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ANGULAR MOMENTUM, CONVEX POLYHEDRA AND ALGEBRAIC GEOMETRY

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1. Introduction

The three families of classical groups of linear transformations (complex, orthogonal, symplectic) give rise to the three great branches of differential geometry (complex analytic, Riemannian and symplectic). Complex analytic geometry derives most of its interest from complex algebraic geometry, while symplectic geometry provides the general framework for Hamiltonian mechanics.

These three classical groups "intersect" in the unitary group and the three branches of differential geometry correspondingly "intersect" in Kähler geometry, which includes the study of algebraic varieties in projective space. This is the basic reason why Hodge was successful in applying Riemannian methods to algebraic geometry in his theory of harmonic forms.

In the past few years it has been realised that some of the ideas from symplectic geometry can also be applied to algebraic geometry. The key notion is that of angular momentum and the main technical result is a convexity theorem [**1**] [**6**] which asserts that the simultaneous values of commuting angular momenta form a convex polyhedron.

My aim in this talk is to illustrate some of these new ideas and I will begin in the next two Sections by describing two results connecting algebra with convex polyhedra. These appear quite unrelated and neither is in a geometric form. Nevertheless I will show in subsequent sections how they both fit elegantly into the symplectic framework, which is explained in Section 4. In Section 7 I will discuss briefly the very important application to geometric invariant theory. Finally in Section 8 I will explain the "exact integration formulae" of Duistermaat and Heckman [**5**]. This is of interest not so much in algebraic geometry but as a prototype of infinite-dimensional counterparts which arise as models in theoretical physics.

2. Eigenvalues of Hermitian matrices

An old result of Schur [**13**] asserts that the diagonal entries $(\mu_1,\dots,\mu_n)$ of an $n\times n$ Hermitian matrix satisfy some inequalities relating them to the eigenvalues $(\lambda_1,\dots,\lambda_n)$. If both sequences are arranged in descending order these inequalities are:

$$\begin{gathered}\lambda_1\geqq\mu_1\\ \lambda_1+\lambda_2\geqq\mu_1+\mu_2\\ \cdots\cdots\\ \lambda_1+\lambda_2+\cdots+\lambda_n=\mu_1+\mu_2+\cdots+\mu_n,\end{gathered}\tag{2.1}$$

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the last equality coming from the trace. Horn [7] proved the converse, i.e. if the λ, μ are related by (2.1) then there exists a Hermitian matrix with the μ_i as diagonal entries and the λ_i as eigenvalues.

The inequalities can be cast in a more geometric form by interpreting λ and μ as vectors in $\mathbb{R}^n$ and considering their orbits $\Sigma_n\lambda$ and $\Sigma_n\mu$ under the symmetric group Σ_n (taking all permutations of the coordinates). Then (2.1) is equivalent to

$$\widehat{\Sigma_n\lambda} \supset \widehat{\Sigma_n\mu} \tag{2.2}$$

where $\hat{C}$ denotes the *convex hull* of C. For a discussion of the equivalence of (2.1) and (2.2) see [7] or [3; Section 12]. For $n=3$ the convex hull $\Sigma_3\lambda$ is in general a hexagon lying in the plane $\lambda_1+\lambda_2+\lambda_3=\text{const.}$ (see Fig.). If two of the eigenvalues coincide this hexagon becomes a triangle. The condition (2.2) asserts that μ lies inside (or on) the hexagon determined by λ.

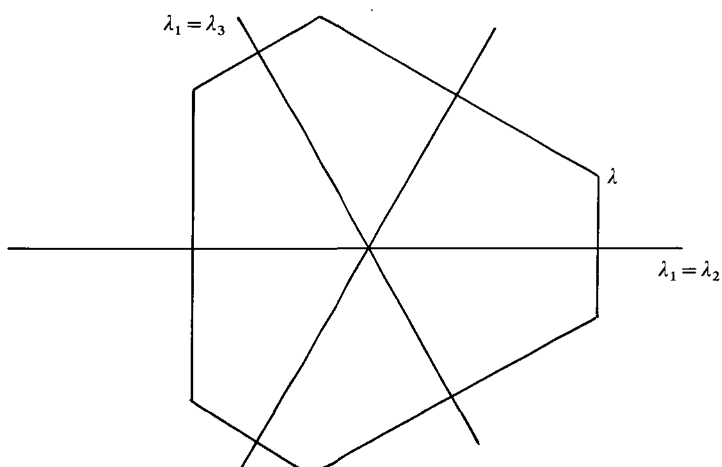

Figure 1

The original proof of Schur-Horn is in [7] while another proof, including a generalisation to Lie groups (other than $U(n)$) was given by Kostant [10]. The proof by symplectic methods which will be outlined later is given in [1] [6].

3. Solutions of polynomial equations

Let $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ be a complex n-vector *with no component zero*, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n$ an integral n-vector and denote by z^α the monomial

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}.$$

Now let S be a finite subset of $\mathbb{Z}^n$ and consider the system of n equations

$$\sum_{\alpha\in S} c_\alpha^j z^\alpha = 0 \quad j=1,\ldots,n, \tag{3.1}$$

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where the coefficients c_α^j are assumed "general". The question we pose is to determine the number $N(S)$ of solutions of (3.1).

Remark. Since the components α_i of α may be negative integers (3.1) is an equation for finite Laurent series. However, this generality is illusory since we can multiply by a suitable monomial z^β to make all equations polynomial. Clearly $N(S+\beta)=N(S)$.

As in Section 2 let $\hat{S}$ denote the convex hull of the finite set $S\subset\mathbb{Z}^n\subset\mathbb{R}^n$. This may be called the *Newton polyhedron* of the set of equations (3.1). Then we have the following elegant formula for $N(S)$:

$$N(S)=n!\,\mathrm{vol}(\hat{S}), \tag{3.2}$$

where vol stands for the standard Euclidean volume in $\mathbb{R}^n$. This formula is due to Koushnirenko [**11**], but the symplectic proof we shall give later is somewhat simpler.

There is an interesting generalisation of this formula due to Bernstein [**4**] for the case when the n equations (3.1) involve different monomials, corresponding to different $S_j\subset\mathbb{Z}^n$. In this case (3.2) gets replaced by

$$N(S_1,\dots,S_n)=V(\hat{S}_1,\dots,\hat{S}_n) \tag{3.3}$$

where V is the *Minkowski mixed volume* of the n Newton polyhedra. This will also be explained in Section 5.

4. Angular momentum

The conservation of angular momentum in ordinary Newtonian mechanics is a consequence of the rotational invariance of Newton's laws of motion: in the same way as translation invariance yields the conservation of linear momentum. Let me recall briefly the general principles of Hamiltonian mechanics underlying such classical conservation laws.

One begins by introducing the basic exterior differential 2-form on phase-space:

$$\omega=\sum_j dp_j\wedge dq_j,$$

where the q_j, p_j are position and momentum coordinates respectively. Then every function $F(p,q)$ has a differential dF which via the 2-form ω gets converted into the vector field

$$X_F=\sum_j \frac{\partial F}{\partial p_j}\frac{\partial}{\partial q_j}-\frac{\partial F}{\partial q_j}\frac{\partial}{\partial p_j}.$$

Formally dF is the interior product of X_F with ω: since ω is non-degenerate this determines X_F uniquely. The fact that dF is a closed 1-form (i.e. $ddF=0$) guarantees that X_F preserves ω. Conversely any vector field preserving ω gives a closed 1-form which (at least locally) is the differential of some function F (unique up to an additive constant). If F is the Hamiltonian H of the system then X_F is the Hamiltonian flow whose integral curves give the dynamical evolution. For example, for a free particle of unit mass $H=\frac{1}{2}\sum_j p_j^2$. Other conserved quantities generate flows which preserve H. Thus (in 3 dimensions)

$$F=q_1p_2-q_2p_1$$

is the angular momentum about the q_3-axis and the corresponding flow X_F generates rotation about this axis. In particular the integral curves are all closed orbits (circles) with the same periods.

Another example, more pertinent for our purposes, is given by taking

$$H=\frac{1}{2}\sum_{j=1}^{n}(p_j^2+q_j^2).$$

This represents n uncoupled *harmonic oscillators*. Each individual

$$H_j=\tfrac{1}{2}(p_j^2+q_j^2)$$

is conserved and the corresponding flow generates rotation in the (p_j, q_j) plane. These flows all commute so that we have an action of the n-dimensional torus T^n on our system. If we introduce the complex variables

$$z_j=p_j+iq_j$$

then the n functions H_j define the map

$$\mu:\mathbb{C}^n\to\mathbb{R}^n$$

given by $(z_1,\ldots,z_n)\to\frac{1}{2}(|z_1|^2,\ldots,|z_n|^2)$. This is an example of the *moment map*. Note that its image is just the positive quadrant in $\mathbb{R}^n$: this is the origin of the convexity results we shall encounter.

If we restrict ourselves to a fixed energy surface, say $H=1$, of this system we get the sphere

$$S^{2n-1}: \sum_{j=1}^{n}|z_j|^2=2.$$

On this the Hamiltonian flow acts, generating the scalar multiplication

$$z_j\to e^{i\theta}z_j$$

so that the quotient space (or orbit space) is just the complex projective space $P_{n-1}(\mathbb{C})$. The original symplectic form ω is of course degenerate when restricted to the energy surface S^{2n-1}, but the degeneracy is just along the orbits of the Hamiltonian flow. Since ω is also invariant under this flow it follows that ω descends to give a non-degenerate closed 2-form ω_1 on $P_{n-1}(\mathbb{C})$. This is the famous Kähler form of algebraic geometry. To get the correct normalisation one computes the Gaussian integral $\int_{\mathbb{R}^{2n}} e^{-H}\omega^n/n!$ in two different ways and finds that

$$\int_{P_{n-1}} \omega_1^{n-1} = (2\pi)^{n-1}.$$

Hence $\Omega = \omega_1/2\pi$ represents the generator of $H^2(P_{n-1}(\mathbb{C}), \mathbb{Z})$.

The functions $H_j = \frac{1}{2}|z_j|^2$, regarded now as functions on $P_{n-1}(\mathbb{C})$, satisfy the constraint $\sum_j H_j = 1$, and the image of $P_{n-1}(\mathbb{C})$ under the moment map is the $(n-1)$-simplex spanned by the points $(1,0,\dots,0)$, $(0,1,0,\dots,0),\dots(0,\dots,0,1)$.

More generally if a torus T^n acts symplectically on a manifold M (i.e. preserving the non-degenerate closed 2-form ω defining the symplectic structure), we have a moment map

$$\mu: M \to \mathbb{R}^n$$

provided the n "Hamiltonians" are globally single-valued (e.g. if M is simply-connected).

Actually, $\mathbb{R}^n$ here should more invariantly be viewed as the dual t^* of the Lie algebra t of T^n. Moreover, under appropriate circumstances one can define moment maps for non-abelian groups. We shall encounter this in Section 6.

The importance of these ideas for algebraic geometry lies in the fact that every non-singular algebraic variety in projective space is symplectic, the symplectic (Kähler) structure being inherited from that of projective space. In the subsequent sections I shall describe a number of applications of the moment map in algebraic geometry and in particular I shall show how the problems of Section 2 and 3 fit into this framework.

5. Complex torus orbits

Let me return now to the problem in Section 3 and rephrase it in more geometric form. We note first that a vector $z \in \mathbb{C}^n$ with no zero components is just a point of the complex n-dimensional torus T_c^n, and $\alpha \in \mathbb{Z}^n$ represents a 1-dimensional (holomorphic) representation V_α. A finite subset $S \subset \mathbb{Z}^n$ therefore defines an n-dimensional representation

$$V = \bigoplus_{\alpha \in S} V_\alpha$$

of T_c^n. Let $\{v_\alpha\}$ be a corresponding system of coordinates for V, so that the n general linear equations

$$\sum_{\alpha \in S} c_\alpha^j v_\alpha = 0 \quad j = 1, \dots, n$$

define a vector subspace $W \subset V$ of codimension n. The solutions of the equations (3.1) represent the intersections of W with the subvariety X' of V given parametrically by

$$v_\alpha = z^\alpha.$$

Note that X' is just the T_c^n-orbit of the unit point $(1,1,\dots,1)$ in V. Moreover if

$$\textit{S generates the lattice } \mathbb{Z}^n \tag{5.1}$$

then X' is a faithful orbit so that the number $N(S)$ of solutions of (3.1) is just the intersection number of X' and W.

We now pass to the projective space $P(V)$, then the T_c^n-orbit X of the unit point in $P(V)$ is just the image of X'. Moreover if

$$\textit{the differences } \alpha-\beta\ (\alpha,\beta\in S) \textit{ generate } \mathbb{Z}^n \tag{5.2}$$

then X is also a faithful orbit and $X'\cong X$. Thus $N(S)$ is also the intersection number of X and $P(W)$ in $P(V)$. But X (or rather its closure $\bar{X}$) is an algebraic subvariety of dimension n and its intersection number with a generic $P(W)$ of codimension n is by definition its *degree* (note that $\dim_C(\bar{X}-X)<n$ and so does not meet a generic $P(W)$). But this degree can also be computed as the integral

$$\int_X \Omega^n$$

where Ω is the normalised Kähler form on $P(V)$, representing the generator of integral cohomology. Equivalently, since $\Omega^n/n!$ is the induced symplectic volume element on X,

$$N(S)=n!\,\mathrm{vol}(X). \tag{5.3}$$

At this point we bring in the moment map

$$\mu_t: P(V)\to t^*$$

relative to the action of the *compact* torus T^n (i.e. all $|z_i|=1$) and the symplectic form Ω. Note that Ω depends on a choice of Hermitian metric on V and we assume this is T^n-invariant. Now μ_t is just the composition of the standard moment map

$$\mu_k: P(V)\to k^*$$

for the standard action of T^N (where $N=\dim_C V=\mathrm{Card}\, S$, and k denotes its Lie algebra) and the linear map

$$s^*: k^*\to t^*$$

dual to the map

$$s: t\to k$$

associated to the representation $S: T^n\to T^N$. Now in Section 4 we saw explicitly that the image of μ_k was the standard $(N-1)$-simplex. Since we are now using the normalised symplectic form $\Omega=\omega/2\pi$ the vertices of the simplex are the generators of the integer lattice $L^*\subset t^*$ dual to the lattice $L\subset t$ which is the kernel of the exponential map $t\to T^n$. Under s^* these vertices become the points of $S\subset T^*$ (the "weights" of the representation). It follows that the image $\mu_t(P(V))$ is the convex hull $\hat{S}$.

In fact a general result proved in [1] shows that the image under μ_t of the generic T_c^n-orbit X is the whole interior of $\hat{S}$, and that $X\to\operatorname{int}(\hat{S})$ (which is the moment map for the action of T^n on the symplectic manifold X) is a fibration with fibre T^n. It follows that

$$\operatorname{vol}(X)=\operatorname{vol}(T^n)\operatorname{vol}(\hat{S}). \tag{5.4}$$

If we normalize so that $\operatorname{vol}(T^n)=1$ then this determines the normalisation of the Euclidean volume on t^*. Together with (5.3) this gives the required formula (3.2), namely

$$N(S)=n!\operatorname{vol}(\hat{S}).$$

Recall finally that this proof was under the assumption (5.2). However this presents no real problem. In the first place if the differences $\alpha-\beta$ generate a lattice of lower rank both sides of (3.2) are zero. On the other hand if this is a lattice of finite index in Z^n it is the weight lattice of a quotient of T^n by a finite group Γ. Working with this new torus and multiplying both sides of (5.4) by the order of Γ then gives (3.2).

The observation that Koushnirenko's formula (3.2) can be deduced from the convexity theorem of [1] is due to V. Arnold and A. G. Hovanski. The same methods also yield Bernstein's generalisation (3.3). For this we consider the representations $V_1,\ldots,V_n$ defined by the subsets $S_1,\ldots,S_n$ of t^* and form

$$M=P(V_1)\times P(V_2)\times\cdots\times P(V_n).$$

For any choice of positive integers $\lambda_1,\ldots,\lambda_n$ we consider M as a symplectic manifold with symplectic form

$$\Omega_\lambda=\lambda_1\Omega_1+\lambda_2\Omega_2+\cdots+\lambda_n\Omega_n.$$

The corresponding moment map

$$\mu_\lambda: M\to t^*$$

is then given by

$$\mu_\lambda=\Sigma\lambda_i\mu_i$$

where μ_i is projection of M onto the i-th factor followed by the natural moment map of $P(V_i)$. Restricting this to a generic T_c^n-orbit X and applying the convexity result of [1] we deduce

$$\deg_\lambda X=n!\operatorname{vol}\mu_\lambda(X)=n!\operatorname{vol}\{\Sigma\lambda_i\mu_i(X)\}=n!\operatorname{vol}\{\Sigma\lambda_i\hat{S}_i\}$$

where $\deg_\lambda$ is the degree relative to the class of Ω_λ, i.e. is the value of the cohomology class Ω_λ^n on X. This is clearly a polynomial in the λ_j of degree n and the coefficient of $\lambda_1\lambda_2\ldots\lambda_n$ is just $n!$ times the number $N(S_1,\ldots,S_n)$ of Section 3. On the other hand the Minkowski mixed volume $V(\hat{S}_1,\ldots,\hat{S}_n)$ can be defined as the coefficient of $\lambda_1\lambda_2\ldots\lambda_n$ in

$$\mathrm{vol}\{\Sigma\lambda_i\hat{S}_i\}.$$

This establishes (3.3). Thus (3.3) is just obtained by "polarising" (3.2).

6. Homogeneous symplectic manifolds

The main convexity result of [1] [6] is that the image under the moment map μ of a compact symplectic manifold M acted on by a torus T^n is a convex polyhedron. When M is a Kähler manifold (e.g. a projective algebraic manifold) there is a refinement, proved in [1], that the image of any T_c^n-orbit X is also (the interior of) a convex polyhedron. Since the closure $\bar{X}$ may have singularities (but is otherwise symplectic) this can be viewed as a generalisation of the first result. Moreover for Kähler manifolds the first result is a consequence of the refinement since, for generic orbits X, one has $\mu(\bar{X})=\mu(M)$.

As we saw in Section 5 Koushnirenko's formula (3.2) is essentially a consequence of the refined convexity theorem about T_c^n-orbits. I shall now explain how the Schur-Horn theorem of Section 2 is a special case of the general convexity theorem for symplectic manifolds.

We first reformulate (2.2) in terms of orbits of $U(n)$ acting by conjugation on the space $\mathcal{H}$ of Hermitian matrices. An orbit M_λ consists of all Hermitian matrices with given eigenvalues $\lambda=(\lambda_1,\ldots,\lambda_n)$. Taking the diagonal part of such a matrix corresponds to the orthogonal projection

$$\pi:\mathcal{H}\to\mathbb{R}^n$$

using the standard Euclidean metric on $\mathcal{H}$ in which

$$\|A\|^2=\mathrm{Tr}\,A^2.$$

The inclusion (2.2), together with its converse, is then equivalent to

$$\pi(\mathcal{H})=\widehat{\Sigma_n\lambda} \qquad (6.1)$$

so that $\pi(\mathcal{H})$ is in particular a convex polyhedron (lying in the subspace $\Sigma\lambda_i=\text{constant}$).

To see that (6.1) is a special case of the convexity theorem of [1] [6] it remains to observe that $\mathcal{H}$ has a natural symplectic structure invariant under $U(n)$, and that π is just the moment map for the action of $T^n\subset U(n)$. This is well-known and is a special case of the general result of Kirillov [8] that orbits M of any Lie group G acting on the dual g^* of its Lie algebra have a natural symplectic structure (and the inclusion $M\to g^*$ is just the moment map for the action of G). When G is compact we can, using the Killing form, identify the Lie algebra with its dual. Finally when $G=U(n)$ its Lie algebra is $i\mathscr{H}$. The symplectic form ω_h at a point $h \in M_\lambda$ is given by

$$\omega_h(\xi,\eta) = \mathrm{Tr}(\xi\eta h - \eta\xi h).$$

Here $\xi, \eta \in i\mathscr{H}$ represent the tangent vectors to M_λ at h given by the infinitesimal action of the Lie algebra of $U(n)$.

In fact the orbits M_λ are even Kähler manifolds and, if $2\pi\lambda$ is an integral vector (so that λ belongs to the integral weight lattice of t^*), then M_λ is actually a projective algebraic variety. If V_λ is the irreducible representation of $U(n)$ indexed by λ (i.e. λ is the Young diagram, partition or maximal weight depending on your preference) then M_λ occurs as an orbit of $U(n)$ in $P(V_\lambda)$: the orbit of the unique "maximal weight vector". From the results of Section 5 we can therefore give an interpretation to the volume of the convex polyhedron $\widehat{\Sigma_n\lambda}$. Namely

$$\frac{(2\pi)^{n-1}}{\sqrt{n}} \mathrm{vol}\,\widehat{\Sigma_n\lambda} = \frac{\deg_\lambda X}{(n-1)!} \tag{6.2}$$

where $X \subset M_\lambda$ is a generic T_c^{n-1}-orbit (T^{n-1} the maximal torus of the projective unitary group $PU(n)$) and $\deg_\lambda$ is the degree in the projective space V_λ.

As an example, and a check on our normalisation, take $2\pi\lambda = (1,0,0,\dots,0)$, then $V_\lambda = \mathbb{C}^n$ with standard action of $U(n)$ and $\bar{X}$ is the whole projective space $P(\mathbb{C}^n)$, so that $\deg_\lambda X = 1$. Since the $(n-1)$-volume V_{n-1} of the standard simplex in $\mathbb{R}^n$ (with vertices $(1,0,\dots,0)$ etc.) is given by

$$V_{n-1} = \frac{\sqrt{n}}{(n-1)!}$$

we see that (6.2) checks in this case.

When the eigenvalues λ_i are distinct the orbit M_λ is the flag manifold $U(n)/T^n$: different general λ now correspond to different symplectic structures. By varying λ in (6.2) we obtain enough numerical formulae to determine completely the homology class of $\bar{X}$ in $U(n)/T^n$. This is because the cohomology of the flag manifold is multiplicatively generated by the different symplectic classes. Moreover, using the polarised form of (6.2) as explained in Section 5, we obtain explicit formulae for the values of all the monomials in these classes on $\bar{X}$. These formulae are in terms of the Minkowski mixed volumes of the polyhedra $\widehat{\Sigma_n\lambda}$.

All the preceding discussion works without essential change when $U(n)$ is replaced by any compact Lie group G and $\mathscr{H}$ by the Lie algebra of G.

7. Invariant theory

We saw in Section 4 that the complex projective space $P_{n-1}(\mathbb{C})$ is naturally the reduced phase-space (or symplectic quotient) for the scalar action of the unit circle S^1 on $\mathbb{C}^n$. That is we fix the energy level $H=1$ where H is the Hamiltonian and then divide out by the Hamiltonian flow. On the other hand, by definition,

$$P_{n-1}(\mathbb{C})=(\mathbb{C}^n-0)/\mathbb{C}^*$$

where $\mathbb{C}^*$ is the group of non-zero complex scalars (and so the complexification of S^1). The fact that the symplectic quotient and the algebro-geometric quotient coincide in this case turns out to have a far-reaching generalisation which I shall now explain.

Let G be a compact Lie group, G_c its complexification (e.g. $G=SU(n)$, $G_c=SL(n,\mathbb{C})$) and let V be a holomorphic representation space of G_c. Then G_c acts on the projective space $P(V)$. Let $X\subset P(V)$ be an algebraic sub-manifold preserved by the action of G_c. In this situation one would like to divide X by G_c and obtain a "quotient projective variety". Because G_c is non-compact there are difficulties in this process (cf. the action of $\mathbb{C}^*$ on $\mathbb{C}^n$). To obtain a quotient one must first restrict to a certain open set $X_{ss}\subset X$ (the semi-stable points which I shall define shortly). The required quotient Y is then a quotient of X_{ss} although some fibres of the map $X_{ss}\to Y$ may consist of more than one G_c-orbit.

This formation of quotients is part of Mumford's Geometric Invariant Theory [**12**]. Classical invariant theory was formulated more algebraically by associating to X its homogeneous coordinate ring

$$A(X)=C[z_1,\dots,z_n]/I(X)$$

$$I(X)=\text{ideal of polynomials vanishing on } X$$

and considering the subring of $A(X)$ *invariant* under the projective action of G_c (i.e. those f with $f(gz)=\lambda(g)f(z)$ for some scalar character λ of G_c).

These invariant polynomials are the natural functions to use to define a projective embedding of the quotient variety Y. This shows at once that the "bad points" in X which should be discarded are the common zeros of the (non-constant) invariant polynomials. The semi-stable part X_{ss} of X is by definition the complement of these bad points.

If we fix a Hermitian metric on V invariant under the compact group this defines a G-invariant Kähler metric on $P(V)$ and hence also on X. We can then form the associated moment map

$$\mu: X\to g^*$$

where g is the Lie algebra of G. Here G need not be abelian but the only difference this makes is that we now have a non-trivial action (by conjugation) of G on g^*. The moment map is compatible with the action of G on both sides.

Now form the symplectic quotient $\mu^{-1}(0)/G$. It turns out that *this can always be identified with Mumford's algebro-geometric quotient* Y [**12**; p. 158].

I shall illustrate this with a simple example. Take $G=SU(2)$, $G_c=SL(2,\mathbb{C})$, $V=\mathbb{C}^{n+1}$ the irreducible representation given by homogeneous polynomials $f(z_1,z_2)$ of degree $n+1$. Then $P(V)$ represents unordered sets of n points on P_1 and G_c acts by projective equivalence. For the trivial case $n=1$ the moment map

$$\mu_1 : P_1 \to su(2)^* = \mathbb{R}^3$$

embeds P_1 as a 2-sphere $S^2 \subset \mathbb{R}^3$ (this is the simplest example of the homogeneous symplectic orbits discussed in Section 6). For general n the moment map† μ (for a suitable normalisation) assigns to n points on $S^2 \subset \mathbb{R}^3$, their *centroid*. Thus $\mu^{-1}(0)$ consists of all "balanced" sets of points on S^2 (i.e. with centroid at 0). The symplectic quotient therefore parametrises *balanced sets up to rotational equivalence* (this being the action of $SU(2)$ on $\mathbb{R}^3$). On the other hand the semi-stable sets are those in which no point has multiplicity greater than $n/2$. When n is odd Mumford's quotient Y is actually the space of semi-stable orbits. The identification of the symplectic and algebro-geometric quotients means (for n odd) that every semi-stable $SL(2,\mathbb{C})$-orbit contains a unique balanced $SU(2)$-orbit.

F. Kirwan [9] has used this symplectic approach to Mumford's quotient to calculate its cohomology. The basic idea is to use, as a Morse function, the norm-square of the moment map. In our simple example this becomes the distance-squared of the centroid (of n points on S^2) from the origin. The semi-stable sets turn out to be those which flow (along paths of steepest descent) towards balanced sets. "Bad sets" flow towards other critical points of the function, and these are easily seen to consist of anti-podal pairs on S^2, one with multiplicity $k > n/2$ and the other with multiplicity $n-k$. Such complete information about the higher critical points then leads, by suitable Morse theory arguments, to information about the minimum (i.e. the balanced sets).

Kirwan's analysis of this Morse function relies heavily on the convexity properties of the moment map for a torus action which I have discussed in earlier sections.

8. Exact integral formulae

The Koushnirenko formula of Section 3 was proved, as an integral formula in Section 5, by using the moment map for a single T^n_c-orbit, while in Section 7 we have been discussing the geometry of the space of orbits. There are also interesting integral formulae in connection with these quotient spaces. These can be formulated in the purely symplectic context so that we do not need a complex Kähler structure (although they will of course apply in the special case of Kähler manifolds).

The main result is due to Duistermaat and Heckman [5] and it applies to the moment map

$$\mu : M \to t^*$$

associated to the action of a torus T on a compact symplectic manifold M. One formulation is the following:

The direct image under μ of the symplectic (Liouville) measure on M is a piece-wise polynomial function (times Lebesgue measure). (8.1)

There is an alternative formulation of (8.1), obtained by taking Fourier transforms,

†Strictly speaking this is the moment map for *ordered* sets of n points.

which is somewhat more enlightening. For simplicity I shall describe this in the special case when the torus T is a circle and the fixed points $\{P\}$ of the action are assumed isolated. We then have a single moment function H (the Hamiltonian of the circle action) and the Fourier transform of (8.1) gives the integral formula:

$$\int_M e^{-itH}\frac{\omega^n}{n!}=\sum_P \frac{e^{-itH(P)}}{(it)^n e(P)}. \tag{8.2}$$

Here ω is the symplectic form on M, $2n = \dim M$ and $e(P)$ is an integer attached to the infinitesimal action of the circle on the tangent space at P (the product of the weights of this representation).

The interest of (8.2) is that the principle of "stationary-phase" gives an asymptotic expansion as $t \to \infty$ in terms of behaviour near the critical points of H, which are just the fixed points P of the circle action. The R.H.S. of (8.2) is precisely what stationary-phase gives (the expansion consists here of just one term for each P). Thus (8.2) asserts that *the stationary-phase approximation is exact for a Hamiltonian coming from a circle action*, and the same holds more generally for torus actions and arbitrary fixed point sets (not necessarily isolated).

It is not hard to see that the polynomial nature of $\mu_*(\omega^n/n!)$ is equivalent to the exactness of stationary-phase so that (8.1) and (the general case of) (8.2) are indeed equivalent.

There are several variants of the proof of (8.1). In addition to that in [**5**] there is an alternative approach in [**2**] which emphasises its essentially cohomological nature.

Stationary-phase approximation is very widely used in mathematical physics, even when M is infinite-dimensional (i.e. some function space), and a general principle telling us when this approximation is exact is clearly of great value. A number of interesting infinite-dimensional examples, including loop spaces of manifolds, fit into the symplectic framework we have been discussing and do give exact formulae. It is therefore an interesting problem to generalise (8.2) to a suitable infinite-dimensional setting. This will of course involve some regularisation of formally infinite quantities as in quantum field theory.

In conclusion I hope that this rather rapid survey has given some indication of the usefulness of the moment map in quite a variety of algebraic and geometric contexts.

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