A}_t^n$ is a nonempty convex set for all $t \in T$ and for all $n \in \mathbb{N}$. Now, let t be an arbitrary but fixed element in T and $n \in \mathbb{N}$. Pick $x_n \in \mathcal{A}_t^n$. Let $\beta_t^n \in \mathrm{P}_{G_\infty}(\alpha_n)$ be the element that exists from Lemma 2.6; that is, $\beta_t^n(t) = x_n$ and U_t is the neighbourhood of t such that $\|\beta_t^n(s) - \beta(s)\|_s < d_n$ for each $s \in U_t$ (such U_t exists by the (u.s.c.) of the norm function). Thus for each s in U_t we have $\beta_t^n(s)$ in $\mathcal{A}_s^n$. The collection $\{U_t : t \in T\}$ is an open covering of T. Let $t_1, \ldots, t_n$ in T be such that $T = \bigcup_{i=1}^{n} U_{t_i}$ and $\{f_i\}_{i=1}^n$ the partition of unity subordinate to $\{U_{t_i}\}_{i=1}^n$. Define $\beta_n = \sum_{i=1}^{n} f_i . \beta_{t_i}^n$. For each t in T $\beta_n(t)$ is a convex combination of elements of $\mathcal{A}_t^n$, and hence $\beta_n(t) \in \mathcal{A}_t^n$. Thus for each t in T we have

$$(2.17)\qquad \begin{cases} \|\beta_n(t) - \alpha_n(t)\| \leqslant r_n \\ \|\beta_n(t) - \beta(t)\| < d_n. \end{cases}$$

The first inequality in (2.17) implies that $\beta_n \in \mathrm{P}(\alpha_n)$ and the second implies that $\beta_n \to \beta$ as $n \to \infty$. ∎

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