Bull. Aust. Math. Soc. **110** (2024), 303–312
doi:10.1017/S0004972724000388

A TROPICAL ANALOGUE OF THE LEMMA ON THE LOGARITHMIC DERIVATIVE

JUHO HALONEN

(Received 2 December 2023; accepted 18 April 2024; first published online 20 May 2024)

Abstract

The tropical analogue of the lemma on the logarithmic derivative is generalised for noncontinuous tropical meromorphic functions, that is, piecewise linear functions that may have discontinuities. In addition, two Borel type results are generalised for piecewise continuous functions. With the generalisation of the tropical analogue of the lemma on the logarithmic derivative, several tropical analogues of Clunie and Mohon'ko type results are also automatically generalised for noncontinuous tropical meromorphic functions.

2020 *Mathematics subject classification*: primary 14T90; secondary 30D35.

Keywords and phrases: tropical Nevanlinna theory, tropical meromorphic functions, piecewise linear functions, lemma on the logarithmic derivative.

1. Introduction

In classical Nevanlinna theory, the lemma on the logarithmic derivative states that for a nonconstant meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$,

$$m\left(r, \frac{f'}{f}\right) = o(T(r, f))$$

as r tends to infinity outside an exceptional set of finite linear measure, where

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$$

and $T(r, f)$ is the Nevanlinna characteristic function for f. The lemma on the logarithmic derivative is an important result that is used to prove several results in Nevanlinna theory. The most important consequence of the lemma on the logarithmic derivative is the second main theorem of Nevanlinna theory [2, 6].

Tropical Nevanlinna theory was first introduced in [4], where Halburd and Southall proved the tropical analogue of the lemma on the logarithmic derivative for finite order tropical meromorphic functions. Later, the growth condition was improved for functions of hyper-order less than one by Laine and Tohge [8], and for subnormal functions by Cao and Zheng [1]. Tropical Nevanlinna theory was extended for noncontinuous tropical meromorphic functions in [5], where the Poisson–Jensen formula and the second main theorem were proved for noncontinuous tropical meromorphic functions.

In this paper, we will prove the following version of the tropical analogue of the lemma on the logarithmic derivative that generalises it for noncontinuous piecewise linear functions.

THEOREM 1.1. *Let f be a piecewise linear function and $f_c(x) \equiv f(x+c)$. If the hyper-order of f is less than one, then*

$$m\Big(r, \frac{f_c}{f} \oslash\Big) = o(T(r, f))$$

holds outside an exceptional set of finite logarithmic measure as r tends to infinity.

By proving the tropical analogue of the lemma on the logarithmic derivative for noncontinuous piecewise linear functions, several Clunie and Mohon'ko type results [7, 9] follow with identical proofs to the continuous case for noncontinuous piecewise linear functions.

2. Basic definitions

The tropical semiring is defined as $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ [10], where addition and multiplication are defined by

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \otimes b = a + b.$$

Additive and multiplicative neutral elements are $0_\circ = -\infty$ and $1_\circ = 0$. Here $\mathbb{T}$ is a semiring, because not all elements have an additive inverse element. For example, there is no $x \in \mathbb{T}$ such that $2 \oplus x = 0_\circ$. For this reason, subtraction is not defined on the tropical semiring. Tropical division is defined as $a \oslash b = a - b$ and exponentiation as $a^{\otimes \alpha} = \alpha a$ for $\alpha \in \mathbb{R}$.

In this paper, we will use the notation

$$f(x_0+) := \lim_{x \to x_0^+} f(x) \quad \text{and} \quad f(x_0-) := \lim_{x \to x_0^-} f(x).$$

A tropical meromorphic function is defined as a continuous piecewise linear function. Here we will drop the assumption of continuity and consider piecewise linear functions that may have discontinuities as in [5]. For piecewise linear functions, we define roots and poles in the same way as for tropical meromorphic functions [7]. We say that x is a pole of f if

$$\omega_f(x) := f'(x+) - f'(x-) < 0$$

and a root if $\omega_f(x) > 0$. The multiplicity of a root or a pole of a tropical meromorphic function f at x is $\tau_f(x) := |\omega_f(x)|$.

For a piecewise linear function f, as in [5], we define

$$\Omega_f(x) := f(x+) - f(x-)$$

https://doi.org/10.1017/S0004972724000388 Published online by Cambridge University Press

and we say that $x \neq 0$ is a positive jump if $x\Omega_f(x) > 0$ and a negative jump if $x\Omega_f(x) < 0$. If there is a discontinuity at 0, we say that it is a positive jump if $f(0) = \max\{f(0+), f(0-)\}$ and a negative jump if $f(0) = \min\{f(0+), f(0-)\}$. We call $h_f(x) = |\omega_f(x)|$ the height of the jump at x. We then define the jump counting function as

$$J(r, f) = \frac{1}{2} \sum_{|\beta_\nu| \leq r} h_f(\beta_\nu),$$

where β_ν are the negative jumps of f. For a piecewise linear function, we define the Nevanlinna proximity function as

$$m(r, f) = \frac{f(r)^+ + f(-r)^+}{2},$$

where $f(x)^+ = \max\{f(x), 0\}$. The Nevanlinna counting function is defined as

$$N(r, f) = \frac{1}{2} \int_0^r n(t, f)\, dt = \frac{1}{2} \sum_{|b_\nu| < r} \tau_f(b_\nu)(r - |b_\nu|),$$

where $n(r, f)$ counts the poles b_ν of f in $(-r, r)$ according to their multiplicities. Finally, the Nevanlinna characteristic function for piecewise linear functions is defined as

$$T(r, f) := m(r, f) + N(r, f) + J(r, f).$$

From [5, Theorem 2.5], for noncontinuous piecewise linear functions, the Poisson–Jensen formula is of the form

$$\begin{aligned} f(x) = {} & \frac{1}{2}(f(r) + f(-r)) + \frac{x}{2r}(f(r) - f(-r)) \\ & - \frac{1}{2r} \sum_{|a_\mu| < r} \tau_f(a_\mu)(r^2 - |a_\mu - x| r - a_\mu x) + \frac{1}{2r} \sum_{|b_\nu| < r} \tau_f(b_\nu)(r^2 - |b_\nu - x| r - b_\nu x) \\ & - S_1(x) + S_2(x) - S_3(x) + S_4(x) - \frac{x}{2r}\Omega_f(0) - \frac{1}{2}(A_f(x) + B_f(x)), \end{aligned} \tag{2.1}$$

where

$$S_1(x) = \frac{x}{2r}\Big(\sum_{-r \leq \beta_\nu \leq 0} h_f(\beta_\nu) + \sum_{0 \leq \alpha_\mu \leq r} h_f(\alpha_\mu) \Big),$$

$$S_2(x) = \frac{x}{2r}\Big(\sum_{-r \leq \alpha_\mu \leq 0} h_f(\alpha_\mu) + \sum_{0 \leq \beta_\nu \leq r} h_f(\beta_\nu) \Big),$$

$$S_3(x) = \frac{1}{2}\Big(\sum_{-r \leq \alpha_\mu \leq \min\{0, x\}} h_f(\alpha_\mu) + \sum_{\max\{x, 0\} \leq \alpha_\mu \leq r} h_f(\alpha_\mu) - \sum_{x \leq \alpha_\mu \leq 0} h_f(\alpha_\mu) - \sum_{0 \leq \alpha_\mu \leq x} h_f(\alpha_\mu) \Big),$$

$$S_4(x) = \frac{1}{2}\Big(\sum_{-r \leq \beta_\nu \leq \min\{0, x\}} h_f(\beta_\nu) + \sum_{\max\{x, 0\} \leq \beta_\nu \leq r} h_f(\beta_\nu) - \sum_{x \leq \beta_\nu \leq 0} h_f(\beta_\nu) - \sum_{0 \leq \beta_\nu \leq x} h_f(\beta_\nu) \Big),$$

https://doi.org/10.1017/S0004972724000388 Published online by Cambridge University Press

$$A_f(x) = \begin{cases} 0 & \text{if } f \text{ is continuous at } x, \\ \Omega_f(x) & \text{if } f \text{ is left-discontinuous at } x, \\ -\Omega_f(x) & \text{if } f \text{ is right-discontinuous at } x \end{cases}$$

and

$$B_f(x) = \begin{cases} 0 & \text{if } x = 0, \\ -\Omega_f(0) & \text{if } x < 0, \\ \Omega_f(0) & \text{if } x > 0. \end{cases}$$

With the special case of $x = 0$, we obtain the Jensen formula

$$T(r, f) = T(r, -f) + f(0).$$

The order $\rho(f)$ and hyper-order $\rho_2(f)$ of a piecewise linear function f are defined by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

3. Lemma on the logarithmic derivative

To prove Theorem 1.1, we need to prove some results for noncontinuous functions. We begin by proving a generalised version of [3, Lemma 8.3]. In this version, the assumption of continuity is replaced with piecewise continuity and the assumption of nondecreasingness with condition (3.1) below. Note that condition (3.1) implies nondecreasingness. The proof closely follows the proof of [3, Lemma 8.3] with some minor changes.

THEOREM 3.1. *Let $s > 0$ and $T : [0, \infty) \to [0, \infty)$ be a piecewise continuous function such that*

$$T(r) \le \frac{R}{r} T(R) \tag{3.1}$$

for all $R \ge r > 0$. If the hyper-order of T is strictly less than one, then

$$T(r + s) = T(r) + o(T(r))$$

outside of a set of finite logarithmic measure as r tends to infinity.

PROOF. Let $\eta > 0$ and assume that the set $F_\eta := \{r \in \mathbb{R}^+ : T(r + s) \ge (1 + \eta)T(r)\} \subset [1, \infty)$ is of infinite logarithmic measure. Define $G : [0, \infty) \to [0, \infty)$ such that $G(r) := T(r+)$ for all $r \in [0, \infty)$. Then $G(r)$ is right-continuous at every point and, therefore,

$$H_\eta = \{r \in \mathbb{R}^+ : G(r + s) \ge (1 + \eta)G(r)\}$$

contains the smallest element r_0. Set $r_n = \min\{H_\eta \cap [r_{n-1} + s, \infty)\}$ for all $n \in \mathbb{N}$. Then the sequence $\{r_n\}_{n\in\mathbb{N}}$ satisfies $r_{n+1} - r_n \ge s$ for all $n \in \mathbb{N}$, $H_\eta \subset \bigcup_{n=0}^{\infty}[r_n, r_n + s]$ and then

$$(1 + \eta)G(r_n) \le G(r_n + s) \le \frac{r_{n+1}}{r_n + s} G(r_{n+1}) \le \frac{r_{n+1}}{r_n} G(r_{n+1}) \tag{3.2}$$

https://doi.org/10.1017/S0004972724000388 Published online by Cambridge University Press

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and suppose that there exists an integer $m \in \mathbb{N}$ such that $r_n \geq n^{1+\varepsilon}$ for all $r_n \geq m$. Then

$$\int_{H_\eta \cap [1,\infty)} \frac{dt}{t} \leq \sum_{n=0}^{\infty} \int_{r_n}^{r_n+s} \frac{dt}{t} \leq \int_1^m \frac{dt}{t} + \sum_{n=1}^{\infty} \log\Big(1 + \frac{s}{r_n}\Big)$$
$$\leq \sum_{n=1}^{\infty} \log(1 + sn^{-(1+\varepsilon)}) + O(1) < \infty,$$

which contradicts the assumption that $H_\eta \cap [1, \infty)$ has infinite logarithmic measure. Therefore, the sequence $\{r_n\}_{n\in\mathbb{N}}$ has a subsequence $\{r_{n_j}\}_{j\in\mathbb{N}}$ such that $r_{n_j} \leq n_j^{1+\varepsilon}$ for all $j \in \mathbb{N}$. By iterating (3.2) along the sequence $\{r_{n_j}\}_{j\in\mathbb{N}}$,

$$G(r_{n_j}) \geq G(r_0) \prod_{\nu=0}^{n_j-1} \Big(\frac{r_\nu}{r_{\nu+1}}(1 + \eta)\Big) = \frac{r_0}{r_{n_j}} G(r_0)(1 + \eta)^{n_j}$$

for all $j \in \mathbb{N}$. Denote the hyper-order of G by ρ_2. Then we can see that

$$\begin{aligned}
\rho_2 &= \limsup_{r\to\infty} \frac{\log\log G(r)}{\log r} \geq \limsup_{j\to\infty} \frac{\log(\log G(r_0) + n_j \log(1 + \eta) + \log r_0 - \log r_{n_j})}{\log r_{n_j}} \\
&\geq \limsup_{j\to\infty} \frac{\log(\log G(r_0) + n_j \log(1 + \eta) + \log r_0 - (1 + \varepsilon) \log n_j)}{(1 + \varepsilon) \log n_j} \\
&= \limsup_{j\to\infty} \frac{\log(n_j((\log G(r_0) + \log r_0 - (1 + \varepsilon) \log n_j)/n_j + \log(1 + \eta)))}{(1 + \varepsilon) \log n_j} \\
&= \limsup_{j\to\infty} \frac{\log n_j + \log((\log G(r_0) + \log r_0 - (1 + \varepsilon) \log n_j)/n_j + \log(1 + \eta))}{(1 + \varepsilon) \log n_j} \\
&= \frac{1}{1 + \varepsilon}.
\end{aligned}$$

Since the hyper-order of T is less than one and $G(r) \neq T(r)$ for at most countably many r, we must have $\rho_2 < 1$. We can now choose $\varepsilon > 0$ such that $1 > 1/(1 + \varepsilon) > \rho_2$ which leads to a contradiction. Therefore, H_η has finite logarithmic measure. As a piecewise continuous function, T has at most countably many discontinuities, so that H_η and F_η differ at at most countably many points and thus also F_η has finite logarithmic measure. □

The following result is a noncontinuous version of [7, Theorem 3.24].

THEOREM 3.2. *Let f be a piecewise linear function. Then for all $\alpha > 1$, as r tends to infinity,*

$$m\Big(r, \frac{f(x + c)}{f(x)} \oslash\Big) \leq \frac{16|c|}{r + |c|} \frac{1}{\alpha - 1}(T(\alpha(r + |c|), f) + |f(0)|/2)$$
$$+ J(r + c, f) - J(r - c, f) + J(r + c, -f) - J(r - c, -f) + O(1).$$

https://doi.org/10.1017/S0004972724000388 Published online by Cambridge University Press

PROOF. Suppose that $\rho > r + |c|$ and $x \in [-r, r]$. Denote by $a_\mu, b_\nu, \alpha_\mu, \beta_\nu$ the roots, poles, positive and negative jumps of f in $[-\rho, \rho]$. By the Poisson–Jensen formula,

$$\begin{aligned} f(x+c) - f(x) &= \frac{c}{2\rho}(f(\rho) - f(-\rho)) \\ &+ \frac{1}{2\rho}\sum_{|a_\mu|<\rho} \tau(a_\mu)((|a_\mu - x - c| - |a_\mu - x|)\rho + ca_\mu) \\ &- \frac{1}{2\rho}\sum_{|b_\nu|<\rho} \tau(b_\nu)((|b_\nu - x - c| - |b_\nu - x|)\rho + cb_\nu) \\ &- \frac{c}{2\rho}\Big(\sum_{-\rho\le\beta_\nu\le 0} h_f(\beta_\nu) + \sum_{0\le\alpha_\mu\le\rho} h_f(\alpha_\mu)\Big) + \frac{c}{2\rho}\Big(\sum_{-\rho\le\alpha_\mu\le 0} h_f(\alpha_\mu) + \sum_{0\le\beta_\nu\le\rho} h_f(\beta_\nu)\Big) \\ &+ \frac{1}{2}\Big(\sum_{\alpha_\mu\in(x,x+c]\cap(-\infty,0]} h_f(\alpha_\mu) - \sum_{\alpha_\mu\in[x,x+c)\cap[0,\infty)} h_f(\alpha_\mu) + \sum_{\alpha_\mu\in[x,x+c)\cap(-\infty,0]} h_f(\alpha_\mu) - \sum_{\alpha_\mu\in(x,x+c]\cap[0,\infty)} h_f(\alpha_\mu)\Big) \\ &- \frac{1}{2}\Big(\sum_{\beta_\nu\in(x,x+c]\cap(-\infty,0]} h_f(\beta_\nu) - \sum_{\beta_\nu\in[x,x+c)\cap[0,\infty)} h_f(\beta_\nu) + \sum_{\beta_\nu\in[x,x+c)\cap(-\infty,0]} h_f(\beta_\nu) - \sum_{\beta_\nu\in(x,x+c]\cap[0,\infty)} h_f(\beta_\nu)\Big) \\ &- \frac{c}{2\rho}\Omega_f(0) - \frac{1}{2}(A_f(x+c) - A_f(x) + B_f(x+c) - B_f(x)). \end{aligned}$$

By the definition of h_f,

$$\begin{aligned} &\sum_{-\rho\le\beta_\nu\le 0} -h_f(\beta_\nu) + \sum_{0\le\alpha_\mu\le\rho} -h_f(\alpha_\mu) + \sum_{-\rho\le\alpha_\mu\le 0} h_f(\alpha_\mu) + \sum_{0\le\beta_\nu\le\rho} h_f(\beta_\nu) \\ &\quad= \sum_{-\rho\le\beta_\nu\le 0} -\Omega_f(\beta_\nu) + \sum_{0\le\alpha_\mu\le\rho} -\Omega_f(\alpha_\mu) + \sum_{-\rho\le\alpha_\mu\le 0} -\Omega_f(\alpha_\mu) + \sum_{0\le\beta_\nu\le\rho} -\Omega_f(\beta_\nu) \\ &\quad= -\Big(\sum_{|\alpha_\mu|\le\rho} \Omega_f(\alpha_\mu) + \sum_{|\beta_\nu|\le\rho} \Omega_f(\beta_\nu)\Big). \end{aligned}$$

By combining terms and by the definition of h_f,

$$\begin{aligned} &\sum_{\alpha_\mu\in(x,x+c]\cap(-\infty,0]} h_f(\alpha_\mu) - \sum_{\alpha_\mu\in[x,x+c)\cap[0,\infty)} h_f(\alpha_\mu) + \sum_{\alpha_\mu\in[x,x+c)\cap(-\infty,0]} h_f(\alpha_\mu) - \sum_{\alpha_\mu\in(x,x+c]\cap[0,\infty)} h_f(\alpha_\mu) \\ &= 2\sum_{\alpha_\mu\in[x,x+c]\cap(-\infty,0]} h_f(\alpha_\mu) - \sum_{\alpha_\mu\in\{x,x+c\}\cap(-\infty,0]} h_f(\alpha_\mu) - 2\sum_{\alpha_\mu\in[x,x+c]\cap[0,\infty)} h_f(\alpha_\mu) + \sum_{\alpha_\mu\in\{x,x+c\}\cap[0,\infty)} h_f(\alpha_\mu) \\ &= -2\sum_{\alpha_\mu\in[x,x+c]} \Omega_f(\alpha_\mu) + \sum_{\alpha_\mu\in\{x,x+c\}} \Omega_f(\alpha_\mu). \end{aligned}$$

Similarly, for the negative jumps,

$$\sum_{\beta_\nu\in(x,x+c]\cap(-\infty,0]} h_f(\beta_\nu) - \sum_{\beta_\nu\in[x,x+c)\cap[0,\infty)} h_f(\beta_\nu) + \sum_{\beta_\nu\in[x,x+c)\cap(-\infty,0]} h_f(\beta_\nu) - \sum_{\beta_\nu\in(x,x+c]\cap[0,\infty)} h_f(\beta_\nu)$$
$$= 2\sum_{\beta_\nu\in[x,x+c]} \Omega_f(\beta_\nu) - \sum_{\beta_\nu\in\{x,x+c\}} \Omega_f(\beta_\nu).$$

By combining these equations,

$$\begin{aligned} f(x+c) - f(x) &= \frac{c}{2\rho}(f(\rho) - f(-\rho)) \\ &+ \frac{1}{2\rho}\sum_{|a_\mu|<\rho} \tau(a_\mu)((|a_\mu - x - c| - |a_\mu - x|)\rho + ca_\mu) \\ &- \frac{1}{2\rho}\sum_{|b_\nu|<\rho} \tau(b_\nu)((|b_\nu - x - c| - |b_\nu - x|)\rho + cb_\nu) \\ &- \frac{c}{2\rho}\Big(\sum_{|\alpha_\mu|\le\rho} \Omega_f(\alpha_\mu) + \sum_{|\beta_\nu|\le\rho} \Omega_f(\beta_\nu)\Big) \\ &- \sum_{\alpha_\mu\in[x,x+c]} \Omega_f(\alpha_\mu) + \frac{1}{2}\sum_{\alpha_\mu\in\{x,x+c\}} \Omega_f(\alpha_\mu) - \sum_{\beta_\nu\in[x,x+c]} \Omega_f(\beta_\nu) + \frac{1}{2}\sum_{\beta_\nu\in\{x,x+c\}} \Omega_f(\beta_\nu) \\ &- \frac{c}{2\rho}\Omega_f(0) - \frac{1}{2}(A_f(x+c) - A_f(x) + B_f(x+c) - B_f(x)). \end{aligned}$$

By taking the proximity function on both sides and by the proof of [7, Theorem 3.23],

$$\begin{aligned} m(r, f(x+c) - f(x)) &\le |c|\Big(\frac{1}{\rho}(m(\rho, f) + m(\rho, -f)) + n(\rho, f) + n(\rho, -f)\Big) \\ &+ m\Big(r, -\frac{c}{2\rho}\Big(\sum_{|\alpha_\mu|\le\rho} \Omega_f(\alpha_\mu) + \sum_{|\beta_\nu|\le\rho} \Omega_f(\beta_\nu)\Big)\Big) \\ &+ m\Big(r, -\Big(\sum_{\alpha_\mu\in[x,x+c]} \Omega_f(\alpha_\mu) + \sum_{\beta_\nu\in[x,x+c]} \Omega_f(\beta_\nu)\Big)\Big) \\ &+ m\Big(r, \frac{1}{2}\sum_{\alpha_\mu\in\{x,x+c\}} \Omega_f(\alpha_\mu)\Big) + m\Big(r, \frac{1}{2}\sum_{\beta_\nu\in\{x,x+c\}} \Omega_f(\beta_\nu)\Big) \\ &+ m\Big(r, -\frac{c}{2\rho}\Omega_f(0) - \frac{1}{2}(A_f(x+c) - A_f(x) + B_f(x+c) - B_f(x))\Big). \end{aligned}$$

Since the quantity below inside the proximity function is constant with respect to x,

$$\begin{aligned} m\Big(r, -\frac{c}{2\rho}\Big(\sum_{|\alpha_\mu|\le\rho} \Omega_f(\alpha_\mu) + \sum_{|\beta_\nu|\le\rho} \Omega_f(\beta_\nu)\Big)\Big) &= \Big(-\frac{c}{2\rho}\Big(\sum_{|\alpha_\mu|\le\rho} \Omega_f(\alpha_\mu) + \sum_{|\beta_\nu|\le\rho} \Omega_f(\beta_\nu)\Big)\Big)^+ \\ &\le \frac{|c|}{2\rho}\Big|\sum_{|\alpha_\mu|\le\rho} \Omega_f(\alpha_\mu) + \sum_{|\beta_\nu|\le\rho} \Omega_f(\beta_\nu)\Big| \end{aligned}$$

https://doi.org/10.1017/S0004972724000388 Published online by Cambridge University Press

$$\le \frac{|c|}{2\rho}\Big(\sum_{|\alpha_\mu|\le\rho}|\Omega_f(\alpha_\mu)| + \sum_{|\beta_\nu|\le\rho}|\Omega_f(\beta_\nu)|\Big)$$
$$= \frac{|c|}{2\rho}(J(\rho,f)+J(\rho,-f)).$$

By the definition of the proximity function,

$$\begin{aligned}
m\Big(r, -\Big(\sum_{\alpha_\mu\in[x,x+c]}\Omega_f(\alpha_\mu) + \sum_{\beta_\nu\in[x,x+c]}\Omega_f(\beta_\nu)\Big)\Big)&\\
= \frac{1}{2}\Big(\sum_{\alpha_\mu\in[r,r+c]}\Omega_f(\alpha_\mu) + \sum_{\beta_\nu\in[r,r+c]}\Omega_f(\beta_\nu)\Big)^+ &+ \frac{1}{2}\Big(\sum_{\alpha_\mu\in[-r,-r+c]}\Omega_f(\alpha_\mu) + \sum_{\beta_\nu\in[-r,-r+c]}\Omega_f(\beta_\nu)\Big)^+\\
\le \frac{1}{2}\sum_{\alpha_\mu\in[r,r+c]}|\Omega_f(\alpha_\mu)| + \frac{1}{2}\sum_{\beta_\nu\in[r,r+c]}|\Omega_f(\beta_\nu)|&\\
+ \frac{1}{2}\sum_{\alpha_\mu\in[-r,-r+c]}|\Omega_f(\alpha_\mu)| + \frac{1}{2}\sum_{\beta_\nu\in[-r,-r+c]}|\Omega_f(\beta_\nu)|&\\
\le \frac{1}{2}\sum_{\alpha_\mu\in[r-c,r+c]}|\Omega_f(\alpha_\mu)| + \frac{1}{2}\sum_{\beta_\nu\in[r-c,r+c]}|\Omega_f(\beta_\nu)|&\\
+ \frac{1}{2}\sum_{\alpha_\mu\in[-r-c,-r+c]}|\Omega_f(\alpha_\mu)| + \frac{1}{2}\sum_{\beta_\nu\in[-r-c,-r+c]}|\Omega_f(\beta_\nu)|&\\
= J(r+c,f) - J(r-c,f) + J(r+c,-f) - J(r-c,-f).&
\end{aligned}$$

The remaining terms are constant almost everywhere, so overall,

$$\begin{aligned}
&m(r, f(x+c)-f(x))\\
&\quad\le |c|\Big(\frac{1}{\rho}\Big(m(\rho,f)+m(\rho,-f)+\frac{1}{2}J(\rho,f)+\frac{1}{2}J(\rho,-f)\Big)+n(\rho,f)+n(\rho,-f)\Big)\\
&\qquad + J(r+c,f) - J(r-c,f) + J(r+c,-f) - J(r-c,-f) + O(1).
\end{aligned}$$

We may choose $\rho = \frac{1}{2}(\alpha+1)(r+|c|) < \alpha(r+|c|)$. Now by [7, Lemma 3.21],

$$n(\rho,f)+n(\rho,-f) \le \frac{4}{\alpha-1}\frac{1}{r+|c|}(T(\alpha(r+|c|),f)+T(\alpha(r+|c|),-f)).$$

Since $1/(\alpha+1) \le 2/(\alpha-1)$, the Jensen formula implies that

$$\begin{aligned}
&m(r, f(x+c)-f(x))\\
&\quad\le |c|\Big(\frac{2}{(\alpha+1)(r+|c|)}(T(\alpha(r+|c|),f)+T(\alpha(r+|c|),-f))\Big)\\
&\qquad + |c|\Big(\frac{4}{\alpha-1}\frac{1}{r+|c|}(T(\alpha(r+|c|),f)+T(\alpha(r+|c|),-f))\Big)\\
&\qquad + J(r+c,f) - J(r-c,f) + J(r+c,-f) - J(r-c,-f) + O(1)
\end{aligned}$$

https://doi.org/10.1017/S0004972724000388 Published online by Cambridge University Press

$$\leq \frac{16|c|}{\alpha - 1} \frac{1}{r + |c|} \Big(T(\alpha(r + |c|), f) + \frac{|f(0)|}{2} \Big)$$
$$+ J(r + c, f) - J(r - c, f) + J(r + c, -f) - J(r - c, -f) + O(1). \qquad \square$$

The following result is a generalisation of [2, Theorem 3.3.1]. It replaces the assumption of continuity with piecewise continuity.

THEOREM 3.3. *Let $F(r)$ be a positive, nondecreasing and piecewise continuous function defined for $r_0 \leq r < \infty$, and let $\phi(r)$ be a positive, nondecreasing, continuous function defined for $r_0 \leq r < \infty$, and assume that $F(r) \geq e$ for $r \geq r_0$. Let $\xi(x)$ be a positive, nondecreasing, continuous function defined for $e \leq x < \infty$. Let $C > 1$ be a constant and let E be the subset of $[r_0, \infty)$ defined by*

$$E = \Big\{ r \in [r_0, \infty) : F\Big(r + \frac{\phi(r)}{\xi(F(r))} \Big) \geq CF(r) \Big\}.$$

Then for all $R < \infty$,

$$\int_{E \cap [r_0, R]} \frac{dr}{\phi(r)} \leq \frac{1}{\xi(e)} + \frac{1}{\log C} \int_e^{F(R+)} \frac{dx}{x\xi(x)}.$$

PROOF. Define $G(r) := F(r+)$ for all $r \in [r_0, \infty)$ so that $F(r) = G(r)$ outside the set of discontinuities of F. Define

$$\hat{E} = \Big\{ r \in [r_0, \infty) : G\Big(r + \frac{\phi(r)}{\xi(G(r))} \Big) \geq CG(r) \Big\}.$$

We can now find the smallest element in $\hat{E}$ and thus we can prove the theorem for $G(r)$ and $\hat{E}$ by following the steps of the proof of [2, Theorem 3.3.1]. Then we can conclude that the statement holds also for the function $F(r)$, since it differs from $G(r)$ only at countably many points. $\square$

Now we have all the tools to prove Theorem 1.1.

PROOF OF THEOREM 1.1. To apply Theorem 3.3, define $\phi(r) := r, \xi(x) := (\log x)^{1+\varepsilon}$ and

$$\alpha = 1 + \frac{\phi(r + |c|)}{(r + |c|)\xi(T(r + |c|, f))} = 1 + \frac{1}{(\log T(r + |c|, f))^{1+\varepsilon}}.$$

By Theorem 3.2,

$$m\Big(r, \frac{f(x + c)}{f(x)} \oslash\Big) \leq \frac{16|c|}{r + |c|} \frac{1}{\alpha - 1} \Big(T(\alpha(r + |c|), f) + \frac{|f(0)|}{2} \Big)$$
$$+ J(r + c, f) - J(r - c, f) + J(r + c, -f) - J(r - c, -f) + O(1)$$
$$= \frac{16|c|}{r + |c|} (\log T(r + |c|, f))^{1+\varepsilon} \Big(T(\alpha(r + |c|), f) + \frac{|f(0)|}{2} \Big)$$
$$+ J(r + c, f) - J(r - c, f) + J(r + c, -f) - J(r - c, -f) + O(1)$$

https://doi.org/10.1017/S0004972724000388 Published online by Cambridge University Press

outside of an exceptional set of finite logarithmic measure. To proceed, fix $\varepsilon > 0$ so that $(\rho_2(f)+\varepsilon)(1+\varepsilon)\leq 1-\gamma$ for some $\gamma<1$. Then, since $(\log T(r+|c|, f))^{1+\varepsilon}\leq (r+|c|)^{1-\gamma}$,

$$\frac{(\log T(r+|c|, f))^{1+\varepsilon}}{r+|c|} \leq (r+|c|)^{-\gamma} \to 0$$

as r approaches to infinity. For the term

$$T(\alpha(r+|c|), f) = T\left(r+|c| + \frac{r+|c|}{(\log T(r+|c|, f))^{1+\varepsilon}}, f\right)$$

above, we may apply Theorem 3.3 to conclude that

$$T(\alpha(r+|c|), f) \leq CT(r+|c|, f)$$

outside an exceptional set of finite logarithmic measure, and further by Theorem 3.1,

$$T(\alpha(r+|c|), f) \leq CT(r, f)$$

and

$$J(r+c, f) - J(r-c, f) + J(r+c, -f) - J(r-c, -f) = o(T(r, f))$$

as r tends to infinity outside an exceptional set of finite logarithmic measure. Therefore, outside an exceptional set of finite logarithmic measure,

$$m\left(r, \frac{f(x+c)}{f(x)}\oslash\right) \leq 32C(r+|c|)^{-\gamma}T(r, f) + o(T(r, f)) = o(T(r, f)). \qquad \square$$

References

[1] T. Cao and J. Zheng, 'Nevanlinna theory for tropical hypersurfaces', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, to appear; doi:10.2422/2036-2145.202211_008.

[2] W. Cherry and Z. Ye, *Nevanlinna's Theory of Value Distribution* (Springer-Verlag, Berlin, 2001).

[3] R. G. Halburd, R. J. Korhonen and K. Tohge, 'Holomorphic curves with shift-invariant hyperplane preimages', *Trans. Amer. Math. Soc.* **366**(8) (2014), 4267–4298.

[4] R. G. Halburd and N. J. Southall, 'Tropical Nevanlinna theory and ultradiscrete equations', *Int. Math. Res. Not. IMRN* **2009**(5) (2009), 887–911.

[5] J. Halonen, R. Korhonen and G. Filipuk, 'Tropical second main theorem and the Nevanlinna inverse problem', Preprint, 2023, arXiv:2305.13939.

[6] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).

[7] R. Korhonen, I. Laine and K. Tohge, *Tropical Value Distribution Theory and Ultra-discrete Equations* (World Scientific Publishing, Hackensack, NJ, 2015).

[8] I. Laine and K. Tohge, 'Tropical Nevanlinna theory and second main theorem', *Proc. Lond. Math. Soc. (3)* **102**(5) (2011), 883–922.

[9] I. Laine and C.-C. Yang, 'Tropical versions of Clunie and Mohon'ko lemmas', *Complex Var. Elliptic Equ.* **55**(1–3) (2010), 237–248.

[10] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, 161 (American Mathematical Society, Providence, RI, 2015).

JUHO HALONEN, Department of Physics and Mathematics,
University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland
e-mail: juho.halonen@uef.fi