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A CLASS OF HYPERGEOMETRIC POLYNOMIALS

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Abstract

The object of the present paper is first to derive an interesting unification (and generalization) of a fairly large number of finite summation formulas including, for example, those that appeared in this Journal recently. We then briefly remark on its various (known or new) special cases which are associated with certain classes of hypergeometric polynomials in one and two variables. We also give several further generalizations (involving multiple series with essentially arbitrary terms) which are shown to be applicable in the derivation of analogous summation formulas for hypergeometric series (and polynomials) in three and more variables. Finally, with a view to presenting relevance of these types of results in various seemingly diverse areas of applied sciences and engineering, some indication of applicability is provided.

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1. Introduction

Making use of the fractional derivative operator $\mathscr{D}_z^\mu$ defined by

$$(1.1) \qquad \mathscr{D}_z^\mu\{z^{\lambda-1}\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} z^{\lambda-\mu-1} \qquad (\mu \neq \lambda;\ \lambda \neq 0, -1, -2, \ldots),$$

Manocha and Sharma ([1], [2]) derived a number of interesting sums involving products of a certain class of Gaussian hypergeometric ${}_2F_1$ polynomials. Recently, while correcting several errors in some of the results of Manocha and Sharma [1], Qureshi and Pathan [4] applied the fractional derivative operator $\mathscr{D}_z^\mu$ along the lines of Manocha and Sharma [1] in order to establish the formula (see

[4, page 178, Equation (2.4)]):

$$(1.2)\qquad \sum_{n=0}^{N}\binom{N}{n}\frac{(a)_{N-n}(c)_n}{(b)_{N-n}(d)_n}\left(-\frac{y}{x}\right)^n$$
$$\cdot {}_2F_1\left[\begin{matrix}-n, & a+N-n; \\ & b+N-n;\end{matrix}\; x\right]{}_2F_1\left[\begin{matrix}-n, & c; \\ & d;\end{matrix}\; y\right]$$
$$= \frac{(a)_N}{(b)_N}\,{}_3F_2\left[\begin{matrix}-N, & 1-b-N, & c; \\ & 1-a-N, & d;\end{matrix}\; \frac{y}{x}\right],$$

which is due to Manocha and Sharma [1, page 475, Equation (31)]; they also gave an alternative proof (without using $\mathscr{D}_z^{\mu}$) of the following result of Manocha and Sharma (see [2, page 233, Equation (15)]; see also [4, page 180, Equation (3.5)]):

$$(1.3)\qquad \sum_{n=0}^{N}(-1)^n\binom{N}{n}{}_2F_1\left[\begin{matrix}-n, & a; \\ & b;\end{matrix}\; x\right]{}_2F_1\left[\begin{matrix}-N+n, & c; \\ & d;\end{matrix}\; y\right]$$
$$= \frac{(a)_N}{(b)_N}x^N\,{}_3F_2\left[\begin{matrix}-N, & 1-b-N, & c; \\ & 1-a-N, & d;\end{matrix}\; \frac{y}{x}\right],$$

where (and throughout this paper) $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$.

In our attempt to present a direct (rather elementary) proof of the summation formula (1.2), without using the fractional derivative operator $\mathscr{D}_z^{\mu}$, we were led naturally to an interesting unification (and generalization) of a fairly large number of finite summation formulas including, for example, (1.2) and (1.3) which happen to be among the main results of [1], [2], and [4]. In Section 2 we state and prove this basic result, and briefly remark on its numerous (known or new) special cases associated with certain classes of hypergeometric polynomials in one and two variables. Finally, in Section 3 we give several further generalizations involving multiple series with essentially arbitrary terms, and show how these general results can be applied with a view to deriving analogous summation formulas for various classes of hypergeometric series (and polynomials) in three and more variables. Some indication of applicability of these classes of summation formulas is also provided.

2. Unification (and generalization) of (1.2) and (1.3)

In terms of the Pochhammer symbol $(\lambda)_n$ used in (1.2) and (1.3), let $F^{p:r;u}_{q:s;v}$ denote the generalized (Kampé de Fériet's) double hypergeometric series (see, for example, [6, page 27, Equation (28) *et seq.*])

$$(2.1)\quad F^{p:r;u}_{q:s;v}\left[\begin{matrix}(\alpha_p):(a_r);(c_u);\\ \\(\beta_q):(b_s);(d_v);\end{matrix}\quad x,y\right] = \sum_{l,m=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_{l+m}\prod_{j=1}^{r}(a_j)_l\prod_{j=1}^{u}(c_j)_m}{\prod_{j=1}^{q}(\beta_j)_{l+m}\prod_{j=1}^{s}(b_j)_l\prod_{j=1}^{v}(d_j)_m}\frac{x^l}{l!}\frac{y^m}{m!},$$

where, for convergence,

(i) $p+r<q+s+1$, $p+u<q+v+1$, $|x|<\infty$, and $|y|<\infty$, or

(ii) $p+r=q+s+1$, $p+u=q+v+1$, and

$$(2.2)\qquad \begin{cases}|x|^{1/(p-q)}+|y|^{1/(p-q)}<1, & \text{if } p>q,\\ \max\{|x|,|y|\}<1, & \text{if } p\leqslant q,\end{cases}$$

unless, of course, the series terminates [that is, when (for example) one of the numerator parameters $\alpha_1,\dots,\alpha_p$ is zero or a negative integer]; here, and in what follows, (α_p) abbreviates the array of p parameters $\alpha_1,\dots,\alpha_p$, with similar interpretations for (β_q), *et cetera*, an empty product is to be interpreted as 1, and none of the denominator parameters is zero or a negative integer.

Our unification (and generalization) of the summation formulas (1.2) and (1.3) is given by

$$(2.3)\quad \sum_{n=0}^{N}\binom{N}{n}\frac{\prod_{j=1}^{\rho}(\gamma_j)_{N-n}\prod_{j=1}^{h}(\xi_j)_n}{\prod_{j=1}^{\sigma}(\delta_j)_{N-n}\prod_{j=1}^{k}(\eta_j)_n}\left(-\frac{Y}{x}\right)^n$$
$$\cdot F^{p:1+r+\rho;1+u+h}_{q:\quad s+\sigma;\quad v+k}\left[\begin{matrix}(\alpha_p):-n, & (a_r),(\gamma_\rho)+N-n;-N+n, & (c_u),(\xi_h)+n;\\ (\beta_q): & (b_s);(\delta_\sigma)+N-n; & (d_v),(\eta_k)+n;\end{matrix}\quad x,y\right]$$
$$=\frac{\prod_{j=1}^{\rho}(\gamma_j)_N}{\prod_{j=1}^{\sigma}(\delta_j)_N}\sum_{l,m=0}^{l+m\leqslant N}\binom{N}{l+m}(l+m)!\frac{\prod_{j=1}^{p}(\alpha_j)_{l+m}\prod_{j=1}^{h}(\xi_j)_{l+m}}{\prod_{j=1}^{q}(\beta_j)_{l+m}\prod_{j=1}^{k}(\eta_j)_{l+m}}$$

$$\cdot \frac{\prod_{j=1}^{r}(a_j)_l \prod_{j=1}^{u}(c_j)_m}{\prod_{j=1}^{s}(b_j)_l \prod_{j=1}^{v}(d_j)_m} \frac{Y^l}{l!} \frac{(-y)^m}{m!}$$

$$\cdot {}_{1+h+\sigma}F_{k+\rho}\left[\begin{matrix} -N+l+m, & (\xi_h)+l+m, 1-(\delta_\sigma)-N; \\ & (\eta_k)+l+m, 1-(\gamma_\rho)-N; \end{matrix} \; (-1)^{\rho-\sigma}\frac{Y}{x}\right],$$

provided that each side exists, it being understood (in addition to the conditions stated already) that $x \neq 0$, and $1-\gamma_j - N \neq 0, -1, -2, \ldots$ $(j = 1, \ldots, \rho)$.

PROOF. For convenience, let

$$(2.4) \qquad \lambda_n = \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}, \qquad \mu_n = \frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n}, \qquad \nu_n = \frac{\prod_{j=1}^{u}(c_j)_n}{\prod_{j=1}^{v}(d_j)_n},$$

$$(2.5) \qquad \theta_n = \frac{\prod_{j=1}^{\rho}(\gamma_j)_n}{\prod_{j=1}^{\sigma}(\delta_j)_n}, \qquad \phi_n = \frac{\prod_{j=1}^{h}(\xi_j)_n}{\prod_{j=1}^{k}(\eta_j)_n}, \qquad n = 0, 1, 2, \ldots,$$

and denote the left-hand side of (2.3) by S. Then, making use of the definition (2.1), we readily have

$$S = \sum_{n=0}^{N}\binom{N}{n}\left(-\frac{Y}{x}\right)^n \sum_{l=0}^{n}\binom{n}{l}\mu_l(-x)^l \sum_{m=0}^{N-n}\binom{N-n}{m}\lambda_{l+m}\nu_m\theta_{N-n+l}\phi_{m+n}(-y)^m$$

$$= N! \sum_{l,m\geqslant 0} \lambda_{l+m}\mu_l\nu_m \frac{(-x)^l}{l!}\frac{(-y)^m}{m!} \sum_{n=l}^{N-m} \frac{\theta_{N-n+l}\,\phi_{m+n}}{(n-l)!(N-m-n)!}\left(-\frac{Y}{x}\right)^n,$$

and, upon replacing n by $n+l$, we find that

$$(2.6) \qquad S = N! \sum_{l,m=0}^{l+m\leqslant N} \frac{\lambda_{l+m}\mu_l\nu_m}{(N-l-m)!}\frac{Y^l}{l!}\frac{(-y)^m}{m!}$$

$$\cdot \sum_{n=0}^{N-l-m}\binom{N-l-m}{n}\theta_{N-n}\,\phi_{l+m+n}\left(-\frac{Y}{x}\right)^n.$$

Since

$$\theta_{N-n} = (-1)^{(\rho-\sigma)N}\,\theta_N \frac{\prod_{j=1}^{\sigma}(1-\delta_j-N)_n}{\prod_{j=1}^{\rho}(1-\gamma_j-N)_n}, \qquad 0 \leqslant n \leqslant N, \tag{2.7}$$

and

$$\phi_{l+m+n} = \phi_{l+m} \frac{\prod_{j=1}^{h}(\xi_j+l+m)_n}{\prod_{j=1}^{k}(\eta_j+l+m)_n}, \tag{2.8}$$

the right-hand side of (2.3) follows when we substitute from (2.7) and (2.8) into (2.6), and rewrite the innermost sum as a generalized hypergeometric polynomial.

Remark 1. In view of the elementary identity

$$\sum_{l,m=0}^{\infty} f(l+m)\frac{x^l}{l!}\frac{y^m}{m!} = \sum_{l=0}^{\infty} f(l)\frac{(x+y)^l}{l!}, \tag{2.9}$$

the second member of (2.3) simplifies considerably when we set $r = s = u = v = 0$, and we thus obtain the summation formula:

$$\sum_{n=0}^{N}\binom{N}{n}\frac{\prod_{j=1}^{\rho}(\gamma_j)_{N-n}\prod_{j=1}^{h}(\xi_j)_n}{\prod_{j=1}^{\sigma}(\delta_j)_{N-n}\prod_{j=1}^{k}(\eta_j)_n}\left(-\frac{Y}{x}\right)^n \tag{2.10}$$

$$\cdot F^{p:1+\rho;1+h}_{q:\ \ \sigma;\ \ k}\left[\begin{matrix}(\alpha_p):-n, & (\gamma_\rho)+N-n;-N+n, & (\xi_h)+n; \\ (\beta_q): & (\delta_\sigma)+N-n; & (\eta_k)+n;\end{matrix}\ x,y\right]$$

$$= \frac{\prod_{j=1}^{\rho}(\gamma_j)_N}{\prod_{j=1}^{\sigma}(\delta_j)_N} F^{1+h:p;\sigma}_{k:q;\rho}\left[\begin{matrix}-N, & (\xi_h):(\alpha_p);1-(\delta_\sigma)-N; \\ & (\eta_k):(\beta_q);1-(\gamma_\rho)-N;\end{matrix}\ -Y+y,(-1)^{\rho-\sigma}\frac{Y}{x}\right],$$

in terms of a terminating version of the generalized Kampé de Fériet series defined by (2.1).

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REMARK 2. For $Y = y$, (2.10) reduces immediately to the elegant form:

$$(2.11)\quad \sum_{n=0}^{N}\binom{N}{n}\frac{\prod_{j=1}^{\rho}(\gamma_j)_{N-n}\prod_{j=1}^{h}(\xi_j)_n}{\prod_{j=1}^{\sigma}(\delta_j)_{N-n}\prod_{j=1}^{k}(\eta_j)_n}\left(-\frac{y}{x}\right)^n$$

$$\cdot F^{p:1+\rho;1+h}_{q:\ \sigma;\ k}\left[\begin{matrix}(\alpha_p): -n, & (\gamma_\rho)+N-n; -N+n, & (\xi_h)+n; \\ (\beta_q): & (\delta_\sigma)+N-n; & (\eta_k)+n;\end{matrix}\ x, y\right]$$

$$= \frac{\prod_{j=1}^{\rho}(\gamma_j)_N}{\prod_{j=1}^{\sigma}(\delta_j)_N}\, {}_{1+h+\sigma}F_{k+\rho}\left[\begin{matrix}-N, & (\xi_h), 1-(\delta_\sigma)-N; \\ & (\eta_k), 1-(\gamma_\rho)-N;\end{matrix}\ (-1)^{\rho-\sigma}\frac{y}{x}\right],$$

which evidently yields the summation formula (1.2) in the further special case when $p = q = 0$ and $h = k = \rho = \sigma = 1$.

REMARK 3. Yet another interesting special case of our formula (2.3) occurs when we set $h = k = \rho = \sigma = 0$ and $Y = x$. Since it is easily verified that

$$(2.12)\quad {}_1F_0\left[\begin{matrix}-N+l+m; \\ \overline{};\end{matrix}\ 1\right] = \sum_{n=0}^{N-l-m}(-1)^n\binom{N-l-m}{n}$$

$$= \begin{cases}1, & \text{if } l+m=N, \\ 0, & \text{otherwise,}\end{cases}$$

it follows from (2.3), in this case, that

$$(2.13)\quad \sum_{n=0}^{N}(-1)^n\binom{N}{n}F^{p:1+r;1+u}_{q:\ s;\ v}\left[\begin{matrix}(\alpha_p): -n, & (a_r); -N+n, & (c_u); \\ (\beta_q): & (b_s); & (d_v);\end{matrix}\ x, y\right]$$

$$= \frac{\prod_{j=1}^{p}(\alpha_j)_N\prod_{j=1}^{r}(a_j)_N}{\prod_{j=1}^{q}(\beta_j)_N\prod_{j=1}^{s}(b_j)_N}x^N\, {}_{1+s+u}F_{r+v}\left[\begin{matrix}-N, & 1-(b_s)-N, (c_u); \\ & 1-(a_r)-N, (d_v);\end{matrix}\ (-1)^{r-s}\frac{y}{x}\right],$$

provided that $1 - a_j - N \neq 0, -1, -2, \ldots$ $(j = 1, \ldots, r)$. The summation formula (1.3) is an obvious further special case of (2.13) when $p = q = 0$ and $r = s = u = v = 1$.

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Indeed, by suitably specializing each of the summation formulas (2.9), (2.11) and (2.13), we can deduce a fairly large number of results which are scattered throughout the literature (see [1], [2], and [4] for details).

3. Further generalizations and multivariable applications

A closer examination of the proof of the general hypergeometric summation formula (2.3), detailed in Section 2, suggests the existence of a much deeper further generalization involving double series with essentially arbitrary terms (subject, of course, to existence and convergence requirements). More generally, for every bounded multiple sequence

$$\{\Omega(k_1,\ldots,k_r;\, l,m,n,t)\}, \qquad k_j,l,m,n,t=0,1,2,\ldots\ (j=1,\ldots,r),$$

we can apply the proof of (2.3) *mutatis mutandis* in order to establish the following multivariable extension of (2.3):

(3.1)

$$\begin{aligned}\sum_{n=0}^{N}\binom{N}{n}\left(-\frac{Y}{x}\right)^{n}&\sum_{k_1,\ldots,k_r=0}^{\infty}\sum_{l=0}^{n}\sum_{m=0}^{N-n}\Omega(k_1,\ldots,k_r;\, l,m,N-n+l,m+n)\\&\cdot(-n)_l(-N+n)_m\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{x^l}{l!}\frac{y^m}{m!}\\=\sum_{k_1,\ldots,k_r=0}^{\infty}&\sum_{l,m=0}^{l+m\leqslant N}\binom{N}{l+m}(l+m)!\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{Y^l}{l!}\frac{(-y)^m}{m!}\\&\cdot\sum_{n=0}^{N-l-m}\binom{N-l-m}{n}\left(-\frac{Y}{x}\right)^{n}\Omega(k_1,\ldots,k_r;\, l,m,N-n,l+m+n),\end{aligned}$$

which holds true whenever each side exists.

Formula (3.1) reduces to the hypergeometric sum (2.3) in the special case when $z_1=\cdots=z_r=0$ and [see Equations (2.4) and (2.5)]

$$\Omega(0,\ldots,0;\, l,m,n,t)=\lambda_{l+m}\,\mu_l\,\nu_m\,\theta_n\,\phi_t \qquad (l,m,n,t=0,1,2,\ldots). \tag{3.2}$$

In view of the elementary series identity (2.9), a special case of (3.1) when

$$\Omega(k_1,\ldots,k_r;\, l,m,n,t)=\Delta(k_1,\ldots,k_r;\, l+m,n,t), \tag{3.3}$$
$$k_j,l,m,n,t=0,1,2,\ldots\ (j=1,\ldots,r),$$

where $\{\Delta(k_1,\ldots,k_r;\, l,m,n)\}$ is a bounded sequence of multiplicity $r+3$, yields the following multivariable extension of the general hypergeometric summation

formula (2.10):
(3.4)

$$\sum_{n=0}^{N}\binom{N}{n}\left(-\frac{Y}{x}\right)^n \sum_{k_1,\dots,k_r=0}^{\infty} \sum_{l=0}^{n}\sum_{m=0}^{N-n} \Delta(k_1,\dots,k_r;\, l+m, N-n+l, m+n)$$

$$\cdot(-n)_l(-N+n)_m \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{x^l}{l!}\frac{y^m}{m!}$$

$$= \sum_{k_1,\dots,k_r=0}^{\infty}\sum_{l=0}^{N}\binom{N}{l} l!\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{(Y-y)^l}{l!}$$

$$\cdot\sum_{n=0}^{N-l}\binom{N-l}{n}\left(-\frac{Y}{x}\right)^n \Delta(k_1,\dots,k_r;\, l, N-n, n+l),$$

which, for $Y = y$, reduces immediately to the elegant form:
(3.5)

$$\sum_{n=0}^{N}\binom{N}{n}\left(-\frac{y}{x}\right)^n \sum_{k_1,\dots,k_r=0}^{\infty} \sum_{l=0}^{n}\sum_{m=0}^{N-n} \Delta(k_1,\dots,k_r;\, l+m, N-n+l, m+n)$$

$$\cdot(-n)_l(-N+n)_m \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{x^l}{l!}\frac{y^m}{m!}$$

$$= \sum_{n=0}^{N}\binom{N}{n}\left(-\frac{y}{x}\right)^n \sum_{k_1,\dots,k_r=0}^{\infty} \Delta(k_1,\dots,k_r;\, 0, N-n, n)\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}.$$

For $z_1 = \cdots = z_r = 0$, this last summation formula (3.5) would provide a generalization of the hypergeometric sum (2.11) to double series with essentially arbitrary terms. A similar multivariable extension of the hypergeometric summation formula (2.13) follows from (3.1) upon setting $Y = x$ and

(3.6) $$\Omega(k_1,\dots,k_r;\, l,m,n,t) = \Lambda(k_1,\dots,k_r;\, l,m),$$
$$k_j, l, m, n, t = 0,1,2,\dots \quad (j = 1,\dots,r),$$

where $\{\Lambda(k_1,\dots,k_r;\, l,m)\}$ is a bounded sequence of multiplicity $r+2$; making use of the identity (2.12), we thus obtain the summation formula:

(3.7) $$\sum_{n=0}^{N}(-1)^n\binom{N}{n} \sum_{k_1,\dots,k_r=0}^{\infty} \sum_{l=0}^{n}\sum_{m=0}^{N-n} \Lambda(k_1,\dots,k_r;\, l,m)$$

$$\cdot(-n)_l(-N+n)_m \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{x^l}{l!}\frac{y^m}{m!}$$

$$= x^N\sum_{n=0}^{N}\binom{N}{n}\left(-\frac{y}{x}\right)^n \sum_{k_1,\dots,k_r=0}^{\infty} \Lambda(k_1,\dots,k_r;\, N-n, n)\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!},$$

provided that both sides exist.

By appropriately specializing the multiple sequences involved, each of our general results (3.1), (3.4), (3.5) and (3.7) can now be applied with a view to deriving the corresponding finite summation formulas for various classes of hypergeometric series (and polynomials) in three and more variables, such as the (Srivastava-Daoust) generalized Lauricella series in $r + 2$ variables (see [6, page 37, Equation (21) *et seq.*]). For example, in terms of Srivastava's general triple hypergeometric series $F^{(3)}[x, y, z]$ (see [5, page 428]; see also [6, page 44, Equation (14) *et seq.*]), the special case $r = 1$ of our last result (3.7) can be readily applied to deduce the following three-variable generalization of the hypergeometric summation formula (2.13):

$$\text{(3.8)}\quad \sum_{n=0}^{N}(-1)^n\binom{N}{n}F^{(3)}\left[\begin{matrix}(\alpha_p)::(\gamma_\rho);\,(\xi_h);\,\text{———}: & -n,\ (a_r);\,-N+n,\ (c_u);\,(e_\tau); \\ (\beta_q)::(\delta_\sigma);\,(\eta_k);\,\text{———}: & (b_s);\quad (d_v);\,(f_\omega);\end{matrix}\ x, y, z\right]$$

$$= \frac{\prod_{j=1}^{p}(\alpha_j)_N\prod_{j=1}^{\rho}(\gamma_j)_N\prod_{j=1}^{r}(a_j)_N}{\prod_{j=1}^{q}(\beta_j)_N\prod_{j=1}^{\sigma}(\delta_j)_N\prod_{j=1}^{s}(b_j)_N}x^N$$

$$\cdot F^{h:1+s+u;p+\tau}_{k:\quad r+v;q+\omega}\left[\begin{matrix}(\xi_h):-N, & 1-(b_s)-N,(c_u); & (\alpha_p)+N,(e_\tau); \\ (\eta_k): & 1-(a_r)-N,(d_v); & (\beta_q)+N,(f_\omega);\end{matrix}\ (-1)^{r-s}\frac{y}{x}, z\right],$$

where a horizontal dash indicates an empty set of parameters, and (as before) $x \neq 0$ and $1 - a_j - N \neq 0, -1, -2, \ldots$ $(j = 1, \ldots, r)$.

The hypergeometric summation formula (3.8), which indeed is contained in such substantially deeper results as (3.1) and (3.7), unifies and generalizes scores of hitherto scattered results in the literature (see [3] for details). It corresponds, when $z = 0$, to the hypergeometric summation (2.13). More importantly, (3.8)

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reduces, when $h = k = 0$, to the interesting form:

$$(3.9)\quad \sum_{n=0}^{N} (-1)^n \binom{N}{n} F^{(3)} \left[\begin{matrix} (\alpha_p)::(\gamma_\rho); \text{———}; \text{———}: & -n, & (a_r); -N+n, & (c_u); (e_\tau); & \\ & & & & x, y, z \\ (\beta_q)::(\delta_\sigma); \text{———}; \text{———}: & & (b_s); & (d_v); (f_\omega); & \end{matrix}\right]$$

$$= \frac{\prod_{j=1}^{p} (\alpha_j)_N \prod_{j=1}^{\rho} (\gamma_j)_N \prod_{j=1}^{r} (a_j)_N}{\prod_{j=1}^{q} (\beta_j)_N \prod_{j=1}^{\sigma} (\delta_j)_N \prod_{j=1}^{s} (b_j)_N} x^N {}_{p+\tau}F_{q+\omega}\left[\begin{matrix} (\alpha_p)+N, (e_\tau); & \\ & z \\ (\beta_q)+N, (f_\omega); & \end{matrix}\right]$$

$$\cdot {}_{1+s+u}F_{r+v}\left[\begin{matrix} -N, & 1-(b_s)-N, (c_u); & \\ & & (-1)^{r-s}\dfrac{y}{x} \\ & 1-(a_r)-N, (d_v); & \end{matrix}\right],$$

which (in conjunction with various known summation theorems for generalized hypergeometric series [6]) is capable of yielding numerous results of possible use in applied mathematics and theoretical physics. Indeed, it is well known that hypergeometrtic series (and hypergeometric polynomials) in one and more variables occur rather frequently in a wide variety of problems in applied mathematics and theoretical physics, and also in engineering sciences, statistics, and operations research (see, for examples, Srivastava and Karlsson [6, Section 1.7] and the various references cited there). For instance, a considerably vast field of physical and quantum chemical situations lead naturally to such special cases of the hypergeometric polynomials

$$(3.10)\qquad \Omega_n(x) = {}_{p+1}F_q\left[\begin{matrix} -n, & \alpha_1, \dots, \alpha_p; & \\ & & x \\ & \beta_1, \dots, \beta_q; & \end{matrix}\right]$$

as the Bessel polynomials and the classical orthogonal polynomials including, for example, Hermite, Jacobi (and, of course, Gegenbauer, Legendre, and Tchebycheff), and Laguerre polynomials, many of which occur, among numerous other places, in Schrödinger's wave mechanics. In particular, in the evaluation of multicentre integrals in certain variational calculations of molecular electron structure, one often encounters such integrals as

$$(3.11)\qquad I_r(x_1, \dots, x_r) = \int_0^\infty t^{\lambda-1} e^{-t} L_{n_1}^{(\alpha_1)}(x_1 t) \cdots L_{n_r}^{(\alpha_r)}(x_r t)\, dt,$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials

$$(3.12) \qquad L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} {}_1F_1\left[\begin{matrix} -n; \\ \alpha+1; \end{matrix} \; x\right],$$

which satisfy a large number of useful summation (or addition) formulas including

$$(3.13) \qquad \sum_{n=0}^{N} L_n^{(\alpha-n)}(x) L_{N-n}^{(\beta+n)}(y) = L_N^{(\alpha+\beta)}(x+y),$$

$$(3.14) \qquad \sum_{n=0}^{N} \frac{(-1)^n}{(\alpha+1)_n(\beta+1)_{N-n}} L_n^{(\alpha)}(x) L_{N-n}^{(\beta)}(y)$$
$$= \frac{(-1)^N (x+y)^N}{(\alpha+1)_N(\beta+1)_N} P_N^{(\alpha,\beta)}\left(\frac{y-x}{y+x}\right),$$

and

$$(3.15) \qquad L_{m_1}^{(\alpha_1)}(x_1 t) \cdots L_{m_r}^{(\alpha_r)}(x_r t) = \sum_{k=0}^{m_1+\cdots+m_r} \gamma_k(x_1, \ldots, x_r) L_k^{(\alpha)}(t),$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomial and $\gamma_k(x_1, \ldots, x_r)$ is a certain hypergeometric polynomial in r variables.

In the special case when $r = 2$, if we replace the Laguerre polynomials by a substantially more general hypergeometric polynomial of the type defined by (3.10), the integral in (3.11) would lead to a double hypergeometric polynomial belonging to the class considered in this paper, and the needed results like (3.13) and (3.14) would follow upon suitably specializing the finite summation formulas which we have presented here.

We conclude by remarking that such summation (or addition) formulas as the ones given in this as well as the preceding sections, involving hypergeometric polynomials in one and more variables, would considerably facilitate the calculation of various multicentre integrals associated with hydrogenic-like functions.

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