ON THE ZEROS OF THE FRESNEL INTEGRALS

ERWIN KREYSZIG

1. Introduction. This paper is concerned with the Fresnel integrals

$$C(u) = \int_0^u \cos(p^2)\,dp, \quad S(u) = \int_0^u \sin(p^2)\,dp \tag{1.1}$$

in the complex domain.

Recent research work in different fields of physical and technical applications of mathematics shows that an increasing number of problems require a detailed knowledge of elementary and higher functions for complex values of the argument. The Fresnel integrals, introduced by A. J. Fresnel (1788–1827) in connection with diffraction problems, are among these functions; a small collection of papers of the above-mentioned kind is included in the bibliography at the end of this paper **(3; 5; 7; 12; 13; 17; 19; 20; 22)**. Moreover, the Fresnel integrals are important since various types of more complicated integrals can be reduced **(6)** to analytic expressions involving $C(u)$ and $S(u)$.

At the present time the detailed investigation of special functions for complex argument is still in its infancy. It has been limited, until now, to certain classes of functions, especially to those which have the advantage of possessing simple functional relations or of satisfying an ordinary differential equation; in the latter case the theory of differential equations can be used when considering these functions. The Gamma function and the Bessel functions, respectively, are of this kind. The Fresnel integrals do not possess these advantageous properties and must therefore be treated by other methods.

The Fresnel integrals have been considered from different points of view **(1; 2; 4; 8; 9; 14; 16; 18)**, but, until a short time ago, for a real argument only. The first two investigations **(10; 11)** of these functions for complex values of the argument include some initial results about the zeros and also two small tables of function values.

In this paper we shall prove some lemmas which yield a much more refined knowledge of the two integrals under consideration. Furthermore, we shall indicate relations to other known functions and develop new methods for investigating and computing the zeros of these integrals. We shall find large domains of the complex plane which cannot contain a zero of the Fresnel integrals. When determining the position of the zeros of a function it is always important to find (more or less accurate) approximate values for those zeros; then the computation up to the desired degree of accuracy can be done

Received May 12, 1956.

This work was supported by the National Research Council of Canada.

schematically by means of the usual iterative methods. We shall see that in the case of the Fresnel integrals such approximate values (which are even of great accuracy) can be obtained in a simple manner. Also the more exact determination of the zeros will turn out to be relatively easy if appropriate representations of these functions are used. A table of the values of some zeros of the Fresnel integrals can be found at the end of the paper.

2. Fundamental relations, asymptotic behaviour. It is advantageous to transform the integrals (1.1) by means of the substitution $p^2 = t$. In this manner we obtain

$$C(u) = \tfrac{1}{2}\int_0^{u^2} t^{-\frac{1}{2}}\cos t\,dt, \quad S(u) = \tfrac{1}{2}\int_0^{u^2} t^{-\frac{1}{2}}\sin t\,dt.$$

Since we will primarily investigate the zeros of those functions the factor $\frac{1}{2}$ becomes unessential, and we will therefore omit it. We write

$$C(z) = \int_0^{z} t^{-\frac{1}{2}}\cos t\,dt, \quad S(z) = \int_0^{z} t^{-\frac{1}{2}}\sin t\,dt \tag{2.1}$$

where $z = x + iy$ denotes a complex variable. The representations (2.1) will be used in what follows.

For finite values of $|z|$ the Taylor series development of the integrands of (2.1) at $z = 0$ may be integrated term by term. We find

$$C(z) = z^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!\,(2m+\frac{1}{2})} z^{2m}, \tag{2.2}$$

$$S(z) = z^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\,(2m+\frac{3}{2})} z^{2m+1}.$$

The functions $z^{-\frac{1}{2}}C(z)$ and $z^{-\frac{1}{2}}S(z)$ are entire transcendental functions. $C(z)$ and $S(z)$ have a branch point at $z = 0$; from (2.2) we obtain the relations

$$C(z\,e^{ik\pi}) = e^{ik\pi/2}C(z), \quad S(z\,e^{ik\pi}) = e^{-ik\pi/2}\,S(z). \tag{2.3}$$

Furthermore,

$$C(\bar{z}) = \overline{C(z)}, \quad S(\bar{z}) = \overline{S(z)}. \tag{2.4}$$

Hence we may limit our considerations to the first quadrant ($x \geqslant 0$, $y \geqslant 0$) of the z-plane.

We now consider the asymptotic behaviour of the Fresnel integrals. In order that the limits

$$C = \lim_{z\to\infty}\int_0^{z} t^{-\frac{1}{2}}\cos t\,dt, \quad S = \lim_{z\to\infty}\int_0^{z} t^{-\frac{1}{2}}\sin t\,dt \tag{2.5}$$

exist, we must choose a path of integration which goes asymptotically parallel to the real axis ($y = 0$) to infinity. Then C and S have a uniquely determined finite value; transforming Euler's integral representation of the Gamma function in a suitable manner we find

$$C = S = 2^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}) = \sqrt{(\pi/2)} = 1.253\,314\,1\ldots \tag{2.6}$$

Using such a path of integration, we have

$$C(z) + c(z) = C, \quad S(z) + s(z) = S, \tag{2.7}$$

where

$$c(z) = \int_z^{\infty} t^{-\frac{1}{2}}\cos t \, dt, \quad s(z) = \int_z^{\infty} t^{-\frac{1}{2}}\sin t \, dt. \tag{2.8}$$

Integrating (2.8) by parts and using (2.7), we obtain the following series representation of the Fresnel integrals:

$$\begin{aligned} &\text{(a)} \quad C(z) \sim C + z^{-\frac{1}{2}}(a(z)\cos z + b(z)\sin z), \\ &\text{(b)} \quad S(z) \sim S + z^{-\frac{1}{2}}(-b(z)\cos z + a(z)\sin z) \end{aligned} \tag{2.9}$$

where

$$a(z) = \sum_{m=1}^{\infty} (-1)^m \frac{1.3\ldots(4m-3)}{(2z)^{2m-1}}, \; b(z) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1.3\ldots(4m-1)}{(2z)^{2m}}.$$

LEMMA 1. *The series* (2.9) *are asymptotic expansions of* $C(z)$ *and* $S(z)$, *respectively, for all complex values of* z *with the exception of pure imaginary ones.*

Proof. Let us assume that we have integrated (2.7) by parts a number of times so that the integrated term finally obtained involves the power $z^{-m+\frac{1}{2}}$. Then, when neglecting some numerical factor which does not interest us, the remaining integral is of the form

$$\int_z^{\infty} t^{-(m+\frac{1}{2})}\cos t \, dt \quad \text{or} \quad \int_z^{\infty} t^{-(m+\frac{1}{2})}\sin t \, dt.$$

We thus have to estimate the integrals

$$K_1 = \int_z^{\infty} t^{-(m+\frac{1}{2})} e^{it} dt, \quad K_2 = \int_z^{\infty} t^{-(m+\frac{1}{2})} e^{-it} dt.$$

Setting $t = z + iw$ and $t = z - iw$, respectively, we obtain

$$K_1 = ie^{iz} z^{-(m+\frac{1}{2})} \int_0^{\infty} (1 + iw/z)^{-(m+\frac{1}{2})} e^{-w} dw,$$

$$K_2 = -\, ie^{-iz} z^{-(m+\frac{1}{2})} \int_0^{\infty} (1 - iw/z)^{-(m+\frac{1}{2})} e^{-w} dw.$$

If z is a not a pure imaginary number, i.e. $|\arg z| \leqslant \frac{1}{2}\pi - \gamma$ or $|\arg z| \geqslant \frac{1}{2}\pi + \gamma$, $\gamma > 0$, the inequality

$$|1 \pm iw/z| \geqslant \sin \gamma$$

holds, and therefore

$$\left| \int_0^{\infty} (1 \pm iw/z)^{-(m+\frac{1}{2})} e^{-w} dw \right| \leqslant (\operatorname{cosec} \gamma)^{m+\frac{1}{2}}.$$

Hence

$$e^{-iz} K_1 = O(z^{-(m+\frac{1}{2})}), \quad e^{iz} K_2 = O(z^{-(m+\frac{1}{2})}).$$

https://doi.org/10.4153/CJM-1957-016-9 Published online by Cambridge University Press

The constant involved in the Landau symbol does not depend on $\arg z$ but depends on γ and tends to infinity if γ tends to zero.

A value of z being given, the greatest possible accuracy is obtained if the number of terms of (2.9) is chosen so that the last term corresponds to the highest value of m for which

$$m \leqslant \tfrac{1}{2}\{(|z|^2 + \tfrac{1}{4})^{\frac{1}{2}} + 1\} \tag{2.10}$$

still holds, as can easily be seen.

3. Relations to other known functions. The relations of the Fresnel integrals to the incomplete gamma functions

$$\text{(3.1)}\quad \text{(a) } P(\phi, z) = \int_0^z e^{-t}t^{\phi-1}dt, \quad \text{(b) } Q(\phi, z) = \int_z^\infty e^{-t}t^{\phi-1}dt = \Gamma(\phi) - P(\phi,z)$$

are of basic importance. Setting $\phi = \frac{1}{2}$ and substituting $t = iw$ and $t = -iw$, respectively, we obtain from (3.1a)

$$\begin{aligned} C(z) &= \tfrac{1}{2}[i^{-\frac{1}{2}}P(\tfrac{1}{2}, iz) + i^{\frac{1}{2}}P(\tfrac{1}{2}, -iz)], \\ S(z) &= \tfrac{1}{2}[i^{\frac{1}{2}}P(\tfrac{1}{2}, iz) + i^{-\frac{1}{2}}P(\tfrac{1}{2}, -iz)]. \end{aligned} \tag{3.2}$$

When computing a table of a special function the situation is very often as follows: For small arguments the Taylor series development at $z = 0$ can be used and for large values of $|z|$ the asymptotic expansion permits a simple calculation. The remaining difficulty consists in determining function values for arguments which are not very close to $z = 0$ but are too small to be calculated exactly enough by means of the asymptotic expansion. With respect to the Fresnel integrals we are just in such a situation, but we can overcome the difficulty by using the Nielsen representation **(16**, p. 84**)**:

$$Q(\phi, z + h) = Q(\phi, z) - e^{-z}\sum_{m=0}^{\infty}(-1)^m\binom{m - \phi}{\phi}\frac{P(m + 1, h)}{z^{m+1-\phi}}. \tag{3.3}$$

Setting $\phi = \frac{1}{2}$, $t = iw$, and $t = -iw$, respectively we obtain from (3.1b),

$$\begin{aligned} c(z) &= \tfrac{1}{2}[i^{-\frac{1}{2}}Q(\tfrac{1}{2}, iz) + i^{\frac{1}{2}}Q(\tfrac{1}{2}, -iz)], \\ s(z) &= \tfrac{1}{2}[i^{\frac{1}{2}}Q(\tfrac{1}{2}, iz) + i^{-\frac{1}{2}}Q(\tfrac{1}{2}, -iz)], \end{aligned} \tag{3.4}$$

and from this and (3.3),

$$\begin{aligned} c(z + h) &= c(z) + i2^{-1}z^{-\frac{1}{2}}(P_1(z) - P_2(z)), \\ s(z + h) &= s(z) - 2^{-1}z^{-\frac{1}{2}}(P_1(z) + P_2(z)) \end{aligned} \tag{3.5}$$

where

$$\text{(3.6)}\quad P_1(z) = e^{-iz}\sum_{m=0}^{\infty} i^m\binom{m - \frac{1}{2}}{\frac{1}{2}}\frac{P(m + 1, ih)}{z^m},$$

$$P_2(z) = e^{iz}\sum_{m=0}^{\infty} i^{-m}\binom{m - \frac{1}{2}}{\frac{1}{2}}\frac{P(m + 1, -ih)}{z^m}.$$

https://doi.org/10.4153/CJM-1957-016-9 Published online by Cambridge University Press

By means of (2.8) and (3.5) we are able to calculate the first differences of the function values of $C(z)$ and $S(z)$. Starting then from function values which can be simply obtained by using (2.2) or (2.9) we can immediately compute the desired function values of the Fresnel integrals. For the above-mentioned "medium" values of the argument this procedure is much better than a direct calculation by means of (2.2). From (3.1) we have

$$(3.7)\qquad \text{(a) } P(1, h) = 1 - e^{-h}, \quad \text{(b) } P(m + 1, h) = \int_0^h e^{-t}t^m dt.$$

Starting from (3.7a) and using the recurrence relation

$$P(m + 1, h) = mP(m, h) - e^{-h}h^m,$$

the functions $P(m + 1, ih)$ and $P(m + 1, -ih)$ occurring in (3.6) can be easily calculated. It is advantageous, of course, to choose a fixed value of h for a certain computation.

For the sake of completeness we mention also the following relations: using the integral representation (**15**, p. 87)

$${}_1F_1(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c - a)} z^{1-c} \int_0^z e^t t^{a-1}(z - t)^{c-a-1}dt$$

of Kummer's function ${}_1F_1(a, c; z)$, setting $a = \frac{1}{2}$, $c = \frac{3}{2}$ and $t = \pm iw$, respectively, we obtain

$$(3.8)\qquad \begin{aligned} C(z) &= \sqrt{z}[{}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}\,; -iz) + {}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}\,; iz)] \\ S(z) &= i\sqrt{z}[{}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}; -iz) - {}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}; iz)]. \end{aligned}$$

It is known that the Fresnel integrals are also related to the error function

$$\phi(z) = \frac{2}{\pi}\int_0^z e^{-t^2}dt;$$

substituting $t = \sqrt{(\pm iw)}$ we have

$$(3.9)\qquad \begin{aligned} C(z) &= \tfrac{1}{2}\pi[i^{\frac{1}{2}}\phi(\sqrt{-iz}) + i^{-\frac{1}{2}}\phi(\sqrt{iz})] \\ S(z) &= \tfrac{1}{2}\pi[i^{-\frac{1}{2}}\phi(\sqrt{-iz}) + i^{\frac{1}{2}}\phi(\sqrt{iz})]. \end{aligned}$$

4. Domains which cannot contain zeros. Let us first give a simple proof of the fact that all zeros $z(\neq 0)$ of the Fresnel integrals must be complex.

LEMMA 2. *The Fresnel integrals do not vanish for any real or purely imaginary argument different from zero. All zeros of these functions are simple and conjugate complex to each other in pairs.*

Proof. In consequence of (2.3) we may consider positive values of x only. From the form of the integrand of (2.1) it follows that

$$(4.1)\qquad \begin{aligned} &\text{(a)} \begin{cases} C((4n + 1)\pi/2) - C((4n - 1)\pi/2) > 0, & n = 1, 2, \ldots, \\ C((4n + 3)\pi/2) - C((4n + 1)\pi/2) < 0, & \\ \end{cases} \\ &\text{(b)} \begin{cases} S((2n + 1)\pi) - S(2n\pi) > 0, & n = 0, 1, \ldots, \\ S(2n\pi) - S((2n - 1)\pi) < 0, & n = 1, 2, \ldots. \end{cases} \end{aligned}$$

Since $x^{-\frac{1}{2}}$ is monotone,

$$
(4.2)\quad
\begin{aligned}
&\text{(a)}\ |C((2n+3)\pi/2) - C((2n+1)\pi/2)| < |C((2n+1)\pi/2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - C((2n-1)\pi/2)| \\
&\text{(b)}\ |S((n+1)\pi) - S(n\pi)| < |S(n\pi) - S((n-1)\pi)|, \quad n = 1, 2, \ldots .
\end{aligned}
$$

From (4.1b), (4.2b), and $S(0) = 0$ it follows that $S(x) \neq 0$ for any real value of $x \neq 0$. In order to draw the same conclusion with respect to $C(x)$ from (4.1a) and (4.2a) we have to prove that $C(3\pi/2) > 0$. Using (2.6), (2.8) and integrating by parts, we find

$$C(3\pi/2) = \sqrt{(\pi/2)} - \sqrt{(2/3\pi)} + \tfrac{3}{4}\int_{3\pi/2}^{\infty} t^{-5/2}\cos t\, dt.$$

Since $t^{-5/2}$ is monotone,

$$\left| \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} t^{-5/2}\cos t\, dt \right| > \left| \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} t^{-5/2}\cos t\, dt \right|, \qquad n = 1, 2, \ldots .$$

Consequently

$$\int_{3\pi/2}^{\infty} t^{-5/2}\cos t\, dt > 0.$$

Hence $C(3\pi/2) > 0$. This completes the proof that $C(z)$ cannot vanish for real values of x ($\neq 0$). If z is a pure imaginary number all terms in (2.2) have the same sign; hence there cannot exist a pure imaginary zero of the Fresnel integrals. The existence of zeros follows from the fact that $z^{\frac{1}{2}}C(z)$ and $z^{\frac{1}{2}}S(z)$ are entire functions which are not of the kind of an exponential function. Since the integrands of the Fresnel integral have real zeros only the zeros of the integrals are simple. Since (2.2) has real coefficients the zeros of the Fresnel integrals are conjugate complex in pairs. This completes the proof of Lemma 2.

We now consider the possibility of limiting the zeros of $C(z)$ and $S(z)$ to certain domains of the complex plane.

THEOREM 3. *The Fresnel integral $C(z)$ cannot possess zeros in any of the strips which are parallel to the y-axis and correspond to the values*

$$0 < x \leqslant \pi, \qquad (4n-1)\pi/2 \leqslant x \leqslant (2n+1)\pi, \qquad n = 1, 2, \ldots .$$

The same is true for $S(z)$ with respect to the strips parallel to the y-axis and corresponding to the values

$$0 < x \leqslant 3\pi/2, \qquad 2n\pi \leqslant x \leqslant (4n+3)\pi/2, \qquad n = 1, 2, \ldots .$$

Proof. Because of (2.3) and (2.4) we may consider the first quadrant ($x < 0$, $y < 0$) of the z-plane only. In order to prove the first of the two statements we start from the integral

$$J = C(x+iy) - C(x) = \int_{x}^{x+iy} t^{-\frac{1}{2}}\cos t\, dt.$$

Setting $t = x + iw$ and $(x + iw)^{-\frac{1}{2}} = a + ib$ we obtain

$$\Re\, J = \sin x \int_0^y a \sinh w\, dw - \cos x \int_0^y b \cosh w\, dw,$$

$$\Im\, J = \sin x \int_0^y b \sinh w\, dw + \cos x \int_0^y a \cosh w\, dw.$$

Since $x > 0$ and $y > 0$ we have $a > 0$ and $b < 0$ for all values of z under consideration. $C(x)$ is real and not negative, cf. Lemma 2. We thus obtain

$$\Re\, J > 0,\ \Re\, C(z) > 0 \qquad (2n\pi \leqslant x \leqslant (4n + 1)\pi/2),$$

and

$$\Im\, J < 0 \qquad ((4n + 1)\pi/2 \leqslant x \leqslant (2n + 1)\pi),$$
$$\Im\, J > 0 \qquad ((4n + 3)\pi/2 \leqslant x \leqslant (2n + 2)\pi).$$

From this the statement on $C(z)$ follows. The second part of Theorem 3 can be proved in a similar manner.

It should be noticed that the idea of the proof of Theorem 3 can be applied to more general integrals of the type

$$C(z, \alpha) = \int_0^z t^{-\alpha} \cos t\, dt, \qquad 0 < \alpha < 1 \tag{4.3}$$

in order to obtain the same result on the zeros. Also the integrals

$$S(z, \alpha) = \int_0^z t^{-\alpha} \sin t\, dt, \qquad 0 < \alpha < 1 \tag{4.4}$$

may be considered in this manner, but the method is not applicable to integrals (4.4) having values of α between 1 and 2 (exclusively), since in this case $\Re\, t^{-\alpha} > 0$ and $\Im\, t^{-\alpha} < 0$ may not hold. Indeed, for sufficiently large values of α (<2), $S(z, \alpha)$ has zeros also outside of the strips defined by

$$(4n - 1)\pi/2 \leqslant x \leqslant 2n\pi.$$

5. Formulas of approximation for the zeros. From Theorem 3 we can draw the important conclusion that all zeros of $C(z)$ and $S(z)$ are at a sufficiently large distance from the origin $z = 0$. This fact enables us to use the asymptotic expansion (2.9) for a more detailed investigation of those zeros.

As was pointed out in the introduction, it is always important to have approximation formulas for the position of the zeros of a function, since approximate values can yield the starting point for applying one of the usual iterative methods for a more accurate determination of those zeros. We will now derive simple approximation formulas for the zeros of $C(z)$ and $S(z)$.

In consequence of (2.9) the zeros of the equation

$$\sin z = -C\sqrt{z} \tag{5.1}$$

are first approximations of the zeros of $C(z)$. Setting $\sqrt{z} = p + iq$ and using (2.6), we obtain from (5.1)

$$p = -\sqrt{(2/\pi)}\, \sin x \cosh y, \qquad q = -\sqrt{(2/\pi)}\, \cos x \sinh y. \tag{5.2}$$

https://doi.org/10.4153/CJM-1957-016-9 Published online by Cambridge University Press

We consider the strip S_n: $(2n - 2)\pi \leqslant x \leqslant 2n\pi$, $y > 0$, which is parallel to the y-axis. In consequence of Theorem 3, only the part $S_n' \subset S_n$, defined by $(2n - 1)\pi \leqslant x \leqslant (4n - 1)\pi/2$, $y > 0$ can contain a complex zero of $C(z)$. Setting

$$H(z, \alpha) = \int_0^z t^{-\alpha} \cos t \, dt, \qquad \tfrac{1}{2} \geqslant \alpha \geqslant 0$$

we have $H(z, 0) = \sin z$ and $H(z, \frac{1}{2}) = C(z)$. $H(z, 0)$ has real zeros and $H(z, \frac{1}{2})$ has complex zeros only. Hence, if α decreases monotonely from $\frac{1}{2}$ to 0 then, for a certain value $\alpha_0 = \alpha_0(n)$, $\frac{1}{2} > \alpha_0 \geqslant 0$ we must have a real zero z_{α_0} of $H(z, \alpha)$ in S_n a first time. Since, for all values of α, $H(z, \alpha)$ has a minimum at $x = (4n - 1)\pi/2$ the zero z_{α_0} must coincide with that point. Hence, when denoting by x_n the real part of the zero of $C(z)$ in S_n' and setting

$$x_n = (4n - 1)\pi/2 - \gamma_n, \tag{5.3}$$

because of continuity γ_n must be a small (positive) quantity. We have

$$\cos x_n = -\gamma_n + 0(\gamma_n^3), \qquad \sin x_n = -1 + 0(\gamma_n^2),$$

where the functions indicated by the Landau symbol are small of higher order. The absolute values of the zeros of $C(z)$—even that of the smallest one—are relatively large, cf. Theorem 3. Hence the same is true for the corresponding quantities $|p|$. Setting

$$\cosh y = \tfrac{1}{2}e^y + O(e^{-y}), \quad \sinh y = \tfrac{1}{2}e^y + O(e^{-y}),$$

the second term is thus small in comparison with the first one. Omitting the functions indicated by the Landau symbols and using

$$x \equiv p^2 - q^2 = \frac{2}{\pi}(\sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y), \tag{5.4}$$

cf. (5.2), we obtain the following approximate expression $y_{0,n}$ for the imaginary part y_n of the zero of $C(z)$ which is located in S_n:

$$y_{0,n} = \log(\pi\sqrt{(4n - 1)}), \qquad n = 1, 2, \ldots. \tag{5.5}$$

From (5.3)–(5.5) and

$$y \equiv 2pq = \frac{1}{\pi} \sin 2x \sinh 2y,$$

cf. (5.2), we obtain a similar approximation $x_{0,n}$ for the real part x_n of the zero of $C(z)$ which is located in S_n:

$$x_{0,n} = \frac{(4n - 1)\pi}{2} - \frac{\log(\pi\sqrt{(4n - 1)})}{(4n - 1)\pi}, \qquad n = 1, 2, \ldots. \tag{5.6}$$

The degree of accuracy of these simple formulas (5.5) and (5.6) is relatively high. For the real and imaginary part of the smallest zero of $C(z)$ the error amounts to 2 per cent and 1 per cent, respectively. The accuracy increases rapidly with increasing values of $|z|$, as can easily be proved.

Comparing (5.5) and (5.6) we find

LEMMA 4. *The imaginary part y_n of the nth zero of $C(z)$ increases monotonely with n. We have*

$$\lim_{n\to\infty} y_n/x_n = 0. \tag{5.7}$$

The difference γ_n between x_n and $(4n - 1)\pi/2$ decreases monotonely with increasing n and tends to zero if n tends to infinity.

Using (5.1), the third term of (2.9) takes the form

$$\tfrac{1}{2} z^{-3/2}\cos z = i(2z)^{-1}\sqrt{(\tfrac{1}{2}\pi - z^{-1})}$$

which becomes arbitrarily small if $|z|$ increases arbitrarily. That is, if the desired accuracy is not too great one may restrict oneself to the first approximation of the zeros.

In consequence of (2.9) the zeros of the equation

$$\cos z = S\sqrt{z} \tag{5.8}$$

are the first approximation of the zeros $z_n{}^* = x_n{}^* + iy_n{}^*$ of $S(z)$. In a manner similar to the preceding one we obtain from (5.8)

$$\overset{*}{y}_{0,n} = \log\,(2\pi\sqrt{n}), \qquad n = 1, 2, \ldots \tag{5.9}$$

and

$$\overset{*}{x}_{0,n} = 2n\pi - \frac{\log\,(2\pi\sqrt{n})}{4n\pi}, \qquad n = 1, 2, \ldots. \tag{5.10}$$

These formulas are also of a relatively high degree of accuracy. For the smallest zero of $S(z)$ the error is about 5 per cent for the imaginary part and 1 per cent for the real part. The error is much smaller for the larger zeros, e.g. about 1 per cent for the imaginary part and 0.3 per cent for the real part of the second zero, etc.

From (5.9) and (5.10) we find

$$\lim_{n\to\infty} \overset{*}{y}_n/\overset{*}{x}_n = 0. \tag{5.11}$$

6. More exact determination of the zeros. In the preceding section first approximations of the zeros of the Fresnel integrals $C(z)$ and $S(z)$ were obtained from (2.9). As follows from Theorem 3 and (2.10) the expansion (2.9) permits also a more exact determination of those zeros. In the case of the smallest zero (2.10) yields $m = 2$, i.e. the greatest possible accuracy is obtained if we take the constant term and the next 4 terms of (2.9) and determine the smallest zero of the function thus obtained. In the case of the second zero we have from (2.10) $m = 6$, i.e. we have to take the constant term and the next 12 terms of (2.9), etc. However, even the simplest of the equations which we obtain in this manner is too complicated and cannot be solved immediately. But there is another way which will turn out to be very simple.

Let us first consider the Fresnel integral $C(z)$. We start from the values obtained from (5.5) and (5.6) and improve those values by applying the Newton method. The values $z_{1,n}$ thus obtained from $z_{0,n}$ are more accurate approximations of the zeros of (5.1). We now apply the Newton method several times and, from step to step, we always take into account one more term of (2.9a). Let us denote by $f_p(z)$ the function which is obtained by taking the constant term and the next p terms of (2.9a). The zero of the equation $f_p(z) = 0$ which is contained in the strip S_n' (cf. §5) will be denoted by $z_{p,n}$. The derivative of $f_p(z)$ is given by the simple expression

$$f_p'(z) = z^{-\frac{1}{2}}(\cos z + h_p(z)), \qquad p = 1, 2, \ldots, \tag{6.1}$$

where $h_p(z)$ is of the form $k_1 z^{-p} \sin z$ or $k_2 z^{-p} \cos z$, k_1 and k_2 denoting certain constants; all other terms drop out in pairs; $h_p(z)$ is small in comparison with $\cos z$. If $z_{p-1,n}$ denotes the zero of the equation $f_{p-1}(z) = 0$ in S_n' then $f_p(z_{p-1,n})$ consists of one term only, namely, of the last term of (2.9a) under consideration. The function $\tan z_{p,n}$, occurring as a factor in some of the Newton quotients f/f', may be replaced by i; this simplification is the same as that in the preceding section where we omitted the functions indicated by Landau symbols.

In the case of the Fresnel integral $S(z)$ the reasoning is exactly the same as in the case of $C(z)$.

The procedure yields a finite sequence $z_{1,n}, z_{2,n}, \ldots$ of approximative values of the zero z_n of $C(z)$. The terms of this sequence are recursively determined by the following simple relation:

$$z_{p+1,n} = z_{p,n} + c_p(2z_{p,n})^{-p}, \qquad p = 1, 2, \ldots, \tag{6.2}$$

where

$$c_{2q+1} = (-1)^q \, 1.3 \ldots (4q + 1), \qquad q = 0, 1, \ldots,$$
$$c_{2q} = (-1)^{q-1} \, 1.3 \ldots (4q - 1)i, \qquad q = 1, 2, \ldots.$$

For $S(z)$ we similarly find

$$\overset{*}{z}_{p+1,n} = \overset{*}{z}_{p,n} + c_p(2\overset{*}{z}_{p,n})^{-p}, \qquad p = 1, 2, \ldots, \tag{6.3}$$

where the constants c_p are the same as in the preceding formula. Of course the numerical values of the different correction terms are entirely different in both cases, since $z_{p,n}$ differs from the corresponding approximate value $z_{p,n}{}^*$. For every fixed value of p the corresponding correction term decreases monotonely with increasing n. Since, for fixed n and p, $|z_{p,n}{}^*| > |z_{p,n}|$, the absolute value of the correction term $c_p(2z_{p,n}{}^*)^{-p}$ is smaller than that of $c_p(2z_{p,n})^{-p}$, but greater than that of $c_p(2z_{p,n+1})^{-p}$.

7. Further properties of the zeros. From the preceding results we can draw some conclusions which might be of interest. Let us compare the zeros of $C(z)$ with those of $S(z)$. From (5.5), (5.6), (5.9), and (5.10) we find that not only the sequences $(y_{0,n})$, $(y_{0,n}{}^*)$, (γ_n), $(\gamma_n{}^*)$, where $\gamma_n{}^* = (4n\pi)^{-1}\log(2\pi\sqrt{n})$, are monotone, but also the sequences $y_{0,1}, y_{0,1}^*, y_{0,2}, y_{0,2}^*, \ldots$ and $\gamma_1, \gamma_1^*, \gamma_2, \gamma_2^*, \ldots$. We thus obtain

THEOREM 5. *The zeros of $C(z)$ and $S(z)$ are (asymptotically) located on one and the same logarithmic curve, in alternating order; this curve can be represented in the form*

$$y = \pm \tfrac{1}{2} \log 2\pi x. \tag{7.1}$$

In consequence of (5.3), (5.7) and (5.11), the series

$$\sum_{n=1}^{\infty} |z_n|^{-1}, \quad \sum_{n=1}^{\infty} |z_n^*|^{-1} \tag{7.2}$$

are minorants of the harmonic series. Hence the series

$$\sum_{n=1}^{\infty} |z_n|^{-1-\epsilon}, \quad \sum_{n=1}^{\infty} |z_n^*|^{-1-\epsilon}, \qquad \epsilon > 0, \tag{7.3}$$

converge, but the series (7.2) diverge. The functions $z^{\frac{1}{2}}C(z)$ and $z^{\frac{1}{2}}S(z)$ thus are entire functions of first order of divergence class.

According to the order of $z^{\frac{1}{2}}C(z)$ and $z^{\frac{1}{2}}S(z)$ the exponent of the exponential function contained in the Weierstrass product of these functions can at most be a linear function of z. It can be proved that, in our case, this function is actually a constant. Since

$$\lim_{|z|\to 0} z^{-1/2}C(z) = 2, \quad \lim_{|z|\to 0} z^{-3/2}S(z) = \tfrac{2}{3},$$

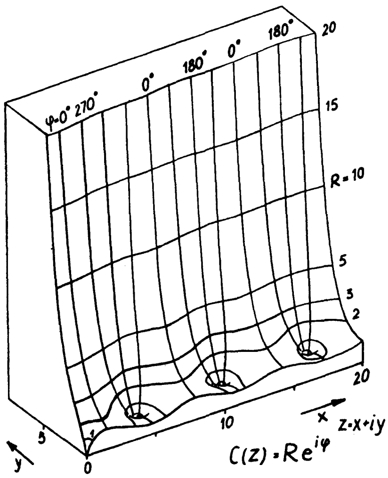
the Weierstrass products of the Fresnel integrals have the form

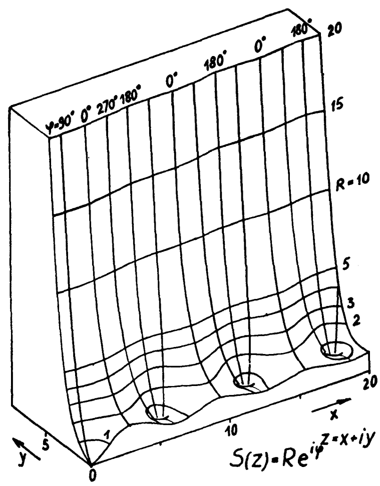
$$C(z) = 2z^{1/2}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{z_n^2}\right)\left(1 - \frac{z^2}{\bar{z}_n^2}\right), \tag{7.4}$$

$$S(z) = \tfrac{2}{3}\, z^{3/2}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{z_n^{*2}}\right)\left(1 - \frac{z^2}{\bar{z}_n^{*2}}\right).$$

8. Distribution of the function values of the Fresnel integrals. Tables of the function values of the Fresnel integrals for complex values of the argument have been communicated in **(10; 11)**. Both functions have a similar behaviour which can be most simply described by characterising the geometric form of the surfaces $F(C)$ and $F(S)$ of the absolute value of $C(z)$ and $S(z)$.

In consequence of the maximum principle, the real extreme values of $C(z)$ and $S(z)$ correspond to saddle points of $F(C)$ and $F(S)$. Because of (2.4), both surfaces are symmetric with respect to the x-axis. Any zero is a singular point of the surfaces which, in a small neighbourhood of such a point, behave like a circular cone Z; the angle between the generators of Z and the xy-plane is about $\frac{1}{4}\pi$, as can be seen from the Taylor series development of $C(z)$ and $S(z)$ at a zero under consideration. Any other point which corresponds to a finite value of z is a regular point of the surfaces. It can be seen from (2.9) that the surfaces ascend rapidly for large y. The tangent of the angle $\alpha(z)$





between the direction of maximum slope and the xy-plane is asymptotically given by the expression

$$\tan\alpha(z) = \tfrac{1}{2}|z|^{-\frac{1}{2}}e^{y}. \tag{8.1}$$

Along the lines y = const., the maximal slope decreases with increasing x. There exist real points of inflection of the curves $C(x)$ and $S(x)$ at $x = n\pi - \delta(n)$ and $x = (2n+1)\pi/2 - \delta^*(n)$, respectively, where $\delta(n)$ and $\delta^*(n)$ are positive and monotonely decreasing functions of n. All these points are isolated parabolic points of the surfaces $F(C)$ and $F(S)$. The surfaces consist of domains of positive and negative Gaussian curvature which are bounded and separated from each other by curves whose points are parabolic ("parabolic curves"). These curves can be obtained from

$$\Re(f'^2/f''f) - 1 = 0, \tag{8.2}$$

(cf. **21**), where $f = C(z)$ or $S(z)$, respectively. Through any zero there passes exactly one of those curves; the curves remain always in a neighbourhood of the curves of constant phase $\pi/2$ and $3\pi/2$ with which they asymptotically coincide.

9. Tables. The methods developed in this paper enable us to investigate and to calculate the zeros of the Fresnel integrals in a simple manner.

We did not consider the functions $c(z)$ and $s(z)$ defined by (2.7). Although these functions are very simply related to the Fresnel integrals, their behaviour is different from that of $C(z)$ and $S(z)$. Since $c(z)$ and $s(z)$ also occur in connection with many practical problems they should eventually be studied more in detail; this will be done in a subsequent paper.

Zeros $z_n = x_n \pm iy_n$ of $C(z)$

n	x_n	y_n	n	x_n	y_n	n	x_n	y_n	n	x_n	y_n	n	x_n	y_n
1	4·62	1·68	11	67·53	3·03	21	130·36	3·35	31	193·20	3·55	41	256·03	3·69
2	10·94	2·11	12	73·81	3·07	22	136·65	3·38	32	199·48	3·57	42	262·32	3·70
3	17·24	2·34	13	80·10	3·11	23	142·93	3·40	33	205·77	3·58	43	268·60	3·72
4	23·53	2·50	14	86·38	3·15	24	149·21	3·42	34	212·05	3·60	44	274·88	3·73
5	29·82	2·62	15	92·66	3·18	25	155·50	3·44	35	218·33	3·61	45	281·17	3·74
6	36·11	2·71	16	98·95	3·22	26	161·78	3·46	36	224·62	3·63	46	287·45	3·75
7	42·40	2·79	17	105·23	3·25	27	168·06	3·48	37	230·90	3·64	47	293·73	3·76
8	48·68	2·86	18	111·51	3·28	28	174·35	3·50	38	237·18	3·65	48	300·02	3·77
9	54·96	2·92	19	117·80	3·30	29	180·63	3·52	39	243·47	3·67	49	306·30	3·78
10	61·25	2·98	20	124·08	3·33	30	186·92	3·53	40	249·75	3·68	50	312·58	3·79

Zeros $z_n{}^* = x_n{}^* + iy_n{}^*$ of $S(z)$

n	$x_n{}^*$	$y_n{}^*$	n	$x_n{}^*$	$y_n{}^*$	n	$x_n{}^*$	$y_n{}^*$	n	$x_n{}^*$	$y_n{}^*$	n	$x_n{}^*$	$y_n{}^*$
1	6·20	1·74	11	69·09	3·04	21	131·93	3·36	31	194·77	3·55	41	257·60	3·69
2	12·51	2·16	12	75·38	3·08	22	138·22	3·38	32	201·05	3·57	42	263·89	3·71
3	18·81	2·37	13	81·66	3·12	23	144·50	3·41	33	207·34	3·59	43	270·17	3·72
4	25·10	2·52	14	87·95	3·16	24	150·79	3·43	34	213·62	3·60	44	276·45	3·73
5	31·38	2·63	15	94·23	3·19	25	157·07	3·45	35	219·90	3·62	45	282·74	3·74
6	37·67	2·72	16	100·52	3·22	26	163·35	3·47	36	226·19	3·63	46	289·02	3·75
7	43·95	2·80	17	106·80	3·25	27	169·64	3·49	37	232·47	3·64	47	295·30	3·76
8	50·24	2·87	18	113·08	3·28	28	175·92	3·50	38	238·75	3·66	48	301·59	3·77
9	56·52	2·93	19	119·37	3·31	29	182·20	3·52	39	245·04	3·67	49	307·87	3·78
10	62·81	2·99	20	125·65	3·34	30	188·49	3·54	40	251·32	3·68	50	314·15	3·79

References

1. A. Bloch, *Sur les intégrales de Fresnel*, Darboux Bull. (2), *46* (1922), 34–35.
2. M. Brelot, *Quelques propriétés des fonctions de Gilbert et de la spirale de Cornu*, Bull. Sci. Math. (2), *61* (1937), 133–160.
3. W. Buerck und H. Lichte, *Untersuchungen über die Laufzeit in Vierpolen und die Verwendbarkeit der Gleitfrequenzmethode*, Elektr. Nachr. Technik, *15* (1938), 78–101.
4. M. F. Egan, *Gamma functions and Fresnel's integrals*, Math. Gazette, *19* (1935), 366–367.
5. F. Emde, *Kurvenlineale*, Instrum. Kunde, *58* (1938), 409–413.
6. B. de Haan, *Nouvelles tables d'intégrales définies* (Leyden, 1867).
7. A. L. Higgins, *The Transition Spiral and its Introduction to Railway Curves* (London, 1921).
8. P. Humbert, *Sur les intégrales de Fresnel*, Bull. Soc. Sc. Cluj, *7* (1934), 530–534.
9. V. Jamet, *Sur les intégrales de Fresnel*, Nouv. Ann. (4), *3* (1903), 357–359.
10. E. Kreyszig, *Ueber den allgemeinen Integralsinus* $\mathrm{Si}(z, \alpha)$, Acta Math., *85* (1951), 117–181.
11. E. Kreyszig, *Der allgemeine Integralkosinus* $\mathrm{Ci}(z, \alpha)$, Acta Math., *89* (1953), 107–131.
12. Th. Laible, *Höhenkarte des Fehlerintegrals*, Z. angew. Math. Physik, *2* (1951), 484–486.
13. D. N. Lehner, *Cornu's spiral as a transition curve*, Cal. Inst. Tech., *3* (1904), 71–82.
14. A. Lindstedt, *Zur Theorie der Fresnelschen Integrale*, Ann. Physik (2), *17* (1882), 720–725.
15. W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (New York, 1949).
16. N. Nielsen, *Theorie des Integrallogarithmus und verwandter Transzendenter* (Leipzig, 1907).
17. L. Oerley, *Übergangsbogen bei Strassenkrümmungen* (Berlin, 1937).
18. N. A. Oumoff, *Interprétation géométrique des intégrales de Fresnel*, J. de Phys. (3), *6* (1897), 281–289.
19. Th. Poeschl, *Das Anlaufen eines einfachen Schwingers*, Ing. Archiv, *4* (1933), 98–102.
20. H. Salinger, *Zur Theorie der Frequenzanalyse mittels Suchtons*, Elektr. Nachr. Technik, *6* (1929), 293–302.
21. E. Ullrich, *Betragflächen mit ausgezeichnetem Krümmungsverhalten*, Math. Z., *54* (1951), 297–328.
22. J. Zbornik, *Asymptotische Entwicklungen für Fresnelsche Integrale und verwandte Funktionen und ihre Anwendungsmöglichkeiten bei der Berechnung spezieller Raketenbahnen*, Z. angew. Math. Physik, *5* (1954), 345–351.

University of Ottawa and
Ohio State University