# ON THE FREQUENCY OF 3-CONNECTED

# SUBGRAPHS OF PLANAR GRAPHS

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The concept of dependence of subgraphs of a plane graph is defined, as a measure of how much they overlap. It is shown that if $M$ is a $3$-connected plane graph, then the number of copies of $M$ in a plane graph which are dependent on a given copy is bounded above by a constant $c(M)$. The number of copies of $M$ in any $n$-vertex plane graph is at most $nc(M)$.

## 1.  Introduction.

Hakimi and Schmeichel [2] gave bounds on the number of $k$-cycles in a maximal planar graph with $n$ vertices. Let $f(n,G)$ denote the maximum number of copies of a graph $G$ (that is, subgraphs isomorphic to $G$) occuring in any planar graph on $n$ vertices. Alon and Caro [1] studied $f(n,G)$ for various $G$, determining it precisely if $G$ is complete bipartite or a maximal planar graph with no non-facial triangles, and obtaining bounds for other triangulations $G$. From their results it follows that $f(n,G)$ is less than $12n$ for any triangulation $G$. The main object of this article is to prove the conjecture, attributed in [1] to Perles, that $f(n,G) \leq c_G n$ for all $3$-connected planar graphs $G$

and all  $n$ .  Here and in what follows,  $c_G$  denotes a positive constant
depending only on the graph  $G$ , perhaps different at different occurrences
of the notation, even within the same formula.  It is interesting to
compare this with the results of [4], which state that if  $F$  is
one of four sets of triangulations (2-connected or 3-connected or 2-face-
colourable 2-or 3-connected), then almost all  $n$ -vertex elements of  $F$
contain at least  $c_G n$  copies of  $G$  for any particular  $G \in F$ .

On the way to deriving the main result in Section 3, we find that
the number of copies of  $G$  which overlap a given copy (in a certain sense)
is at most  $c_G$ .  In Section 4, bounds on this constant are given if  $G$
is a wheel, and a general bound is conjectured.

## 2.  Notation.

Unless otherwise specified,  $G$  denotes a  3-connected planar graph
and  $\nu(G)$  is the cardinality of its vertex set.  Our graphs have no
loops or multiple edges.  A <u>plane</u> graph is as usual a proper embedding of
a planar graph in the plane, and its <u>faces</u> are the connected components of
the plane remaining when the vertices and edges are removed.  In a 2-
connected plane graph, the cycle of vertices and edges around the boundary
of a face is called the <u>bounding cycle</u> of that face;  the 2-connectedness
ensures that no vertex appears at two different places in the bounding
cycle.  The bounding cycle of the unbounded face of a plane graph  $M$  is
called the <u>bounding cycle</u> of  $M$ .  The vertices of  $M$  not on its bounding
cycle are called <u>interior</u> vertices.  Also,  $M$  is <u>internally 3-connected</u>
if it is 2-connected and removal of any two or fewer vertices (together
with incident edges) leaves every remaining vertex of  $M$  connected (by
a path along vertices and edges) to at least one vertex in the bounding
cycle of  $M$ .

By an <u>embedding</u> of a plane graph  $M$  in a plane graph  $N$ , we mean
an isomorphism from the underlying graph of  $M$  to a subgraph of the under-
lying graph of  $N$ , which is induced by a homeomorphism of the plane from
$M$  to  $N$ .  We may regard the embedding as a superposition of  $M$  onto  $N$ ,
and accordingly we may speak of the vertices and faces of the embedding.
Given two embeddings  $M_1$  and  $M_2$  of a plane graph in a plane graph, the
distribution of the interior vertices of  $M_2$  into the faces and vertices

of $M_1$ determines a function between these two sets. We denote this function by $D(M_1, M_2)$ .

## 3. The main results.

First a lemma is given which is the key to the main result. Then in Theorem 1 the number of copies of a plane graph which can overlap a given copy is bounded. Theorem 2 relates this bound to the maximum number of copies of $G$ is a plane graph.

LEMMA. *Let $M$ be an internally 3-connected plane graph, and suppose an embedding of the bounding cycle of $M$ in a plane graph $N$ is specified. Then there are at most $c_M$ different ways to complete this embedding to an embedding of $M$ of $N$.*

Proof. This is by induction on the number, say $m$ , of interior vertices of $M$ . If $m = 0$ then the result is immediate, since it only remains to map certain edges of $M$ to edges of $N$ , and since our graphs have no multiple edges this can be done in at most one way. So suppose $m \geq 1$.

Let $M_1$ and $M_2$ denote two distinct completions of the embedding of the bounding cycle to an embedding of $M$ . Then the number of possibilities for $D(M_1, M_2)$ , for a given $M_1$ , is at most $c_M$ . Also, for any face $F$ of $M_1$ , the portion of $M_2$ lying within $F$ , together with the bounding cycle of $F$ , determine a plane graph $P(F)$ which is clearly internally 3-connected. Given $M_1$ and $D(M_1, M_2)$ , the isomorphism type of $P(F)$ and the embedding of its bounding cycle in $N$ are determined. Since $M$ is internally 3-connected, not all of its interior vertices can lie in $F$ . Hence $P(F)$ has fewer than $m$ internal vertices, and so by induction there are at most $c_{P(F)}$ distinct ways to complete the embedding of the bounding cycle of $P(F)$ to an embedding of $P(F)$ in $N$. Since this holds for each face $F$ , it follows that there are at most $c_M$ ways to complete the embedding of $M_2$ , given $D(M_1, M_2)$ . Hence, given $M_1$ , the number of possibilities for $M_2$ is at most $c_M$ , and the lemma follows.                    ☐

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We say that two embeddings $M_1$ and $M_2$ of plane graphs in a plane graph are <u>independent</u> (of each other) if the vertices of one are contained entirely in the unbounded face and bounding cycle of the other. Otherwise they are <u>dependent</u> (on each other).

THEOREM 1. *For each 3-connected plane graph* $M$ *, there is a constant* $r(M)$ *with the following property: For any plane graph* $N$ *and any embedding* $M_1$ *of* $M$ *in* $N$ *, there are at most* $r(M)$ *distinct embeddings of* $M$ *in* $N$ *which are dependent on* $M_1$ *.*

Proof. Let $M_2$ be an embedding of $M$ in $N$ dependent on $M_1$. Then by the 3-connectedness of $M$, for each face $F$ of $M_1$, the portion of $M_2$ lying within $F$, together with the bounding cycle of $F$, can be assumed to determine a plane graph $P(F)$ which is internally 3-connected. (This is not immediate when $F$ is the unbounded face of $M_1$. In this case, we can either re-define "internally 3-connected" appropriately or argue that with regard to the examination of $F$, $M_1$ can be projected onto the sphere and then re-projected onto the plane so that $F$ becomes bounded and the assumption holds.) Hence, it now follows by the Lemma that, given $M_1$ and $D(M_1,M_2)$, there are at most $c_{P(F)}$ ways to complete the embedding of $P(F)$. Since this holds for each face $F$ of $M_1$, there are at most $c_M$ ways to choose $M_2$. As the number of potential $D(M_1,M_2)$ is at most $c_M$, the theorem follows. $\square$

If $\nu(G) = p$ then $G$ can be embedded in the plane in at most $4(p-2)$ ways. This is because it has at most $2(p-2)$ faces, and each face can be the unbounded face in at most two distinct ways in view of Whitney's theorem [6] that $G$ is uniquely embeddable in the sphere. (For our present purposes, we regard the plane as possessing an orientation.) Let $r(G)$ denote the sum of $r(M)$ over all the different embeddings of $G$ in the plane. An upper bound on $f(n,G)$ can now be given as follows.

THEOREM 2. *If* $\nu(G) = p$ *then* $f(n,G) \leq (n-p+1)r(G)$ *.*

Proof. Let $H$ be a planar graph with $\nu(H) = n$ and let $N$ be any

proper planar embedding of $H$ . Then each copy of $G$ in $H$ corresponds to an embedding of some plane graph $M$ in $N$ , where the underlying graph of $M$ is $G$ . We bound the maximum number, say $g(n,M)$ , of embeddings of $M$ in <u>any</u> plane graph with $n$ vertices, and then sum over all plane graphs $M$ with underlying graph $G$ to bound $f(n,G)$ .

Let $M_1$ denote an embedding of $M$ in $N$ , and denote the faces of $M_1$ by $F_1,\ldots,F_f$ . For $i = 1,\ldots, f$ , let $k_i$ denote the number of vertices of $N$ lying in $F_i$ or on its bounding cycle. Then the number of embeddings of $M$ in $N$ which are independent of $M_1$ is at most $\sum g(k_i,M)$ . This summation and the following ones are for $i = 1,\ldots, f$ unless otherwise indicated.

As $M$ is 3-connected, we have $k_i \leq n-1$ for each $i$ . Let $t_1,\ldots, t_f$ denote the valencies of $F_1,\ldots, F_f$ respectively. Then $k_i \geq t_i$ for each $i$ , and as $M$ is 3-connected, $p > t_i$ for each $i$ and so $g(k_i,M) = 0$ if $k_i = t_i$ . Also $\sum (k_i - t_i) = n-p$ . Thus by Theorem 1,

$$g(n,M) \leq r(M) + \max \sum g(k_i,M)$$

where the maximum is over all sequences $k_1,\ldots,k_f$ for which $\sum k_i = n-p+ \sum t_i$ , and $k_i \geq t_i$ for each $i$ . Let $t = \max t_i$, so $t \leq p-1$ , and say $t_w = t$ . Define a function $d$ on the non-negative integers by $d(j) = 0$ for $j < p$ and

$$d(j) = r(M) + \max_j \sum d(k_i) \qquad (3.1)$$

for $j \geq p$ , where the maximum of the empty set is taken as $0$ , and $\max_j$ denotes the maximum over all sequences $k_1,\ldots,k_f$ with $k_i \geq t_i$ for each $i$ and $\sum k_i \leq j-t-1 + \sum t_i$ .

Then

$$d(j) \geq g(j,M) . \qquad (3.2)$$

For $j \geq p$ we have

$$d(j) - d(j-1) \geq \max_j \sum d(k_i) - \max_{j-1} \sum d(k_i)$$

$$\geq \max_{j-1}(d(k_w+1) + \sum_{i \neq w} d(k_i) - \sum d(k_i))$$

$$= \max_{j-1}(d(k_w+1) - d(k_w))$$

$$\geq d(j-1) - d(j-2)$$

since if $k_i = t_i$ $(i \neq w)$ and $k_w = j-2$ , then $\sum k_i = j-t-2+\sum t_i$ .
Hence $d$ is convex, and it follows that the maximum in (3.2) is achieved
when $k_w = j-1$ and $k_i = t_i$ otherwise, and is thus equal to $d(j-1)$.
So $d(n) = (n-p+1) r(M)$ . The theorem now follows from (3.2), as $f(n,G)$
is at most the sum of $g(n,M)$ over all $M$ whose underlying graph is $G$. □

## 4.   Refinements.

Although the proof of Theorem 2 can give a slightly better result
than that stated, it seems unlikely that the result can be improved
considerably without substantial modifications of the proof.  Clearly
$f(n,G) \geq \lfloor n/p \rfloor$ , and so our bounds on $f(n,G)$ are perhaps best summarised
as $c_G n \leq f(n,G) \leq c_G n$ .

It would be interesting to know the value of $c(G) \equiv \sup\{f(n,G)/n : n \in \mathbb{N}\}$
for various $G$ . As pointed out in [1] , it is easy to see that for the
$k$-spoked wheel $W_k$ (defined as in Tutte [5,p.78]), we have $c(W_k) > c^k$
for some constant $c > 1$ (and perhaps $k$ large·enough). Hence $c(G)$ is
not bounded independently of $G$ even though it is less than $12$ if $G$ is
any triangulation. Even if $G$ has all faces of valency $4$ , $c(G)$ is
arbitrarily large. For instance, let $G$ be the $1$-skeleton of a stack
of $k$ cubes (that is the graph $P_{k+1} \times C_4$) . Then if $H$ is a suitable
triangulation of $G$ (for example with all diagonals of the faces in the
sides of the stack sloping in the same direction around the stack), we can
find $2^k$ copies of $G$ in $H$ . Thus $c(G) \geq 2^k/4(k+1)$.

All upper bounds on $r(G)$ are upper bounds on $c(G)$ by Theorem 2,
but tight upper bounds seem to be difficult to obtain. However, in view
of the comment above, the next result shows that $r(W_k)$ grows essentially

exponentially.

THEOREM 3. *For* $k \geq 2$, $r(W_k) \leq 3.2^{k-1}$ .

Proof. Denote the hub of $W_k$ *(k $\geq$ 3)* by $u$ , and let $v_1, \ldots, v_k$ denote the rim vertices in cyclic order. Let $U_k$ denote the graph obtained by removing the edge $v_1 v_k$ from $W_k$ , and let $M$ denote the embedding of $U_k$ in the plane with all vertices in the bounding cycle. Let $h(k)$ denote the maximum number of ways of completing an embedding of $M$ in a planar graph, given an embedding of $u$, $v_1$ and $v_k$ , and given the orientation of $M$ (which determines whether $v_1 v_2 \ldots$ are embedded in clockwise or anticlockwise order). We find an upper bound on $h(k)$ , thereby proving its finiteness, and use this to obtain the upper bound on $r(W_k)$ .

Let $M_1$ denote an embedding of $M$ in a plane graph $N$ . We use $v_{i,j}$ to denote the image of $v_j$ in $M_i$ , and (by a slight abuse of notation) $v_1$ , $v_k$ and $u$ denote the images of $v_1$ , $v_k$ and $u$ . Let $\ell$ denote the greatest integer less than $k$ for which some edge of $N$ joins the vertices $v_1$ and $v_\ell$ of $M_1$ . Then $2 \leq \ell \leq k-1$ . If $M_2$ is an embedding of $M$ which coincides with $M_1$ on $u$, $v_1$ and $v_k$ and has the same orientation, then $v_{1,\ell}$ must equal $v_{2,j}$ for some $2 \leq j \leq k-1$ . Hence $M_2$ is determined by an embedding of $U_j$ (with $u$, $v_1$ and $v_j$ mapped onto $u$, $v_{1,1}$ and $v_{1,\ell}$ respectively) and an embedding of $U_{k+1-j}$ (with $u$, $v_1$ and $v_{k+1-j}$ mapped onto $u$, $v_{1,\ell}$ and $v_k$ respectively). If $\ell$ is $2$ or $k-1$ , we refer here to $U_2$ , which we define to be the triangle $u v_1 v_2$ , and $h(2)$ is accordingly defined to be $1$ . It is now evident that

$$h(k) \leq \sum_{\ell=2}^{k-1} h(\ell) h(k+1-\ell) . \qquad (4.1)$$

Assume for the moment that equality holds here. Then since

$h(2) = 1$ , this recurrence defines the Catalan numbers and we have

$h(k) = \frac{1}{k-1}\binom{2k-4}{k-2}) = q_k$  say (see Liu [3,p. 75] for example).  Therefore

$q_k$  is an upper bound on  $h(k)$  even without assuming equality in (4.1).

Now suppose that  $M$  is the plane graph with underlying graph  $W_k$
and with unbounded face of valency  $k$ , and  $M_1$  and  $M_2$  are dependent
embeddings of  $M$  in a plane graph  $N$ .  Then the hub of  $M_2$  must coin-
cide with a vertex of  $M_1$ : either the hub or one of the rim vertices.  We
consider these two cases separately.

In the first case, some two adjacent vertices in the rim of  $M_1$
must be vertices (not necessarily adjacent) in the rim of  $M_2$ .  Hence
the number of possibilities for  $M_2$ , given  $M_1$ , is in  this case at most

$k \sum_{i=2}^{k} h(i)h(k+2-i) \le kq_{k+1}$ .  This count includes the event that  $M_2 = M_1$ .

In the second case, the hub,  $u$ , and another vertex, say  $v$ , of
$M_1$  are in the rim of  $M_2$ .  The edge  $uv$  lies in the unbounded face of
$M_1$  or in its bounding cycle.  There can be at most  $2k-3$  such edges in
$N$ .  So the number of possibilities for  $M_2$  in this case is at most
$(2k-3)q_{k+1}$ , by an argument similar to that used in the first case.

Next suppose  $M$  is the other plane embedding of  $W_k$ , so that the
unbounded face is a triangle.  Then a similar argument again gives
$r(M) \le 3(k-1)q_{k+1}$ .  The theorem follows.                      ☐

Since  $r(G) \le 12$  for any triangulation  $G$  by the results of [1],
and since  $W_k$  has a face of largest possible valency amongst those planar
graphs on  $k+1$  vertices, a plausible conjecture is that  $r(G)$  is
maximised for  $p$-vertex graphs when  $G = W_{p-1}$.  This would yield  $c(G) \le 2^p$
whenever  $\nu(G) = p$ , but so far the evidence for this is weak.  The
following much weaker conjecture would still, if true, give a nice
description of the nature of  $c(G)$ .

**Conjecture.** There exists an absolute constant $c > 1$ such that if $\nu(G) = p$ then $f(n,G) < c^p n$ .

## References

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