This is a ``preproof'' accepted article for *Canadian Mathematical Bulletin*
This version may be subject to change during the production process.
DOI: 10.4153/S0008439525000141

ON BOUNDARY-NON-PRESERVING MAPPINGS WITH POLETSKY INEQUALITY

VICTORIA DESYATKA, EVGENY SEVOST'YANOV

January 28, 2025

Abstract

The manuscript is devoted to the boundary behavior of mappings with bounded and finite distortion. We consider mappings of domains of the Euclidean space that satisfy weighted Poletsky inequality. Assume that, the definition domain is finitely connected on its boundary and, in addition, on the set of all points which are pre-images of the cluster set of this boundary. Then the specified mappings have a continuous boundary extension provided that the majorant in the Poletsky inequality satisfies some integral divergence condition, or has a finite mean oscillation at every boundary point.

2010 Mathematics Subject Classification: Primary 30C65; Secondary 31A15, 31B25

1 Introduction

This paper is devoted to the study of mappings with bounded and finite distortion, see, e.g., [Cr], [MRV_1], [MRSY_1]–[MRSY_2], [PSS], [RV], [SalSt], [Vu] and [Va]. Observe that, the most of known results concerning the continuous boundary extension of mappings use the assumption of boundary preservation under the mapping. Given a mapping between domains D and D' in $\mathbb{R}^n$, it means that $C(f,\partial D)\subset\partial D'$, where $C(f,\partial D)$ is a cluster set of f at ∂D. Exceptions from this rule are, say, theorems on continuous extension to an isolated point of the boundary, or to sets of "small measure", see e.g. [MRV_1, Theorem 4.1], [KM, Theorem 4.1], [MRSY_1, Theorem 4.5] and [Ra, Theorem 3.7]. In other cases, mappings are either assumed to be homeomorphisms (and therefore preserve the boundary), or open, discrete and closed, which also implies the boundary preservation. In this regard, we point out publications [Cr], [GRSY], [MRSY_2], [Na_1]–[Na_2], [SSD], [Sm], [Sr] and [Vu], as well as many others from the same series. The theory of boundary behavior of mappings that do not preserve the boundary seems not to have been constructed even in the quasiregular case, and this manuscript contains some studies in this direction.

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

From this position, it should be said that the (conditional) majority of known mappings, including analytic functions, are not closed, and the study of their boundary behavior may turn out to be more important than the study of homeomorphisms or (more general) closed mappings. For example, an analytic function $f(z) = z^4$, on the complex plane is a closed mapping in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ but in the disk $B(1,1) = \{z \in \mathbb{C} : |z-1| < 1\}$ it is no longer such. Due to the well-known theorems of Casorati-Weierstrass or Picard, even an isolated singularity of an analytic function will be its essential singularity only in the case when the mapping is not closed (closed mappings, in particular, preserve the boundary of the domain, and, at the same time, the cluster set of an analytic function with an essential singularity is the entire extended complex plane).

Given some known results in the context of this manuscript, the following results are valid.

Theorem A. *Suppose that $f : D \to D'$ is a quasiconformal mapping and that D has property P_1 at $b \in \partial D$. Then $C(f, b)$ contains at most one point at which D' is finitely connected (see [Va, Theorem 17.13]).*

Here we should explain a little the meaning of the property P_1, see [Va, Definition 17.5]. Let D be a domain in $\overline{\mathbb{R}^n}$ and let $b \in \partial D$. Then D has property P_1 at b if the following condition is satisfied: If E and F are connected subsets of D such that $b \in \overline{E} \cup \overline{F}$, then $M(\Gamma(E, F, D)) = \infty$, where M denotes the (conformal) modulus of families of paths in $\mathbb{R}^n$ (see the definition below), and $\Gamma(E, F, D)$ is a family of paths joining E and F in D.

An analogue of Theorem **A** was also obtained for quasiregular mappings (which, in particular, are not homeomorphisms; see, for example, [Sr, Theorem 4.2], cf. [Vu, Theorem 4.2]).

Theorem B. *Let $f : D \to \mathbb{R}^n$ be quasiregular mapping with $C(f, \partial D) \subset \partial f(D)$. If D is locally connected at a point $b \in \partial D$ and $D' = f(D)$ is qc accessible at some point $y \in C(f, b)$, then $C(f, b) = \{y\}$.*

Subsequent studies are related to the concept of the modulus of families of paths, the definition of which is given below. A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1$$

holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm} \Gamma$. Let $p \geqslant 1$, then *p-modulus* of Γ is defined by the equality

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\,dm(x)\,.$$

We set $M(\Gamma) := M_n(\Gamma)$. The Poletsky classical inequality is the inequality

$$M(f(\Gamma)) \leqslant K \cdot M(\Gamma)\,, \tag{1.1}$$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

where $K \geqslant 1$ is some constant, and Γ is an arbitrary family of paths in D. It was obtained by E. Poletsky in the early 70s last century, see [Pol, Theorem 1]. Our further research is related to the fact that we are considering mappings of a more general nature, see below. Let $x_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, \quad B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\} \tag{1.2}$$

and

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,. \tag{1.3}$$

Given sets E, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. Let $S_i = S(x_0, r_i)$, $i = 1, 2$, where spheres $S(x_0, r_i)$ centered at x_0 of the radius r_i are defined in (1.2). Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. Let $p \geqslant 1$. A mapping $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at the point $x_0 \in \overline{D} \setminus \{\infty\}$ with respect to p-modulus*, if the condition

$$M_p(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A\cap D} Q(x) \cdot \eta^p(|x - x_0|)\, dm(x) \tag{1.4}$$

holds for some $r_0(x_0) > 0$, all $0 < r_1 < r_2 < r_0$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{1.5}$$

Note that, (1.1) implies (1.4)–(1.5) whenever $p = n$ and $Q(x) \equiv K$. So, (1.4) may be interpreted as Poletsky weighed inequality with the weight $Q(x)$.

Remark 1.1. Note that all quasiregular mappings $f : D \to \mathbb{R}^n$ satisfy the condition

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{D\cap A(x_0,r_1,r_2)} K_I \cdot \eta^n(|x - x_0|)\, dm(x) \tag{1.6}$$

at each point $x_0 \in \overline{D} \setminus \{\infty\}$ with some constant $K_I = K_I(f) \geqslant 1$ and an arbitrary Lebesgue-dimensional function $\eta : (r_1, r_2) \to [0, \infty]$, which satisfies condition (1.5). Indeed, quasiregular mappings satisfy the condition

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{D\cap A(x_0,r_1,r_2)} K_I \cdot \rho^n(x)\, dm(x) \tag{1.7}$$

for an arbitrary function $\rho \in \operatorname{adm} \Gamma(S_1, S_2, D)$, see [Ri, Theorem 8.1.II]. Put $\rho(x) := \eta(|x - x_0|)$ for $x \in A(x_0, r_1, r_2) \cap D$, and $\rho(x) = 0$ otherwise. By Luzin theorem, we may assume that the function ρ is Borel measurable (see, e.g., [Fe, Section 2.3.6]). Due to [Va, Theorem 5.7],

$$\int\limits_{\gamma} \rho(x)\, |dx| \geqslant \int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

for each (locally rectifiable) path γ in $\Gamma(S(x_0,r_1),S(x_0,r_2),A(x_0,r_1,r_2))$. By substituting the function ρ mentioned above into (1.7), we obtain the desired ratio (1.6).

Recall that a mapping $f:D\rightarrow {\Bbb R}^n$ is called *discrete* if the pre-image $\{f^{-1}\left(y\right)\}$ of each point $y\,\in\,{\Bbb R}^n$ consists of isolated points, and *is open* if the image of any open set $U\subset D$ is an open set in ${\Bbb R}^n$. Later, in the extended space $\overline{{\Bbb R}^n}={\Bbb R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is a stereographic projection $\overline{{\Bbb R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$ in ${\Bbb R}^{n+1}$, namely,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2} \sqrt{1+{|y|}^2}}\,, \quad x\ne \infty\ne y$$

(see [Va, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary ∂A of the set $A\subset\overline{{\Bbb R}^n}$ we understand relative to the chordal metric h in $\overline{{\Bbb R}^n}$. Given a mapping $f:D\rightarrow {\Bbb R}^n$, we denote

$$C(f, x):=\{y\in \overline{{\Bbb R}^n}:\exists\,x_k\in D: x_k\rightarrow x, f(x_k)\rightarrow y, k\rightarrow\infty\}$$

and

$$C(f, \partial D)=\bigcup\limits_{x\in \partial D}C(f, x)\,.$$

In what follows, ${\rm Int\,}A$ denotes the set of inner points of the set $A\subset \overline{{\Bbb R}^n}$. Recall that the set $U\subset\overline{{\Bbb R}^n}$ is neighborhood of the point z_0, if $z_0\in {\rm Int\,}A$.

Due to [IR], we say that a function $\varphi:D\rightarrow{\Bbb R}$ has a *finite mean oscillation* at a point $x_0\in D$, write $\varphi\in FMO(x_0)$, if

$$\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} |{\varphi}(x)-\overline{{\varphi}}_{\varepsilon}|\ dm(x)<\infty\,,$$

where

$$\overline{{\varphi}}_{\varepsilon}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} {\varphi}(x) dm(x)\,.$$

Let $Q:{\Bbb R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function. We set

$$Q^{\,\prime}(x)=\left\{\begin{array}{rr} Q(x), & Q(x)\geqslant 1\,,\\ 1, & Q(x)<1\,.\end{array}\right.$$

Denote by $q^{\,\prime}_{x_0}$ the mean value of $Q^{\,\prime}(x)$ over the sphere $|x-x_0|=r$, that means,

$$q^{\,\prime}_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q^{\,\prime}(x)\,d\mathcal{H}^{n-1}\,. \qquad (1.8)$$

Note that, using the inversion $\psi(x)=\frac{x}{|x|^2}$, we may give the definition of FMO as well as the quantity in (1.8) for $x_0=\infty$.

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

We say that the boundary ∂D of a domain D in $\mathbb{R}^n$, $n \geqslant 2$, is *strongly accessible at a point $x_0 \in \partial D$ with respect to the p-modulus* if for each neighborhood U of x_0 there exist a compact set $E \subset D$, a neighborhood $V \subset U$ of x_0 and $\delta > 0$ such that

$$M_p(\Gamma(E, F, D)) \geqslant \delta \tag{1.9}$$

for each continuum F in D that intersects ∂U and ∂V. When $p = n$, we will usually drop the prefix in the "p-modulus" when speaking about (1.9).

Some analogues of the following result were established for the case of homeomorphisms in [MRSY$_1$, Lemma 5.20, Corollary 5.23], [RS, Lemma 6.1, Theorem 6.1] and [Sm, Lemma 5, Theorem 3]. For open discrete and closed mappings, see, e.g., in [SSD], [Sev$_1$] and [Sev$_2$].

Theorem 1.1. *Let $p \geqslant 1$, let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$, $f : D \to D'$ be an open discrete mapping satisfying relations (1.4)–(1.5) at the point $b \in \partial D$ with respect to p-modulus, $f(D) = D'$. In addition, assume that*

1) the set

$$E := f^{-1}(C(f, \partial D))$$

is nowhere dense in D and D is finitely connected on E, i.e., for any $z_0 \in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components.

2) for any neighborhood U of b there is a neighborhood $V \subset U$ of b such that:

2a) $V \cap D$ is connected,

2b) $(V \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$,

3) $D' \setminus C(f, \partial D)$ consists of finite components, each of them has a strongly accessible boundary with respect to p-modulus.

Suppose that at least one of the following conditions is satisfied: 4_1) a function Q has a finite mean oscillation at the point b; 4_2) $q_b(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right)$ as $r \to 0$; 4_3) the condition

$$\int\limits_0^{\delta(b)} \frac{dt}{t^{\frac{n-1}{p-1}} {q_b'}^{\frac{1}{p-1}}(t)} = \infty \tag{1.10}$$

holds for some $\delta(b) > 0$. Then f has a continuous extension to b.

If the above is true for any point $b \in \partial D$, the mapping f has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$, moreover, $\overline{f}(\overline{D}) = \overline{D'}$.

Further in the text we will construct relevant examples that show how important it is that there is no requirement to preserve the boundary in a mapping. It should be understood that conditions 1), 2b) and 3) always hold for homeomorphisms, as well as open discrete

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

and closed mappings. In addition, conditions 4_1)–4_3) hold for quasiregular mappings. Thus, Theorem 1.1 implies the following important consequence.

Corollary 1.1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$, $f : D \to D'$ be a quasiregular mapping such that $f(D) = D'$. In addition, assume that the conditions 1)–3) from Theorem 1.1 hold for $p = n$. Then f has a continuous extension to b. If the above is true for any point $b \in \partial D$, the mapping f has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$, moreover, $\overline{f}(\overline{D}) = \overline{D'}$.*

2 Lemma on paths

Lemma 2.1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $x_0 \in \partial D$. Assume that E is closed and nowhere dense in D, and D is finitely connected on E, i.e., for any $z_0 \in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components.*

In addition, assume that the following condition holds: for any neighborhood U of x_0 there is a neighborhood $V \subset U$ of x_0 such that:

a) $V \cap D$ is connected,

b) $(V \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$.

Let $x_k, y_k \in D \setminus E$, $k = 1, 2, \ldots$, be a sequences converging to x_0 as $k \to \infty$. Then there are subsequences x_{k_l} and y_{k_l}, $l = 1, 2, \ldots$, belonging to some sequence of neighborhoods V_l, $l = 1, 2, \ldots$, of the point x_0 such that $V_l \subset B(x_0, 2^{-l})$, $l = 1, 2, \ldots$, and, in addition, any pair x_{k_l} and y_{k_l} may be joined by a path γ_l in $V_l \cap D$, where γ_l contains at most $m-1$ points in E.

Proof. From the conditions of the lemma it follows that there exists a sequence of neighborhoods $V_k \subset B(x_0, 2^{-k})$, $k = 1, 2, \ldots$, such that $V_k \cap D$ is connected and $(V_k \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$. Now, there are subsequences x_{k_l} and y_{k_l} which belong to $V_l \cap D$. To simplify the notation, we will assume that the sequences x_k and y_k themselves satisfy this condition, i.e., $x_k, y_k \in V_k$, $k = 1, 2, \ldots$. Since $V_k \cap D$ is connected, there is a path $\gamma_k : [0, 1] \to V_k$ such that $\gamma_k(0) = x_k$ and $\gamma_k(1) = y_k$.

Let K_1 be a component of $(V_k \cap D) \setminus E$ containing x_k. If $y_k \subset K_1$, the proof is complete. In the contrary case, $|\gamma_k| \cap (D \setminus K_1) \neq \varnothing$. Let us to show that, in this case,

$$|\gamma_k| \cap (D \setminus \overline{K_1}) \neq \varnothing\,. \tag{2.1}$$

Indeed, since $y_k \in (V_k \cap D) \setminus E$, there is a component K_* of $(V_k \cap D) \setminus E$ such that $y_k \in K_*$. Observe that $y_k \notin \overline{K_1}$. Indeed, in the contrary case there is a sequence $z_s \in K_1$, $s = 1, 2, \ldots$, such that $z_s \to y_k$ as $s \to \infty$. But all components of $(V_k \cap D) \setminus E$ are closed in $(V_k \cap D) \setminus E$ and disjoint (see, e.g., [Ku, Theorem 1.5.III]). Thus $y_k \in K_1$, as well. It is possible only if

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

$K_1 = K_*$, that contradicts the assumption mentioned above. Therefore, the relation (2.1) holds, as required.

By (2.1), there is $t_1 \in (0,1)$ such that

$$t_1 = \sup_{t\geqslant 0:\gamma_k(t)\in \overline{K_1}} t\,. \tag{2.2}$$

Set

$$z_1 = \gamma_k(t_1) \in V_k \cap D\,. \tag{2.3}$$

Observe that $z_1 \in E$. Indeed, in the contrary case there is a component K_{**} of $(V_k\cap D)\setminus E$ such that $z_1 \in K_{**}$. Since K_1 is closed, $K_{**} = K_1$. Since K_1 is open set (see [Ku, Theorem 4.6.II]), there is $t_1^* > t_1$ such that a path $\gamma_k|_{[t_1,t_1^*]}$ belongs to K_1 yet. But this contradicts with the definition of t_1 in (2.2).

Let us to show that, there is another component K_2 of $(V_k \cap D) \setminus E$ such that $z_1 \in \partial K_2$. Indeed, the points $\gamma_k(t)$, $t > t_1$, do not belong to $\overline{K_1}$, so that there is a sequence $z_1^l \in (V_k \cap D) \setminus K_1$, $l = 1, 2 \ldots$, such that $z_1^l \to z_1$ as $l \to \infty$. Since E is nowhere dense in D, $V_k \cap D$ is open and $z_1 \in V_k \cap D$, we may consider that $z_1^l \in (V_k \cap D) \setminus E$ for any $l \in \mathbb{N}$. Since there are m components of $(V_k \cap D) \setminus E$, we may choose a component K_2 of them which contains infinitely many elements of the sequence z_1^l. Without loss of generality, passing to a subsequence, if need, we may consider that all elements $z_1^l \in K_2$, $l = 1, 2, \ldots$. Thus, $z_1 \in \partial K_2$, as required.

Observe that, any component K of $(V_k \cap D) \setminus E$ is finitely connected at any point $z_0 \in \partial K$. Indeed, since D is finitely connected on E, for any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components. Now, $(K \cap \widetilde{V}) \setminus E$ consists of finite number of components, as well.

Now, due to Lemma 3.10 in [Vu] we may replace $\gamma_k|_{[0,t_1]}$ by a path belonging to K_1 for any $t \in [0, t_1)$ and tending to z_1 as $t \to t_1 - 0$. (Here z_1 is defined in (2.3)). In order to simplify the notation, without limiting the generality of the reasoning, we may assume that the path $\gamma_k|_{[0,t_1]}$ already has the indicated property. Now, there are two cases:

a) $y_k \in K_2$. Then we join the points y_k and z_1 by a path $\alpha_1 : (t_1, 1] \to K_2$ belonging to K_2 and tending to z_1 as $t \to t_1 + 0$. This is possible due to finitely connectedness of K_2 proved above and by Lemma 3.10 in [Vu]. Uniting paths $\gamma_k|_{[0,t_1]}$ and α_1, we obtain the desired path. In particular, only one point z_1 belonging to this path also belongs to E.

b) $y_k \notin K_2$. Arguing similarly to mentioned above, we may prove that there is $t_2 \in (t_1, 1)$ such that

$$t_2 = \sup_{t\geqslant t_1:\gamma_k(t)\in \overline{K_2}} t\,. \tag{2.4}$$

Reasoning similarly to the case with point t_1, we may show that:

$$(b_1) \qquad z_2 = \gamma_k(t_2) \in (V_k \cap D) \cap E\,,$$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

(b_2) there is another component K_3 of $(V_k\cap D)\setminus E$, $K_1\ne K_3\ne K_2$ such that $z_2\in\partial K_3$,

(b_3) due to Lemma 3.10 in [Vu] we may replace $\gamma_k|_{[t_1,t_2]}$ by a path belonging to K_2 for any $t\in(t_1,t_2)$, tending to z_1 as $t\to t_1+0$ and tending to z_2 as $t\to t_2-0$. In order to simplify the notation, without limiting the generality of the reasoning, we may assume that the path $\gamma_k|_{[t_1,t_2]}$ already has the indicated property.

Now, there are two cases:

a) $y_k\in K_3$. Then we join the points y_k and z_3 by a path $\alpha_2:(t_2,1]\to K_3$ belonging to K_3 and tending to z_2 as $t\to t_2+0$. This is possible due to finitely connectedness of K_3 proved above and by Lemma 3.10 in [Vu]. Uniting paths $\gamma_k|_{[0,t_2]}$ and α_2, we obtain the desired path. In particular, precisely two points z_1 and z_2 belonging to this path also belong to E.

b) $y_k\not\in K_2$. Arguing similarly to mentioned above, we may prove that there is $t_3\in(t_2,1)$ such that

$$t_3=\sup\limits_{t\geqslant t_2:\gamma_k(t)\in\overline{K_3}}t\,. \tag{2.5}$$

And so on. Continuing this process, we will obtain a certain number of points $z_1=\gamma_k(t_1), z_2=\gamma_k(t_2), z_3=\gamma_k(t_3),\ldots,z_{k_{\widetilde{p}-1}}=\gamma_k(t_{\widetilde{p}-1})$ in E and a certain number of components $K_1,K_2,\ldots,K_{\widetilde{p}}$ in $(V_k\cap D)\setminus E$, $K_1\ne K_2\ne\ldots\ne K_{\widetilde{p}}$. The corresponding path $\gamma_k|_{[0,t_{\widetilde{p}-1}]}$ is a part of a path γ_k which joins the point x_k and $z_{k_{\widetilde{p}-1}}\in\partial K_{\widetilde{p}}$ in $V_k\cap D$ and such that

$$|\gamma_k|_{[0,t_{k_{\widetilde{p}-1}}]}|\cap E=\{z_1,z_2,z_3,\ldots,z_{k_{\widetilde{p}-1}}\}\,.$$

Since at each step the remaining part of the path does not belong to the union of the closures of the previous components $K_1,K_2,\ldots,K_{\widetilde{p}-1}$, and there are only a finite number of these components does not exceed m, then $y_k\in K_{\widetilde{p}}$ for some $1\leqslant\widetilde{p}\leqslant m$. Then we join the points y_k and $z_{\widetilde{p}-1}$ by a path $\alpha_{\widetilde{p}-1}:(t_{\widetilde{p}-1},1]\to K_{\widetilde{p}}$ belonging to $K_{\widetilde{p}}$ and tending to $z_{\widetilde{p}-1}$ as $t\to t_{\widetilde{p}-1}+0$. This is possible due to finitely connectedness of $K_{\widetilde{p}}$ proved above and by Lemma 3.10 in [Vu]. Uniting paths $\gamma_k|_{[0,t_{\widetilde{p}-1}]}$ and $\alpha_{\widetilde{p}-1}$, we obtain the desired path. In particular, only the points $z_1,z_2,\ldots,z_{\widetilde{p}-1}$ belonging to this path also belongs to E. The number of these points does not exceed $m-1$. Lemma is proved. $\Box$

3 Main Lemma

Let $D\subset\mathbb{R}^n$, $f:D\to\mathbb{R}^n$ be a discrete open mapping, $\beta:[a,\,b)\to\mathbb{R}^n$ be a path, and $x\in f^{-1}(\beta(a))$. A path $\alpha:[a,\,c)\to D$ is called a *maximal f-lifting* of β starting at x, if (1) $\alpha(a)=x$; (2) $f\circ\alpha=\beta|_{[a,\,c)}$; (3) for $c<c'\leqslant b$, there is no a path $\alpha':[a,\,c')\to D$ such that $\alpha=\alpha'|_{[a,\,c)}$ and $f\circ\alpha'=\beta|_{[a,\,c')}$. Here and in the following we say that a path $\beta:[a,b)\to\overline{\mathbb{R}^n}$ converges to the set $C\subset\overline{\mathbb{R}^n}$ as $t\to b$, if $h(\beta(t),C)=\sup\limits_{x\in C}h(\beta(t),C)\to 0$ at $t\to b$. The following is true (see [MRV$_2$, Lemma 3.12]).

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

Proposition 3.1. *Let $f : D \to \mathbb{R}^n$, $n \geqslant 2$, be an open discrete mapping, let $x_0 \in D$, and let $\beta : [a, b) \to \mathbb{R}^n$ be a path such that $\beta(a) = f(x_0)$ and such that either $\lim\limits_{t\to b}\beta(t)$ exists, or $\beta(t) \to \partial f(D)$ as $t \to b$. Then β has a maximal f-lifting $\alpha : [a, c) \to D$ starting at x_0. If $\alpha(t) \to x_1 \in D$ as $t \to c$, then $c = b$ and $f(x_1) = \lim\limits_{t\to b}\beta(t)$. Otherwise $\alpha(t) \to \partial D$ as $t \to c$.*

Versions of the following lemma have been repeatedly proven by different authors, including the second co-author of this work in the situation where the maps are homeomorphisms or open discrete closed maps, see, for example, [IR, Lemma 2.1], [MRSY$_1$, Lemma 5.16], [MRSY$_2$, Theorem 4.6, Theorem 13.1], [RS, Theorrem 5.1], [Sev$_3$] and [Sm, Theorem 1]. For quasiconformal mappings, results of this kind were proved in [Va, Theorem 17.15] and [Na$_1$, Section 3]. For closed quasiregular mappings it was proved by Vuorinen and Srebro with some differences in the formulation, see, for example, [Vu, Theorem 4.10.II] and cf. [Sr, Theorem 4.2]. Apparently, in the situation where the mappings are not closed, this lemma is published for the first time, even for quasiregular mappings.

Lemma 3.1. *Let $p \geqslant 1$, Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$, $f : D \to D'$ be an open discrete mapping satisfying relations (1.4)–(1.5) at the point $b \in \partial D$, $b \neq \infty$, with respect to p-modulus, $f(D) = D'$.*

In addition, assume that

1) the set

$$E := f^{-1}(C(f, \partial D))$$

is nowhere dense in D and D is finitely connected on E;

2) for any neighborhood U of b there is a neighborhood $V \subset U$ of b such that:

2a) $V \cap D$ is connected,

2b) $(V \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$,

3) $D' \setminus C(f, \partial D)$ consists of finite components, each of them has a strongly accessible boundary with respect to p-modulus.

Suppose that there is $\varepsilon_0 = \varepsilon_0(b) > 0$ and some positive measurable function $\psi : (0, \varepsilon_0) \to (0, \infty)$ such that

$$0 < I(\varepsilon, \varepsilon_0) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \tag{3.1}$$

for sufficiently small $\varepsilon \in (0, \varepsilon_0)$ and, in addition,

$$\int\limits_{A(\varepsilon,\varepsilon_0,b)} Q(x) \cdot \psi^p(|x-b|)\, dm(x) = o(I^p(\varepsilon, \varepsilon_0))\,, \qquad \varepsilon \to 0\,, \tag{3.2}$$

where $A := A(b, \varepsilon, \varepsilon_0)$ is defined in (1.3). Then f has a continuous extension to b.

Proof. Suppose the opposite. Due to the compactness of $\overline{\mathbb{R}^n}$, then there are at least two sequences x_k, $y_k \in D$, $k = 1, 2, \ldots$, such that x_k, $y_k \in D$, $x_k \to b$, $y_k \to b$ as $k \to \infty$, and $f(x_k) \to y$, $f(y_k) \to y'$ as $k \to \infty$, while $y' \neq y$, see Figure 1. In particular,

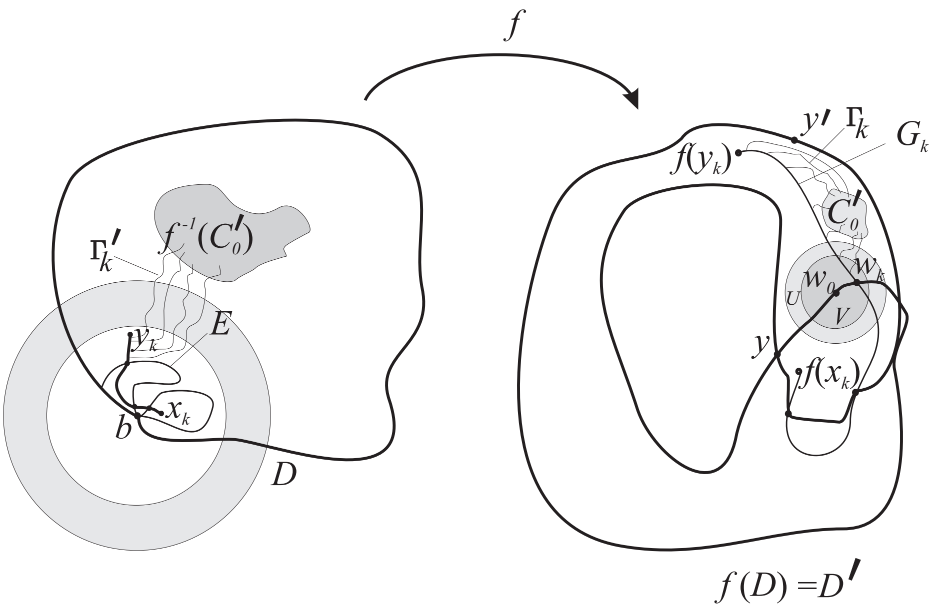

Figure 1: To the proof of Theorem 1.1

$$h(f(x_k), f(y_k)) \geqslant \delta > 0 \tag{3.3}$$

for some $\delta > 0$ and all $k \in \mathbb{N}$, where h is chordal (spherical) metric. By the assumption 2), there exists a sequence of neighborhoods $V_k \subset B(b, 2^{-k})$, $k = 1, 2, \ldots$, such that $V_k \cap D$ is connected and $(V_k \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$.

We note that the points x_k and y_k, $k = 1, 2, \ldots$, may be chosen such that $x_k, y_k \notin E$. Indeed, since under condition 1) the set E is nowhere dense in D, there exists a sequence $x_{ki} \in D \setminus E$, $i = 1, 2, \ldots$, such that $x_{ki} \to x_k$ as $i \to \infty$. Put $\varepsilon > 0$. Due to the continuity of the mapping f at the point x_k, for the number $k \in \mathbb{N}$ there is a number $i_k \in \mathbb{N}$ such that $h(f(x_{ki_k}), f(x_k)) < \frac{1}{2^k}$. So, by the triangle inequality

$$h(f(x_{ki_k}), y) \leqslant h(f(x_{ki_k}), f(x_k)) + h(f(x_k), y) \leqslant \frac{1}{2^k} + \varepsilon \,,$$

$k \geqslant k_0 = k_0(\varepsilon)$, since $f(x_k) \to y$ as $k \to \infty$ and by the choice of x_k and y. Therefore, $x_k \in D$ may be replaced by $x_{ki_k} \in D \setminus E$, as required. We may reason similarly for the sequence y_k.

Now, by Lemma 2.1 there are subsequences x_{k_l} and y_{k_l}, $l = 1, 2, \ldots$, belonging to some sequence of neighborhoods V_l, $l = 1, 2, \ldots$, of the point x_0 such that $\operatorname{diam} V_l \to 0$ as $l \to \infty$ and, in addition, any pair x_{k_l} and y_{k_l} may be joined by a path γ_l in $V_l \cap D$, where γ_l contains at most $m-1$ points in E. Without loss of generality, we may assume that the same sequences x_k and y_k satisfy properties mentioned above. Let $\gamma_k : [0, 1] \to D$, $\gamma_k(0) = x_k$ and $\gamma_k(1) = y_k$, $k = 1, 2, \ldots$.

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

Observe that, the path $f(\gamma_k)$ contains not more than $m-1$ points in $C(f,\partial D)$. In the contrary case, there are at least m such points $b_1=f(\gamma_k(t_1)), b_2=f(\gamma_k(t_2)),\ldots,b_m=f(\gamma_k(t_m))$, $0\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_m\leqslant 1$. But now the points $a_1=\gamma_k(t_1), a_2=\gamma_k(t_2),\ldots,a_m=\gamma_k(t_m)$ are in $E=f^{-1}(C(f,\partial D))$ and simultaneously belong to γ_k. This contradicts the definition of γ_k.

Let

$$b_1=f(\gamma_k(t_1)),\, b_2=f(\gamma_k(t_2))\quad,\ldots,\quad b_l=f(\gamma_k(t_l))\,,$$

$$0:=t_0\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_l\leqslant 1:=t_{l+1},\qquad 1\leqslant l\leqslant m-1\,,$$

be points in $f(\gamma_k)\cap C(f,\partial D)$. By the relation (3.3) and due to the triangle inequality,

$$\delta\leqslant h(f(x_k),f(y_k))\leqslant\sum_{r=0}^{l}h(f(\gamma_k(t_r)),f(\gamma_k(t_{r+1}))\,. \tag{3.4}$$

It follows from (3.4) that, there is $0\leqslant r=r(k)\leqslant m$ such that such that

$$h(f(\gamma_k(t_{r(k)})),f(\gamma_k(t_{r(k)+1}))\geqslant\delta/(l+1)\geqslant\delta/m\,. \tag{3.5}$$

Observe that, the set $G_k:=|\gamma_k|_{(t_{r(k)},t_{r(k)+1})}|$ belongs to $D'\setminus C(f,\partial D)$, because it does not contain any point in E. Since $D'\setminus C(f,\partial D)$ consists only of finite components, there exists at least one a component of $D'\setminus C(f,\partial D)$, containing infinitely many components of G_k. Without loss of generality, going to a subsequence, if need, we may assume that all G_k belong to one component K of $D'\setminus C(f,\partial D)$.

Due to the compactness of $\overline{\mathbb{R}^n}$, we may assume that the sequence $w_k:=f(\gamma_k(t_{r(k)}))$, $k=1,2,\ldots$, converges to some a point $w_0\in\overline{D'}$. Let us to show that $w_0\in C(f,\partial D)$. Indeed, there are two cases: either $w_k=f(\gamma_k(t_{r(k)}))\in C(f,\partial D)$ for infinitely many k, or $w_k\not\in E$ for infinitely many $k\in\mathbb{N}$. In the first case, the inclusion $w_0\in C(f,\partial D)$ is obvious, because $C(f,\partial D)$ is closed. In the second case, obviously, $w_k=f(x_k)$, but this sequence converges to $y\in C(f,\partial D)$ by the assumption.

By the assumption, each component of the set $D'\setminus C(f,\partial D)$ has a strongly accessible boundary with respect to p-modulus. In this case, for any neighborhood U of the point $w_0\in\partial K$ there is a compact set $C_0'\subset D'$, a neighborhood V of a point w_0, $V\subset U$, and a number $P>0$ such that

$$M_p(\Gamma(C_0',F,K))\geqslant P>0 \tag{3.6}$$

for any continua F, intersecting ∂U and ∂V. Choose a neighborhood U of w_0 with $h(U)<\delta/2m$, where δ is from (3.3). Let C_0' and V be a compact set and a neighborhood corresponding to w_0.

Observe that, G_k contains some a continuum $\widetilde{G}_k$ in K intersecting ∂U and ∂V for sufficiently large k. Indeed, by the construction of G_k, there is a sequence of points $a_{s,k}:=$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

$f(\gamma_k(p_s))\rightarrow w_k:=f(\gamma_k(t_{r(k)}))$ as $s\rightarrow\infty$ and $b_{s,k}=f(\gamma_k(q_s))\rightarrow f(\gamma_k(t_{r(k)+1}))$ as $s\rightarrow\infty$, where $t_{r(k)}<p_s<q_s<t_{r(k)+1}$ and $a_{s,k}, b_{s,k}\in K$. By the triangle inequality and by (3.5)

$$\begin{gathered} h(a_{s,k},b_{s,k})\geqslant h(b_{s,k},w_k)-h(w_k,a_{s,k})\geqslant \\ h(f(\gamma_k(t_{r(k)+1})),w_k)-h(f(\gamma_k(t_{r(k)+1})),b_{s,k})-h(w_k,a_{s,k})\geqslant \\ \delta/m-h(f(\gamma_k(t_{r(k)+1}),b_{s,k}))-h(w_k,a_{s,k})\,. \end{gathered}\tag{3.7}$$

Since $a_{s,k}:=f(\gamma_k(p_s))\rightarrow w_k:=f(\gamma_k(t_{r(k)}))$ as $s\rightarrow\infty$ and $b_{s,k}=f(\gamma_k(q_s))\rightarrow f(\gamma_k(t_{r(k)+1}))$ as $s\rightarrow\infty$, it follows from the last inequality that there is $s=s(k)\in\mathbb{N}$ such that

$$h(a_{s(k),k},b_{s(k),k})\geqslant \delta/(2m)\,.\tag{3.8}$$

Since V is open, there is some neighborhood U_k of w_k such $U_k\subset V$. Since $a_{s,k}\rightarrow w_k:=f(\gamma_k(t_{r(k)}))$ as $s\rightarrow\infty$, we may assume that $a_{s(k),k}\in V$. Now, we set

$$\widetilde{G}_k:=f(\gamma_k)|_{[p_{s(k)},q_{s(k)}]}\,.$$

In other words, $\widetilde{G}_k$ is a part of the path $f(\gamma_k)$ between points $a_{s(k),k}$ and $b_{s(k),k}$. Let us to show that $\widetilde{G}_k$ intersects ∂U and ∂V. Indeed, by the mentioned above, $a_{s(k),k}\subset V$, so that $V\cap\widetilde{G}_k\ne\varnothing$. In particular, $U\cap\widetilde{G}_k\ne\varnothing$, because $V\subset U$. On the other hand, by (3.8) $h(\widetilde{G}_k)\geqslant\delta/2m$, however, $h(U)<\delta/2m$ by the choice of U. In particular, $h(V)<\delta/2m$. It follows from this that

$$(\overline{\mathbb{R}^n}\setminus U)\cap\widetilde{G}_k\ne\varnothing\,,\qquad(\overline{\mathbb{R}^n}\setminus V)\cap\widetilde{G}_k\ne\varnothing\,.$$

Now, by [Ku, Theorem 1.I.5, §46]

$$\partial U\cap\widetilde{G}_k\ne\varnothing\,,\qquad \partial V\cap\widetilde{G}_k\ne\varnothing\,,$$

as required. Now, by (3.6),

$$M_p(\Gamma(\widetilde{G}_k,C_0^{\prime},K))\geqslant P>0\,,\qquad k=1,2,\ldots\,.\tag{3.9}$$

Let us to show that, the relation (3.9) contradicts with the definition of f in (1.4) together with the conditions (3.1)–(3.2). Indeed, let us denote by Γ_k the family of all half-open paths $\beta_k:[a,b)\rightarrow\overline{\mathbb{R}^n}$ such that $\beta_k(a)\in\widetilde{G}_k$, $\beta_k(t)\in K$ for all $t\in[a,b)$ and, moreover, $\lim\limits_{t\rightarrow b-0}\beta_k(t):=B_k\in C_0^{\prime}$. Obviously, by (3.9)

$$M_p(\Gamma_k)=M_p(\Gamma(\widetilde{G}_k,C_0^{\prime},K))\geqslant P>0\,,\qquad k=1,2,\ldots\,.\tag{3.10}$$

Consider the family $\Gamma_k^{\prime}$ of maximal f-liftings $\alpha_k:[a,c)\rightarrow D$ of the family Γ_k starting at $|\gamma_k|$; such a family exists by Proposition 3.1.

Observe that, the situation when $\alpha_k\rightarrow\partial D$ as $k\rightarrow\infty$ is impossible. Suppose the opposite: let $\alpha_k(t)\rightarrow\partial D$ as $t\rightarrow c$. Let us choose an arbitrary sequence $\varphi_m\in[0,c)$ such

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

that $\varphi_m \to c-0$ as $m\to\infty$. Since the space $\overline{\mathbb{R}^n}$ is compact, the boundary ∂D is also compact as a closed subset of the compact space. Then there exists $w_m\in\partial D$ such that

$$h(\alpha_k(\varphi_m),\partial D)=h(\alpha_k(\varphi_m),w_m)\to 0\,,\qquad m\to\infty\,. \tag{3.11}$$

Due to the compactness of ∂D, we may assume that $w_m\to w_0\in\partial D$ as $m\to\infty$. Therefore, by the relation (3.11) and by the triangle inequality

$$h(\alpha_k(\varphi_m),w_0)\leqslant h(\alpha_k(\varphi_m),w_m)+h(w_m,w_0)\to 0\,,\qquad m\to\infty\,. \tag{3.12}$$

On the other hand,

$$f(\alpha_k(\varphi_m))=\beta_k(\varphi_m)\to\beta(c)\,,\quad m\to\infty\,, \tag{3.13}$$

because by the construction, the path $\beta_k(t)$, $t\in[a,b]$, lies in $K\subset D'\backslash C(f,\partial D)$ together with its finite ones points. At the same time, by (3.12) and (3.13) we have that $\beta_k(c)\in C(f,\partial D)$. The inclusions $\beta_k\subset D'\setminus C(f,\partial D)$ and $\beta_k(c)\in C(f,\partial D)$ contradict each other.

Therefore, by Proposition 3.1 $\alpha_k\to x_1\in D$ as $t\to c-0$, and $c=b$ and $f(\alpha_k(b))=f(x_1)$. In other words, the f-lifting α_k is complete, i.e., $\alpha_k:[a,b]\to D$. Besides that, it follows from that $\alpha_k(b)\in f^{-1}(C_0^{\prime})$.

Observe that, there is $r_0>0$ such that

$$h(f^{-1}(C_0^{\prime}),\partial D)\geqslant r_0>0\,. \tag{3.14}$$

In the contrary case, there is $z_k\in f^{-1}(C_0^{\prime})$ such that $z_k\to z_0\in\partial D$ as $k\to\infty$. Now, due to the compactness of $\overline{\mathbb{R}^n}$ the sequence $f(z_k)$ converges to some ω_0 as $k\to\infty$. But, in this case, ω_0 in $C(f,\partial D)$ and simultaneously $\omega_0\in K\subset D'\setminus C(f,\partial D)$ because $f(z_k)\in C_0^{\prime}\subset K$ and $C_0^{\prime}$ is closed. We have obtained a contradiction, so that the relation (3.14) holds.

Since $b\neq\infty$, it follows from (3.14) that

$$f^{-1}(C_0^{\prime})\subset D\setminus B(b,r_0)\,. \tag{3.15}$$

Let k be such that $2^{-k}<\varepsilon_0$. We may consider that $\varepsilon_0<r_0$. Due to (3.15), we may show that

$$\Gamma_k^{\prime}>\Gamma(S(b,2^{-k}),S(b,\varepsilon_0),D) \tag{3.16}$$

(see [Ku, Theorem 1.I.5, §46]). Observe that the function

$$\eta(t)=\begin{cases}\psi(t)/I(2^{-k},\varepsilon_0), & t\in(2^{-k},\varepsilon_0),\\ 0, & t\in\mathbb{R}\setminus(2^{-k},\varepsilon_0)\,,\end{cases}$$

where $I(2^{-k},\varepsilon_0)$ is defined in (3.1), satisfies (1.5) for $r_1:=2^{-k}$, $r_2:=\varepsilon_0$. Therefore, by the definition of the mapping in (1.4) and by (3.16) we obtain that

$$M_p(f(\Gamma_k^{\prime}))\leqslant\Delta(k)\,, \tag{3.17}$$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

where $\Delta(k)\rightarrow 0$ as $k\rightarrow\infty$. However, $\Gamma_k=f(\Gamma_k^{\prime})$. Thus, by (3.17) and (3.1)–(3.2) we obtain that

$$M_p(\Gamma_k)=M_p(f(\Gamma_k^{\prime}))\leqslant \Delta(k)\rightarrow 0 \tag{3.18}$$

as $k\rightarrow\infty$. However, the relation (3.18) together with the inequality (3.10) contradict each other, which proves the lemma. $\Box$

The proof of the following statement may be found in [Sev$_4$, Lemmas 1.3 and 1.4].

Lemma 3.2. *Let $Q:{\Bbb R}^n\rightarrow [0,\infty]$, $n\geqslant 2$, be a Lebesgue measurable function and let $x_0\in {\Bbb R}^n$. Let $0<p\leqslant n$. Assume that either of the following conditions holds*

(a) $Q\in FMO(x_0)$,

(b) $q_{x_0}(r)\,=\,O\left(\left[\log{\frac1r}\right]^{n-1}\right)$ as $r\rightarrow 0$,

(c) for some small $\delta_0=\delta_0(x_0)>0$ we have the relations

$$\int\limits_{\delta}^{\delta_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(t)}<\infty,\qquad 0<\delta<\delta_0,$$

and

$$\int\limits_{0}^{\delta_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(t)}=\infty\,.$$

Then there exist a number $\varepsilon_0\in(0,1)$ and a function $\psi(t)\geqslant 0$ such that the relation

$$\int\limits_{\varepsilon<|x-b|<\varepsilon_0}Q(x)\cdot\psi^p(|x-b|)\;dm(x)=o(I^p(\varepsilon,\varepsilon_0))\,,$$

holds as $\varepsilon\rightarrow 0$, where $\psi:(0,\varepsilon_0)\rightarrow[0,\infty)$ is some function such that, for some $0<\varepsilon_1<\varepsilon_0$,

$$0<I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty\qquad\forall\quad \varepsilon\in(0,\varepsilon_1)\,.$$

Proof of Theorem 1.1 immediately follows by Lemmas 3.1 and 3.2. Let us to prove the equality $f(\overline{D})=\overline{D^{\prime}}$, cf. the last part of the proof of Theorem 3.1 in [SSD]. Indeed, obviously, $\overline{f}(\overline{D})\subset\overline{D^{\prime}}$. Let us to show that, $\overline{D^{\prime}}\subset\overline{f}(\overline{D})$. Indeed, let $y_0\in\overline{D^{\prime}}$. Then either $y_0\in D^{\prime}$, or $y_0\in\partial D^{\prime}$. If $y_0\in D^{\prime}$, then $y_0=f(x_0)$ and $y_0\in\overline{f}(\overline{D})$, because f maps D onto $D^{\prime}$ by the assumption. Finally, let $y_0\in\partial D^{\prime}$. Then there is a sequence $y_k\in D^{\prime}$ such that $y_k=f(x_k)\rightarrow y_0$ as $k\rightarrow\infty$ and $x_k\in D$. Since $\overline{{\Bbb R}^n}$ is a compact space, we may consider that $x_k\rightarrow x_0$, where $x_0\in\overline{D}$. Observe that, $x_0\in\partial D$, because f is open. Then $f(x_0)=y_0\in\overline{f}(\partial D)\subset\overline{f}(\overline{D})$. Theorem is completely proved. $\Box$

4 Some remarks on uniform domains

A domain $D\subset{\Bbb R}^n$, $n\geqslant 2$, is called a *uniform* domain with respect to p-modulus, $p\geqslant 1$, if, for each $r>0$, there is $\delta>0$ such that $M_p(\Gamma(F,F^*,D))\geqslant\delta$ whenever F and F^* are continua

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

of D with $h(F)\geqslant r$ and $h(F^*)\geqslant r$. Let I be some set of indices. Domains D_i, $i\in I$, are said to be *equi-uniform* domains with respect to p-modulus, if, for $r>0$, the modulus condition above is satisfied by each D_i with the same number δ. It should be noted that the proposed concept of a uniform domain has, generally speaking, no relation to definition, introduced for the uniform domain in Martio-Sarvas sense [MSa]. Note that, uniform domains with respect to p-modulus have strongly accessible boundaries with respect to p-modulus, see [SevSkv$_1$, Remark 1].

Remark 4.1. The statement of Lemma 3.1 remains true if condition 3) in this lemma is replaced by the conditions:

($3_{\mathbf{a}}$) the family of components of $D'\setminus C(f,\partial D)$ is *equi-uniform* with respect to p-modulus and, besides that,

($3_{\mathbf{b}}$) there is $\delta_*>0$ such that for each component K of $D'\setminus C(f,\partial D)$ there is a nondegenerate continuum $F\subset K$ such that $h(F)\geqslant\delta_*$ and $h(f^{-1}(F),\partial D)\geqslant\delta_*>0$.

Indeed, just as in the proof of Lemma 3.1, we will prove this statement by contradiction. The proof up to relation (3.5) is repeated verbatim. Further, just as in the proof of relations (3.7)–(3.8), it may be shown that there is a subcontinuum $\widetilde{G}_k$ in G_k such that $h(\widetilde{G}_k)\geqslant\delta/2m$.

Let K_k be a component of $D'\setminus C(f,\partial D)$ containing $\widetilde{G}_k$, and let $F_k:=F$ be compactum in K_k from the condition $3_{\mathbf{b}}$. Due to the equi-uniformity of K_k (see the condition $3_{\mathbf{b}}$), there exists $P>0$ such that

$$M_p(\Gamma_k)=M_p(\Gamma(\widetilde{G}_k,F_k,K_k))\geqslant P>0\,,\qquad k=1,2,\ldots\,. \tag{4.1}$$

Here Γ_k denotes the family of all half-open paths $\beta_k:[a,b)\rightarrow\overline{\mathbb{R}^n}$ such that $\beta_k(a)\in\widetilde{G}_k$ $\beta_k(t)\in K_k$ for all $t\in[a,b)$ and, moreover, $\lim\limits_{t\rightarrow b-0}\beta_k(t):=B_k\in F_k$.

Consider the family Γ_k' of maximal f-liftings $\alpha_k:[a,c)\rightarrow D$ of the family Γ_k starting at $|\gamma_k|$; such a family exists by Proposition 3.1. Similarly to the proof of Lemma 3.1 we may prove that all these liftings are complete, i.e., any path $\alpha_k:[a,c)\rightarrow D$ is such that $c=b$, $\alpha_k(b)\in f^{-1}(F_k)$. Due to the condition $3_{\mathbf{b}}$,

$$f^{-1}(K_k)\subset D\setminus B(b,r_0)\,.$$

Arguing further in a similar way to the rest of the proof of Lemma 3.1, we arrive at the relation

$$M_p(\Gamma_k)=M_p(f(\Gamma_k'))\leqslant\Delta(k)\rightarrow 0 \tag{4.2}$$

as $k\rightarrow\infty$. The relations (4.1) and (4.2) contradict each other. The desired statement is proved.

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

5 Some examples

Example 1. Let $p = n = 2$. Consider the disk $D = B(1,1) = \{z \in \mathbb{C} : |z-1| < 1\}$ on the complex plane. Let us define a mapping $f : D \to \mathbb{C}$ as follows: $f(z) = z^4$. The Figure 2 schematically shows the image of the domain D under the mapping f; the complete image is marked in gray, the domain that is mapped twice in the disk $B(1,1)$ is marked in a darker color. The image of the boundary of the domain D has one point of self-intersection

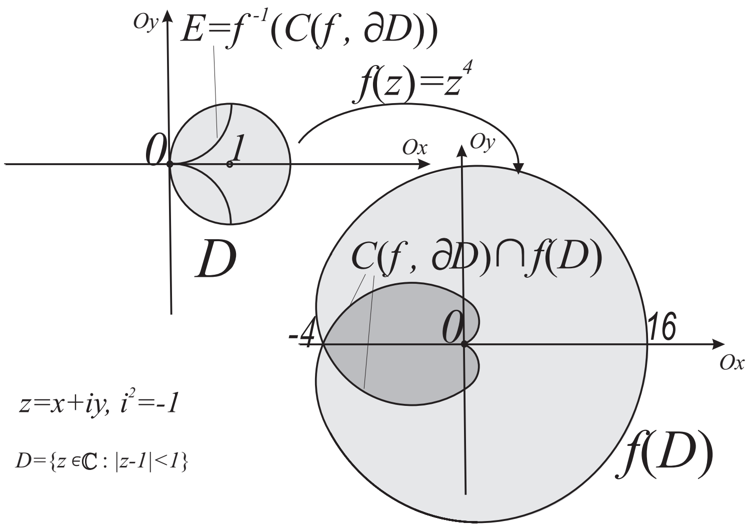

Figure 2: An open quasiregular mapping that satisfies conditions of Theorem 1.1

$w = (-4, 0)$. Paths $z_1(\varphi) = 16\cos^4\varphi \cdot e^{i\cdot 4\varphi}$, $\frac{\pi}{4} < \varphi \leqslant \frac{\pi}{2}$, and $z_2(\varphi) = 16\cos^4\varphi \cdot e^{i\cdot 4\varphi}$, $-\frac{\pi}{2} \leqslant \varphi < -\frac{\pi}{4}$, is a part of the image of the boundary of D that lies inside the mapped domain. It is not difficult to see that, the complete pre-image in D of this paths consists of two paths $\widetilde{z_1}(\varphi) = 2\cos\varphi \cdot e^{i\left(\varphi+\frac{\pi}{2}\right)}$, $\frac{\pi}{4} < \varphi \leqslant \frac{\pi}{2}$, and $\widetilde{z_2}(\varphi) = 2\cos\varphi \cdot e^{i\left(\varphi-\frac{\pi}{2}\right)}$, $-\frac{\pi}{2} \leqslant \varphi < -\frac{\pi}{4}$, and there are no others.

Note that, the mapping f satisfies all conditions of Theorem 1.1. Indeed, the set $E := f^{-1}(C(f,\partial D))$ is nowhere dense in D, because E consists only on paths $\widetilde{z_1}(\varphi)$ and $\widetilde{z_2}(\varphi)$ mentioned above. Also, D is finitely connected on E, because D is two-connected at any inner point of D which belong to the paths $\widetilde{z_1}(\varphi)$ and $\widetilde{z_2}(\varphi)$. Obviously, D is locally connected at any point of ∂D, while it is three-connected with respect to E at the origin, two-connected with respect to E at the points $1 \pm i$ and locally connected with respect to E at the rest points. In addition, $f(D) \setminus C(f, \partial D)$ consists of two components each of them is strongly accessible because it is uniform (see comments made before Remark 4.1). The uniformity of these components follows from the fact that, each plane domain having a finite number of components, which is finitely connected on the boundary is uniform (see [Na_3, Corollary 6.8]). Finally, $f(z) = z^4$ is a quasiregular mapping, therefore satisfies the relation (1.4) for the function $Q(y) \equiv 1$ (see, e.g., [Ri, Theorem 8.1.II]. Observe that, $Q(x) \equiv 1$ satisfies simultaneously the conditions $Q \in FMO(x_0)$ at any $x_0 \in \mathbb{R}^n$ and (1.10), as well.

Example 2. It is not difficult to construct a similar mapping with some unbounded function Q in (1.4). For this purpose, let D and f be a domain and a mapping from the previous example, correspondingly. Let us consider an arbitrary point $z_0\in D\setminus E$, where $E:=f^{-1}(C(f,\partial D))$, and let $0<r_0<\min\{{\rm dist}\,(z_0,\partial D), {\rm dist}\,(z_0, E)\}$. Put

$$h(z)=\frac{r_0(z-z_0)}{|z-z_0|\log\frac{r_0e}{|(z-z_0)|}}+z_0, \qquad h(z_0)=z_0\,.$$

Reasoning similarly to [MRSY$_2$, Proposition 6.3], we may show that h satisfies the relations (1.4)–(1.5) in each point $b\in\partial D$ for $Q=Q(z)=\log\left(\frac{r_0e}{|z-z_0|}\right)$ with $p=2$.

Put

$$F(z)=\begin{cases}(f\circ h)(z)\,, & z\in B(z_0, r_0)\,,\\ (f(z)-z_0)/r_0\,, & z\in D\setminus B(z_0, r_0)\end{cases}.$$

Since f satisfies the relations (1.4)–(1.5) with $Q\equiv 1$, the mapping F satisfies the relations (1.4)–(1.5) in each point $b\in\partial D$ for $Q=Q(x)=\log\left(\frac{e}{|x|}\right)$ with $p=2$. Note that, Q satisfies the condition (1.10) for any point $b\in\partial D$, because Q is locally bounded in ∂D. Observe that, the mapping $F:D\rightarrow {\Bbb C}$ satisfies all conditions of Theorem 1.1. In particular, F is constructed so that it does not change the geometry of D. Besides that, F is open and discrete as a superposition of a homeomorphism g with some quasiregular mapping. Now, by Theorem 1.1 F has a continuous extension to any point $b\in\partial D$. All conditions regarding the geometry of a domain are also hold, see Example 1.

6 On the equicontinuity of families in the closure of a domain

For mappings with branching satisfying (1.4)–(1.5), we have obtained several different theorems on the equicontinuity of families of mappings in the closure of a domain. In particular, in [SevSkv$_1$] we are talking about families of mappings with a certain condition on the preimage of a continuum, the diameter of which does not decrease, and in [SevSkv$_2$] we assumed the presence of one normalization condition. In these papers, the mappings were assumed to be closed (=boundary preserving). Now we extend these results to mappings that are not closed.

Following to [Ri], a *condenser* in ${\Bbb R}^n$ is a pair $E=(A, C)$, where A is open in ${\Bbb R}^n$ and $C\ne\varnothing$ is a compact subset of A. If $1\leqslant p<\infty$, the *p-capacity of E* is defined by

$${\rm cap}_pE=\inf_u\int\limits_A|\nabla u|^p\,dm\,,$$

where the infimum is taken over all nonnegative functions u in ${\rm ACL}^p(A)$ with compact support in A and $u|_C\geqslant 1$. Let F be a compact set in ${\Bbb R}^n$. We say that F is of *p-capacity*

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

zero if $\operatorname{cap}_p(A, F) = 0$ for some (and hence for all) bounded open set $A \supset F$. An arbitrary set $E \subset \mathbb{R}^n$ is of *p-capacity zero* if the same is true for every compact subset of E. In this case we write $\operatorname{cap}_p E = 0$ ($\operatorname{cap} E = 0$ if $p = n$), otherwise $\operatorname{cap}_p E > 0$.

Given $p \geqslant 1$, $\delta > 0$, closed sets E, E_*, F in $\overline{\mathbb{R}^n}$, $n \geqslant 2$, a domain $D \subset \mathbb{R}^n$ and a Lebesgue measurable function $Q : D \to [0, \infty]$ let us denote by $\mathfrak{R}^{E_*, E, F}_{Q, \delta, p}(D)$ the family of all open discrete mappings $f : D \to \overline{\mathbb{R}^n} \setminus F$ satisfying the conditions (1.4)–(1.5) at the point $x_0 \in \overline{D}$ such that:

1) $C(f, \partial D) \subset E_*$,

2) for each component K of $D_f^{\prime} \setminus E_*$, $D_f^{\prime} := f(D)$, there is a continuum $K_f \subset K$ such that $h(K_f) \geqslant \delta$ and $h(f^{-1}(K_f), \partial D) \geqslant \delta > 0$,

3) $f^{-1}(E_*) \subset E$.

Theorem 6.1. *Let $p \geqslant 1$, let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$. Assume that:*

1) the set E is nowhere dense in D, and D is finitely connected on E, i.e., for any $z_0 \in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components;

2) for any $x_0 \in \partial D$ there is $m = m(x_0) \in \mathbb{N}$, $1 \leqslant m < \infty$ such that the following is true: for any neighborhood U of x_0 there is a neighborhood $V \subset U$ of x_0 and such that:

2a) $V \cap D$ is connected,

2b) $(V \cap D) \setminus E$ consists at most of m components.

Let for $p = n$ the set F have positive capacity, and for $n - 1 < p < n$ it is an arbitrary closed set.

Suppose that, for any $x_0 \in \partial D$ at least one of the following conditions is satisfied: 3_1) a function Q has a finite mean oscillation at x_0; 3_2) $q_{x_0}(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right)$ as $r \to 0$; 3_3) the condition

$$\int\limits_0^{\delta(x_0)} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\prime\,\frac{1}{p-1}}(t)} = \infty$$

holds for some $\delta(x_0) > 0$, where $q_{x_0}^{\prime}(t)$ is defined in (1.8).

Let the family of all components of $D_f^{\prime} \setminus E_*$ is equi-uniform over $f \in \mathfrak{R}^{E_*, E, F}_{Q, \delta, p}(D)$ with respect to p-modulus. Then every $f \in \mathfrak{R}^{E_*, E, F}_{Q, \delta, p}(D)$ has a continuous extension to ∂D and the family $\mathfrak{R}^{E_*, E, F}_{Q, \delta, p}(\overline{D})$, consisting of all extended mappings $\overline{f} : \overline{D} \to \overline{\mathbb{R}^n}$, is equicontinuous in $\overline{D}$.

The proof of Theorem 1.1 is based on the following lemma.

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

Lemma 6.1. *Assume that, under conditions of Theorem 6.1 we replace the assumptions* $3_1)$–$3_3)$ *by the following assumption: Suppose that, for any* $x_0\in \partial D$ *there is* $\varepsilon_0=\varepsilon_0(x_0)>0$ *and some positive measurable function* $\psi:(0, \varepsilon_0)\rightarrow (0,\infty)$ *such that*

$$0< I(\varepsilon, \varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt < \infty \tag{6.1}$$

for sufficiently small $\varepsilon\in(0, \varepsilon_0)$ *and, in addition,*

$$\int\limits_{A(\varepsilon, \varepsilon_0, x_0)} Q(x)\cdot\psi^{\,p}(|x-x_0|)\,dm(x) = o(I^p(\varepsilon, \varepsilon_0))\,, \tag{6.2}$$

where $A:=A(x_0, \varepsilon, \varepsilon_0)$ *is defined in (1.3).*

Let the family of all components of $D^{\,\prime}_f\setminus E_*$ *is equi-uniform over* $f\in\mathfrak{R}_{Q,\delta,p}^{E_*, E, F}(D)$ *with respect to p-modulus. Then every* $f\in\mathfrak{R}_{Q,\delta,p}^{E_*, E, F}(D)$ *has a continuous extension to* ∂D *and the family* $\mathfrak{R}_{Q,\delta,p}^{E_*, E, F}(\overline{D})$, *consisting of all extended mappings* $\overline{f}:\overline{D}\rightarrow \overline{{\Bbb R}^n}$, *is equicontinuous in* $\overline{D}$.

Proof. The equicontinuity of $\mathfrak{R}_{Q,\delta,p}^{E_*, E, F}(D)$ in D follows by [SalSev, Lemma 4.2] for the case of $p=n$ and [GSS, Lemma 2.4] for $n-1<p<n$.

Since any component of $D^{\,\prime}_f\setminus E_*$ is uniform with respect to p-modulus, it has strongly accessible boundary with respect to p-modulus (see [SevSkv$_1$, Remark 1]). Now the possibility of continuous extension of any $f\in\mathfrak{R}_{Q,\delta,p}^{E_*, E, F}(D)$ to ∂D follows from Lemma 3.1 and Remark 4.1. It remains to prove the equicontinuity of the extended family $\mathfrak{R}_{Q,\delta,p}^{E_*, E, F}(\overline{D})$ in ∂D.

Suppose the opposite. Then there is $x_0\in \partial D$, $\varepsilon_0>0$, a sequence $x_k\rightarrow x_0$, $x_k\in \overline{D}$, and $f_k\in \mathfrak{R}_{Q,\delta,p}^{E_*, E, F}(\overline{D})$ such that

$$h(f_k(x_k), f_k(x_0))\geqslant \varepsilon_0\,. \tag{6.3}$$

Since f_k has a continuous extension to ∂D, we may consider that $x_k\in D$. In addition, it follows from (6.3) that we may find a sequence $x^{\,\prime}_k\in D$, $k=1,2,\ldots$, such that $x^{\,\prime}_k\rightarrow x_0$ and

$$h(f_k(x_k), f_k(x^{\,\prime}_k))\geqslant \varepsilon_0/2\,. \tag{6.4}$$

By the assumption 2), there exists a sequence of neighborhoods $V_k\subset B(x_0, 2^{-k})$, $k=1,2,\ldots$, such that $V_k\cap D$ is connected and $(V_k\cap D)\setminus E$ consists of m components, $1\leqslant m<\infty$.

We note that the points x_k and $x^{\,\prime}_k$, $k=1,2,\ldots$, may be chosen such that $x_k, x^{\,\prime}_k\not\in E$. Indeed, since under condition 1) the set E is nowhere dense in D, there exists a sequence $x_{ki}\in D\setminus E$, $i=1,2,\ldots$, such that $x_{ki}\rightarrow x_k$ as $i\rightarrow\infty$. Put $\varepsilon>0$. Due to the continuity of the mapping f_k at the points x_k and $x^{\,\prime}_k$, for the number $k\in {\Bbb N}$ there are numbers $i_k, j_k\in {\Bbb N}$ such that $h(f_k(x_{ki_k}), f_k(x_k))<\frac{1}{2^k}$ and $h(f_k(x^{\,\prime}_{kj_k}), f_k(x^{\,\prime}_k))<\frac{1}{2^k}$. Now, by (6.4) and by the triangle inequality,

$$h(f_k(x_{ki_k}), f_k(x^{\,\prime}_{kj_k}))\geqslant h(f_k(x^{\,\prime}_k), f_k(x_{ki_k}))-h(f_k(x^{\,\prime}_k), f_k(x^{\,\prime}_{kj_k}))\geqslant$$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

$$\geqslant h(f_k(x_k), f_k(x'_k)) - h(f_k(x_k), f_k(x_{ki_k})) - h(f_k(x'_k), f_k(x'_{kj_k})) \geqslant \tag{6.5}$$
$$\geqslant \varepsilon_0/2 - \frac{2}{2^k} \geqslant \varepsilon_0/4$$

for sufficiently large k. Due to (6.5), we may assume that $x_k, x'_k \notin E$, as required.

Now, by Lemma 3.1 there are subsequences x_{k_l} and x'_{k_l}, $l = 1, 2, \ldots$, belonging to some sequence of neighborhoods V_l, $l = 1, 2, \ldots$, of the point x_0 such that $\operatorname{diam} V_l \to 0$ as $l \to \infty$ and, in addition, any pair x_{k_l} and x'_{k_l} may be joined by a path γ_l in $V_l \cap D$, where γ_l contains at most $m-1$ points in E. Without loss of generality, we may assume that the same sequences x_k and y_k satisfy properties mentioned above. Let $\gamma_k : [0, 1] \to D$, $\gamma_k(0) = x_k$ and $\gamma_k(1) = y_k$, $k = 1, 2, \ldots$.

Observe that, the path $f_k(\gamma_k)$ contains not more than $m - 1$ points in E_*. In the contrary case, there are at least m such points $b_1 = f_k(\gamma_k(t_1)), b_2 = f_k(\gamma_k(t_2)), \ldots, b_m = f_k(\gamma_k(t_m))$, $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m \leqslant 1$. But now the points $a_1 = \gamma_k(t_1), a_2 = \gamma_k(t_2), \ldots, a_m = \gamma_k(t_m)$ are in $E = f^{-1}(E_*)$ and simultaneously belong to γ_k. This contradicts the definition of γ_k.

Let

$$b_1 = f_k(\gamma_k(t_1)), b_2 = f_k(\gamma_k(t_2)) \quad , \ldots , \quad b_l = f_k(\gamma_k(t_l)) \,,$$
$$t_0 := 0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_l \leqslant 1 = t_{l+1}, \qquad 0 \leqslant l \leqslant m - 1 \,,$$

be points in $f_k(\gamma_k) \cap E_*$. By the relation (6.4) and due to the triangle inequality,

$$\varepsilon_0/2 \leqslant h(f_k(x_k), f_k(y_k)) \leqslant \sum_{r=0}^{l} h(f_k(\gamma_k(t_r)), f_k(\gamma_k(t_{r+1}))) \,. \tag{6.6}$$

It follows from (6.6) that, there is $1 \leqslant r = r(k) \leqslant m - 1$ such that such that

$$h(f(\gamma_k(t_{r(k)})), f_k(\gamma_k(t_{r(k)+1}))) \geqslant \varepsilon_0/(2(l + 1)) \geqslant \varepsilon_0/(2m) \,. \tag{6.7}$$

Observe that, the set $G_k := |\gamma_k|_{(t_{r(k)}, t_{r(k)+1})}|$ belongs to $D' \setminus E$.

Observe that, G_k contains some a continuum $\widetilde{G}_k$ with $h(\widetilde{G}_k) \geqslant \varepsilon_0/4m$ for any $k \in \mathbb{N}$. Indeed, by the construction of G_k, there is a sequence of points $a_{s,k} := f_k(\gamma_k(p_s)) \to w_k := f(\gamma_k(t_{r(k)}))$ as $s \to \infty$ and $b_{s,k} = f_k(\gamma_k(q_s)) \to f(\gamma_k(t_{r(k)+1}))$ as $s \to \infty$, where $t_{r(k)} < p_s < q_s < t_{r(k)+1}$ and $a_{s,k}, b_{s,k} \in G_k$. By the triangle inequality and by (6.7)

$$h(a_{s,k}, b_{s,k}) \geqslant h(b_{s,k}, w_k) - h(w_k, a_{s,k}) \geqslant$$
$$h(f_k(\gamma_k(t_{r(k)+1})), w_k) - h(f_k(\gamma_k(t_{r(k)+1})), b_{s,k}) - h(w_k, a_{s,k}) \geqslant$$
$$\varepsilon_0/(2m) - h(f_k(\gamma_k(t_{r(k)+1}), b_{s,k})) - h(w_k, a_{s,k}) \,.$$

Since $a_{s,k} := f_k(\gamma_k(p_s)) \to w_k := f_k(\gamma_k(t_{r(k)}))$ as $s \to \infty$ and $b_{s,k} = f_k(\gamma_k(q_s)) \to f_k(\gamma_k(t_{r(k)+1}))$ as $s \to \infty$, it follows from the last inequality that there is $s = s(k) \in \mathbb{N}$ such that

$$h(a_{s(k),k}, b_{s(k),k}) \geqslant \varepsilon_0/(4m) \,.$$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

Now, we set

$$\widetilde{G}_k := f_k(\gamma_k)|_{[p_{s(k)},q_{s(k)}]}\,.$$

In other words, $\widetilde{G}_k$ is a part of the path $f_k(\gamma_k)$ between points $a_{s(k),k}$ and $b_{s(k),k}$. Since $\widetilde{G}_k$ is a continuum in $D^{\prime}_{f_k}\setminus E_*$, there is a component K_k of $D^{\prime}_{f_k}\setminus E_*$, containing K_k. Let us apply the definition of equi-uniformity for he sets $\widetilde{G}_k$ and K_{f_k} in K_k (here K_{f_k} is a continuum from the definition of the class $\mathfrak{R}_{Q,\delta,p}^{E_*,E,F}(D)$, in particular, $h(K_{f_k})\geqslant\delta$). Due to this definition, for the number $\delta_*:=\min\{\delta,\varepsilon/4m\}>0$ there is $P>0$ such that

$$M_p(\Gamma(\widetilde{G}_k, K_{f_k}, K_k))\geqslant P>0\,,\qquad k=1,2,\ldots\,. \tag{6.8}$$

Let us to show that, the relation (6.8) contradicts with the definition of f in (1.4) together with the conditions (6.1)–(6.2). Indeed, let us denote by Γ_k the family of all half-open paths $\beta_k:[a,b)\rightarrow\overline{{\mathbb R}^n}$ such that $\beta_k(a)\in\widetilde{G}_k$, $\beta_k(t)\in K_k$ for all $t\in[a,b)$ and, moreover, $\lim\limits_{t\rightarrow b-0}\beta_k(t):=B_k\in K_{f_k}$. Obviously, by (6.8)

$$M_p(\Gamma_k)=M_p(\Gamma(\widetilde{G}_k, K_{f_k}, K_k))\geqslant P>0\,,\qquad k=1,2,\ldots\,. \tag{6.9}$$

Consider the family $\Gamma^{\prime}_k$ of all maximal f_k-liftings $\alpha_k:[a,c)\rightarrow D$ of the family Γ_k starting at $|\gamma_k|$; such a family exists by Proposition 3.1.

Observe that, the situation when $\alpha_k\rightarrow\partial D$ as $k\rightarrow\infty$ is impossible. Suppose the opposite: let $\alpha_k(t)\rightarrow\partial D$ as $t\rightarrow c$. Let us choose an arbitrary sequence $\varphi_m\in[0,c)$ such that $\varphi_m\rightarrow c-0$ as $m\rightarrow\infty$. Since the space $\overline{{\mathbb R}^n}$ is compact, the boundary ∂D is also compact as a closed subset of the compact space. Then there exists $w_m\in\partial D$ such that

$$h(\alpha_k(\varphi_m),\partial D)=h(\alpha_k(\varphi_m),w_m)\rightarrow 0\,,\qquad m\rightarrow\infty\,. \tag{6.10}$$

Due to the compactness of ∂D, we may assume that $w_m\rightarrow w_0\in\partial D$ as $m\rightarrow\infty$. Therefore, by the relation (6.10) and by the triangle inequality

$$h(\alpha_k(\varphi_m),w_0)\leqslant h(\alpha_k(\varphi_m),w_m)+h(w_m,w_0)\rightarrow 0\,,\qquad m\rightarrow\infty\,. \tag{6.11}$$

On the other hand,

$$f_k(\alpha_k(\varphi_m))=\beta_k(\varphi_m)\rightarrow\beta(c)\,,\quad m\rightarrow\infty\,, \tag{6.12}$$

because by the construction the path $\beta_k(t)$, $t\in[a,b]$, lies in $K_k\subset D^{\prime}_{f_k}\setminus E_*$ together with its finite ones points. At the same time, by (6.11) and (6.12) we have that $\beta_k(c)\in C(f_k,\partial D)\subset E_*$ by the definition of the class $\mathfrak{R}_{Q,\delta,p}^{E_*,E,F}(D)$. The inclusions $\beta_k\subset D^{\prime}\setminus E_*$ and $\beta_k(c)\in E_*$ contradict each other.

Therefore, by Proposition 3.1 $\alpha_k\rightarrow x_1\in D$ as $t\rightarrow c-0$, and c_b and $f_k(\alpha_k(b))=f_k(x_1)$. In other words, the f_k-lifting α_k is complete, i.e., $\alpha_k:[a,b]\rightarrow D$. Besides that, it follows from that $\alpha_k(b)\in f_k^{-1}(K_{f_k})$.

Again, by the definition of the class $\mathfrak{R}_{Q,\delta,p}^{E_*,E,F}(D)$,

$$h(f_k^{-1}(K_{f_k}),\partial D)\geqslant\delta>0\,. \tag{6.13}$$

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

Since $x_0\ne\infty$, it follows from (6.13) that

$$f_k^{-1}(K_{f_k})\subset D\setminus B(x_0, r_0) \tag{6.14}$$

for any $k\in\mathbb{N}$ and some $r_0>0$. Let k be such that $2^{-k}<\varepsilon_0$. We may consider that $\varepsilon_0<r_0$. Due to (6.14), we may show that

$$\Gamma_k^{\prime}>\Gamma(S(x_0, 2^{-k}), S(x_0, \varepsilon_0), D) \tag{6.15}$$

(see [Ku, Theorem 1.I.5, §46]). Observe that the function

$$\eta_k(t)=\left\{\begin{array}{rr} \psi(t)/I(2^{-k}, \varepsilon_0), & t\in (2^{-k}, \varepsilon_0),\\ 0, & t\in {\mathbb R}\setminus (2^{-k}, \varepsilon_0)\,, \end{array} \right.$$

where $I(2^{-k}, \varepsilon_0)$ is defined in (3.1), satisfies (1.5) for $r_1:=2^{-k}$, $r_2:=\varepsilon_0$. Therefore, by the definition of the mapping in (1.4) and by (6.15) we obtain that

$$M_p(f(\Gamma_k^{\prime}))\leqslant \Delta(k)\,, \tag{6.16}$$

where $\Delta(k)\rightarrow 0$ as $k\rightarrow\infty$. However, $\Gamma_k=f_k(\Gamma_k^{\prime})$. Thus, by (6.16), (6.1)–(6.2) we obtain that

$$M_p(\Gamma_k)=M_p(f_k(\Gamma_k^{\prime}))\leqslant \Delta(k)\rightarrow 0 \tag{6.17}$$

as $k\rightarrow\infty$. However, the relation (6.17) together with the inequality (6.9) contradict each other, which proves the lemma. □

Proof of Theorem 6.1 directly follows from Lemmas 6.1 and 3.2. □

References

[Cr] Cristea, M.: Open discrete mappings having local ACL^n inverses. - Complex Variables and Elliptic Equations 55: 1–3, 2010, 61–90.

[Fe] Federer, H.: Geometric Measure Theory. - Springer, Berlin etc., 1969.

[GSS] Golberg, A., R. Salimov and E. Sevost'yanov: Normal Families of Discrete Open Mappings with Controlled p-Module. - Contemporary Mathematics 667, 2016, 83–103.

[GRSY] Gutlyanskii V., V. Ryazanov, U. Srebro, E. Yakubov: The Beltrami equation. A geometric approach. Developments in Mathematics, 26. - Springer, New York, 2012, xiv+301 pp., ISBN: 978-1-4614-3190-9.

[IR] Ignat'ev, A., V. Ryazanov: To the theory of the boundary behavior of space mappings. - Ukr. Math. Bull. 3:2, 2006, 189—201.

[Ku] Kuratowski, K.: Topology, v. 2. – Academic Press, New York–London, 1968.

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press

[KM] Koskela, P., O. Martio: Removability theorems for quasiregulat mappings. - Annales Academia Scientiarum Fennicre Series A. I. Mathematica Volumen 15, 1990, 381–399.

[MRV_1] Martio, O., S. Rickman, and J. Väisälä: Distortion and singularities of quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A1. 465, 1970, 1–13.

[MRV_2] Martio, O., S. Rickman, and J. Väisälä: Topological and metric properties of quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A1. 488, 1971, 1–31.

[$MRSY_1$] Martio, O., V. Ryazanov, U. Srebro, and E. Yakubov: On Q-homeomorphisms. - Ann. Acad. Sci. Fenn. Ser. A1 30:1, 2005, 49–69.

[$MRSY_2$] Martio, O., V. Ryazanov, U. Srebro, and E. Yakubov: Moduli in modern mapping theory. - Springer Science + Business Media, LLC, New York, 2009.

[MSa] Martio, O., J. Sarvas: Injectivity theorems in plane and space. - Ann. Acad. Sci. Fenn. Ser. A1 Math. 4, 1978/1979, 384–401.

[Na_1] Näkki, R.: Boundary behavior of quasiconformal mappings in n-space. - Ann. Acad. Sci. Fenn. Ser. A. 484, 1970, 1–50.

[Na_2] Näkki, R.: Continuous boundary extension of quasiconformal mappings. - Ann. Acasd. Sci. Fenn. Ser. A I. Math. 511, 1970, 1–10.

[Na_3] Näkki, R.: Extension of Loewner's capacity theorem. - Trans. Amer. Math. Soc. 180, 1973, 229–236.

[Pol] Poleckii, E.A.: The modulus method for nonhomeomorphic quasiconformal mappings. - Mat. Sb. 83(125):2(10), 1970, 261—272.

[PSS] Petkov, I., R. Salimov, M. Stefanchuk, Nonlinear Beltrami equation: Lower estimates of Schwarz lemma's type. - 67:3, 2024, 533—543.

[Ra] Rajala, K.: Mappings of finite distortion: removability of Cantor sets. - Annales Academiæ Scientiarum Fennicæ Mathematica 29, 2004, 269–281.

[Ri] Rickman, S.: Quasiregular mappings. Springer-Verlag, Berlin, 1993.

[RS] Ryazanov, V.I. and R.R. Salimov: Weakly flat spaces and bondaries in the mapping theory . - Ukr. Math. Bull. 4:2, 2007, 199–233.

[RV] Ryazanov, V., S. Volkov: On the Boundary Behavior of Mappings in the Class $W^{1,1}_{loc}$ on Riemann Surfaces. - Complex Analysis and Operator Theory 11, 2017, 1503–1520.

[SalSev] Salimov, R.R. and E.A. Sevost'yanov: The theory of shell-based Q-mappings in geometric function theory. - Sb. Math. 201:6, 2010, 909–934.

[SalSt] Salimov, R.R., M.V. Stefanchuk, Functional Asymptotics of Solutions of the Nonlinear Cauchy–Riemann–Beltrami System. - Journal of Mathematical Sciences (United States) 277:2, 2023, 311–328.

[Sev_1] Sevost'yanov, E.A: Boundary extensions of mappings satisfying the Poletsky inverse in terms of prime ends. - Ukrainian Mathematical Journal 73:7, 2021,1107-1121.

[Sev_2] Sevost'yanov, E.A: On boundary discreteness of mappings with a modulus conditions. - Acta Mathematica Hungarica 171:1, 2023, 67–87.

[Sev_3] Sevost'yanov, E.A: On boundary behavior of the open discrete mappings with unbounded characteristics. - Ukr. Mat. J. 64: 6, 2012, 979-984.

[Sev_4] Sevost'yanov, E.A: Mappings with Direct and Inverse Poletsky Inequalities. Developments in Mathematics (DEVM, volume 78). - Springer Nature Switzerland AG, Cham, 2023.

[SevSkv_1] Sevost'yanov, E.A., S.A. Skvortsov: On the equicontinuity of families of mappings in the case of variable domains. - Ukrainian Mathematical Journal 71:7, 2019, 1071–1086.

[SevSkv_2] Sevost'yanov, E.A., S.A. Skvortsov: Equicontinuity of Families of Mappings with One Normalization Condition. - Math. Notes 109:4, 2021, 614–622.

[SSD] Sevost'yanov, E.A., S.O. Skvortsov, O.P. Dovhopiatyi: On nonhomeomorphic mappings with the inverse Poletsky inequality. - Journal of Mathematical Sciences 252:4, 2021, 541–557.

[Sm] Smolovaya, E.S.: Boundary behavior of ring Q-homeomorphisms in metric spaces. - Ukrainian Math. J. 62:5, 2010, 785—793.

[Sr] Srebro, U.: Conformal capacity and quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I. Math. 529, 1973, 1–8.

[Va] Väisälä J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin etc., 1971.

[Vu] Vuorinen, M.: Exceptional sets and boundary behavior of quasiregular mappings in n-space. - Ann. Acad. Sci. Fenn. Ser. A 1. Math. Dissertationes 11, 1976, 1–44.

Victoria Desyatka
Zhytomyr Ivan Franko State University,
40 Velyka Berdychivs'ka Str., 10 008 Zhytomyr, UKRAINE
victoriazehrer@gmail.com

Evgeny Sevost'yanov
1. Zhytomyr Ivan Franko State University,
40 Velyka Berdychivs'ka Str., 10 008 Zhytomyr, UKRAINE
2. Institute of Applied Mathematics and Mechanics
of NAS of Ukraine,
19 Henerala Batyuka Str., 84 116 Slov'yans'k, UKRAINE
esevostyanov2009@gmail.com

https://doi.org/10.4153/S0008439525000141 Published online by Cambridge University Press