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ON THE AREA GROWTH OF A HYPERBOLIC SURFACE

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We conjecture that if the rate of area growth of a geodesic disc of radius r on a smooth simply-connected complete surface with non-positive Gaussian curvature is faster than $r^2(\log r)^{1+\varepsilon}$ for some $\varepsilon > 0$, then the surface is hyperbolic. We prove this under an additional assumption that the surface is rotationally symmetric.

In this note we consider a smooth surface M with a smooth Riemannian metric ds^2. We assume that M is simply-connected complete and its Gaussian curvature K satisfies $K \leqslant 0$. Therefore M is diffeomorphic to the plane. Through the existence of isothermal coordinates M becomes a Riemann surface, Hicks [**2**, p.139]. Hence, from function theory, we know that M is either parabolic (that is conformally equivalent to the plane) or hyperbolic (that is conformally equivalent to the open unit disc). It is natural to ask how does the metric property of the surface influence its function theoretic property. Let us fix a point on M as origin and introduce geodesic polar coordinates (r, θ). Then we know the following:

THEOREM 1. (Greene and Wu [**1**, p.120])

(i) *If $K(r, \theta) \geqslant (-1)/(r^2 \log r)$ for $r \geqslant$ some constant and for all θ in $[0, 2\pi]$, then M is parabolic.*

(ii) *If $K(r, \theta) \leqslant (-(1+\varepsilon))/(r^2 \log r)$ for a constant $\varepsilon > 0$, for $r \geqslant$ some constant and for all θ in $[0, 2\pi]$, then M is hyperbolic.*

Remark. The above theorem was first proved by Milnor under the additional assumption that M is rotationally symmetric (Milnor [**4**, p.43]).

Let us write $A(r)$ for the area of the geodesic disc of radius r centred at the origin. Then we know the following:

THEOREM 2. (Karp [**3**, p.545]) *Let $\phi\colon [0, \infty) \to [0, \infty)$ denote any non-decreasing function which satisfies $\int_a^\infty (dr)/(r\phi(r)) = \infty$ for some $a > 0$. If $A(r)$ satisfies*

$$A(r) \leqslant cr^2\phi(r)$$

for some constant c and for $r \geqslant$ some constant, then M is parabolic.

For example if $A(r) \leqslant cr^2 \log r$ for $r \geqslant$ some constant, then M is parabolic.

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Remark. (Karp [**3**, p.544]) Since $K(r,\theta) \geqslant -1/(r^2\log r)$ implies that $A(r) \leqslant cr^2\log r$, Karp's theorem generalises Theorem 1, (i).

Now we observe the following:

PROPOSITION. *If $K(r,\theta) \leqslant -(1+2\varepsilon)/(r^2\log r)$ for $\varepsilon > 0$ and for $r \geqslant$ some constant and any θ in $[0, 2\pi]$, then*

$$A(r) \geqslant cr^2(\log r)^{1+\varepsilon}$$

for some constant $c > 0$ and for $r \geqslant$ some constant.

PROOF: Write our metric in geodesic polar coordinates as $dr^2 + g(r,\theta)^2 d\theta^2$ and let $f(r) = r(\log r)^{1+\varepsilon}$. Then the Gaussian curvature is given by (Struik [**5**, p.138]) $K(r,\theta) = -g_{rr}/g$. We have for large r, say $r \geqslant a$,

$$\begin{aligned} -\frac{f''}{f} &= -\frac{(1+\varepsilon)(\varepsilon+\log r)}{r^2(\log r)^2} \\ &\geqslant -\frac{(1+2\varepsilon)}{r^2\log r} \geqslant -\frac{g_{rr}}{g} \end{aligned}$$

and so $gf'' \leqslant fg_{rr}$ for $r \geqslant a$. After multiplying g by a large constant if necessary, we may assume that $g(a,\theta) > f(a)$ and $g_r(a,\theta) > f'(a)$ for all θ in $[0, 2\pi]$. We now claim that $g(r,\theta) > f(r)$ for all $r \geqslant a$ and all θ in $[0, 2\pi]$. For suppose b is the smallest number $> a$ such that $g(b,\alpha) \leqslant f(b)$ for some α in $[0, 2\pi]$, then we will have $g(r,\alpha) \geqslant f(r)$ for all $a \leqslant r \leqslant b$ and hence $f''(r) \leqslant g_{rr}(r,\alpha)$ for all $a \leqslant r \leqslant b$. Integrate this from a to r for r in $[a, b]$ and using $g_r(a,\alpha) > f'(a)$, we obtain $g_r(r,\alpha) \geqslant f'(r)$ for any r in $[a, b]$. Hence, integrating this last inequality from a to b and using $g(a,\alpha) > f(a)$, we have $g(b,\alpha) > f(b)$, which is a contradiction. Therefore we have for $r \geqslant \max\{\sqrt{\pi}a, e^{\pi(1+\varepsilon)}\}$,

$$\begin{aligned} A(r) &= \int_0^{2\pi}\int_0^r g(r,\theta)\,dr\,d\theta \\ &\geqslant 2\pi\int_a^r f(r)\,dr \\ &= \pi r^2(\log r)^{1+\varepsilon} - \pi a^2(\log a)^{1+\varepsilon} - \pi z(1+\varepsilon)(\log z)^{\varepsilon}(r-a) \\ &\qquad\qquad \text{for some } z \in (a, r) \\ &\geqslant (\pi-2)r^2(\log r)^{1+\varepsilon} + (r^2 - \pi a^2)(\log a)^{1+\varepsilon} \\ &\qquad\qquad + r^2(\log r)^{\varepsilon}(\log r - \pi(1+\varepsilon)) \\ &\geqslant (\pi-2)r^2(\log r)^{1+\varepsilon}. \end{aligned}$$

∎

Hence it is natural to conjecture the following:

Conjecture A. Let $f\colon [0, \infty) \to [0, \infty)$ denote any non-decreasing function which satisfies $\int_a^\infty (dr)/(rf(r)) < \infty$ for some $a > 0$. If $A(r) \geqslant cr^2 f(r)$ for some constant $c > 0$ and for $r \geqslant$ some constant, then M is hyperbolic.

In particular, we have

Conjecture A′. If $A(r) \geqslant cr^2(\log r)^{1+\epsilon}$ for some constant $\epsilon > 0$ and $c > 0$ and for $r \geqslant$ some constant, then M is hyperbolic.

From now on we will assume in addition that M is rotationally symmetric so that $g(r, \theta) = g(r)$ is independent of θ. Then Milnor proved the following:

THEOREM 3. (Milnor [**4**, p.43]) *Let M be a complete, non-compact, simply-connected smooth surface with a rotationally symmetric Riemannian metric $ds^2 = dr^2 + g(r)^2 d\theta^2$. Then M is hyperbolic if and only if $\int_a^\infty (dr)/(g(r)) < \infty$ for some $a > 0$.*

We shall now prove the following:

THEOREM A. *Let M be a complete, non-compact, simply-connected smooth surface with a rotationally symmetric Riemannian metric $ds^2 = dr^2 + g(r)^2 d\theta^2$ such that $g(r)$ is an increasing function. Then M is hyperbolic if and only if $\int_a^\infty r/(A(r))dr < \infty$ for some $a > 0$.*

PROOF: Sufficiency:

Using the Mean Value Theorem, we have

$$\begin{aligned} A(r) &= \int_0^r 2\pi g(s)ds \\ &= 2\pi g(z)r \text{ for some } z \in (0, r) \\ &\leqslant 2\pi g(r)r \quad \text{(since } g \text{ is an increasing function).} \end{aligned}$$

Hence $(2\pi r)/(A(r)) \geqslant 1/(g(r))$ for $r > 0$.

Therefore $\int_a^\infty r/(A(r))dr < \infty$ implies that $\int_a^\infty 1/(g(r)) < \infty$ and so by Theorem 3, we conclude that M is hyperbolic.

Necessity:

Again using the Mean Value Theorem, we have

$$\begin{aligned} A(r) &= \int_0^r 2\pi g(s)ds \\ &\geqslant \int_{\frac{r}{2}}^r 2\pi g(s)ds \\ &= \pi g(z)r \text{ for some } z \in \left(\frac{r}{2}, r\right) \\ &\geqslant \pi g\left(\frac{r}{2}\right)r \text{ (since } g \text{ is an increasing function).} \end{aligned}$$

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Hence $(\pi r)/(A(r)) \leqslant 1/(g(r/2))$.

Therefore $\pi \int_a^\infty r/(A(r))dr \leqslant \int_a^\infty (dr)/(g(r/2)) = 2\int_{a/2}^\infty (dr)/(g(r))$.

If M is hyperbolic, then by Theorem 3 we have $\int_{a/2}^\infty (dr)/(g(r)) < \infty$ and hence $\int_a^\infty r/(A(r))dr < \infty$. ∎

Observe that if $K \leqslant 0$, then we have $g''(r) \geqslant 0$ for all $r \geqslant 0$ and so integrate from 0 to r we obtain $g'(r) \geqslant g'(0) = 1 > 0$ so that g is an increasing function. Therefore we have the following:

COROLLARY A. *Let M be a complete, simply-connected smooth surface with a smooth rotationally symmetric Riemannian metric and non-positive Gaussian curvature. Let $f\colon [0, \infty) \to [0, \infty)$ denote any non-decreasing function which satisfies $\int_a^\infty (dr)/(rf(r)) < \infty$ for some $a > 0$. If $A(r) \geqslant cr^2 f(r)$ for some constant $c > 0$ and for $r \geqslant$ some constant, then M is hyperbolic.*

In particular, we have

COROLLARY A$'$. *M as in Corollary A. If $A(r) \geqslant cr^2(\log r)^{1+\varepsilon}$ for some constants $\varepsilon > 0$ and $c > 0$ and for $r \geqslant$ some constant, then M is hyperbolic.*

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