Canad. J. Math. Vol. **58** (3), 2006 pp. 492–528

Extension Theorems on Weighted Sobolev Spaces and Some Applications

Seng-Kee Chua

Abstract. We extend the extension theorems to weighted Sobolev spaces $L^p_{w,k}(\mathcal{D})$ on (ε,δ) domains with doubling weight w that satisfies a Poincaré inequality and such that $w^{-1/p}$ is locally $L^{p'}$. We also make use of the main theorem to improve weighted Sobolev interpolation inequalities.

1 Introduction

By a weight w, we mean a non-negative locally integrable function that is positive almost everywhere on $\mathbb{R}^n$. By an abuse of notation, we will also write w for the measure induced by w. Sometimes we write dw to denote wdx. We usually assume w is doubling, by which we mean $w(2Q) \le Cw(Q) = C\int_Q w(x)\,dx$ for every cube Q, where $2Q$ denotes the cube with the same center as Q and twice the edgelength of Q. All cubes in this paper are assumed to be closed and with edges parallel to the axes. Q will always be a cube and $l(Q)$ will be its edgelength. $Q_r(x)$ will be the cube with center x and $l(Q_r(x)) = r$. Let μ be another weight. By $w/\mu \in A_p(\mu)$ (the Muckenhoupt A_p condition with respect to μ), we mean

$$\frac{1}{\mu(Q)}\Big(\int_Q \frac{w}{\mu}\,d\mu\Big)^{1/p}\Big(\int_Q \Big(\frac{w}{\mu}\Big)^{-1/(p-1)}\,d\mu\Big)^{1/p'} \le C$$

when $1 < p < \infty$, $1/p + 1/p' = 1$, and

$$\frac{\mu(x)}{\mu(Q)} \le C\frac{w(x)}{w(Q)}$$

for almost every x in Q when $p = 1$ for all cubes Q in $\mathbb{R}^n$.

When $\mu = 1$, we will just write it as A_p. Note that w is doubling when it is in A_p and clearly $w/w \in A_p(w)$.

Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$. If α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_+$, we will denote $\sum_{j=1}^n \alpha_j$ by $|\alpha|$ and $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$. By $\alpha \ge \beta$, we mean $\alpha_j \ge \beta_j$ for all $1 \le j \le n$. Moreover we write $\alpha > \beta$ if $\alpha \ge \beta$ and $\alpha \ne \beta$. We denote by ∇ the vector $\big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\big)$ and by ∇^m the vector of all possible m-th order derivatives for $m \in \mathbb{N}$. A locally integrable function f on $\mathcal{D}$ (we will write

Received by the editors June 1, 2004; revised May 5, 2005.
AMS subject classification: 46E35.
Keywords: Poincaré inequalities, A_p weights, doubling weights, (ε,δ) and (ε,∞) domains.

$f \in L^1_{\mathrm{loc}}(\mathcal{D})$) has a weak derivative of order α if there is a locally integrable function (denoted by $D^\alpha f$) such that

$$\int_{\mathcal{D}} f(D^\alpha \varphi)\,dx = (-1)^{|\alpha|} \int_{\mathcal{D}} (D^\alpha f)\varphi\,dx$$

for all C^∞ functions φ with compact support in $\mathcal{D}$ (we will write $\varphi \in C_0^\infty(\mathcal{D})$).

For $1 \le p < \infty$, $k \in \mathbb{N}$, and any weight w, $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ are the spaces of functions having weak derivatives of all orders α, $|\alpha| \le k$, and satisfying

$$\|f\|_{L^p_{w,k}(\mathcal{D})} = \sum_{0\le|\alpha|\le k} \|D^a f\|_{L^p_w(\mathcal{D})} = \sum_{0\le|\alpha|\le k} \Big(\int_{\mathcal{D}} |D^\alpha f|^p\,dw\Big)^{1/p} < \infty,$$

and

$$\|f\|_{E^p_{w,k}(\mathcal{D})} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p_w(\mathcal{D})} < \infty,$$

respectively. Moreover, in the case when $w \equiv 1$, we will denote $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ by $L^p_k(\mathcal{D})$ and $E^p_k(\mathcal{D})$, respectively. We let $C^{k-1,1}_{\mathrm{loc}}(\mathcal{D})$ be the collection of all functions on $\mathcal{D}$ such that all their derivatives of order $k-1$ are locally Lipschitz continuous on $\mathcal{D}$. In case $k = 1$, we will just denote it by $\mathrm{Lip}_{\mathrm{loc}}(\mathcal{D})$. Furthermore, by $f \in C^{k-1,1}_{\mathrm{loc}}(\overline{\mathcal{D}})$, we mean $f = \tilde f|_{\overline{\mathcal{D}}}$ with $\tilde f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Definition 1.1 An open set $\mathcal{D}$ is an (ε,δ) domain if for all $x, y \in \mathcal{D}, \mid x - y \mid < \delta$, there exists a rectifiable curve γ connecting x to y such that γ lies in $\mathcal{D}$ and

$$l(\gamma) < \frac{|x-y|}{\varepsilon}, \tag{1.1}$$

$$d(z,\partial\mathcal{D}) > \frac{\varepsilon|x-z||y-z|}{|x-y|} \quad \forall z \in \gamma. \tag{1.2}$$

Here $l(\gamma)$ is the length of γ and $d(z,\partial\mathcal{D})$ is the distance between z and the boundary of $\mathcal{D}$. Moreover, we will write $d(Q,S) = \inf_{x\in Q, y\in S} \mid x-y \mid$, $d(Q) = d(Q,\partial\mathcal{D})$ and $d(z) = d(\{z\},\partial\mathcal{D})$.

In 1981, P. Jones [27] extended a famous extension theorem on Lipschitz domains to (ε,δ) domains.

Theorem 1.2 *If $\mathcal{D}$ is a connected (ε,δ) domain and $1 \le p \le \infty$, then $C^\infty(\mathbb{R}^n) \cap L^p_k(\mathcal{D})$ is dense in $L^p_k(\mathcal{D})$ and $L^p_k(\mathcal{D})$ has a bounded extension operator,* i.e., *there exists $\Lambda\colon L^p_k(\mathcal{D}) \to L^p_k(\mathbb{R}^n)$ such that $\Lambda f|_{\mathcal{D}} = f$ a.e. and $\|\Lambda\|$ is bounded. Moreover, the norm of the extension operator depends only on $\varepsilon, \delta, k, p$, $\mathrm{rad}(\mathcal{D})$, and the dimension n.*

Furthermore, he proved that

Theorem 1.3 *If $\mathcal{D}$ is an (ε,∞) domain in $\mathbb{R}^n$, then $E^n_1(\mathcal{D})$ has a bounded extension operator.*

Let $\mathcal{D}$ be a bounded (ε,∞) domain with radius $r = \inf_{x\in\mathcal{D}} \sup_{y\in\mathcal{D}} |x-y| = \mathrm{rad}(\mathcal{D})$ and let Ω be a bounded open set containing $\mathcal{D}$. Let W_2 be the collection of cubes in the Whitney decomposition of $(\mathcal{D}^c)^o$ and define

$$W_3 = \Big\{ Q \in W_2 : l(Q) \le \frac{\varepsilon r}{16nL} \Big\}, \quad L = 2^{-m}, m \in \mathbb{Z}_+,$$

where L is chosen so that $\Omega \subset (\bigcup_{Q\in W_3} Q) \cup \overline{\mathcal{D}}$.

In 1992, the author [10] extended Theorems 1.2 and 1.3 to weighted Sobolev spaces $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ when the weight is in A_p. Moreover, in the case of (ε,∞) domains, the author showed that:

Theorem 1.4 ([10, Theorems 1.4 and 1.5]) *Let $w_i \in A_{p_i}, 1 \le p_i < \infty$ for $i = 0, 1, \dots, N$. Let Ω be an open set containing an (ε,∞) domain $\mathcal{D}$ and let L and r be defined as above such that $\Omega \subset (\bigcup_{Q\in W_3} Q)\cup\overline{\mathcal{D}}$. Then there exists an extension operator Λ on $\mathcal{D}$ such that*

$$\|\nabla^{k_i}\Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^{k_i} f\|_{L^{p_i}_{w_i}(\mathcal{D})} \quad \text{for all } i \tag{1.3}$$

for all $f \in \bigcap_{i=0}^N E^{p_i}_{w_i,k_i}(\mathcal{D})$. Here C_i depends only on $\varepsilon, p_i, w_i, k_i, n, L$ and $\max_i k_i$. Moreover, if $\mathcal{D}$ is unbounded, then (1.3) holds for $\Omega = \mathbb{R}^n$.

Furthermore, in 1994, Theorems 1.2 and 1.3 were further extended by relaxing the A_p condition on the weight w to just doubling weights that satisfy a Poincaré inequality [12, Theorems 1.2 and 1.3]. However, the extension operator obtained there was only on $C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. The author also extended Theorem 1.4 to more general weights:

Theorem 1.5 ([12, Theorem 1.4]) *Let $1 \le p_i < \infty$ for $i = 0, 1, \dots, N$. Let Ω be a bounded open set containing an (ε,∞) domain $\mathcal{D}$ and let L and r be as before. Let μ be a weight and suppose that w_i are doubling weights such that ($f_{Q,\mu} = \frac{1}{\mu(Q)}\int_Q f\,d\mu$)*

$$\|f - f_{Q,\mu}\|_{L^{p_i}_{w_i}(Q)} \le A_i l(Q) \|\nabla f\|_{L^{p_i}_{w_i}(Q)} \quad \forall Q \subset \mathcal{D} \tag{1.4}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{D})$ and $i = 0, 1, \dots, N$. Then there exists an extension operator on $\mathcal{D}$ such that $\Lambda f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|\nabla^k \Lambda f\|_{L^{p_i}_{w_i}(\mathbb{R}^n)} \le C_i \|\nabla^k f\|_{L^{p_i}_{w_i}(\mathcal{D})}$$

for all i and $f \in C^{k-1,1}_{\mathrm{loc}}(\overline{\mathcal{D}})$; in addition, if $w_i/\mu \in A_{p_i}(\mu)$ for some i, then for that i,

$$\|\Lambda f\|_{L^{p_i}_{w_i,k-1}(\Omega)} \le C_i \|f\|_{L^{p_i}_{w_i,k-1}(\mathcal{D})}$$

and

$$\|\nabla^{k-1}\Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^{k-1} f\|_{L^{p_i}_{w_i}(\mathcal{D})}.$$

C_i depends only on $w_i, \mu, \varepsilon, L, p_i, A_i, k$ and n.

Moreover, if $\mathcal{D}$ is a bounded (ε, δ) domain and w is a doubling weight such that $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$ and w is locally A_p in $\mathcal{D}$, then it is obtained in [15] that the extension theorem holds for $L^p_{w,k}(\mathcal{D})$ if (1.5) below holds. In this paper, we will further study the extension problem when the weights are just doubling and satisfy a Poincaré inequality. Note that in those previous studies, the standard approach is to extend functions in $C^{k-1,1}_{\mathrm{loc}}(\overline{\mathcal{D}})$ and then apply density theorems. However, in general, one does not have density theorems for weighted Sobolev space when the weight is only doubling and satisfies a Poincaré inequality. Note that even though (ε, δ) domains need not be connected, one can always consider each of its connected components. Thus we will just consider connected (ε, δ) domains. Let us now state our main theorems and results.

Theorem 1.6 *Let $\mathcal{D}$ be a connected (ε, δ) domain and let $1 \le p < \infty$, $k \in \mathbb{N}$. Suppose w is a doubling weight such that $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$ and the following Poincaré inequality holds (where $f_{Q,w} = \frac{1}{w(Q)} \int_Q f dw$):*

$$\|f - f_{Q,w}\|_{L^p_w(Q)} \le A l(Q) \|\nabla f\|_{L^p_w(Q)}, \tag{1.5}$$

for all $f \in E^p_{w,1}(\mathcal{D})$ and cubes $Q \subset \mathcal{D}$, $\frac{l(Q)}{d(Q)} \le A_0$, $A_0 > 0$. Then for any $f \in L^p_{w,k}(\mathcal{D})$, there exists an extension $\Lambda f \in L^p_{w,k}(\mathbb{R}^n)$ such that

$$\|\Lambda f\|_{L^p_{w,k}(\mathbb{R}^n)} \le C \|f\|_{L^p_{w,k}(\mathcal{D})}$$

where C depends only on $A, A_0, \varepsilon, \delta, \mathrm{rad}(\mathcal{D}), w$ and the dimension n. Moreover, if in addition $f \in C^{k-1,1}_{\mathrm{loc}}(\overline{\mathcal{D}})$, then indeed $\Lambda f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, we have the following.

Theorem 1.7 *Let $1 \le p_i < \infty$ for $i = 1, \dots, N$, and $k \in \mathbb{N}$. Let Ω be a bounded open set containing an (ε, ∞) domain $\mathcal{D}$ and let L and r be as in Theorem 1.4. Let μ and w_i be doubling weights such that*

$$\|f - f_{Q,\mu}\|_{L^1_\mu(Q)} \le A_i \frac{l(Q)\mu(Q)}{w_i(Q)^{1/p_i}} \|\nabla f\|_{L^{p_i}_{w_i}(Q)} \tag{1.6}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{D})$ and for all cubes $Q \subset \mathcal{D}$ such that $\frac{l(Q)}{d(Q)} \le A_0, A_0 > 0$. Then for any $f \in C^{k-1,1}_{\mathrm{loc}}(\overline{\mathcal{D}})$, there exists an extension $\Lambda f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ such that ($\Lambda f = f$ on $\mathcal{D}$)

$$\|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i \|\nabla^l f\|_{L^{p_i}_{w_i}(\mathcal{D})}, \quad 1 \le l \le k. \tag{1.7}$$

Here C_i depends only on $w_i, \mu, \varepsilon, L, p_i, A_i, k$ and n. Furthermore, for any doubling weight v such that $v/\mu \in A_p(\mu)$, $1 \le p < \infty$, we have

$$\|\Lambda f\|_{L^p_v(\Omega)} \le C \|f\|_{L^p_v(\mathcal{D})}. \tag{1.8}$$

Here C depends only on $v, \mu, \varepsilon, L, p, k$ and n.

Remark 1.8

(1) Theorem 1.7 is indeed stronger than Theorem 1.5 except when $l = k$ in (1.7). Note that (1.6) will imply (1.4) (see Remark 2.9). However, in case $w_i/\mu \in A_{p_i}(\mu)$, then (1.6) is indeed equivalent to (1.4). Thus in case $0 < l < k$, the conclusion of Theorem 1.7 is strictly stronger with slightly weaker conditions (since we do not assume $w_i/\mu \in A_{p_i}(\mu)$ here). Moreover, even though we will only prove that

$$\|\nabla^k \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \leq C_i \|\nabla^k f\|_{L^{p_i}_{w_i}(\mathcal{D})},$$

one can indeed replace Ω by $\mathbb{R}^n$ in the above inequality by modifying the extension of functions outside $\cup W_3$; see the proof of [12, Theorem 1.4] for the detail.

(2) Since (1.6) implies (1.4), by repeated applications of (1.4), we have for all $0 \leq |\alpha| < l$,

$$\|D^\alpha(f - P^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)} \leq Cl(Q)\|\nabla D^\alpha(f - P^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)} \leq Cl(Q)^{l-|\alpha|}\|\nabla^l f\|_{L^{p_i}_{w_i}(Q)} \tag{1.9}$$

if $P^l_\mu(Q)f$ is the unique polynomial of degree $< l$ such that

$$\int_Q D^\beta(f - P^l_\mu(Q)f)\, d\mu = 0, \quad \text{for all } 0 \leq |\beta| < l.$$

(3) It is easy to check that (1.5) holds for distant-type weights $w(x) = \operatorname{dist}(M, x)^a$, $M \subset \partial\mathcal{D}$. Note that clearly such weights need not be in A_p. Moreover, there is a class of domains with $\operatorname{dist}(x, \partial\mathcal{D})^{-1/p} \in L^{p'}(\mathcal{D})$, [24, Theorem 6]. Also, see [7] for another class of non-A_p weights such that (1.5) holds.

2 Preliminaries

In what follows, C denotes various positive constants, which may differ even in a sequence of consecutive estimates. Moreover, sometimes we will use $C(\alpha, \beta, \dots)$ instead of C to emphasize that the constant depends on $\alpha, \beta, \dots$.

In this section, we will collect some useful results that will be needed in the proof of our main theorem. First of all, since we will need to project functions into spaces of polynomials, we will state some results about polynomials.

Theorem 2.1 ([10, Lemma 2.3]) *Let F, Q be cubes such that $F \subset Q$ and $|F| > \gamma|Q|$. If w is a doubling weight, $1 \leq q < \infty$, and p is a polynomial of degree less than k, then*

$$\|p\|_{L^q_w(E)} \leq C(\gamma, k, n, w)\Big(\frac{w(E)}{w(F)}\Big)^{1/q}\|p\|_{L^q_w(F)}$$

for all measurable sets $E \subset Q$.

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

Lemma 2.2 ([37, Chapter 3, Lemma 7]) *If w is a doubling measure and k is a positive integer, then there exists $s_0(n,k,w)$ such that if $s < s_0$, then for all cubes Q, $\lambda > 0$ such that*

$$w\big(\{x \in Q : |p(x)| > \lambda\}\big) \le sw(Q)$$

we have

$$\sup_{x\in Q} |p(x)| \le C\lambda,$$

where p is any polynomial of degree less than k and C is a constant independent of λ, Q and p.

It follows from Chebyshev's inequality and this lemma that given k and a polynomial p of degree less than k,

$$\|p\|_{L^\infty(Q)} \le \frac{C}{w(Q)} \|p\|_{L^1_w(Q)}, \tag{2.1}$$

with C independent of Q and p.

The following is now a consequence of Markov's inequality (see [2]) and Lemma 2.2.

Theorem 2.3 *Let p be a polynomial of order less than k and $1 \le q \le \infty$. If w is a doubling weight, then*

$$\|\nabla p\|_{L^q_w(Q)} \le Cl(Q)^{-1} \|p\|_{L^q_w(Q)}$$

for all cubes Q in $\mathbb{R}^n$, where C depends only on k, w, q and n.

We now prove an interesting fact about projection of functions into polynomials.

Proposition 2.4 *For any $k \in \mathbb{N}$, cube $Q \subset \mathbb{R}^n$ and doubling weight σ, there exists a projection $\pi^k_\sigma(Q) : L^1_\sigma(Q) \to \mathcal{P}_{k-1}$ (space of polynomials of degree $< k$) such that*

$$\|\pi^k_\sigma(Q) f\|_{L^\infty(Q)} \le \frac{C}{\sigma(Q)} \|f\|_{L^1_\sigma(Q)}$$

where C depends only on k, n and the doubling constant of σ. When $\sigma = 1$, we just denote the projection by $\pi^k(Q) f$.

Proof This proposition can indeed be found in [17]. However, as the proof is quite short and the reference may not be available, we will prove it here.

First note that $\mathcal{P}_{k-1}$ is a finite dimensional vector space over $\mathbb{R}$ and $\int_Q p_1 p_2 \, d\sigma$ defines an inner product on $\mathcal{P}_{k-1}$. Hence there exists an orthonormal basis $\{\varphi_1, \varphi_2, \dots, \varphi_m\} \subset \mathcal{P}_{k-1}$ with respect to this inner product. Then $\|\varphi_i\|_{L^2_\sigma(Q)} = 1$ and

$$p(x) = \sum_{i=1}^m \varphi_i(x) \int_Q p(y)\varphi_i(y) \, d\sigma$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

if $p \in \mathcal{P}_{k-1}$. We now define

$$\pi_\sigma^k(Q)f(x) = \sum_{i=1}^m \varphi_i(x) \int_Q f(y)\varphi_i(y)d\sigma \quad \text{for } f \in L^1_\sigma(Q).$$

It is clear that $\pi_\sigma^k(Q)$ is a projection to $\mathcal{P}_{k-1}$. Next, by (2.1) and Hölder's inequality, we have

$$\|\varphi_i\|_{L^\infty(Q)} \le \frac{C}{\sigma(Q)^{1/2}}\|\varphi_i\|_{L^2_\sigma(Q)} = C/\sigma(Q)^{1/2}$$

where C depends only on the doubling constant of σ, k and the dimension n. It is now clear that

$$\|\pi_\sigma^k(Q)f\|_{L^\infty(Q)} \le \sum_{i=1}^m \|\varphi_i\|_{L^\infty(Q)}\|\varphi_i\|_{L^\infty(Q)}\|f\|_{L^1_\sigma(Q)} \le \frac{C}{\sigma(Q)}\|f\|_{L^1_\sigma(Q)}. \qquad \blacksquare$$

Consequently, we have

Lemma 2.5 *Let $1 \le p \le \infty$ and $k \in \mathbb{N}$. Let w be a doubling weight on $\mathbb{R}^n$ such that for any weakly differentiable function f and cube Q, there exists a constant $a(f,Q)$ such that*

$$\|f - a(f,Q)\|_{L^p_w(Q)} \le Cl(Q)\|\nabla f\|_{L^p_w(Q)}. \tag{2.2}$$

Then

$$\|D^\alpha(f - \pi_w^k(Q)f)\|_{L^p_w(Q)} \le Cl(Q)^{l-|\alpha|}\|\nabla^l f\|_{L^p_w(Q)} \tag{2.3}$$

and

$$\|D^\alpha \pi_w^k(Q)f\|_{L^p_w(Q)} \le C\|\nabla^{|\alpha|} f\|_{L^p_w(Q)} \tag{2.4}$$

for $|\alpha| \le l \le k$ and $f \in E^p_{w,k}(Q)$.

Proof Let $f \in E^p_{w,k}(Q)$. First note that then $f \in L^p_{w,k}(Q)$ by repeated applications of (2.2). Next note that by the triangle inequality, Hölder's inequality, and (2.2), we have

(2.5)
$$\begin{aligned}\|f - f_{Q,w}\|_{L^p_w(Q)} &\le \|f - a(f,Q)\|_{L^p_w(Q)} + \|f_{Q,w} - a(f,Q)\|_{L^p_w(Q)}\\ &= \|f - a(f,Q)\|_{L^p_w(Q)} + w(Q)^{1/p}\Big|\frac{1}{w(Q)}\int_Q (f - a(f,Q))\,dw\Big|\\ &\le 2\|f - a(f,Q)\|_{L^p_w(Q)} \quad \text{(by Hölder's inequality)}\\ &\le Cl(Q)\|\nabla f\|_{L^p_w(Q)}.\end{aligned}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

For each $l \in \mathbb{N}$, $l \le k$, let $P_w^l(Q)f$ be the polynomial of degree $< l$ such that

$$\int_Q D^\alpha(f - P_w^l(Q)f)\,dw = 0 \quad \text{for all } 0 \le |\alpha| < l.$$

Then by repeated applications of (2.5), we have

$$\|D^\alpha(f - P_w^l(Q)f)\|_{L_w^p(Q)} \le Cl(Q)^{l-|\alpha|}\|\nabla^l f\|_{L_w^p(Q)} \tag{2.6}$$

for $0 \le |\alpha| < l$. Also, (2.6) clearly holds if $|\alpha| = l$ as $P_w^l(Q)f$ is a polynomial of degree $< l$. Hence if $0 \le |\alpha| \le l \le k$,

$$\begin{aligned}
&\|D^\alpha(f - \pi_w^k(Q)f)\|_{L_w^p(Q)} \\
&\quad\le \|D^\alpha(f - P_w^l(Q)f)\|_{L_w^p(Q)} + Cl(Q)^{-|\alpha|}\|\pi_w^k(Q)[f - P_w^l(Q)f]\|_{L_w^p(Q)} \\
&\qquad\qquad \text{(by the triangle inequality and Theorem 2.3)} \\
&\quad\le \|D^\alpha(f - P_w^l(Q)f)\|_{L_w^p(Q)} + Cl(Q)^{-|\alpha|}w(Q)^{1/p}\|\pi_w^k(Q)[f - P_w^l(Q)f]\|_{L^\infty(Q)} \\
&\qquad\qquad \text{(by Hölder's inequality)} \\
&\quad\le \|D^\alpha(f - P_w^l(Q)f)\|_{L_w^p(Q)} + Cl(Q)^{-|\alpha|}\|f - P_w^l(Q)f\|_{L_w^p(Q)} \\
&\qquad\qquad \text{(by Proposition 2.4 and Hölder's inequality)} \\
&\quad\le Cl(Q)^{l-|\alpha|}\|\nabla^l f\|_{L_w^p(Q)}
\end{aligned}$$

by (2.6). Next, by the triangle inequality and the previous inequality,

$$\begin{aligned}
\|D^\alpha \pi_w^k(Q)f\|_{L_w^p(Q)} &\le \|D^\alpha(\pi_w^k(Q)f - f)\|_{L_w^p(Q)} + \|D^\alpha f\|_{L_w^p(Q)} \\
&\le C\|\nabla^{|\alpha|} f\|_{L_w^p(Q)} + \|D^\alpha f\|_{L_w^p(Q)} \\
&\le C\|\nabla^{|\alpha|} f\|_{L_w^p(Q)}.
\end{aligned}$$

■

Remark 2.6 Inequality (2.3) has been established before. However, only recently did we realize that (2.4) is indeed just a consequence of (2.3).

Next, let us state a consequence of [16, Theorem 1.6].

Theorem 2.7 *Let $0 < p, s < \infty$, $1 < \lambda < \infty$. Let u be a measurable function defined on a cube Q_0 and let "a" be a nonnegative set function on all cubes Q with $\lambda Q \subset Q_0$. Let μ be a doubling weight with doubling constant C_μ. Suppose there exists a doubling* *weight σ such that for any cube Q with $\lambda Q \subset Q_0$, there exists a polynomial P_Q of degree $< k$ so that*

$$\frac{1}{\sigma(Q)^{1/p}} \|u - P_Q\|_{L^p_\sigma(Q)} \le a(Q) \tag{2.7}$$

and there exists $0 < \delta < 1$ such that

$$\sum_{Q \in \mathcal{F}} a(Q)^s \mu(Q)^{1-\delta} \le a_0^s \mu(Q_0)^{1-\delta} \tag{2.8}$$

for any collection $\mathcal{F}$ of nonoverlapping cubes Q such that $\lambda Q \subset Q_0$. If there exists $F \subset Q_0$ such that $\mu(Q_0 \backslash F) = 0$ and for all $x \in F$, $P_{Q_r(x)}(x) \to u(x)$ as $r \to 0$ (recall that $Q_r(x)$ is the cube with center x and $l(Q_r(x)) = r$), then for $0 < q < s$, we have

$$\frac{1}{\mu(Q_0)^{1/q}} \|u - P_{Q'}\|_{L^q_\mu(Q_0)} \le C a_0 \tag{2.9}$$

where $\lambda Q' \subset Q_0 \subset \lambda^2 Q'$.

It follows from the preceding theorem that we have the following lemma.

Lemma 2.8 *Let $1 \le p, q < \infty$, $A_0 > 0$. Let $\mathcal{D}$ be any open connected set. If σ and w are doubling weights such that*

$$\frac{1}{\sigma(Q)} \|f - f_{Q,\sigma}\|_{L^1_\sigma(Q)} \le C \frac{l(Q)}{w(Q)^{1/p}} \|\nabla f\|_{L^p_w(Q)} \tag{P}$$

for all cubes $Q \subset \mathcal{D}$ such that $\frac{l(Q)}{d(Q)} \le A_0$ and weakly differentiable functions f, then

$$\|f - f_{Q,\sigma}\|_{L^p_w(Q)} \le C l(Q) \|\nabla f\|_{L^p_w(Q)}$$

for all cubes $Q \subset \mathcal{D}$ such that $\frac{l(Q)}{d(Q)} \le A_0$ and weakly differentiable functions f.

Proof First note that since w is doubling, there exists $k > 1$ such that

$$\left(\frac{w(Q)}{w(\tilde{Q})}\right)^{1-1/k^2} \ge C \left(\frac{l(Q)}{l(\tilde{Q})}\right)^p \quad \text{for all cubes } Q \subset \tilde{Q}.$$

Also, note that for almost all x, $f_{Q_r(x),\sigma} = \frac{1}{\sigma(Q_r(x))} \int_{Q_r(x)} f \, d\sigma \to f(x)$ as $r \to 0$. Let $a(Q) = \frac{l(Q)}{w(Q)^{1/p}} \|\nabla f\|_{L^p_w(Q)}$. If $\delta = 1 - \frac{1}{k}$ and $s = kp$, then for any collection $\mathcal{F}$ of

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

nonoverlapping cubes in a cube $\tilde{Q} \subset \mathcal{D}$, $l(\tilde{Q})/d(\tilde{Q}) \le A_0$, we have

$$\begin{aligned}\sum_{Q\in\mathcal{F}} a(Q)^s w(Q)^{1-\delta} &= \sum_{Q\in\mathcal{F}} \frac{l(Q)^{kp}}{w(Q)^{k-\frac{1}{k}}} \|\nabla f\|^{kp}_{L^p_w(Q)} \\ &\le C \frac{l(\tilde{Q})^{kp}}{w(\tilde{Q})^{k-\frac{1}{k}}} \sum_{Q\in\mathcal{F}} \|\nabla f\|^{kp}_{L^p_w(Q)} \\ &\le C \frac{l(\tilde{Q})^{kp}}{w(\tilde{Q})^{k-\frac{1}{k}}} \Big(\sum_{Q\in\mathcal{F}} \|\nabla f\|^{p}_{L^p_w(Q)}\Big)^k \\ &\le C \frac{l(\tilde{Q})^{kp}}{w(\tilde{Q})^{k-\frac{1}{k}}} \|\nabla f\|^{kp}_{L^p_w(\tilde{Q})} \\ &= Ca(\tilde{Q})^s w(\tilde{Q})^{1-\delta}.\end{aligned}$$

Note that if $Q \subset \tilde{Q}$, then $\frac{l(Q)}{d(Q)} \le \frac{l(\tilde{Q})}{d(\tilde{Q})}$. Hence, fixing any $\lambda > 1$, since $0 < p < kp = s$, it follows from the previous theorem that

$$\|f - f_{Q',\sigma}\|_{L^p_w(Q)} \le Cl(Q)\|\nabla f\|_{L^p_w(Q)} \quad \text{when } \lambda Q' \subset Q \subset \lambda^2 Q'$$

and

$$\begin{aligned}\|f - f_{Q,\sigma}\|_{L^p_w(Q)} &\le \|f - f_{Q',\sigma}\|_{L^p_w(Q)} + \|f_{Q,\sigma} - f_{Q',\sigma}\|_{L^p_w(Q)} \\ &\qquad\qquad \text{(by the triangle inequality)} \\ &= \|f - f_{Q',\sigma}\|_{L^p_w(Q)} + w(Q)^{1/p}\left|\frac{1}{\sigma(Q')}\int_{Q'} (f - f_{Q,\sigma})\, d\sigma\right| \\ &\le \|f - f_{Q',\sigma}\|_{L^p_w(Q)} + Cw(Q)^{1/p}\frac{1}{\sigma(Q)}\int_{Q} |f - f_{Q,\sigma}|\, d\sigma \\ &\le \|f - f_{Q',\sigma}\|_{L^p_w(Q)} + Cl(Q)\|\nabla f\|_{L^p_w(Q)}\end{aligned}$$

by (P). The conclusion of the Lemma is now clear. ■

Remark 2.9

(1) Theorem 2.7 and Lemma 2.8 are indeed results in "self-improving inequalities", see [21, 24] for details.

(2) If (1.6) holds and $\pi^l_\mu(Q)f$ is the polynomial in Proposition 2.4, then (1.9)

holds and

$$
\begin{aligned}
&\|D^\alpha(f-\pi^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)}\\
&\quad\le \|D^\alpha(f-P^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)}+\|D^\alpha(P^l_\mu(Q)f-\pi^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)}\\
&\quad\le \|D^\alpha(f-P^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)}+Cl(Q)^{-|\alpha|}w_i(Q)^{1/p_i}\|\pi^l_\mu(Q)(P^l_\mu(Q)f-f)\|_{L^\infty(Q)}\\
&\qquad\qquad\text{(by Hölder's inequality and Theorem 2.3)}\\
&\quad\le \|D^\alpha(f-P^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)}+Cl(Q)^{-|\alpha|}\frac{w_i(Q)^{1/p_i}}{\mu(Q)}\|f-P^l_\mu(Q)f\|_{L^1_\mu(Q)}\\
&\qquad\qquad\text{(by Proposition 2.4)}\\
&\quad\le \|D^\alpha(f-P^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)}+Cl(Q)^{-|\alpha|+1}\|\nabla(f-P^l_\mu(Q)f)\|_{L^{p_i}_{w_i}(Q)}\ \text{(by (1.6))}\\
&\quad\le Cl(Q)^{l-|\alpha|}\|\nabla^l f\|_{L^{p_i}_{w_i}(Q)}\quad\text{(by (1.9) when } |\alpha|<l).
\end{aligned}
$$

Next, the following lemma is indeed a special case of a result in [11].

Lemma 2.10 ([11, Theorem 2.1]) *Let f be a measurable function on $\mathbb{R}^n$ and let w be a doubling weight. Also, let $1\le p\le\infty$, $k\in\mathbb{N}$ and $L>0$. For each cube Q in $\mathbb{R}^n$, let $P(f,Q)$ be a polynomial of degree $<k$ associated to f on Q. Suppose that $\{Q_i\}_{i=0}^l$ is a sequence of cubes such that $Q_i\cap Q_{i+1}$ contains a cube Q^i with $|Q^i|\ge L\max\{|Q_i|,|Q_{i+1}|\}$ for each $i=0,1,\dots,l-1$. Then*

$$\|f-P(f,Q_0)\|_{L^p_w(Q_l)}\le C\sum_i\|f-P(f,Q_i)\|_{L^p_w(Q_i)},\tag{2.10}$$

where C depends only on L,l,w,k,p and the dimension n.

3 Facts About (ε,δ) Domains

Let $\mathcal{D}$ be a connected (ε,δ) domain. Recall that $r=\mathrm{rad}(\mathcal{D})=\inf_{x\in\mathcal{D}}\sup_{y\in\mathcal{D}}|x-y|$. Following the terminology used in [27], we say that two cubes touch if a face of one cube is contained in a face of the other. In particular, the union of two touching cubes of equal size is a rectangle. A collection of cubes $\{S_i\}_{i=0}^m$ is called a chain if S_i touches S_{i+1} for all i. Also let W_1 be the cubes in the Whitney decomposition of $\mathcal{D}$ and W_2 be the cubes in the Whitney decomposition of $(\mathcal{D}^c)^o$; see [36] for the definition of the Whitney decomposition.

Next let us recall some properties of the cubes in the Whitney decomposition of the open set $\mathcal{D}$ or $(\mathcal{D}^c)^o$. Since these properties are well known, we will often make

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

use of them without explicitly mentioning them.

$$l(Q) = 2^{-k} \quad \text{for some } k \in \mathbb{Z},$$

$$Q_1^o \cap Q_2^o = \varnothing \quad \text{if } Q_1 \neq Q_2,$$

$$1/4 \leq \frac{l(Q_1)}{l(Q_2)} \leq 4 \quad \text{if } Q_1 \cap Q_2 \neq \varnothing,$$

$$(*) \qquad \sqrt{n} \leq \frac{d(Q, \partial\mathcal{D})}{l(Q)} \leq 4\sqrt{n}.$$

If necessary, we will subdivide all the Whitney cubes l times so that the above hold except that $(*)$ will be replaced by

$$\sqrt{n}2^l \leq \frac{d(Q, \partial\mathcal{D})}{l(Q)} \leq 4\sqrt{n}2^l,$$

where l is a fixed given positive integer. We will call such a decomposition a Whitney l-decomposition.

Next, let us collect some facts concerning (ε, δ) domains. The reader can find the proofs in [27]. More details can be found in [10, 12, 15].

Let $\mathcal{D}$ be an (ε, δ) domain. Recall that W_1 and W_2 are the Whitney decompositions of $\mathcal{D}$ and $(\mathcal{D}^c)^o$, respectively. Then there exists a positive constant L' depending only on $\varepsilon, \delta, \operatorname{rad}(\mathcal{D})$ and the dimension n such that if $W_3 = \{Q \in W_2 : l(Q) \leq L'\}$, then the following five properties hold.

(A) There exists $C > 0$ such that for all $Q \in W_3$, there exists $S \in W_1$ such that $1 \leq \frac{l(S)}{l(Q)} \leq 4$ and $d(S, Q) \leq Cl(Q)$. We will choose such an S and write $S = Q^*$.

(B) There exists $C > 0$ such that for all $Q \in W_3$, and $S_1, S_2 \in W_1$ such that $S_1, S_2 = Q^*$, then $d(S_1, S_2) \leq Cl(Q)$.

(C) There exists $C > 0$ such that for all $S \in W_1$, there are at most C cubes $Q \in W_3$ with $Q^* = S$.

(D) There exists $C > 0$ such that for all $Q_1, Q_2 \in W_3$ with $Q_1 \cap Q_2 \neq \varnothing$, we have $d(Q_1^*, Q_2^*) \leq Cl(Q_1)$.

(E) There exists $C > 0$ such that for all $Q_j, Q_k \in W_3$ with $Q_j \cap Q_k \neq \varnothing$, there exists a chain $F_{j,k} = \{Q_j^* = S_0, S_1, S_2, \dots, S_m = Q_k^*\}$ of cubes in W_1 connecting Q_j^* to Q_k^* with $m \leq C$. (Then $l(S_i), l(Q_j)$ are comparable and $d(S_i, Q_j^*) \leq Cl(Q_j)$.)

Remark 3.1

(1) Note that even if W_1 and W_2 are just Whitney l-decompositions, there still exists a constant L' such that (A)–(E) hold.

(2) The constants in (A)–(E) depend only on ε, δ and n. Moreover, when $\mathcal{D}$ is an (ε, ∞) domain, given any $0 < L \leq 1$, we may also take $W_3 = \{Q \in W_2 : l(Q) \leq \varepsilon r/(16nL)\}$ so that properties (A)–(E) hold except that now $L \leq l(Q^*)/l(Q) \leq 4L$ for $Q \in W_3$. Of course, the constants now in (A)–(E) also depend on L. Again, it remains valid even if W_1 and W_2 are just Whitney l-decompositions.

Finally, let us state an important property that was proved by Jones [27].

Proposition 3.2 ([27, Lemma 2.3]) *If $\mathcal{D}$ is an (ε, δ) domain, then $|\partial \mathcal{D}| = 0$.*

4 Proof of the Main Theorems

We will follow the approach by Jones [27] and our previous approach in [10, 12]. However, as $C^{k-1,1}_{\text{loc}}(\mathbb{R}^n)$ may not be dense in our weighted Sobolev spaces, we need to consider the extension of all functions directly.

Recall that W_1 is the Whitney decomposition of $\mathcal{D}$ and W_2 is the Whitney decomposition of $(\mathcal{D}^c)^o$. Choose $W_3 \subset W_2$ such that properties (A)–(E) hold. Note that $l(Q) \le C$ for all $Q \in W_3$ and $l(Q) \ge C(r)$ if $Q \in W_2 \backslash W_3$. For each $Q_j \in W_3$, choose $0 \le \varphi_j \le \chi_{\frac{17}{16}Q_j}$, $\varphi_j \in C^\infty(\mathbb{R}^n)$, such that

$$\sum_{Q_j \in W_3} \varphi_j \equiv 1 \text{ on } \bigcup W_3, \quad 0 \le \sum_{Q_j \in W_3} \varphi_j \le 1,$$

and $|D^\alpha \varphi_j| \le C l(Q_j)^{-|\alpha|}$ for $0 \le |\alpha| \le k$.

Instead of proving Theorem 1.6 directly, let us first establish a more general result by assuming the existence of a "nice" projection of $E^p_{w,k}$ functions into polynomials.

Proposition 4.1 *Let w be a doubling weight, $1 \le p < \infty$, $k \in \mathbb{N}$, and let $\mathcal{D}$ be a connected (ε, δ) domain. Suppose for each $l \in \mathbb{N}$, $l \le k$ and cube $Q \subset \mathcal{D}$ such that $\frac{l(Q)}{d(Q)} \le A_0, A_0 > 0$, there exists a projection $P^l(Q)\colon E^p_{w,l}(Q) \to \mathcal{P}_{l-1}$ (space of polynomials of degree $\le l-1$) such that for all $0 \le |\alpha| < l$,*

$$\|D^\alpha(P^l(Q)f - f)\|_{L^p_w(Q)} \le C l(Q)^{l-|\alpha|} \|\nabla^l f\|_{L^p_w(Q)}. \tag{4.1}$$

If for $f \in L^p_{w,k}(\mathcal{D})$, we define $P_j = P^k(Q_j^*)f$ (see (B) for Q^*) and

$$\Lambda f(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{D}, \\ \sum_{Q_j \in W_3} P_j(x)\varphi_j(x) & \text{if } x \in (\mathcal{D}^c)^o, \end{cases}$$

then $\|\Lambda f\|_{L^p_{w,k}(\mathcal{D}^c)} \le C\|f\|_{L^p_{w,k}(\mathcal{D})}$. Moreover, if in addition $w^{-1/p} \in L^{p'}_{\text{loc}}(\mathbb{R}^n)$, then $\Lambda f \in L^p_{w,k}(\mathbb{R}^n)$.

Before we begin, we will first establish some inequalities regarding chains of touching cubes. Recall that two cubes touch if a face of one cube is contained in a face of the other.

Lemma 4.2 *Let w be a doubling weight and $1 \le p < \infty$, $k \in \mathbb{N}$. Let $P^k(Q)$ be as in the previous proposition and satisfy (4.1). If $\{Q_0, Q_1, \dots, Q_m\}$ is a chain of touching Whitney cubes or touching cubes of same size such that $\frac{l(Q_i)}{d(Q_i)} \le A_0, A_0 > 0$, for all i, then for all $0 \le |\alpha| \le k$,*

$$\|D^\alpha(P^k(Q_0)f - P^k(Q_m)f)\|_{L^p_w(Q_0)} \le C(m, p, w, k) l(Q_0)^{k-|\alpha|} \|\nabla^k f\|_{L^p_w(\cup_i Q_i)}. \tag{4.2}$$

Proof First, let $\{S_0, S_1, \dots, S_l\}$, $l = 2m$, be a chain of cubes that satisfies the condition of Lemma 2.10 and

$$\bigcup S_j = \bigcup Q_i, \quad S_0 = Q_0, \quad S_l = Q_m, \quad \frac{l(S_j)}{d(S_j)} \le A_0 \text{ for all } j, \quad \sum_{j=0}^{l} \chi_{S_j} \le 2 \ a.e.$$

Then by the triangle inequality and Lemma 2.10, we have

$$\begin{aligned}
&\|D^\alpha(P^k(Q_0)f - P^k(Q_m)f)\|_{L^p_w(Q_0)} \\
&\quad = \|D^\alpha(P^k(S_0)f - P^k(S_l)f)\|_{L^p_w(S_0)} \\
&\quad \le Cl(S_0)^{-|\alpha|}\|P^k(S_0)f - P^k(S_l)f\|_{L^p_w(S_0)} && \text{(by Lemma 2.3)} \\
&\quad \le Cl(S_0)^{-|\alpha|}\big(\|f - P^k(S_l)f\|_{L^p_w(S_0)} + \|f - P^k(S_0)f\|_{L^p_w(S_0)}\big) \\
&\qquad && \text{(by the triangle inequality)} \\
&\quad \le Cl(S_0)^{-|\alpha|}\Big(\sum_{j=0}^{l}\|f - P^k(S_j)f\|_{L^p_w(S_j)} + \|f - P^k(S_0)f\|_{L^p_w(S_0)}\Big) \\
&\qquad && \text{(by Lemma 2.10)} \\
&\quad \le Cl(Q_0)^{k-|\alpha|}\sum_j \|\nabla^k f\|_{L^p_w(S_j)} && \text{(by (4.1))} \\
&\quad \le Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p_w(\cup S_j)} \\
&\quad = Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p_w(\cup Q_i)}.
\end{aligned}$$

■

Proof of Proposition 4.1 First recall that $|\partial\mathcal{D}| = 0$ by Proposition 3.2. To simplify the proof, we will just consider the case $A_0 \ge 1$. Note that in case $A_0 < 1$, we will just consider Whitney l-decomposition (instead of Whitney decomposition) where $l \in \mathbb{N}$ is such that $A_0 \ge 2^{-l}$.

Claim 1 If $Q_0 \in W_3$ then

$$\|D^\alpha \Lambda f\|_{L^p_w(Q_0)} \le C\|D^\alpha f\|_{L^p_w(Q_0^*)} + Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p_w(\cup F(Q_0))}, \tag{4.3}$$

where $0 \le |\alpha| \le k$ and $F(Q_0)$ is the collection of cubes that belong to any of the chains $F_{0,j}$ (guaranteed by (E)) for which $Q_j \cap Q_0 \ne \varnothing$. And if $Q_0 \in W_2 \backslash W_3$, then

$$\|D^\alpha \Lambda f\|_{L^p_w(Q_0)} \le C(r) \sum_{\substack{Q_j \cap Q_0 \ne \varnothing \\ Q_j \in W_3}} \Big[\|\nabla^k f\|_{L^p_w(Q_j^*)} + \|f\|_{L^p_w(Q_j^*)}\Big]. \tag{4.4}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

Proof of Claim 1 The following is just a modification of what we have done in [10, (7.1), (7.2)]. While proof of (4.3) is about the same, the proof of (4.4) required a few more steps. First, let $Q_0 \in W_3$. Then

$$\begin{aligned}
&\|D^\alpha\Big(\sum P_j\varphi_j\Big)\|_{L^p_w(Q_0)} \\
&\quad\le \|D^\alpha(\sum P_j\varphi_j - \sum P_0\varphi_j)\|_{L^p_w(Q_0)} + \|D^\alpha(P_0\sum\varphi_j)\|_{L^p_w(Q_0)} \\
&\qquad\qquad\text{(by the triangle inequality)} \\
&\quad= \|D^\alpha(\sum P_j\varphi_j - \sum P_0\varphi_j)\|_{L^p_w(Q_0)} + \|D^\alpha P_0\|_{L^p_w(Q_0)} \\
&\qquad\qquad\text{(since } \textstyle\sum\varphi_j = 1 \text{ on } Q_0) \\
&\quad\le C\sum_{\beta\le\alpha}\|\sum_j D^{\alpha-\beta}(P_j - P_0)D^\beta\varphi_j\|_{L^p_w(Q_0)} + C\|D^\alpha P_0\|_{L^p_w(Q_0^*)} \\
&\qquad\qquad\text{(by the triangle inequality, (A), and Theorem 2.1)} \\
&\quad\le C\sum_{\beta\le\alpha}\sum_{Q_j\cap Q_0\ne\varnothing} l(Q_0)^{-|\beta|}\|D^{\alpha-\beta}(P_j - P_0)\|_{L^p_w(Q_0)} + C\|D^\alpha P_0\|_{L^p_w(Q_0^*)} \\
&\qquad\qquad\text{(by the triangle inequality)} \\
&\quad\le C\sum_{\beta\le\alpha}\sum_{Q_j\cap Q_0\ne\varnothing} l(Q_0)^{-|\beta|}\|D^{\alpha-\beta}(P_j - P_0)\|_{L^p_w(Q_0^*)} + C\|D^\alpha(P_0 - f)\|_{L^p_w(Q_0^*)} \\
&\qquad + C\|D^\alpha f\|_{L^p_w(Q_0^*)} \qquad\text{(by Theorem 2.1 and the triangle inequality)} \\
&\quad\le C\sum_{\beta\le\alpha}\sum_{Q_j\cap Q_0\ne\varnothing} l(Q_0)^{-|\beta|} l(Q_0)^{k-|\alpha-\beta|}\|\nabla^k f\|_{L^p_w(\cup F_{0,j})} \\
&\qquad + Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p_w(Q_0^*)} + C\|D^\alpha f\|_{L^p_w(Q_0^*)} \\
&\qquad\qquad\text{(by (E), Lemma 4.2 and (4.1))} \\
&\quad\le Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p_w(\cup F(Q_0))} + C\|D^\alpha f\|_{L^p_w(Q_0^*)}.
\end{aligned}$$

Next, if $Q_0 \in W_2\backslash W_3$, recall that $l(Q_0) \ge C(r)$, and observe that

$$\begin{aligned}
\|D^\alpha\Big(\sum P_j\varphi_j\Big)\|_{L^p_w(Q_0)} &\le C\sum_{\beta\le\alpha}\sum_{\substack{Q_j\cap Q_0\ne\varnothing\\ Q_j\in W_3}} l(Q_0)^{-|\beta|}\|D^{\alpha-\beta}P_j\|_{L^p_w(Q_0)} \\
&\qquad\qquad\text{(by the triangle inequality)} \\
&\le C\sum_{\beta\le\alpha}\sum_{\substack{Q_j\cap Q_0\ne\varnothing\\ Q_j\in W_3}} l(Q_0)^{-|\beta|}\|D^{\alpha-\beta}P_j\|_{L^p_w(Q_j^*)} \\
&\qquad\qquad\text{(by (A), (D) and Theorem 2.1)}
\end{aligned}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

$$\leq C \sum_{\beta \leq \alpha} \sum_{\substack{Q_j \cap Q_0 \neq \varnothing \\ Q_j \in W_3}} l(Q_0)^{-|\beta|} l(Q_j^*)^{-|\alpha-\beta|} \|P_j\|_{L^p_w(Q_j^*)}$$

(by Theorem 2.3)

$$\leq C(r) \sum_{\substack{Q_j \cap Q_0 \neq \varnothing \\ Q_j \in W_3}} \|P_j\|_{L^p_w(Q_j^*)}$$

(since $l(Q_j^*) \geq Cl(Q_0) \geq C(r)$ by (A))

$$\leq C(r) \sum_{\substack{Q_j \cap Q_0 \neq \varnothing \\ Q_j \in W_3}} \big(\|P_j - f\|_{L^p_w(Q_j^*)} + \|f\|_{L^p_w(Q_j^*)} \big)$$

(by the triangle inequality)

$$\leq C(r) \sum_{\substack{Q_j \cap Q_0 \neq \varnothing \\ Q_j \in W_3}} \Big[\|\nabla^k f\|_{L^p_w(Q_j^*)} + \|f\|_{L^p_w(Q_j^*)} \Big]$$

by (4.1). This completes the proof of Claim 1.

Next, observe that

$$\Big\| \sum_{Q_j \in W_2 \setminus W_3} \sum_{\substack{Q_l \in W_3 \\ Q_l \cap Q_j \neq \varnothing}} \chi_{Q_l^*} \Big\|_{L^\infty} \leq C, \tag{4.5}$$

$$\Big\| \sum_{Q_j \in W_3} \chi_{\cup F(Q_j)} \Big\|_{L^\infty} \leq C. \tag{4.6}$$

Combining these facts with (4.3), (4.4) and using $l(Q_j) \leq C(r)$ if $Q_j \in W_3$, we obtain that for $0 \leq |\alpha| \leq k$,

$$\begin{aligned}
\|D^\alpha \Lambda f\|^p_{L^p_w((\mathcal{D}^c)^o)} &= \sum_{Q_j \in W_3} \|D^\alpha \Lambda f\|^p_{L^p_w(Q_j)} + \sum_{Q_j \in W_2 \setminus W_3} \|D^\alpha \Lambda f\|^p_{L^p_w(Q_j)} \\
&\leq \sum_{Q_j \in W_3} C\big(\|D^\alpha f\|_{L^p_w(Q_j^*)} + \|\nabla^k f\|_{L^p_w(\cup F(Q_j))} \big)^p + \\
&\quad \sum_{Q_j \in W_2 \setminus W_3} \Big(\sum_{\substack{Q_l \in W_3 \\ Q_j \cap Q_l \neq \varnothing}} C(r) \big[\|\nabla^k f\|_{L^p_w(Q_l^*)} + \|f\|_{L^p_w(Q_l^*)} \big] \Big)^p \\
&\leq \sum_{Q_j \in W_3} C\big(\|D^\alpha f\|^p_{L^p_w(Q_j^*)} + \|\nabla^k f\|^p_{L^p_w(\cup F(Q_j))} \big) + \\
&\quad \sum_{Q_j \in W_2 \setminus W_3} \sum_{\substack{Q_l \in W_3 \\ Q_j \cap Q_l \neq \varnothing}} C(r) \big(\|\nabla^k f\|^p_{L^p_w(Q_l^*)} + \|f\|^p_{L^p_w(Q_l^*)} \big) \\
&\leq C(r) \|f\|^p_{L^p_{w,k}(\mathcal{D})}.
\end{aligned}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

Hence

$$\|\Lambda f\|_{L^p_{w,k}(\mathcal{D}^c)^o} \le C(r)\|f\|_{L^p_{w,k}(\mathcal{D})}.$$

We now show that indeed $\Lambda f \in L^p_{w,k}(\mathbb{R}^n)$ if $f \in L^p_{w,k}(\mathcal{D})$ and $w^{-1/p} \in L^{p'}_{\text{loc}}(\mathbb{R}^n)$. We will show that for any $h \in C_0^\infty(\mathbb{R}^n)$ and $0 \le |\alpha| \le k$, we have

$$\int (D^\alpha \Lambda f) h \, dx = (-1)^{|\alpha|} \int (\Lambda f) D^\alpha h \, dx$$

where

$$D^\alpha \Lambda f(x) = \begin{cases} D^\alpha f(x) & \text{if } x \in \mathcal{D}, \\ D^\alpha \big(\sum_{Q_j \in W_3} P_j(x) \varphi_j(x) \big) & \text{if } x \in (\mathcal{D}^c)^o. \end{cases}$$

To this end, it suffices to show that for any bounded set Ω and any $\eta > 0$, we can find C^∞ function f_η such that

$$\int_\Omega |\nabla^l (\Lambda f - f_\eta)| \, dx < C\eta$$

for $0 \le l \le k$ with C independent of η. We will first choose an open bounded set $\mathcal{D}_0 \subset \mathcal{D}$ such that

$$\{x \in \mathcal{D} : d(x, \Omega \cap \mathcal{D}) < 1\} \subset \mathcal{D}_0; \tag{4.7}$$

(4.8) $\cup F(Q_0) \subset \mathcal{D}_0$ for any $Q_0 \in W_3, Q_0 \cap \Omega \ne \varnothing$

(see (4.3) for the definition of $F(Q_0)$);

$$\{x \in \mathcal{D} : d(x, \Omega \cap \mathcal{D}^c) < 1\} \subset \mathcal{D}_0. \tag{4.9}$$

We then choose a compact set $K \subset \mathcal{D}$ such that

$$\|\nabla^l f\|_{L^p_w(\mathcal{D}_0 \setminus K)} < \eta \quad \text{and hence } \|\nabla^l f\|_{L^1(\mathcal{D}_0 \setminus K)} < C\eta \text{ for } 0 \le l \le k. \tag{4.10}$$

Next, we choose $0 < s < 1$ such that

$$K^{3s} = \{x + y : x \in K, y \in \mathbb{R}^n, |y| \le 3s\} \subset \mathcal{D},$$

and then choose a function $\Psi \in C_0^\infty(\mathbb{R}^n)$ such that (since $D^\alpha f \in L^1_{\text{loc}}(\mathcal{D})$)

$$\chi_{K^s} \le \Psi \le \chi_{K^{2s}} \quad \text{and} \quad |D^\alpha \Psi| \le cs^{-|\alpha|} \text{ for all } \alpha.$$

Let us fix a function $\xi \in C_0^\infty\big(\{x \in \mathbb{R}^n : |x| \le 1\}\big)$ such that $\int \xi = 1$. Let $\xi_t(x) = t^{-n}\xi(x/t)$ for $t > 0$. We now note that there exists $0 < t < s$ such that

$$\|D^\alpha(f - f * \xi_t)\|_{L^1(K^{2s})} = \|D^\alpha f - (D^\alpha f) * \xi_t\|_{L^1(K^{2s})} \le \eta s^{k-|\alpha|}, \ 0 \le |\alpha| \le k. \tag{4.11}$$

Let $\varrho = 2^{-m}$, $m \in \mathbb{Z}_+$. Recall that W_1 is the Whitney decomposition of $\mathcal{D}$. Define

$$\mathfrak{R}' = \{\text{dyadic cubes } R \text{ with edgelength } \varrho, R \subset \mathcal{D}\},$$

$$\mathfrak{R} = \{R \in \mathfrak{R}' : R \subset S \text{ for some } S \in W_1, l(S) \geq 32n^3\varrho/\varepsilon\}$$

(we may assume $\varepsilon \leq 1$). Moreover, for each $R \in \mathfrak{R}$ let $\tilde{R}, \tilde{\tilde{R}}$ be cubes concentric with R with sides parallel to the axes and $l(\tilde{R}) = 1300n^4\varrho/\varepsilon^2$ and $l(\tilde{\tilde{R}}) = 2562n^4\varrho/\varepsilon^2$. First, let us make the following three observations:

(I) $\mathcal{D}^{10\varrho} = \{x + y : x \in \mathcal{D}, y \in \mathbb{R}^n, |y| \leq 10\varrho\} \subset \bigcup_{R\in\mathfrak{R}} \tilde{R}$ provided ϱ is small enough.

(II) For all $R_0, R_j \in \mathfrak{R}$ with $\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \varnothing$ and $\tilde{R}_0 \cap (\mathcal{D}\backslash K^s) \neq \varnothing$, there exists a chain $G_{0,j} = \{R_0 = S_1, S_2, \dots, S_m = R_j\}$ in $\mathfrak{R}'$ connecting R_0, R_j with $m \leq C$ that depends only on ε, δ and n, and $\cup G_{0,j} \subset \mathcal{D}\backslash K$, $d(\cup G_{0,j}) > \varrho$, provided ϱ is small enough. Moreover, if in addition that $\tilde{R}_0 \cap (\mathcal{D} \cap \Omega\backslash K^s) \neq \varnothing$, then indeed $\cup G_{0,j} \subset \mathcal{D}_0\backslash K$.

(III) Cubes in $W_2\backslash W_3$ will not intersect $\bigcup_{R_j\in\mathfrak{R}} \tilde{\tilde{R}}_j$ when ϱ is small enough. Moreover, if $Q_0 \in W_3$ intersects $\bigcup_{R_j\in\mathfrak{R}} \tilde{R}_j$ and Ω, we may assume that $\cup F(Q_0) \subset \mathcal{D}_0\backslash K$.

A similar conclusion to (I) was first stated in [27] (with $\mathcal{D} \subset \bigcup_{R\in\mathfrak{R}} \tilde{R}$) without proof. Nevertheless, the reader can refer to the proof of Theorem 6.1 in [10] (with $\mathcal{D} \subset \bigcup_{R\in\mathfrak{R}} \tilde{R}$). A similar conclusion to (II) can be found in [27, Lemma 4.1] or [15].

Next let $R_0, R_j \in \mathfrak{R}$, R_0, R_j be as in (II). Suppose that $G_{0,j}$ is the chain connecting R_0, R_j guaranteed by (II). Similar to the proof of Lemma 4.2, by the Poincaré inequality, if $\pi^k(R_0)f, \pi^k(R_j)f$ are the polynomials as in Proposition 2.4, we can show that

$$\|D^\alpha(\pi^k(R_0)f - \pi^k(R_j)f)\|_{L^1(R_0)} \leq C\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^1(\cup G_{0,j})} \forall\, 0 \leq |\alpha| \leq k \tag{4.12}$$

where C is independent of f, R_0, R_j and ϱ.

For each $R_j \in \mathfrak{R}$, let us choose $\psi_j \in C_0^\infty(\mathbb{R}^n)$ with

$$0 \leq \psi_j \leq \chi_{\tilde{\tilde{R}}_j}$$

such that $\sum_{R_j\in\mathfrak{R}} \psi_j \equiv 1$ on $\bigcup_{R_j\in\mathfrak{R}} \tilde{R}_j$, $0 \leq \sum_{R_j\in\mathfrak{R}} \psi_j \leq 1$ and $|D^\alpha\psi_j| \leq C\varrho^{-|\alpha|}$ for all α. We now let $q_j = \pi^k(R_j)f$ be the polynomial as in Proposition 2.4. Also, we will need a function $\Phi \in C^\infty(\mathbb{R}^n)$ such that

$$0 \leq \Phi \leq \chi_{\mathcal{D}^c}, \Phi = 1 \text{ on } Q_0 \text{ if } Q_0 \in W_2,\ Q_0 \not\subset \bigcup_{R_j\in\mathfrak{R}} \tilde{R}_j.$$

Next, since $\mathcal{D}^{10\varrho} \subset \bigcup_{R_j\in\mathfrak{R}} \tilde{R}_j$, we may assume $|D^\alpha\Phi| \leq c\varrho^{-|\alpha|}$.

We define

$$f_\eta = (f * \xi_t)\Psi + \sum_{R_j\in\mathfrak{R}} q_j\psi_j(1 - \Psi - \Phi) + \sum_{Q_i\in W_3} P_i\varphi_i\Phi.$$

Claim 2 If ϱ is small enough, then for $0 \le l \le k$,

$$\|\nabla^l(\Lambda f - f_\eta)\|_{L^1(\mathcal{D}\cap\Omega)} < C\eta, \tag{4.13}$$

$$\|\nabla^l(\Lambda f - f_\eta)\|_{L^1(\mathcal{D}^c\cap\Omega)} < C\eta, \tag{4.14}$$

where C is independent of η.

Proof of Claim 2

The proof of (4.13) is indeed just a slight modification of the proof of density of $C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ (or $C^\infty(\mathbb{R}^n)$) in the Sobolev space on the domain $\mathcal{D}$ in [10, 27]. First, recall that

$$\begin{aligned} f - f_\eta &= (f - f * \xi_t)\Psi + (f - \sum_{R_j \in \Re} q_j\psi_j)(1 - \Psi - \Phi) + (f - \sum_{Q_i \in W_3} P_i\varphi_i)\Phi \\ &= (f - f * \xi_t)\Psi + (f - \sum_{R_j \in \Re} q_j\psi_j)(1 - \Psi) \end{aligned}$$

on $\mathcal{D} \cap \Omega$ since $\Phi = 0$ on $\mathcal{D}$. Hence for any $0 \le |\alpha| \le k$,

$$\begin{aligned} &\|D^\alpha(f - f_\eta)\|_{L^1(\mathcal{D}\cap\Omega)} \\ &\qquad \le \|D^\alpha(\Psi(f - f * \xi_t))\|_{L^1(\mathcal{D}\cap\Omega)} + \Big\|D^\alpha\big[(1-\Psi)(f - \sum q_j\psi_j)\big]\Big\|_{L^1(\mathcal{D}\cap\Omega)}. \end{aligned}$$

To show the above is less than $C\eta$, we now follow the technique we have used to show the density of $C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^p_{w,k}(\mathcal{D})$ in $L^p_{w,k}(\mathcal{D})$ in [10, 15]. It is easy to see that

$$\begin{aligned} &\|D^\alpha[(\Psi)(f - f * \xi_t)]\|_{L^1(\mathcal{D}\cap\Omega)} \\ &\quad = \Big\|\sum_{0\le\beta\le\alpha} C_{\alpha,\beta} D^{\alpha-\beta}\Psi D^\beta(f - f * \xi_t))\Big\|_{L^1(K^{2s})} && \text{(since } \Psi = 0 \text{ outside } K^{2s}) \\ &\quad \le C \sum_{0\le\beta\le\alpha} s^{-|\alpha-\beta|}\|D^\beta(f - f * \xi_t)\|_{L^1(K^{2s})} && \text{(by the triangle inequality)} \\ &\quad \le C \sum_{0\le\beta\le\alpha} s^{-|\alpha-\beta|}\eta s^{k-|\beta|} \le C\eta && \text{(by (4.11)).} \end{aligned}$$

Next, since $(1 - \Psi) = 0$ on K^s, we need only to prove that

$$\Big\|D^\alpha\big[(1-\Psi)\big(f - \sum q_j\psi_j\big)\big]\Big\|_{L^1(\mathcal{D}\cap\Omega\setminus K^s)} \le C\eta.$$

To this end, first note that if $R_0 \in \Re$ such that $\tilde{R}_0 \cap (\mathcal{D} \cap \Omega \setminus K^s) \ne \varnothing$, then

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

$$(4.15)\quad \sum_{R_j\in\Re}\|D^\beta((q_0-q_j)\psi_j)\|_{L^1(\tilde R_0)}$$

$$\le C\sum_{R_j\in\Re}\sum_{\gamma\le\beta}\|D^\gamma\psi_j D^{\beta-\gamma}(q_0-q_j)\|_{L^1(\tilde R_0)}\qquad\text{(by the triangle inequality)}$$

$$\le C\sum_{\tilde{\tilde R}_0\cap\tilde{\tilde R}_j\neq\varnothing}\sum_{\gamma\le\beta}\varrho^{-|\gamma|}\|D^{\beta-\gamma}(q_0-q_j)\|_{L^1(\tilde R_0)}$$

$$\le C\sum_{\tilde{\tilde R}_0\cap\tilde{\tilde R}_j\neq\varnothing}\sum_{\gamma\le\beta}\varrho^{-|\gamma|}\|D^{\beta-\gamma}(q_0-q_j)\|_{L^1(R_0)}\qquad\text{(by Theorem 2.1)}$$

$$\le C\sum_{\tilde{\tilde R}_0\cap\tilde{\tilde R}_j\neq\varnothing}\varrho^{k-|\beta|}\|\nabla^k f\|_{L^1(\cup G_{0,j})}$$

by (4.12). Also, note that on $\mathcal{D}$ we have

$$(4.16)\qquad |D^\beta(f-\sum_{R_j\in\Re}q_j\psi_j)|\le|D^\beta(f-q_0)|+|D^\beta\sum_{R_j\in\Re}(q_0-q_j)\psi_j|.$$

We now consider two cases:

Case 1 $\beta<\alpha$. Then $D^{\alpha-\beta}(1-\Psi)=0$ outside K^{2s} and note that $K^{2s}\backslash K^s\subset\bigcup_{R_0\in\Re}R_0$ if ϱ is small enough, hence

$$\|D^{\alpha-\beta}(1-\Psi)D^\beta(f-\sum_j q_j\psi_j)\|^p_{L^1(\mathcal{D}\cap\Omega\backslash K^s)}$$

$$\le Cs^{-|\alpha-\beta|}\sum_{\substack{R_0\in\Re\\ R_0\cap(K^{2s}\cap\Omega\backslash K^s)\neq\varnothing}}\|D^\beta(f-\sum_j q_j\psi_j)\|_{L^1(R_0)}$$

(by the triangle inequality)

$$\le Cs^{-|\alpha-\beta|}\sum_{\substack{R_0\in\Re\\ R_0\cap(K^{2s}\cap\Omega\backslash K^s)\neq\varnothing}}\big(\|D^\beta(f-q_0)\|_{L^1(R_0)}+\|D^\beta\sum_{R_j\in\Re}(q_0-q_j)\psi_j\|_{L^1(R_0)}\big)$$

(by the triangle inequality and the fact that $\sum\psi_j=1$ on R_0)

$$\le Cs^{-|\alpha-\beta|}\sum_{\substack{R_0\in\Re\\ R_0\cap(K^{2s}\cap\Omega\backslash K^s)\neq\varnothing}}\varrho^{k-|\beta|}\|\nabla^k f\|_{L^1(R_0)}$$

$$+\,Cs^{-|\alpha-\beta|}\sum_{\substack{R_0\in\Re\\ R_0\cap(K^{2s}\cap\Omega\backslash K^s)\neq\varnothing}}\sum_{\tilde{\tilde R}_0\cap\tilde{\tilde R}_j\neq\varnothing}\varrho^{k-|\beta|}\|\nabla^k f\|_{L^1(\cup G_{0,j})}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

by (4.16) and (4.15). Next note that $\|\sum_{R_0\in\Re}\sum_{\tilde{\tilde{R}}_j\cap\tilde{\tilde{R}}_0\neq\varnothing}\chi_{\cup G_{0,j}}\|_{L^\infty}\leq C$ where C is independent of ϱ. Moreover by (II), if $R_0\cap(K^{2s}\cap\Omega\backslash K^s)\neq\varnothing$, and $\tilde{\tilde{R}}_j\cap\tilde{\tilde{R}}_0\neq\varnothing$, then $\cup G_{0,j}\subset\mathcal{D}_0\backslash K$, and in particular $R_0\subset\mathcal{D}_0\backslash K$. Hence if $\alpha>\beta$ then $|\beta|<k$, and

$$\|D^{\alpha-\beta}(1-\Psi)D^\beta(f-\sum_j q_j\psi_j)\|_{L^1(\mathcal{D}\cap\Omega\backslash K^s)}\leq Cs^{-|\alpha-\beta|}\varrho^{k-|\beta|}\|\nabla^k f\|_{L^1(\mathcal{D}_0\backslash K)}\leq C\eta.$$

Case 2 $\beta=\alpha$. First observe that for each $R_0\in\Re$ such that $\tilde{R}_0\cap(\mathcal{D}\cap\Omega\backslash K^s)\neq\varnothing$, we have

$$\begin{aligned}
&\|D^\alpha\sum q_j\psi_j\|_{L^1(\tilde{R}_0)}\\
&\quad\leq\|D^\alpha q_0\|_{L^1(\tilde{R}_0)}+\|D^\alpha\sum(q_j-q_0)\psi_j\|_{L^1(\tilde{R}_0)}\\
&\qquad\qquad\text{by the triangle inequality and the fact that }\textstyle\sum_j\psi_j=1\text{ on }\tilde{R}_0\\
&\quad\leq C\|D^\alpha q_0\|_{L^1(R_0)}+C\sum_{\substack{R_j\in\Re\\ \tilde{\tilde{R}}_0\cap\tilde{\tilde{R}}_j\neq\varnothing}}\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^1(\cup G_{0,j})}\\
&\qquad\qquad\text{by Theorem 2.1 and (4.15)}\\
&\quad\leq C\|D^\alpha(q_0-f)\|_{L^1(R_0)}+C\|D^\alpha f\|_{L^1(R_0)}+C\sum_{\substack{R_j\in\Re\\ \tilde{\tilde{R}}_0\cap\tilde{\tilde{R}}_j\neq\varnothing}}\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^1(\cup G_{0,j})}\\
&\quad\leq C\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^1(R_0)}+C\|D^\alpha f\|_{L^1(R_0)}+C\varrho^{k-|\alpha|}\sum_{\substack{R_j\in\Re\\ \tilde{\tilde{R}}_0\cap\tilde{\tilde{R}}_j\neq\varnothing}}\|\nabla^k f\|_{L^1(\cup G_{0,j})}.
\end{aligned}$$

Next, note that again by (II), if $\tilde{R}_0\cap(\mathcal{D}\cap\Omega\backslash K^s)\neq\varnothing$ and $\tilde{\tilde{R}}_0\cap\tilde{\tilde{R}}_j\neq\varnothing$, then $\cup G_{0,j}\subset\mathcal{D}_0\backslash K$, and in particular $R_0\subset\mathcal{D}_0\backslash K$. Hence by the triangle inequality and the previous estimate,

$$\begin{aligned}
&\|(1-\Psi)D^\alpha(f-\sum q_j\psi_j)\|_{L^1(\mathcal{D}\cap\Omega\backslash K^s)}\\
&\qquad\leq\big\|D^\alpha\big(f-\sum q_j\psi_j\big)\big\|_{L^1(\mathcal{D}\cap\Omega\backslash K^s)}\\
&\qquad\leq\|D^\alpha f\|_{L^1(\mathcal{D}\cap\Omega\backslash K^s)}+\sum_{\substack{R_0\in\Re\\ \tilde{R}_0\cap(\mathcal{D}\cap\Omega\backslash K^s)\neq\varnothing}}C\big\|D^\alpha\sum_{R_j\in\Re}q_j\psi_j\big\|_{L^1(\tilde{R}_0)}.
\end{aligned}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

However,

$$\sum_{\substack{R_0\in\Re\\ \tilde{R}_0\cap(\mathcal{D}\cap\Omega\setminus K^s)\neq\varnothing}} \Big\| D^\alpha \sum_{R_j\in\Re} q_j\psi_j \Big\|_{L^1(\tilde{R}_0)}$$

$$\le C \sum_{\substack{R_0\in\Re\\ \tilde{R}_0\cap(\mathcal{D}\cap\Omega\setminus K^s)\neq\varnothing}} \Big[\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^1(R_0)} + \|D^\alpha f\|_{L^1(R_0)} + \varrho^{k-|\alpha|} \sum_{\substack{R_j\in\Re\\ \tilde{\tilde{R}}_0\cap\tilde{\tilde{R}}_j\neq\varnothing}} \|\nabla^k f\|_{L^1(\cup G_{0,j})}\Big]$$

$$\le C\|D^\alpha f\|_{L^1(\mathcal{D}_0\setminus K)} + C\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^1(\mathcal{D}_0\setminus K)}$$

since $\|\sum_{R_0\in\Re}\sum_{\tilde{\tilde{R}}_j\cap\tilde{\tilde{R}}_0\neq\varnothing}\chi_{\cup G_{0,j}}\|_{L^\infty} < C$. Thus

$$\Big\|(1-\Psi)D^\alpha(f-\sum q_j\psi_j)\Big\|_{L^1(\mathcal{D}\cap\Omega)} \le C\eta$$

and hence

$$\|D^\alpha(f-f_\eta)\|_{L^1(\mathcal{D}\cap\Omega)} < C\eta.$$

This completes the proof of (4.13).

To prove (4.14), first note that

$$\begin{aligned}
&\|D^\alpha(f_\eta-\Lambda f)\|_{L^1(\Omega\cap\mathcal{D}^c)}\\
&\quad= \Big\| D^\alpha\Big[\sum q_j\psi_j(1-\Phi)+\sum P_i\varphi_i\Phi-\Lambda f\Big]\Big\|_{L^1(\Omega\cap\mathcal{D}^c)}\\
&\quad= \Big\| D^\alpha\Big[\Big(\sum q_j\psi_j-\sum P_i\varphi_i\Big)(1-\Phi)\Big]\Big\|_{L^1(\Omega\cap\mathcal{D}^c)}\\
&\quad\le \sum_{0\le\beta\le\alpha}\Big\|\Big[\sum(D^\beta q_j)\psi_j-\sum(D^\beta P_i)\varphi_i\Big]D^{\alpha-\beta}(1-\Phi)\Big\|_{L^1(\mathcal{D}^c\cap\Omega)}\\
&\qquad+\sum_{0\le\beta\le\alpha}\sum_{0\le\gamma<\beta}\Big\|\Big(\sum D^\gamma q_j D^{\beta-\gamma}\psi_j\\
&\qquad\qquad-\sum D^\gamma P_i D^{\beta-\gamma}\varphi_i\Big)D^{\alpha-\beta}(1-\Phi)\Big\|_{L^1(\mathcal{D}^c\cap\Omega)}\\
&\quad=: I+II.
\end{aligned}$$

We now let $W_\varrho = \{Q_0\in W_2 : Q_0\subset\bigcup_{R_j\in\Re}\tilde{R}_j, \Omega\cap Q_0\neq\varnothing\}$. Note that if $Q_0\in W_\varrho$, then $Q_0\in W_3$, $l(Q_0)\le C\varrho$ and $\sum_i\varphi_i=\sum_j\psi_j=1$ on Q_0. Also recall that $\Phi=1$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

on $Q_0 \in W_2$ if $Q_0 \notin W_\varrho$. Hence

$$
\begin{aligned}
I &\le C \sum_{0\le\beta\le\alpha} \varrho^{-|\alpha-\beta|} \sum_{Q_0\in W_\varrho} \Big\| \sum_{R_j\in\Re} (D^\beta q_j)\psi_j - \sum_{Q_i\in W_3} (D^\beta P_i)\varphi_i \Big\|_{L^1(Q_0)} \\
&= C \sum_{0\le\beta\le\alpha} \varrho^{-|\alpha-\beta|} \sum_{Q_0\in W_\varrho} \Big\| \sum D^\beta(q_j - P_0)\psi_j - \sum D^\beta(P_i - P_0)\varphi_i \Big\|_{L^1(Q_0)} \\
&\qquad\qquad (\text{since } \textstyle\sum \varphi_i = \sum \psi_j = 1 \text{ on any } Q_0 \text{ in } W_\varrho) \\
&\le C \sum_{0\le\beta\le\alpha} \varrho^{-|\alpha-\beta|} \Big(\sum_{Q_0\in W_\varrho} \sum_{\tilde{\tilde{R}}_j\cap Q_0\neq\varnothing} \|D^\beta(q_j - P_0)\|_{L^1(Q_0)} \\
&\qquad + \sum_{Q_0\in W_\varrho} \sum_{Q_i\cap Q_0\neq\varnothing} \|D^\beta(P_i - P_0)\|_{L^1(Q_0)} \Big) \\
&= C \sum_{0\le\beta\le\alpha} \varrho^{-|\alpha-\beta|} (I'_\beta + I''_\beta).
\end{aligned}
$$

Next, note that by Hölder's inequality and Lemma 4.2,

$$
\begin{aligned}
I''_\beta &\le \sum_{Q_0\in W_\varrho} \sum_{\substack{Q_i\cap Q_0\neq\varnothing \\ Q_i\in W_3}} \|w^{-1/p}\|_{L^{p'}(Q_0)} \|D^\beta(P_i - P_0)\|_{L^p_w(Q_0)} \\
&\le C \sum_{Q_0\in W_\varrho} \sum_{\substack{Q_i\cap Q_0\neq\varnothing \\ Q_i\in W_3}} l(Q_0)^{k-|\beta|} \|w^{-1/p}\|_{L^{p'}(Q_0)} \|\nabla^k f\|_{L^p_w(\cup F_{i,0})} \\
&\le C \sum_{Q_0\in W_\varrho} l(Q_0)^{k-|\beta|} \|w^{-1/p}\|_{L^{p'}(Q_0)} \|\nabla^k f\|_{L^p_w(F(Q_0))} \\
&\le C \|w^{-1/p}\|_{L^{p'}(\cup W_\varrho)} \Big(\sum_{Q_0\in W_\varrho} \|\nabla^k f\|^p_{L^p_w(\cup F_{i,0})} \Big)^{1/p} \\
&\qquad\qquad (\text{by Hölder's inequality and since } l(Q_0) \le C\,) \\
&\le C \|\nabla^k f\|_{L^p_w(\mathcal{D}_0\setminus K)} \qquad\qquad (\text{by (4.6) and (4.8)}).
\end{aligned}
$$

On the other hand, note that if $\tilde{\tilde{R}}_j \cap Q_0 \neq \varnothing$ with $Q_0 \in W_\varrho$, then since $l(Q_0) < C\varrho$, $l(Q_0^*) \le Cl(Q_0)$ and $d(Q_0, Q_0^*) < C\varrho$, there exists $c > 0$ (independent of ϱ) such that $Q_0, Q_0^* \subset cR_j$. Let us choose c sufficiently large such that $\tilde{R}_0 \subset cR_0$ and let

$$\Re_\varrho = \{R_j \in \Re : cR_j \cap \mathcal{D}^c \cap \Omega \neq \varnothing\}.$$

Note that unlike C, the constant c is fixed. If $\tilde{\tilde{R}}_j \cap Q_0 \neq \varnothing$ with $Q_0 \in W_\varrho$, then $R_j \in \Re_\varrho$ since $Q_0 \cap \Omega \neq \varnothing$ and $Q_0 \subset cR_j$. Moreover, recall that if $R_j \in \Re$, there exists $Q_{j'} \in W_1$ such that $R_j \subset Q_{j'}$. If in addition that $R_j \in \Re_\varrho$, then since $d(Q_{j'}) \le d(R_j) < C\varrho$, we have $l(Q_{j'}) \le C\varrho$. Furthermore since

$$d(Q_{j'}, \Omega \cap \mathcal{D}^c) \le d(R_j, \Omega \cap \mathcal{D}^c) < c\sqrt{n}\varrho,$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

we may assume that $C'Q_{j'} \subset \mathcal{D}_0 \backslash K$ (in particular, $R_j \subset Q_{j'} \subset \mathcal{D}_0 \backslash K$) for any fixed constant C' if ϱ is small enough by (4.9) and (III).

Next, by the triangle inequality,

$$I'_\beta \le \sum_{R_j \in \Re_\varrho} \sum_{\substack{Q_i^* \subset cR_j \\ Q_i \in W_\varrho}} \|D^\beta (q_j - P_i)\|_{L^1(Q_i)}$$

$$\le \sum_{R_j \in \Re_\varrho} \sum_{\substack{Q_i^* \subset cR_j \\ Q_i \in W_\varrho}} \Big(\|D^\beta (q_j - P^k(Q_{j'})f)\|_{L^1(Q_i)}$$

$$+ \|D^\beta (P^k(Q_{j'})f - P_i\|_{L^1(Q_i)} \Big) = I_A + I_B.$$

The estimate of I_A is quite straightforward,

$$I_A \le C \sum_{R_j \in \Re_\varrho} \sum_{\substack{Q_i^* \subset cR_j \\ Q_i \in W_\varrho}} \|D^\beta (q_j - P^k(Q_{j'})f)\|_{L^1(Q_i^*)} \qquad \text{(by Theorem 2.1)}$$

$$\le C \sum_{R_j \in \Re_\varrho} \|D^\beta (q_j - P^k(Q_{j'})f)\|_{L^1(cR_j)}$$

(since there are only a bounded number (independent of ϱ) of Q_i with the same Q_i^*)

$$\le C \sum_{R_j \in \Re_\varrho} \|D^\beta (q_j - P^k(Q_{j'})f)\|_{L^1(R_j)} \qquad \text{(by Theorem 2.1)}$$

$$\le C \sum_{R_j \in \Re_\varrho} (\|D^\beta (q_j - f)\|_{L^1(R_j)} + \|D^\beta (f - P^k(Q_{j'})f)\|_{L^1(R_j)})$$

(by the triangle inequality)

$$\le C \sum_{R_j \in \Re_\varrho} (\|D^\beta (q_j - f)\|_{L^1(R_j)} + \|D^\beta (f - P^k(Q_{j'})f)\|_{L^1(Q_{j'})})$$

$$\le C \sum_{R_j \in \Re_\varrho} (\|D^\beta (q_j - f)\|_{L^1(R_j)} + \|w^{-1/p}\|_{L^{p'}(Q_{j'})} \|D^\beta (f - P^k(Q_{j'})f)\|_{L^p_w(Q_{j'})})$$

(by Hölder's inequality)

$$\le C \sum_{R_j \in \Re_\varrho} (\varrho^{k-|\beta|} \|\nabla^k f\|_{L^1(R_j)} + \|w^{-1/p}\|_{L^{p'}(Q_{j'})} \varrho^{k-|\beta|} \|\nabla^k f\|_{L^p_w(Q_{j'})})$$

(by Lemma 2.5)

$$\le C\|\nabla^k f\|_{L^1(\mathcal{D}_0 \backslash K)} + C\|w^{-1/p}\|_{L^{p'}(\bigcup_{R_j \in \Re_\varrho} Q_{j'})} \Big(\sum_{R_j \in \Re_\varrho} \|\nabla^k f\|^p_{L^p_w(Q_{j'})} \Big)^{1/p}$$

(since there are only a bounded number (independent of ϱ) of R_j inside each $Q_{j'}$)

$$\le C\|\nabla^k f\|_{L^1(\mathcal{D}_0 \backslash K)} + C\|\nabla^k f\|_{L^p_w(\mathcal{D}_0 \backslash K)}.$$

However, the estimate of I_B is much more complicated. We will use an idea used in the estimate of weighted inequalities on Boman chains; see [11, Proof of Theorem 1.5].

Next, let

$$W_{1,\varrho} = \{Q_{j'} \in W_1 : R_j \subset Q_{j'}, \text{ for some } R_j \in \Re_\varrho\}.$$

Note that for each $Q_{j'} \in W_{1,\varrho}$, by choosing ϱ sufficiently small, as $d(Q_{j'}) < C\varrho$, we can make sure that $d(Q_{j'}, Q_i^*) < \delta$ whenever $Q_i^* \subset cQ_{j'}$, $Q_i^* \in W_1$. Recall that $\varrho \le l(Q_{j'}) \le C\varrho$. Let $x_i, x_{j'}$ be the center of Q_i^* and $Q_{j'}$, respectively. Since $\mathcal{D}$ is an (ε, δ) domain, there exists a rectifiable curve γ that connects x_i and $x_{j'}$ such that

$$d(z) > \frac{\varepsilon|x_{j'} - z||x_i - z|}{|x_i - x_{j'}|} \quad \text{for all } z \in \gamma \text{ and } l(\gamma) < \frac{|x_i - x_{j'}|}{\varepsilon}.$$

However, if $z \notin Q_{j'}$, then $|x_{j'} - z| > l(Q_{j'})/2$. On the other hand, recall that $|x_{j'} - x_i| \le Cl(Q_{j'})$ since $Q_i^* \subset cQ_{j'}$. Thus $d(z) > C|x_i - z|$ when $z \in \gamma$ and $z \notin Q_{j'}$. Hence, if $z \in \gamma$, $z \notin Q_{j'}$, and $z \in Q$, $Q \in W_1$, then $d(Q) > C|x_i - z|$ since $d(Q) \ge d(z)/(1 + \sqrt{n})$. In particular, $Q_i^* \subset NQ$ for some constant N independent of ϱ, since $l(Q_i^*) \le 2|x_i - z|$ as we may assume $z \notin Q_i^*$. Also, since

$$d(Q, Q_{j'}) \le d(z, x_{j'}) \le l(\gamma) < \frac{|x_i - x_{j'}|}{\varepsilon} \le Cl(Q_{j'})$$

and

$$l(Q) \le d(Q) \le d(Q, Q_{j'}) + d(Q_{j'}) + \sqrt{n}l(Q_{j'}) < Cl(Q_{j'}),$$

we have $Q \subset C'Q_{j'}$ with C' independent of ϱ. We can now choose ϱ small enough such that $C'Q_{j'} \subset \mathcal{D}_0 \backslash K$. Hence if $Q_{j'} \in W_{1,\varrho}$ and $Q_i^* \subset cQ_{j'}$, we can find an appropriate chain $\{Q_{j'} = S_0, S_1, \dots, S_m = Q_i^*\}$ of touching cubes in W_1 which intercept γ and connect $Q_{j'}$ to Q_i^*. Now, similar to the proof of Lemma 4.2, we can find a chain of cubes $\{\hat{S}_0, \dots, \hat{S}_{2m}\}$ that satisfies the condition of Lemma 2.10 and

$$\textstyle\bigcup_{l=0}^{2m} \hat{S}_l = \bigcup_{i=0}^{m} S_i, \quad \hat{S}_0 = S_0, \quad \hat{S}_{2m} = S_m.$$

Indeed, we will choose them such that $\hat{S}_{2i} = S_i$, $\hat{S}_{2i+1} \subset S_i \cup S_{i+1}$, and

$$|\hat{S}_{2i+1}| = \min\{|S_i|, |S_{i+1}|\} = 2|S_i \cap \hat{S}_{2i+1}| = 2|S_{i+1} \cap \hat{S}_{2i+1}| \quad \text{for } i = 0, 1, \dots, m-1.$$

It is then clear that there exists a constant N independent of ϱ, $Q_{j'}$ and Q_i^* such that

$$N|\hat{S}_l \cap \hat{S}_{l+1}| \ge |\hat{S}_l \cup \hat{S}_{l+1}| \quad \text{and} \quad \hat{S}_{2m} = Q_i^* \subset N\hat{S}_l, S_l \subset C'Q_{j'} \quad \text{for all } l.$$

We now let $\hat{W}_1$ be the collection of all cubes in W_1 together with above mentioned types of cubes ($\hat{S}_{2i+1}$). Then

(4.17)

$$\begin{aligned} &\|D^\beta(P^k(Q_{j'})f - f)\|_{L^1(Q_i^*)} \\ &\quad \le \sum_{l=0}^{2m-1} \|D^\beta(P^k(\hat{S}_l)f - P^k(\hat{S}_{l+1})f)\|_{L^1(Q_i^*)} + \|D^\beta(P^k(Q_i^*)f - f)\|_{L^1(Q_i^*)} \end{aligned}$$

(by the triangle inequality)

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

$$\leq C\sum_{l}\frac{|Q_i^*|}{|\hat{S}_{l+1}\cap\hat{S}_l|}\|D^\beta(P^k(\hat{S}_l)f-P^k(\hat{S}_{l+1})f)\|_{L^1(\hat{S}_l\cap\hat{S}_{l+1})}$$
$$+\|D^\beta(P^k(\hat{S}_{2m})f-f)\|_{L^1(\hat{S}_{2m})} \qquad \text{(by Theorem 2.1)}$$
$$\leq C\sum_{l}\frac{|Q_i^*|}{|\hat{S}_{l+1}\cap\hat{S}_l|}\Big(\|D^\beta(P^k(\hat{S}_l)f-f)\|_{L^1(\hat{S}_l\cap\hat{S}_{l+1})}$$
$$+\|D^\beta(f-P^k(\hat{S}_{l+1})f)\|_{L^1(\hat{S}_l\cap\hat{S}_{l+1})}\Big)+\|D^\beta(P^k(\hat{S}_{2m})f-f)\|_{L^1(\hat{S}_{2m})}$$
$$\text{(by the triangle inequality)}$$
$$\leq C\sum_{l}\frac{|Q_i^*|}{|\hat{S}_{l+1}\cap\hat{S}_l|}\big(\|D^\beta(P^k(\hat{S}_l)f-f)\|_{L^1(\hat{S}_l)}+\|D^\beta(f-P^k(\hat{S}_{l+1})f)\|_{L^1(\hat{S}_{l+1})}\big)$$
$$+\|D^\beta(P^k(\hat{S}_{2m})f-f)\|_{L^1(\hat{S}_{2m})}$$
$$\leq C\sum_{l=0}^{2m}\frac{|Q_i^*|}{|\hat{S}_l|}\|D^\beta(P^k(\hat{S}_l)f-f)\|_{L^1(\hat{S}_l)}.$$

We now note that

$$(4.18)\quad \sum_{R_j\in\Re_\varrho}\sum_{\substack{Q_i^*\subset cR_j\\ Q_i\in W_\varrho}}\|D^\beta(P^k(Q_{j'})f-f)\|_{L^1(Q_i^*)}$$
$$=\sum_{R_j\in\Re_\varrho}\sum_{\substack{Q_i^*\subset cR_j\\ Q_i\in W_\varrho}}\int_{\mathbb{R}^n}\|D^\beta(P^k(Q_{j'})f-f)\|_{L^1(Q_i^*)}\chi_{Q_i^*}(x)\frac{dx}{|Q_i^*|}.$$

But by (4.17) and recall that $W_{1,\varrho}=\{Q_{j'}\in W_1 : R_j\subset Q_{j'},\ \text{for some } R_j\in\Re_\varrho\}$, for any $Q_i\in W_\varrho$, we have

$$\|D^\beta(P^k(Q_{j'})f-f)\|_{L^1(Q_i^*)}\chi_{Q_i^*}(x)/|Q_i^*|$$
$$\leq C\sum_{Q_{j'}\in W_{1,\varrho}}\sum_{\substack{S\subset C'Q_{j'}\\ \hat{S}\in\hat{W}_1}}\|D^\beta(P^k(\hat{S})f-f)\|_{L^1(\hat{S})}\chi_{N\hat{S}}(x)/|\hat{S}|.$$

As there are only a bounded number of Q_i with the same Q_i^*, we have

$$\Big\|\sum_{R_j\in\Re_\varrho}\sum_{\substack{Q_i^*\subset cR_j\\ Q_i\in W_\varrho}}\chi_{Q_i^*}\Big\|_{L^\infty}\leq C$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

and hence,

$$\sum_{R_j\in\Re_\varrho}\sum_{\substack{Q_i^*\subset cR_j\\ Q_i\in W_\varrho}}\|D^\beta(P^k(Q_{j'})f-f)\|_{L^1(Q_i^*)}\chi_{Q_i^*}(x)/|Q_i^*| \leq C\sum_{Q_{j'}\in W_{1,\varrho}}\sum_{\substack{\hat S\subset C'Q_{j'}\\ \hat S\in\hat W_1}}\|D^\beta(P^k(\hat S)f-f)\|_{L^1(\hat S)}\chi_{N\hat S}/|\hat S|.$$

Thus,

$$\begin{aligned}
&\sum_{R_j\in\Re_\varrho}\sum_{\substack{Q_i^*\subset cR_j\\ Q_i\in W_\varrho}}\|D^\beta(P^k(Q_{j'})f-f)\|_{L^1(Q_i^*)}\\
&\leq C\sum_{Q_{j'}\in W_{1,\varrho}}\sum_{\substack{\hat S\subset C'Q_{j'}\\ \hat S\in\hat W_1}}\int_{\mathbb{R}^n}\|D^\beta(P^k(\hat S)f-f)\|_{L^1(\hat S)}\frac{\chi_{N\hat S}(x)}{|\hat S|}\,dx\\
&= C\sum_{Q_{j'}\in W_{1,\varrho}}\sum_{\substack{\hat S\subset C'Q_{j'}\\ \hat S\in\hat W_1}}\|D^\beta(P^k(\hat S)f-f)\|_{L^1(\hat S)}\\
&\leq C\sum_{Q_{j'}\in W_{1,\varrho}}\sum_{\substack{\hat S\subset C'Q_{j'}\\ \hat S\in\hat W_1}}\|w^{-1/p}\|_{L^{p'}(\hat S)}\|D^\beta(P^k(\hat S)f-f)\|_{L^p_w(\hat S)}\\
&\qquad\qquad\text{(by Hölder's inequality)}\\
&\leq C\sum_{Q_{j'}\in W_{1,\varrho}}\sum_{\substack{\hat S\subset C'Q_{j'}\\ \hat S\in\hat W_1}}\|w^{-1/p}\|_{L^{p'}(\hat S)}l(S)^{k-|\beta|}\|\nabla^k f\|_{L^p_w(\hat S)} \qquad \text{(by (4.1))}\\
&\leq C\varrho^{k-|\beta|}\|w^{-1/p}\|_{L^{p'}(\bigcup_{\hat S\subset C'Q_{j'},\hat S\in\hat W_1,Q_{j'}\in W_{1,\varrho}}\hat S)}\Big(\sum_{Q_{j'}\in W_{1,\varrho}}\sum_{\substack{\hat S\subset C'Q_{j'}\\ \hat S\in\hat W_1}}\|\nabla^k f\|^p_{L^p_w(\hat S)}\Big)^{1/p}\\
&\qquad\qquad\big(\text{by Hölder's inequality and since } \{C'Q_{j'}: Q_{j'}\in W_1, l(Q_{j'})\geq\varrho\}\\
&\qquad\qquad\text{has bounded overlap}\big)\\
&\leq C\varrho^{k-|\beta|}\|\nabla^k f\|_{L^p_w(\mathcal{D}_0\setminus K)}.
\end{aligned}$$

Recall that we have chosen ϱ small enough such that $C'Q_{j'}\subset D_0\setminus K$.

We can now estimate I_B.

$$I_B\leq C\sum_{R_j\in\Re_\varrho}\sum_{\substack{Q_i^*\subset cR_j\\ Q_i\in W_\varrho}}\|D^\beta(P^k(Q_{j'})f-P_i)\|_{L^1(Q_i^*)}$$

$$\leq C \sum_{R_j \in \Re_\varrho} \sum_{\substack{Q_i^* \subset cR_j \\ Q_i \in W_\varrho}} \left[\|D^\beta(f - P^k(Q_{j'})f)\|_{L^1(Q_i^*)} + \|D^\beta(f - P_i)\|_{L^1(Q_i^*)} \right]$$

$$\leq C\varrho^{k-|\beta|} \|\nabla^k f\|_{L^p_w(\mathcal{D}_0 \setminus K)} + C \sum_{R_j \in \Re_\varrho} \sum_{\substack{Q_i^* \subset cR_j \\ Q_i \in W_\varrho}} \varrho^{k-\beta|} \|w^{-1/p}\|_{L^{p'}(Q_i^*)} \|\nabla^k f\|_{L^p_w(Q_i^*)}$$

(by the previous estimate, Hölder's inequality and (4.1))

$$\leq C\varrho^{k-|\beta|} \|\nabla^k f\|_{L^p_w(\mathcal{D}_0 \setminus K)}$$

by similar argument as in the previous estimate. Finally, let us look at the estimate of II. First, by the triangle inequality,

$$\begin{aligned} II \leq \sum_{0 \leq \beta \leq \alpha} \sum_{0 \leq \gamma < \beta} \Bigg(\Big\| \sum_j D^\gamma q_j D^{\beta-\gamma} \varphi_j D^{\alpha-\beta}(1 - \Phi) \Big\|_{L^1(\mathcal{D}^c \cap \Omega)} \\ + \Big\| \sum_i D^\gamma P_i D^{\beta-\gamma} \psi_i D^{\alpha-\beta}(1 - \Phi) \Big\|_{L^1(\mathcal{D}^c \cap \Omega)} \Bigg) \\ = \sum_{0 \leq \beta \leq \alpha} \sum_{0 \leq \gamma < \beta} (II_A + II_B). \end{aligned}$$

Next recall that $\Phi = 1$ outside $\bigcup_{R_0 \in \Re} \tilde{R}_0$. Thus

$$II_A \leq C\varrho^{-|\alpha-\beta|} \sum_{R_0 \in \Re} \Big\| \sum_j D^\gamma q_j D^{\beta-\gamma} \varphi_j \Big\|_{L^1(\tilde{R}_0 \cap \mathcal{D}^c \cap \Omega)}.$$

Moreover, since $\tilde{R}_0 \subset cR_0$, if $R_0 \in \Re \setminus \Re_\varrho$, then $\tilde{R}_0 \cap \mathcal{D}^c \cap \Omega \subset cR_0 \cap \mathcal{D}^c \cap \Omega = \varnothing$. Hence,

$$\begin{aligned} II_A &\leq C\varrho^{-|\alpha-\beta|} \sum_{R_0 \in \Re_\varrho} \Big\| \sum_j D^\gamma q_j D^{\beta-\gamma} \varphi_j \Big\|_{L^1(\tilde{R}_0 \cap \mathcal{D}^c \cap \Omega)} \\ &\leq C\varrho^{-|\alpha-\beta|} \sum_{R_0 \in \Re_\varrho} \Big\| \sum_j D^\gamma q_j D^{\beta-\gamma} \varphi_j \Big\|_{L^1(\tilde{R}_0)}. \end{aligned}$$

Note that by choosing ϱ small enough, similar to property (II), we may assume that if $R_0 \in \Re_\varrho$, then $R_0 \subset \mathcal{D}_0 \setminus K^s$ and $\cup G_{0,j} \subset \mathcal{D}_0 \setminus K$ whenever $\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \varnothing$ and $R_j \in \Re$. Thus,

$$II_A \leq C\varrho^{-|\alpha-\beta|} \sum_{R_0 \in \Re_\varrho} \Big\| \sum_j D^\gamma (q_j - q_0) D^{\beta-\gamma} \varphi_j \Big\|_{L^1(\tilde{R}_0)}$$

(since $\sum D^{\beta-\gamma} \varphi_j = D^{\beta-\gamma} \sum_j \varphi_j = 0$ on $\tilde{R}_0$ as $\beta > \gamma$)

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

$$\leq C\varrho^{-|\alpha-\beta|} \sum_{R_0 \in \mathfrak{R}_\varrho} \varrho^{-|\beta-\gamma|} \sum_{\tilde{\tilde{R}}_j \cap \tilde{R}_0 \neq \varnothing} \|D^\gamma(q_j - q_0)\|_{L^1(\tilde{R}_0)}$$

$$\leq C\varrho^{-|\alpha-\beta|} \sum_{R_0 \in \mathfrak{R}_\varrho} \varrho^{k-|\beta|} \|\nabla^k f\|_{L^1(\cup G_{0,j})} \qquad \text{(by Theorem 2.1 and (4.12))}$$

$$\leq C\varrho^{k-|\alpha|} \|\nabla^k f\|_{L^1(\mathcal{D}_0 \setminus K)}.$$

Also, recall that $\Phi = 1$ outside $\bigcup_{Q_0 \in W_\varrho} Q_0$, hence

$$II_B \leq C\varrho^{-|\alpha-\beta|} \sum_{Q_0 \in W_\varrho} \Big\| \sum_i D^\gamma P_i D^{\beta-\gamma} \psi_i \Big\|_{L^1(Q_0)}$$

$$= C\varrho^{-|\alpha-\beta|} \sum_{Q_0 \in W_\varrho} \Big\| \sum_i D^\gamma (P_i - P_0) D^{\beta-\gamma} \psi_i \Big\|_{L^1(Q_0)} \qquad \text{(since } \textstyle\sum D^{\beta-\gamma}\psi_i = 0)$$

$$\leq C\varrho^{-|\alpha-\beta|} \sum_{Q_0 \in W_\varrho} \sum_{Q_i \cap Q_0 \neq \varnothing} l(Q_0)^{-|\beta-\gamma|} \|D^\gamma(P_i - P_0)\|_{L^1(Q_0)}$$

$$\leq C\varrho^{-|\alpha-\beta|} \sum_{Q_0 \in W_\varrho} \sum_{Q_i \cap Q_0 \neq \varnothing} l(Q_0)^{-|\beta-\gamma|} \|D^\gamma(P_i - P_0)\|_{L^1(Q_0^*)} \qquad \text{(by Theorem 2.1)}$$

$$\leq C\varrho^{-|\alpha-\beta|} \sum_{Q_0 \in W_\varrho} \sum_{Q_i \cap Q_0 \neq \varnothing} l(Q_0)^{-|\beta-\gamma|} \|w^{-1/p}\|_{L^{p'}(Q_0^*)} \|D^\gamma(P_i - P_0)\|_{L^p_w(Q_0^*)}$$

$$\text{(by Hölder's inequality)}$$

$$\leq C\varrho^{-|\alpha-\gamma|} \sum_{Q_0 \in W_\varrho} \sum_{Q_i \cap Q_0 \neq \varnothing} l(Q_0)^{k-|\gamma|} \|w^{-1/p}\|_{L^{p'}(Q_0^*)} \|\nabla^k f\|_{L^p_w(\cup F_{0,i})}$$

$$\text{(by Lemma 4.2)}$$

$$\leq C\varrho^{k-|\alpha|} \|\nabla^k f\|_{L^p_w(\mathcal{D}_0 \setminus K)}$$

by similar argument as before. The proof of (4.14) is now completed by (4.10), and this concludes the proof of Proposition 4.1. ■

Next, we will show that under one additional condition, then indeed the extension of a $C^{k-1,1}_{\text{loc}}$ function is still a $C^{k-1,1}_{\text{loc}}$ function.

Lemma 4.3 *Under the assumption of Proposition* 4.1*, if in addition*

$$\|D^\alpha(f - P^k(Q)f)\|_{L^\infty(Q)} \leq Cl(Q)^{k-|\alpha|} \|\nabla^k f\|_{L^\infty(Q)} \tag{4.19}$$

for all $f \in C^{k-1,1}_{\text{loc}}(\mathcal{D})$ *and cubes* $Q \subset \mathcal{D}$ *such that* $\frac{l(Q)}{d(Q)} \leq A_0, A_0 > 0$*, then* $D^\alpha \Lambda f$ *is locally Lipschitz for all* α*,* $0 \leq |\alpha| < k$ *if* $f \in C^{k-1,1}_{\text{loc}}(\overline{\mathcal{D}})$.

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

Proof Again, we will just consider the case $A_0 \geq 1$. We can proceed as in the proof of (4.4) and (4.3) to obtain

$$\|\Lambda f\|_{L_k^\infty(Q)} \leq C\big(\|\nabla^k f\|_{L^\infty(\cup F(Q))} + \sum_{\substack{Q_j \in W_3 \\ Q_j \cap Q \neq \varnothing}} \|f\|_{L_k^\infty(Q_j^*)}\big) \ \forall Q \in W_2. \tag{4.20}$$

(If $Q \notin W_3$, we take $\cup F(Q) = \varnothing$). To prove (4.20), we only need to replace p by ∞ in (4.4) and (4.3) since if Ω is a bounded set in $(\mathcal{D}^c)^o$, then there exists $G \subset W_2$ such that $\Omega \subset \cup G$ and $\cup G$ is bounded. Thus

$$\|\Lambda f\|_{L_k^\infty(\Omega)} \leq \|\Lambda f\|_{L_k^\infty(\cup G)} \leq C\|f\|_{L_k^\infty(K)} < \infty,$$

where K is a compact set containing $\bigcup F(Q)\ \forall Q \in G$ and containing $Q_j^*\ \forall Q_j \in W_3$ with $Q_j \cap Q \neq \varnothing, Q \in G$. We now show that $D^\alpha \Lambda f$ is continuous for all α, $0 \leq |\alpha| < k$. To this end, one only need to show that

$$\lim_{\substack{x \to x_0 \\ x \in (\mathcal{D}^c)^o}} D^\alpha \Lambda f(x) = D^\alpha f(x_0)\ \forall x_0 \in \partial\mathcal{D}, 0 \leq |\alpha| < k.$$

Nevertheless, it suffices to show that if $Q_j \in W_3$ and $d(Q_j, \partial\mathcal{D}) \to 0$ then

$$\|D^\alpha \Lambda f - \frac{1}{|Q_j^*|}\int_{Q_j^*} D^\alpha f\, dx\|_{L^\infty(Q_j)} \to 0.$$

However, the proof is again quite standard. For the details, see [27, 10]. This concludes the proof of Lemma 4.3. ∎

We can now prove our main theorems.

Proof of Theorem 1.6 First, by repeated applications of (1.5), we know (4.1) in Proposition 4.1 holds with $P^l(Q)f = P_w^l(Q)f$ which is the polynomial of degree $< l$ such that

$$\int_Q D^\alpha(f - P_w^l(Q)f)\, dw = 0 \text{ for } 0 \leq |\alpha| < l.$$

Moreover, it is obvious that

$$\|f - f_{Q,w}\|_{L^\infty(Q)} \leq Cl(Q)\|\nabla f\|_{L^\infty(Q)}$$

and hence

$$\|\mathcal{D}^\alpha(P_w^k(Q)f - f)\|_{L^\infty(Q)} \leq Cl(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^\infty(Q)}.$$

The theorem now follows from Proposition 4.1 and Lemma 4.3. ∎

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

Proof of Theorem 1.7 First recall that

$$W_3 = \left\{ Q \in W_2 : l(Q) \le \frac{\varepsilon r}{16nL} \right\}, \quad L = 2^{-m}, m \in \mathbb{Z}_+,$$

where L is chosen so that $\Omega \subset (\bigcup_{Q \in W_3} Q) \cup \overline{\mathcal{D}}$.

We will now prove the following lemma.

Lemma 4.4 *Under the assumption of Proposition* 4.1*, if* $P^k(Q)f = \pi^k_\mu(Q)f$ *and*

$$\|f - f_{Q,\mu}\|_{L^1_\mu(Q)} \le Al(Q)\mu(Q)w(Q)^{-1/p}\|\nabla f\|_{L^p_w(Q)}, \tag{4.21}$$

for all cubes $Q \subset \mathcal{D}$ *such that* $l(Q)/d(Q) \le A_0$, $A_0 > 0$, *then for any* $1 \le |\alpha| \le k$ *and* $Q_0 \in W_3$,

$$\|D^\alpha \Lambda f\|_{L^p_w(Q_0)} \le C\|\nabla^{|\alpha|} f\|_{L^p_w(\bigcup F(Q_0))} \tag{4.22}$$

where $F(Q_0)$ *is the collection of cubes which belong to any of the chains* $F_{0,j}$ *(guaranteed by* (E)*) for which* $Q_j \cap Q_0 \ne \varnothing$*. Here the constant* C *depends only on* $A, \varepsilon, w, p, k, L$ *and the dimension* n*.*

Proof Let $Q_j \cap Q_0 \ne \varnothing$, $Q_j, Q_0 \in W_3$ and $\alpha > 0$. Then similar to the argument of the proof of Lemma 4.2, there exists a chain of cubes $\{S_0, S_1, \dots, S_M\}$ that satisfies the condition of Lemma 2.10 and such that

$$\cup S_i = \bigcup F_{0,j}, \quad S_0 = Q_0^*, \quad S_M = Q_j^*, \frac{l(S_i)}{d(S_i)} \le A_0 \quad \text{and} \quad \sum_i \chi_{S_i} \le 2 \; a.e.$$

Again similar to the proof of Lemma 4.2, we have

$$\begin{aligned}
&\|D^{\alpha-\beta}(\pi^k_\mu(Q_0^*)f - \pi^k_\mu(Q_j^*)f)\|_{L^p_w(Q_0)} \\
&\quad \le C\|D^{\alpha-\beta}(\pi^k_\mu(S_0)f - \pi^k_\mu(S_M)f)\|_{L^p_w(S_0)} && \text{(by (A) and Theorem 2.1)} \\
&\quad \le Cl(S_0)^{-|\alpha-\beta|}\|\pi^k_\mu(S_0)f - \pi^k_\mu(S_M)f\|_{L^p_w(S_0)} && \text{(by Theorem 2.3)} \\
&\quad \le Cl(S_0)^{-|\alpha-\beta|}\big[\|\pi^k_\mu(S_0)f - f\|_{L^p_w(S_0)} + \|f - \pi^k_\mu(S_M)f\|_{L^p_w(S_0)}\big] \\
& && \text{(by the triangle inequality)} \\
&\quad \le Cl(S_0)^{-|\alpha-\beta|}\sum_{i=0}^{M}\|f - \pi^k_\mu(S_i)f)\|_{L^p_w(S_i)} && \text{(by Lemma 2.10)} \\
&\quad \le Cl(Q_0)^{-|\alpha-\beta|}\sum_{i=0}^{M}\big[\|f - P^l_\mu(S_i)f\|_{L^p_w(S_i)} + \|\pi^k_\mu(S_i)(f - P^l_\mu(S_i)f\|_{L^p_w(S_i)}\big] \\
& && \text{(by the triangle inequality)}
\end{aligned}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

$\big($where $P^l_\mu(S_i)f$ is the unique polynomial of degree $< |\alpha|$ such that

$\int_{S_i} D^\gamma(f - P^l_\mu(S_i)f)\, d\mu = 0$ for all $0 \le |\gamma| < l.\big)$

$$\le Cl(Q_0)^{-|\alpha-\beta|} \sum_{i=0}^{M} \Big[\|f - P^l_\mu(S_i)f\|_{L^p_w(S_i)} + \frac{w(S_i)^{1/p}}{\mu(S_i)} \|f - P^l_\mu(S_i)f\|_{L^1_\mu(S_i)} \Big]$$

(by Proposition 2.4)

$$\le Cl(Q_0)^{-|\alpha-\beta|} \sum_{i=0}^{M} \Big[l(S_i)\|\nabla(f - P^l_\mu(S_i)f)\|_{L^p_w(S_i)}$$

$$+ l(S_i)\|\nabla(f - P^l_\mu(S_i)f)\|_{L^p_w(S_i)} \Big] \text{ (by Lemma 2.8 and (4.21))}$$

$$\le Cl(Q_0)^{-|\alpha-\beta|} \sum_{i} l(S_i)^{|\alpha|} \|\nabla^{|\alpha|} f\|_{L^p_w(S_i)}$$

$$\le Cl(Q_0)^{|\beta|} \|\nabla^{|\alpha|} f\|_{L^p_w(\bigcup S_i)} = Cl(Q_0)^{|\beta|} \|\nabla^{|\alpha|} f\|_{L^p_w(\bigcup F_{0,j})}.$$

Next, again let $P^{|\alpha|}_\mu(Q_0^*)f$ be the unique polynomial of degree $< |\alpha|$ such that

$$\int_{Q_0^*} D^\gamma(f - P^{|\alpha|}_\mu(Q_0^*)f)\, d\mu = 0 \text{for all } 0 \le |\gamma| < |\alpha|.$$

Then

$$\begin{aligned} \|D^\alpha P_0\|_{L^p_w(Q_0^*)} &= \|D^\alpha(P_0 - P^{|\alpha|}_\mu(Q_0^*)f)\|_{L^p_w(Q_0^*)} \\ &\le Cl(Q_0)^{-|\alpha|} \|\pi^k_\mu(Q_0^*)f - P^{|\alpha|}_\mu(Q_0^*)f\|_{L^p_w(Q_0^*)} \\ &\le Cl(Q_0)^{-|\alpha|} w(Q_0^*)\mu(Q_0^*)^{-1} \|f - P^{|\alpha|}_\mu(Q_0^*)f\|_{L^1_\mu(Q_0^*)} \\ &\le C\|\nabla^{|\alpha|} f\|_{L^p_w(Q_0^*)} \end{aligned}$$

by (4.21) and repeated applications of Lemma 2.8.

Let us now look at the proof of (4.3) again. Recall that

$$\begin{aligned} &\|D^\alpha\Big(\sum P_j\varphi_j\Big)\|_{L^p_w(Q_0)} \\ &\quad\le \|D^\alpha(\sum P_j\varphi_j - \sum P_0\varphi_j)\|_{L^p_w(Q_0)} + \|D^\alpha P_0\|_{L^p_w(Q_0)} \\ &\quad\le C \sum_{\beta\le\alpha} \sum_{Q_j\cap Q_0\ne\varnothing} l(Q_0)^{-|\beta|} \|D^{\alpha-\beta}(P_j - P_0)\|_{L^p_w(Q_0)} + \|D^\alpha P_0\|_{L^p_w(Q_0^*)} \\ &\quad\le C \sum_{\beta\le\alpha} \sum_{Q_j\cap Q_0\ne\varnothing} l(Q_0)^{-|\beta|} \|D^{\alpha-\beta}(P_j - P_0)\|_{L^p_w(Q_0^*)} + C\|\nabla^{|\alpha|} f\|_{L^p_w(Q_0^*)} \end{aligned}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

(by Theorem 2.1 and the previous estimate)

$$\le C \sum_{Q_j \cap Q_0 \neq \varnothing} \|\nabla^{|\alpha|} f\|_{L^p_w(\bigcup F_{0,j})}$$

$$\le C \|\nabla^{|\alpha|} f\|_{L^p_w(\bigcup F(Q_0))}.$$

This completes the proof of Lemma 4.4. ∎

We now return to the proof of Theorem 1.7. We will let $P^k(Q)f = \pi^k_\mu(Q)f$. For any $0 < l \le k$, since (1.6) holds, by the previous lemma, we have

$$\begin{aligned}\|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\Omega\cap\mathcal{D}^c)} &\le \sum_{Q_0 \in W_3} \|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(Q_0)} \\ &\le C \sum_{Q_0 \in W_3} \|\nabla^l f\|_{L^{p_i}_{w_i}(\bigcup F(Q_0))} \\ &\le C \|\nabla^l f\|_{L^{p_i}_{w_i}(\mathcal{D})}\end{aligned}$$

by (4.6). We now note that, similar to Lemma 4.3, we can show that $\Lambda f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ when $f \in C^{k-1,1}_{\mathrm{loc}}(\overline{\mathcal{D}})$. Thus, we have

$$\|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le \|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\mathcal{D})} + \|\nabla^l \Lambda f\|_{L^{p_i}_{w_i}(\Omega\cap\mathcal{D}^c)} \le C \|\nabla^l f\|_{L^{p_i}_{w_i}(\mathcal{D})}.$$

Finally, if ν is any doubling weight such that $\nu/\mu \in A_p(\mu)$, we have by Proposition 2.4,

$$\|P^k(Q)f\|_{L^p_\nu(Q)} \le \nu(Q)^{1/p} \|P^l(Q)f\|_{L^\infty(Q)} \le C \frac{\nu(Q)^{1/p}}{\mu(Q)} \|f\|_{L^1_\mu(Q)} \le C \|f\|_{L^p_\nu(Q)}.$$

Again, let us look at the proof of (4.3). For any $Q_0 \in W_3$,

$$\begin{aligned}\Big\| \sum P_j \varphi_j \Big\|_{L^p_\nu(Q_0)} &\le C \Big\| \sum_{Q_j \cap Q_0 \neq \varnothing} P_j \Big\|_{L^p_\nu(Q_0)} \\ &\le C \sum_{Q_j \cap Q_0 \neq \varnothing} \|P_j\|_{L^p_\nu(Q_j^*)} && \text{(by Theorem 2.1)} \\ &\le C \sum_{Q_j \cap Q_0 \neq \varnothing} \|f\|_{L^p_\nu(Q_j^*)} && \text{(by the previous estimate)} \\ &\le C \|f\|_{L^p_\nu(\bigcup F(Q_0))}.\end{aligned}$$

Thus, just as before, we have

$$\|\Lambda f\|_{L^p_\nu(\Omega)} \le C \|f\|_{L^p_\nu(\mathcal{D})}.$$

This concludes the proof of Theorem 1.7. ∎

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

Remark 4.5

(i) If for some i, we have $w_i^{-1/p_i} \in L^{p_i'}_{\mathrm{loc}}(\mathbb{R}^n)$, then the conclusion of Theorem 1.7 also holds for all functions $f \in \bigcap E^{p_i}_{w_i,k_i}(\mathcal{D})$, $1 \le k_i \le k$ for all i.

(ii) In case $\mathcal{D}$ is an unbounded (ε, ∞) domain, Theorem 1.7 will hold for $\Omega = \mathbb{R}^n$ with the same extension operator. Note that now $\mathrm{rad}(\mathcal{D}) = \infty$ and hence $\mathbb{R}^n \subset \overline{\mathcal{D}} \cup (\bigcup_{Q \in W_3} Q)$ with any choice of $L > 0$.

5 Applications

We will now use our main result to extend some weighted interpolation inequalities. First, let us recall a weighted interpolation inequality in [17].

Theorem 5.1 ([17, Theorem 1.7]) *Let $1 \le p, r, q < \infty$, $r \le q$, $h > 1$, $\frac{1}{q} \le \frac{h-1}{rh} + \frac{1}{ph}$, $0 \le i < k$, $i, k \in \mathbb{Z}$. Let σ, v, v_0, w be doubling weights such that $v_0/\sigma \in A_r(\sigma)$. Suppose*

$$\|f - f_{Q,w}\|_{L^q_w(Q)} \le C w(Q)^{1/q} v(Q)^{-1/p} l(Q) \|\nabla f\|_{L^p_v(Q)}; \tag{5.1}$$

$$\|f - f_{Q,\sigma}\|_{L^1_\sigma(Q)} \le C \sigma(Q) v(Q)^{-1/p} l(Q) \|\nabla f\|_{L^p_v(Q)} \tag{5.2}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ and cube Q in $\mathbb{R}^n$. If

$$\Big(\frac{w(\tilde{Q})}{w(Q)}\Big)^{1/q} \le C \Big(\frac{v_0(\tilde{Q})}{v_0(Q)}\Big)^{\frac{h-1}{rh}} \Big(\frac{l(\tilde{Q})}{l(Q)}\Big)^{i} \Big[\Big(\frac{v_0(\tilde{Q})}{v_0(Q)}\Big)^{1/rh} + \Big(\frac{l(\tilde{Q})}{l(Q)}\Big)^{-k/h} \Big(\frac{v(\tilde{Q})}{v(Q)}\Big)^{1/ph}\Big] \tag{5.3}$$

for all cubes $\tilde{Q} \subset Q$ in $\mathbb{R}^n$, then

$$\|\nabla^i f\|_{L^q_w(Q)} \le C w(Q)^{1/q} l(Q)^{-i} \Big(\frac{\|f\|_{L^r_{v_0}(3Q)}}{v_0(Q)^{1/r}}\Big)^{\frac{h-1}{h}} \Big(\frac{\|f\|_{L^r_{v_0}(3Q)}}{v_0(Q)^{1/r}} + \frac{l(Q)^k \|\nabla^k f\|_{L^p_v(3Q)}}{v(Q)^{1/p}}\Big)^{1/h} \tag{5.4}$$

for all $f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

With the help of extension theorem, we can replace $3Q$ by Q in (5.4).

Theorem 5.2 *Under the assumption of the previous theorem,* (5.3) *holds for all cubes $\tilde{Q} \subset Q$ in $\mathbb{R}^n$ if and only if*

$$\|\nabla^i f\|_{L^q_w(Q)} \le C w(Q)^{1/q} l(Q)^{-i} \Big(\frac{\|f\|_{L^r_{v_0}(Q)}}{v_0(Q)^{1/r}}\Big)^{\frac{h-1}{h}} \Big(\frac{\|f\|_{L^r_{v_0}(Q)}}{v_0(Q)^{1/r}} + \frac{l(Q)^k \|\nabla^k f\|_{L^p_v(Q)}}{v(Q)^{1/p}}\Big)^{1/h} \tag{5.5}$$

for all $f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof First, let us observe that (5.5) implies (5.3). For any cubes $\tilde{Q} \subset Q$, we can let $f = \chi_{\tilde{Q}} P$ where P is a polynomial of degree at least k such that $D^\alpha P = 0$ on $\partial\tilde{Q}$ for

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

$0 \le |\alpha| \le k-1$. Note that then $f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. It then follows from Lemma 2.2, Theorem 2.3 and the nonweighted Poincaré inequality that for any doubling weight w, there exist constant $C_1, C_2 > 0$ such that

$$C_1 \|P\|_{L^\infty(\tilde{Q})} \le w(\tilde{Q})^{-1/p} l(Q)^i \|\nabla^i P\|_{L^p_w(\tilde{Q})} \le C_2 \|P\|_{L^\infty(\tilde{Q})}.$$

It is now clear that (since v, v_0 and w are doubling weights)

$$w(\tilde{Q})^{1/q} l(\tilde{Q})^{-i} \le C w(Q)^{1/q} l(Q)^{-i} \Big(\frac{v_0(\tilde{Q})}{v_0(Q)}\Big)^{\frac{h-1}{rh}} \Big(\frac{v_0(\tilde{Q})^{1/r}}{v_0(Q)^{1/r}} + \frac{l(Q)^k}{l(\tilde{Q})^k} \frac{v(\tilde{Q})^{1/p}}{v(Q)^{1/p}}\Big)^{1/h}.$$

It is now easy to see that (5.3) holds.

Next, if (5.3) holds, then by the previous theorem we know that for any function $f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ (5.4) holds. We will then apply Theorem 1.7 with $\Omega = 3Q$ and $\mathcal{D} = Q$. Note that the constant L will be independent of the cube Q. Hence, there exists $\Lambda f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ (with $P^k(Q)f = \pi^k_\sigma(Q)f$), such that

$$\Lambda f = f \text{ on } Q, \quad \|\nabla^k \Lambda f\|_{L^p_v(3Q)} \le C \|\nabla^k f\|_{L^p_v(Q)} \quad \text{and} \quad \|\Lambda f\|_{L^r_{v_0}(3Q)} \le C \|f\|_{L^r_{v_0}(Q)}$$

since $v_0/\sigma \in A_r(\sigma)$ and (5.2) holds. Finally, note that since $\Lambda f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, (5.4) holds for Λf. It is now easy to see that (5.5) holds. ■

Next, it is easy to extend the weighted interpolation inequality to (ε, ∞) domains.

Theorem 5.3 *Under the assumption of the Theorem* 5.1*,* (5.3) *holds for all cubes $\tilde{Q} \subset Q$ in $\mathbb{R}^n$ if and only if*

$$\|\nabla^i f\|_{L^q_w(\mathcal{D})} \le C w(\mathcal{D})^{1/q} \operatorname{rad}(\mathcal{D})^{-i} \Big(\frac{\|f\|_{L^r_{v_0}(\mathcal{D})}}{v_0(Q)^{1/r}}\Big)^{\frac{h-1}{h}} \times \Big(\frac{\|f\|_{L^r_{v_0}(\mathcal{D})}}{v_0(\mathcal{D})^{1/r}} + \frac{\operatorname{rad}(\mathcal{D})^k \|\nabla^k f\|_{L^p_v(\mathcal{D})}}{v(\mathcal{D})^{1/p}}\Big)^{1/h} \tag{5.6}$$

for all $f \in C^{k-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and bounded (ε, ∞) domains $\mathcal{D}$.

We now look at the extension of weighted Sobolev interpolation inequality to unbounded (ε, ∞) domains. The following theorem is indeed an extension of [17, Theorem 1.5]

Theorem 5.4 *Let p, r, q, h, i, k, and doubling weights v, v_0, σ, w be as in Theorem* 5.1*. If $v(Q)^{1/p} v_0(Q)^{-1/r} l(Q)^{-k} \to 0$ as $l(Q) \to \infty$, then*

$$\|\nabla^i f\|_{L^q_w(\mathcal{D})} \le C \|f\|^{\frac{h-1}{h}}_{L^r_{v_0}(\mathcal{D})} \|\nabla^k f\|^{1/h}_{L^p_v(\mathcal{D})} \tag{5.7}$$

https://doi.org/10.4153/CJM-2006-021-0 Published online by Cambridge University Press

for all $f \in C^{k-1,1}_{\text{loc}}(\mathbb{R}^n)$ *such that* $\|\nabla^k f\|_{L^p_v(\mathcal{D})} \neq 0$ *and any unbounded* (ε, ∞) *domain* $\mathcal{D}$ *if and only if*

$$w(Q)^{1/q} \leq C v_0(Q)^{\frac{h-1}{rh}} l(Q)^{i-\frac{k}{h}} v(Q)^{\frac{1}{ph}} \tag{5.8}$$

for all cubes Q *in* $\mathbb{R}^n$.

Proof Instead of using Theorem 1.7, we will use the extension theorem for unbounded (ε, ∞) domains; see Remark 4.5(ii). The theorem will then follow from [17, Theorem 1.5]. ■

Acknowledgement The author would like to thank the referee for his suggestions.

References

[1] R. Adams and J. Fournier, *Sobolev Spaces.* Academic Press, New York 2003.

[2] B. Bojarski, *Remarks on Sobolev imbedding inequalities.* In: Complex Analysis. Lecture Notes in Math. 1351, Springer-Verlag, 1989, pp. 52–68.

[3] R. C. Brown and D. B. Hinton, *Sufficient conditions for weighted inequalities of sum form.* J. Math. Anal. Appl. **112**(1985), no. 2, 563–578.

[4] ———, *Weighted interpolation inequalities of sum and product form in* $\mathbb{R}^n$. Proc. London Math. Soc. **56**(1988), no. 3, 261–280.

[5] ———, *Weighted interpolation inequalities and embeddings in* $\mathbb{R}^n$. Canad. J. Math. **62**(1990), no. 6, 959–980.

[6] A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions.* Proc. Symp. Pure Math. 4, American Mathematical Society, Providence, RI, 1961, pp. 33–49.

[7] S. Chanillo and R. L. Wheeden, *Poincaré inequalities for a class of non-A_p weights.* Indiana Math. J. **41**(1992), no. 3, 605–623.

[8] F. Chiarenza and M. Frasca, *A note on a weighted Sobolev inequality.* Proc. Amer. Math. Soc. **93**(1985), no. 4, 703–704.

[9] M. Christ, *The extension problem for certain function spaces involving fractional orders of differentiability.* Ark. Mat. **22**(1984), no. 1, 63–81.

[10] S.-K. Chua, *Extension theorems on weighted Sobolev spaces.* Indiana Math. J. **41**(1992), no. 4, 1027–1076.

[11] ———, *Weighted Sobolev's inequalities on domains satisfying the chain condition.* Proc. Amer. Math. Soc. **117**(1993), 449–457.

[12] ———, *Some remarks on extension theorems for weighted Sobolev spaces.* Illinois J. Math. **38**(1994), 95–126.

[13] ———, *On weighted Sobolev interpolation inequalities.* Proc. Amer. Math. Soc. **121**(1994), no. 2, 441–449.

[14] ———, *Weighted Sobolev interpolation inequalities on certain domains.* J. London Math. Soc. (2) **51**(1995), no, 3, 532–544.

[15] ———, *On weighted Sobolev spaces.* Canad. J. Math. **48**(1996), 527–541.

[16] ———, *Weighted inequalities on John Domains.* Jour. Math. Anal. Appl. **258**(2001), 763–776.

[17] ———, *Sharp conditions for weighted interpolation inequalities.* Forum Math. **17**(2005), 461–478.

[18] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals.* Studia Math. **51**(1974), 241–250.

[19] D. E. Edmunds, A. Kufner, and J. Sun, *Extension of functions in weighted Sobolev spaces.* Rend. Accad. Naz. Sci. XL Mem. Mat. (5) **14**(1990), 327–339.

[20] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations.* Comm. Partial Differential Equations **7**(1982), no. 1, 77–116.

[21] B. Franchi, C. Pérez, and R. L. Wheeden, *Self-improving properties of John-Nirenberg and Poincaré inequalities on spaces of homogeneous type.* J. Funct. Anal. **153**(1998), 67–97.

[22] V. M. Gol'dshtein and Yu. G. Reshetnyak, *Quasiconformal Mappings and Sobolev Spaces.* Kluwer Academic Publishers, Dordrecht, 1990.

[23] C. E. Gutierrez and R. L. Wheeden, *Sobolev interpolation inequalities with weights.* Trans. Amer. Math. Soc. **323**(1991), no. 1, 263–281.
[24] P. Hajłasz and P. Koskela, *Isoperimetric inequalities and imbedding theorems in irregular domains* J. London Math. Soc. (2) **58**(1998), no. 2, 425–450.
[25] ______, *Sobolev met Poincaré.* Mem. Amer. Math. Soc. **145**(2000), no. 688.
[26] R. A. Hunt, D. S. Kurtz, and C. I. Neugebauer, *A note on the equivalence of A_p and Sawyer's condition for equal weights.* In: Conference on Harmonic Analysis in Honor of A. Zymund. Wadsworth, Belmont, CA, 1983, pp. 156–158.
[27] P. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces.* Acta Math. **147**(1981), no. 1-2, 71–88.
[28] A. Kufner, *Weighted Sobolev Spaces.* John Wiley, New York, 1985.
[29] ______, *How to define reasonable Sobolev spaces.* Comment. Math. Univ. Carolin. **25**(1984), no. 3, 537–554.
[30] O. Martio and J. Sarvas, *Injectivity theorems in plane and space.* Ann. Acad. Sci. Fenn. Ser. A / Math. **4**(1979), no. 2, 383–401.
[31] V. G. Maz'ja, *Sobolev Spaces.* Springer Series in Soviet Mathematics, Springer-Verlag, New York 1985.
[32] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function.* Trans. Amer. Math. Soc. **165**(1972), 207–226.
[33] B. Muckenhoupt and R. Wheeden, *On the dual of weighted H^1 of the half-space.* Studia Math. **63**(1978), no. 1, 57–79.
[34] E. Sawyer, *A characterization of a two-weight norm inequality for maximal operators.* Studia Math. **75**(1982), no. 1, 1-11.
[35] E. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces.* Amer. J. Math. **114**(1992), no. 4, 813–874.
[36] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton Mathematical Series 30, Princeton University Press, Princeton, NJ, 1970.
[37] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces.* Lecture Notes in Mathematics 1381, Springer-Verlag, Berlin, 1989.
[38] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis.* Pure and Applied Mathematics 123, Academic Press, Orlando, FL, 1986.

National University of Singapore
Department of Mathematics
2, Science Drive 2
Singapore 117543
e-mail: matcsk@nus.edu.sg