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# The affirmative answer to Singer's conjecture on the algebraic transfer of rank four

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During the last decades, the structure of mod-2 cohomology of the Steenrod ring $\mathscr{A}$ became a major subject in Algebraic topology. One of the most direct attempt in studying this cohomology by means of modular representations of the general linear groups was the surprising work [*Math. Z.* **202** (1989), 493–523] by William Singer, which introduced a homomorphism, the so-called *algebraic transfer*, mapping from the coinvariants of certain representation of the general linear group to mod-2 cohomology group of the ring $\mathscr{A}$. He conjectured that this transfer is a monomorphism. In this work, we prove Singer's conjecture for homological degree 4.

*Keywords:* Adams spectral sequences; primary cohomology operations; Steenrod algebra; lambda algebra; hit problem; stable homotopy of spheres; algebraic transfer

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## 1. Introduction

Everywhere in the text of this article, we will be working over the field $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ of characteristic 2 and taking (co)homology with coefficients in $\mathbb{F}_2$. It is well-known that the calculation of the stable homotopy groups of spheres $\pi_*^S(\mathbb{S}^0)$ is one of the most central and intractable problems in Algebraic topology. Historically, in the 1950s, Serre [24] used his spectral sequence to study this problem. In the late 1950s, Adams [1] constructed his celebrated spectral sequence that converges to $\pi_*^S(\mathbb{S}^0)$, completed at prime 2. He claimed that $E_2$-page of that spectral sequence could be identified with

$$\mathrm{Ext}^q_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = \{\mathrm{Ext}^{q,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = H^{q,t}(\mathscr{A})\}_{(q,t)\in\mathbb{Z}^2, q\geqslant 0,\, t\geqslant 0},$$

the bigraded cohomology algebra of the classical, singly-graded Steenrod algebra $\mathscr{A}$ over $\mathbb{F}_2$. This cohomology has been explicitly computed by Adem [3] for $q = 1$, by Adams [2] and Wall [31] for $q = 2$, by Adams [2] and Wang [32] for $q = 3$, by Lin [12] for $q = 4$, by Lin [12] and Chen [6] for $q = 5$. However, it is still largely mysterious for all $q > 5$. With an idea that we can study the structure of $\mathrm{Ext}^q_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ through the modular invariant theory, in 1989, W. Singer

[25] introduced a 'transfer' homomorphism of rank $q$, which passes from coinvariants of a certain representation of the general linear group $GL(q)$ over $\mathbb{F}_2$ to mod-2 cohomology of $\mathscr{A}$. It has been shown that this transfer is highly nontrivial (see the works by Boardman [4], Minami [13], Bruner, Hà and Hu'ng [5], Hu'ng [9], Hà [8], Nam [15], Hu'ng and Quỳnh [10], Cho'n and Hà [7], Sum [29], the author [17–23], and others). In order to better understand it, we offer some related issues. Let us denote by $V^{\oplus q} \cong \prod_{1\leqslant i\leqslant q}(\mathbb{Z}/2\mathbb{Z})$ a rank $q$ elementary abelian 2-group, which is considered as $q$-dimensional vector $\mathbb{F}_2$-space. It is known, $H^*(V^{\oplus q}) \cong S(V_*^{\oplus q})$, the symmetric algebra over the dual space $V_*^{\oplus q} \equiv H^1(V^{\oplus q})$ of $V^{\oplus q}$. Pick $u_1, \ldots, u_q$ to be a basis of $H^1(V^{\oplus q})$. Then, it has been shown that $\mathcal{P}_q := H^*(V^{\oplus q}) \cong \mathbb{F}_2[u_1, \ldots, u_q]$, the connected $\mathbb{Z}$-graded polynomial algebra on generators of degree 1, equipped with the canonical unstable algebra structure over $\mathscr{A}$. By dualizing, $H_*(V^{\oplus q}) \cong \Gamma(a_1, \ldots, a_q)$, the divided power algebra generated by $a_1, \ldots, a_q$, each of degree one, where $a_i \equiv a_i^{(1)}$ is dual to $u_i$. It is to be noted that this algebra and the polynomial algebra $\mathcal{P}_q$ are not in general isomorphic as $\mathbb{F}_2GL(q)$-modules. Now, let us recall that the algebra $\mathscr{A}$ consists of the Steenrod squaring operations $Sq^i$ for $i \geqslant 0$. The operations $Sq^0$ and $Sq^{2^i}$, $i \geqslant 0$, constitute a system of multiplicative generators for $\mathscr{A}$ (see also Walker and Wood [30]). Emphasizing that these $Sq^i$ are the cohomology operations satisfying the naturality property. Moreover, they commute with the suspension maps, and therefore, they are stable. Let $P_{\mathscr{A}}H_*(V^{\oplus q}) \cong \mathrm{Ext}^0_{\mathscr{A}}(\mathcal{P}_q, \mathbb{F}_2)$ be the subspace of $H_*(V^{\oplus q})$ consisting of all elements that are annihilated by all $Sq^i$ for $i > 0$. The group $GL(q)$ acts regularly on $V^{\oplus q}$ and therefore on $\mathcal{P}_q$ and $H_*(V^{\oplus q})$. This action commutes with that of the algebra $\mathscr{A}$ and so acts $\mathbb{F}_2 \otimes_{\mathscr{A}} \mathcal{P}_q$ and $P_{\mathscr{A}}H_*(V^{\oplus q})$. Singer [25] constructed a homomorphism from $P_{\mathscr{A}}H_n(V^{\oplus q})$ to $\mathrm{Ext}^{q,q+n}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$, which commutes with two $Sq^0$'s on $P_{\mathscr{A}}H_n(V^{\oplus q})$ and $\mathrm{Ext}^{q,q+n}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ (see also [4, 14]). He shows that this map factors through the quotient of its domain's $GL(q)$-coinvariants to give rise to the so-called *algebraic transfer of rank q*

$$Tr_q(\mathbb{F}_2) : \mathbb{F}_2 \otimes_{GL(q)} P_{\mathscr{A}}H_n(V^{\oplus q}) \longrightarrow \mathrm{Ext}^{q,q+n}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

In fact, this transfer is induced over the $E_2$-term of the Adams spectral sequence by the geometrical transfer map $\Sigma^\infty(B(V^{\oplus q})_+) \longrightarrow \Sigma^\infty(\mathbb{S}^0)$ between the suspension spectrum in stable homotopy category. Singer [25] demonstrated that the 'total' transfer $\{Tr_q(\mathbb{F}_2)\}_{q\geqslant 0}$ is an algebra homomorphism and that $Tr_q(\mathbb{F}_2)$ is an isomorphism for $q = 1, 2$. Afterwards, Boardman [4] stated that $Tr_3(\mathbb{F}_2)$ is also an isomorphism. Remarkably, in mostly all the decade 1980s, Singer believed that $Tr_q(\mathbb{F}_2)$ is an isomorphism for all $q$. However, in the rank 5 case, he himself claimed in [25] that it is not an isomorphism by showing that the indecomposable element $Ph_1 \in \mathrm{Ext}^{5,14}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ does not belong to the image of the transfer homomorphism, where $P$ denotes the Adams periodicity operator. Thence, he proposed the following yet-left open.

Conjecture 1.1 [25, Conj. 1.1]. *$Tr_q(\mathbb{F}_2)$ is a monomorphism for any $q$.*

As shown above, Singer's transfer is an isomorphism in ranks $\leqslant 3$, and so, conjecture 1.1 is true in these ranks. The rank 4 case is the subject of this paper.

Remarkably, the investigation of the image of the algebraic transfer of rank 4 was completed by the authors in [**5**, **8**, **9**, **15**] and [**10**]. More precisely, in [**5**], Bruner, Hà and Hu'ng proved that

Theorem 1.2 [**5**, Thm. 1.1]. *For each* $s \geqslant 1$, *the non-zero element* $g_s \in \mathrm{Ext}_{\mathscr{A}}^{4,12.2^s}(\mathbb{F}_2, \mathbb{F}_2)$ *is not in the image of* $Tr_4(\mathbb{F}_2)$.

This result refuted a prediction by Minami in [**14**] that the localization of the algebraic transfer

$$(Sq^0)^{-1}Tr_q(\mathbb{F}_2) : (Sq^0)^{-1}\mathbb{F}_2 \otimes_{GL(q)} P_{\mathscr{A}}H_n(V^{\oplus q}) \longrightarrow (Sq^0)^{-1}\mathrm{Ext}_{\mathscr{A}}^{q,q+n}(\mathbb{F}_2, \mathbb{F}_2)$$

given by inverting $Sq^0$ is an isomorphism. More explicitly, when $q = 4$ and $n = 12 \cdot 2^s - 4$, following [**5**, Cor. 1.2], the localization of the fourth transfer given by inverting $Sq^0$ is not an epimorphism. As a continuation of the work [**5**], Hu'ng proved in [**9**] that

Theorem 1.3 [**9**, Thm. 1.9]. *Any element in the* $Sq^0$*-families* $\{D_3(s) |\, s \geqslant 0\}$ *and* $\{p'_s |\, s \geqslant 0\}$ *does not belong to the image of* $Tr_4(\mathbb{F}_2)$.

Theorems 1.2 and 1.3 imply that there are infinitely many degrees in which the fourth algebraic transfer is not an isomorphism. Further, those theorems together with the following establish a prediction in Hu'ng [**9**, Conj. 1.10].

Theorem 1.4 [**8**, **10**, **15**]. *Every element in the following families belongs to the image of the fourth algebraic transfer* $Tr_4(\mathbb{F}_2)$*:*

(i) $\{d_s |\, s \geqslant 0\}$, $\{e_s |\, s \geqslant 0\}$ (see Hà [**8**, Thm. 1.1])*;*

(ii) $\{f_s |\, s \geqslant 0\}$ (see Nam [**15**, Thm. 1.4])*;*

(iii) $\{p_s |\, s \geqslant 0\}$ (see Hu'ng and Quỳnh [**10**, Thm. 1.1])*.*

Alternatively, since the total transfer $\{Tr_q(\mathbb{F}_2)\}_{q\geqslant 0}$ is a homomorphism of algebras and $Tr_q(\mathbb{F}_2)$ is an isomorphism for $q = 1, 2, 3$, all decomposable elements in $\mathrm{Ext}_{\mathscr{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ belong to the image of $Tr_4(\mathbb{F}_2)$.

Now, based on the results of [**5**, **8–10**, **15**] on the image of $Tr_4(\mathbb{F}_2)$, Singer's conjecture 1.1 for $Tr_4(\mathbb{F}_2)$ turns out to be equivalent to the following.

Conjecture 1.5 Stated by the referee.

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}}H_n(V^{\oplus 4}) = \begin{cases} \dim \mathrm{Ext}_{\mathscr{A}}^{4,4+n}(\mathbb{F}_2, \mathbb{F}_2) - 1 & \textit{if } n \textit{ is bad,} \\ \dim \mathrm{Ext}_{\mathscr{A}}^{4,4+n}(\mathbb{F}_2, \mathbb{F}_2) & \textit{if } n \textit{ is not bad.} \end{cases}$$

*Here* $n$ *is called bad if it equals to the stem of one element in the three families* $\{g_s |\, s \geqslant 1\}$, $\{D_3(s) |\, s \geqslant 0\}$ *and* $\{p'_s |\, s \geqslant 0\}$, *whose every element is not in the image of the algebraic transfer of rank four. Otherwise,* $n$ *is said to be not bad.*

Thus, by verifying this conjecture, we will get the answer for conjecture 1.1 on the fourth transfer. This will be presented in the sequel.

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The algebraic transfer we are describing is closely related to the *hit problem* in literature [16] of determination of a minimal generating set for the $\mathbb{F}_2$-algebra $\mathcal{P}_q$, considered as an unstable $\mathscr{A}$-module. The reader is familiar with an event that if $\mathbb{F}_2$ is an $\mathscr{A}$-module concentrated in degree 0, then solving the hit problem is equivalent to finding a basis consisting of all equivalence classes of homogeneous polynomials for the $\mathbb{Z}$-graded vector space over the field $\mathbb{F}_2$ :

$$Q\mathcal{P}_q := \mathbb{F}_2 \otimes_{\mathscr{A}} \mathcal{P}_q = \{(\mathbb{F}_2 \otimes_{\mathscr{A}} \mathcal{P}_q)_n\}_{n\in\mathbb{Z},\, n\geqslant 0} = \mathcal{P}_q / \sum_{i\geqslant 0} \mathrm{Im}(Sq^{2^i}) = \mathrm{Tor}_0^{\mathscr{A}}(\mathbb{F}_2, \mathcal{P}_q),$$

where the homogeneous components $(Q\mathcal{P}_q)_n := (\mathbb{F}_2 \otimes_{\mathscr{A}} \mathcal{P}_q)_n$ of degrees $n$ are $\mathbb{F}_2 GL(q)$-submodules of $Q\mathcal{P}_q$. Usually, one would investigate this tensor product. Its structure was systematically depicted by Peterson [16] for $q = 1, 2$, by Kameko's thesis [11] for $q = 3$, and by Sum [27, 28] for $q = 4$. So far it has been thoroughly studied for more than three decades by many topologist (see also [4, 5, 11, 17–19, 21, 23, 26–28, 30, 33]), but it remains unanswered for $q \geqslant 5$. We also emphasize that in general, it is not easy to compute or even estimate the dimension of $Q\mathcal{P}_q$ in each positive degree. Most notably, in his thesis [11], Kameko conjectured that an upper bound on the dimension of $(Q\mathcal{P}_q)_n$ is the order of the factor group of $GL(q)$ by the Borel subgroup $B_q$, i.e.,

$$\dim(Q\mathcal{P}_q)_n \leqslant \mathrm{ord}(GL(q)/B_q) = \frac{2^{q(q-1)/2}\prod_{1\leqslant j\leqslant q}(2^j-1)}{2^{q(q-1)/2}} = \prod_{1\leqslant j\leqslant q}(2^j-1),$$

for any $n \geqslant 0$. However, in 2010, the famous work [26] of Sum refuted the above prediction. In order to reduce the process of the calculation of $Q\mathcal{P}_q$ in each certain degree, one considers the arithmetic function $\mu : \mathbb{N} \to \mathbb{N}$, which is defined by

$$\mu(n) = \min\{k \in \mathbb{N} \,|\, n = \sum_{1\leqslant i\leqslant k}(2^{d_i}-1),\ d_i > 0, \forall i,\ 1 \leqslant i \leqslant k\},\ \text{for all } n \in \mathbb{N}.$$

THEOREM 1.6. *For each non-negative integer* $n$, *the following assertions are true:*

(i) $(Q\mathcal{P}_q)_n = 0$ *if and only if* $\mu(n) > q$ (see Peterson's conjecture [16], Wood [33])*;*

(ii) $(Q\mathcal{P}_q)_n \cong (Q\mathcal{P}_q)_{(n-q)/2}$ *if and only if* $\mu(n) = q$ (see Kameko's thesis [11]).

The statement (i) is given by Peterson's conjecture. Peterson himself confirmed it for $q \leqslant 2$. Afterwards, Wood proved the general case under a stronger form.

To close this section, we recall the already known results on $\mathrm{Ext}_{\mathscr{A}}^{q,*}(\mathbb{F}_2, \mathbb{F}_2)$ for $q \leqslant 4$.

THEOREM 1.7 [2, 3, 12, 31, 32]. *The following hold*:

(i) $\mathrm{Ext}_{\mathscr{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ *is generated by* $h_i$ *for* $i \geqslant 0$ (see Adem [3])*;*

(ii) $\mathrm{Ext}_{\mathscr{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$ *is generated by* $h_ih_j$ *for* $j \geqslant i \geqslant 0$ *and* $j \neq i+1$ (see Adams [2] and Wall [31])*;*

(iii) $\mathrm{Ext}^{3,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ *is generated by* $h_ih_jh_\ell$, $c_t$ *for* $t \geqslant 0$; $\ell \geqslant j \geqslant i \geqslant 0$, *and subject only to the relations* $h_ih_{i+1} = 0$, $h_ih^2_{i+2} = 0$ *and* $h^3_i = h^2_{i-1}h_{i+1}$ (see Adams [2] and Wang [32])*;*

(iv) $\mathrm{Ext}^{4,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ *is generated by* $h_ih_jh_\ell h_m$, $h_uc_v$, $d_t$, $e_t$, $f_t$, $g_{t+1}$, $p_t$, $D_3(t)$, $p'_t$ *for* $m \geqslant \ell \geqslant j \geqslant i \geqslant 0$, $u$, $v$, $t \geqslant 0$, *and subject to the relations in (iii) together with* $h^2_ih^2_{i+3} = 0$, $h_{v-1}c_v = 0$, $h_vc_v = 0$, $h_{v+2}c_v = 0$ *and* $h_{v+3}c_v = 0$ (see Lin [12])*.*

## 2. A solution to Singer's conjecture on the rank 4 transfer

As mentioned above, the goal of this section is to verify Singer's conjecture for the algebraic transfer of rank four. To make this, we prove conjecture 1.5. Firstly, let us recall that the domain of $Tr_4(\mathbb{F}_2)$ is isomorphic to the $GL(4)$-invariants space $(QP_4)^{GL(4)}_n$ as vector spaces for all $n$. By this and theorem 1.6, we shall compute the dimension of the domain of $Tr_4(\mathbb{F}_2)$ in the internal degrees $n$ satisfying $\mu(n) < q = 4$. In these cases, due to Sum [27], $n$ is of the following 'generic' forms:

$$\begin{array}{lll} (i) & n & = 2^{s+1} - t,\ t \in \{1, 2, 3\}; \\ (ii) & n & = 2^{s+t+1} + 2^{s+1} - 3; \\ (iii) & n & = 2^{s+t} + 2^s - 2; \\ (iv) & n & = 2^{s+t+u} + 2^{s+t} + 2^s - 3, \end{array} \tag{2.1}$$

whenever $s$, $t$, $u$ are the positive integers. We are now in a position to present our main result.

Main Theorem. *Let us consider generic degrees in* (2.1)*. Then, conjecture 1.5 is true in these degrees. Further,* $Tr_4(\mathbb{F}_2)$ *is an isomorphism in these internal degrees, except item (i) with* $(s, t) \in \{(6, 2), (5, 3)\}$, *item (ii) with* $(s, t) = (2, 3)$ *and item (iii) with* $(s, t) \in \{(4, 3), (1, 7)\}$*. In these items,* $Tr_4(\mathbb{F}_2)$ *is a monomorphism, but it is not an epimorphism.*

The theorem has been proved by Bruner, Hà and Hu'ng [5, Prop. 4.4] for item (iii) with $(s, t) = (1, 2)$, by Hu'ng [9, Thm. 7.3] for item (iii) with $(s, t) \in \{(1, 7), (4, 3)\}$, by Sum [29, Thm. 4.1] for item (i) and by the present author for items (ii), (iii) where $t \neq 3$ (see [20, 21]) and item (iv) (see [22]). It is remarkable that in [20, 21], we have proved the theorem for the cases $(s, t) \in \{(1, 2), (1, 7)\}$ by another method. Thus, we need only to prove the theorem for items (ii) and (iii) with $t = 3$. Note again that the case $(s, t) = (4, 3)$ has been proved by Hu'ng [9]. However, it will be proved in this paper using other techniques. Before going into detail, the known results will be briefly presented as well for the reader's convenience.

We first discuss the theorem for item (i), which has been proved by Sum [29].

*Case* $n := n_{s,\,t} = 2^{s+1} - t$, $1 \leqslant t \leqslant 3$, $s \geqslant 1$. According to Sum [29], the dimension of the domain of $Tr_4(\mathbb{F}_2)$ in degree $n_{s\,t}$ is determined by

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,\,t}}(V^{\oplus 4}) = \begin{cases} 0 & \text{if } t=1, \text{ and } s=1,\\ 1 & \text{if } t=1, \text{ and } s \geqslant 2,\\ 0 & \text{if } t=2, \text{ and } 1 \leqslant s \leqslant 2,\\ 1 & \text{if } t=2, \text{ and } s \geqslant 3,\\ 0 & \text{if } t=3, \text{ and } s \geqslant 1. \end{cases} \tag{2.2}$$

On the other side, by theorem 1.7, one gets

$$\mathrm{Ext}_{\mathscr{A}}^{4,4+n_{s,\,t}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 0 & \text{if } t=1, \text{ and } s=1,\\ \langle h_0^3 h_{s+1}\rangle & \text{if } t=1, \text{ and } s \geqslant 2,\\ 0 & \text{if } t=2, \text{ and } 1 \leqslant s \leqslant 2,\\ \langle d_0\rangle & \text{if } t=2, \text{ and } s=3,\\ \langle h_0^2 h_s^2\rangle & \text{if } t=2, \text{ and } s \geqslant 4, s \neq 6,\\ \langle h_0^2 h_6^2, D_3(1)\rangle & \text{if } t=2, \text{ and } s=6,\\ \langle D_3(0)\rangle & \text{if } t=3, \text{ and } s=5,\\ 0 & \text{if } t=3, \text{ and } s \geqslant 6. \end{cases} \tag{2.3}$$

We see that $n_{s,\,t}$ is bad for $(s,\,t) = (6,\,2),\,(5,\,3)$ and that by the equalities (2.2), (2.3), conjecture 1.5 holds for the degrees $n_{s,\,t}$ for $1 \leqslant t \leqslant 3$ and arbitrary positive integer $s$.

Notice that the non-zero elements $h_0^3 h_{s+1}$, $h_0^2 h_s^2$, and $h_0^2 h_6^2$ are decomposable in the fourth cohomology groups $\mathrm{Ext}_{\mathscr{A}}^{4,4+n_{s,\,t}}(\mathbb{F}_2,\,\mathbb{F}_2)$, and so they are in the image of $Tr_4(\mathbb{F}_2)$. It is well-known that by Hu'ng [9], the indecomposable elements $D_3(0)$ and $D_3(1)$ are not in the image of $Tr_4(\mathbb{F}_2)$ (see also theorem 1.3) and that by Hà [8], the indecomposable element $d_0$ is in the image of $Tr_4(\mathbb{F}_2)$ (see also theorem 1.4(i)). These results together with the equalities (2.2) and (2.3) show that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_{s,\,t}$, except the degrees $n_{6,2}$ and $n_{5,3}$. In the degrees $n_{6,2}$ and $n_{5,3}$, the fourth transfer is a monomorphism, but it is not an epimorphism, since

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{6,\,2}}(V^{\oplus 4}) = 1 < 2 = \dim \mathrm{Ext}_{\mathscr{A}}^{4,4+n_{6,\,2}}(\mathbb{F}_2, \mathbb{F}_2),$$
$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H n_{5,\,3}(V^{\oplus 4}) = 0 < 1 = \dim \mathrm{Ext}_{\mathscr{A}}^{4,4+n_{5,\,3}}(\mathbb{F}_2, \mathbb{F}_2).$$

Next, we discuss the theorem for items (ii), (iii) where $t \neq 3$ and item (iv). In item (iii), the cases $(s,\,t) = (1,\,2)$ and $(1,\,7)$ have been proved by Bruner, Hà and Hu'ng [5] and Hu'ng [9], respectively. The remaining cases have been proved by the present author in [20–22]. More precisely, with item (ii), we have the following case.

*Case* $n := n_{s,\,t} = 2^{s+t+1} + 2^{s+1} - 3,\ \ t \geqslant 1, t \neq 3, s \geqslant 1.$ Our previous works [20, 21] have shown that

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,\,t}}(V^{\oplus 4}) = \begin{cases} 1 & \text{if } t=1, \text{ and } s=1,\\ 1 & \text{if } t=1, \text{ and } s \geqslant 2,\\ 1 & \text{if } t=2, \text{ and } s=1,\\ 0 & \text{if } t=2, \text{ and } s \geqslant 2,\\ 1 & \text{if } t \geqslant 4, \text{ and } 1 \leqslant s \leqslant 2,\\ 2 & \text{if } t \geqslant 4, \text{ and } s \geqslant 3. \end{cases} \tag{2.4}$$

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On the other hand, from theorem 1.7, we get

$$\mathrm{Ext}_{\mathscr{A}}^{4,4+n_{s,\,t}}(\mathbb{F}_2,\mathbb{F}_2)=\begin{cases}\langle h_1c_0\rangle & \text{if } t=1, \text{ and } s=1,\\ \langle h_0h_{s+1}^3\rangle & \text{if } t=1, \text{ and } s\geqslant 2,\\ \langle e_0\rangle & \text{if } t=2, \text{ and } s=1,\\ 0 & \text{if } t=2, \text{ and } s\geqslant 2,\\ \langle h_0h_{s+1}h_{s+t}^2\rangle & \text{if } t\geqslant 4, \text{ and } 1\leqslant s\leqslant 2,\\ \langle h_0h_s^2h_{s+t+1}, h_0h_{s+1}h_{s+t}^2\rangle & \text{if } t\geqslant 4, \text{ and } s\geqslant 3.\end{cases} \tag{2.5}$$

Then, the equalities (2.4) and (2.5) show that $n_{s,\,t}$ is not bad and conjecture 1.5 also holds for the degree $n_{s,\,t}$ whenever $t\geqslant 1$, $t\neq 3$, and $s\geqslant 1$.

As is well known, $\{h_j|\,j\geqslant 0\}\subset \mathrm{Im}(Tr_1(\mathbb{F}_2))$ (see Singer [**25**]), $\{c_j|\,j\geqslant 0\}\subset \mathrm{Im}(Tr_3(\mathbb{F}_2))$ (see Boardman [**4**]), $\{e_j|\,j\geqslant 0\}\subset \mathrm{Im}(Tr_4(\mathbb{F}_2))$ (see Hà [**8**] and theorem 1.4(i)), and the 'total' transfer $\{Tr_q(\mathbb{F}_2)\}_{q\geqslant 0}$ is an algebra homomorphism (see Singer [**25**]). So, together with the equalities (2.4) and (2.5), we assert that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_{s,t}$ for each $t\geqslant 1$, $t\neq 3$ and all $s\geqslant 1$.

Next, for item (iii) with $t\neq 3$, we have the following case.

<u>*Case* $n:=n_{s,\,t}=2^{s+t}+2^s-2,\ \ t\geqslant 1, t\neq 3,\ s\geqslant 1.$</u> By theorem 1.7, we obtain

$$\mathrm{Ext}_{\mathscr{A}}^{4,4+n_{s,\,t}}(\mathbb{F}_2,\mathbb{F}_2)=\begin{cases}0 & \text{if } t=1, \text{ and } 1\leqslant s\leqslant 4,\ s\neq 3,\\ \langle h_2c_1\rangle & \text{if } t=1, \text{ and } s=3,\\ \langle h_1h_s^3\rangle & \text{if } t=1, \text{ and } s\geqslant 5,\\ 0 & \text{if } t=2, \text{ and } s=1,\\ \langle h_0^2h_2h_4, f_0\rangle & \text{if } t=2, \text{ and } s=2,\\ \langle h_0^2h_3h_5, e_1\rangle & \text{if } t=2, \text{ and } s=3,\\ \langle h_0^2h_sh_{s+2}\rangle & \text{if } t=2, \text{ and } s\geqslant 4,\\ \langle d_1\rangle & \text{if } t=4, \text{ and } s=1,\\ \langle h_1^3h_6\rangle & \text{if } t=4, \text{ and } s=2,\\ \langle h_0^2h_3h_7, h_1h_3h_6^2\rangle & \text{if } t=4, \text{ and } s=3,\\ \langle h_1h_4h_7^3\rangle & \text{if } t=4, \text{ and } s=4,\\ \langle h_1h_{s-1}^2h_{s+4}, h_1h_sh_{s+3}^2\rangle & \text{if } t=4, \text{ and } s\geqslant 5,\\ \langle h_1^2h_t^2\rangle & \text{if } t\geqslant 5,\ t\neq 7, \text{ and } s=1,\\ \langle h_1^2h_7^2, D_3(2)\rangle & \text{if } t=7, \text{ and } s=1,\\ \langle h_1^3h_{t+2}\rangle & \text{if } t\geqslant 5, \text{ and } s=2,\\ \langle h_1h_3h_{t+2}^2, h_0^2h_3h_{t+3}\rangle & \text{if } t\geqslant 5, \text{ and } s=3,\\ \langle h_1h_3h_{t+3}^2, h_0^2h_4h_{t+4}\rangle & \text{if } t\geqslant 5, \text{ and } s=4,\\ \langle h_1h_sh_{s+t-1}^2, h_0^2h_sh_{s+t}, & \\ \quad h_1h_{s-1}^2h_{s+t}\rangle & \text{if } t\geqslant 5, \text{ and } s\geqslant 5.\end{cases} \tag{2.6}$$

Based upon the results in Bruner, Hà and Hu'ng [**5**, Prop. 4.4], Hu'ng [**9**, Thm. 7.3] and our previous papers [**20**, **21**], the dimension of the domain of $Tr_4(\mathbb{F}_2)$ in

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degree $n_{s,t}$ is determined by

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,t}}(V^{\oplus 4}) = \begin{cases} 0 & \text{if } t = 1, \text{ and } 1 \leqslant s \leqslant 4,\ s \neq 3, \\ 1 & \text{if } t = 1, \text{ and } s \geqslant 3,\ s \neq 4, \\ 0 & \text{if } t = 2, \text{ and } s = 1 \text{ (see [5, Prop. 4.4])}, \\ 2 & \text{if } t = 2, \text{ and } 2 \leqslant s \leqslant 3, \\ 1 & \text{if } t = 2, \text{ and } s \geqslant 4, \\ 1 & \text{if } t = 4, \text{ and } 1 \leqslant s \leqslant 4,\ s \neq 3, \\ 2 & \text{if } t = 4, \text{ and } s \geqslant 3,\ s \neq 4, \\ 1 & \text{if } t = 7, \text{ and } s = 1 \text{ (see [9, Thm. 7.3])}, \\ 1 & \text{if } t \geqslant 5,\ t \neq 7, \text{ and } s = 1, \\ 1 & \text{if } t \geqslant 5, \text{ and } s = 2, \\ 2 & \text{if } t \geqslant 5, \text{ and } 3 \leqslant s \leqslant 4, \\ 3 & \text{if } t \geqslant 5, \text{ and } s \geqslant 5. \end{cases} \tag{2.7}$$

From the equalities (2.6) and (2.7), the only degree $n_{1,7}$ is bad and conjecture 1.5 is true for the degree $n_{s,t}$ whenever $t \geqslant 1$, $t \neq 3$ and $s \geqslant 1$.

Based on theorems 1.3, 1.4, the equalities (2.6), (2.7) and the facts that $\{h_s |\, s \geqslant 0\} \subset \operatorname{Im}(Tr_1(\mathbb{F}_2))$ and the total algebraic transfer $\{Tr_q(\mathbb{F}_2)\}_{q\geqslant 0}$ is a homomorphism of algebras (see Singer [**25**]), we may claim that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_{s,t}$ for all $t \geqslant 1$, $t \notin \{3, 7\}$, and any $s \geqslant 1$. When $s = 1$ and $t = 7$, it has been shown in the proof of theorem 7.3 in Hu'ng [**9**] that the fourth transfer is a monomorphism, but it is not an epimorphism in the internal degree $n_{1,7}$.

<u>*Case* $n := n_{s,u,t} = 2^{s+t+u} + 2^{s+t} + 2^s - 3,\ \ s \geqslant 1,\ u \geqslant 1,\ t \geqslant 1.$</u> From theorem 1.7 and our previous work in [**22**], we have the following results:

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,u,t}}(V^{\oplus 4}) = \begin{cases} 1 & \text{if } s = 1, u = 2, \text{ and } t \geqslant 2, \\ 1 & \text{if } s = 2, u \geqslant 1, \text{ and } t = 1, \\ 1 & \text{if } s \geqslant 2, u \geqslant 2, \text{ and } t \geqslant 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

$$\operatorname{Ext}_{\mathscr{A}}^{4,4+n_{s,u,t}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_0 c_t \rangle & \text{if } s = 1, u = 2, \text{ and } t \geqslant 2, \\ \langle h_{u+3} c_0 \rangle & \text{if } s = 2, u \geqslant 1, \text{ and } t = 1, \\ \langle h_0 h_t h_{s+t} h_{s+t+u} \rangle & \text{if } s \geqslant 2, u \geqslant 2, \text{ and } t \geqslant 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

We should note that the remaining cases of $s, u, t$ where $\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,u,t}}(V^{\oplus 4}) = 0 = \dim \operatorname{Ext}_{\mathscr{A}}^{4,4+n_{s,u,t}}(\mathbb{F}_2, \mathbb{F}_2)$ are described as follows: $s = 1$, $u = 1$ and $t \geqslant 1$; $s = 2$, $u = 1$ and $t \geqslant 2$; $s \geqslant 3$, $u = 1$ and $t \geqslant 1$; $s = 1$, $u = 2$ and $t = 1$; $s = 1$, $u \geqslant 3$ and $t \geqslant 1$; $s \geqslant 3$, $u \geqslant 2$ and $t = 1$. It is straightforward to see that $n_{s,u,t}$ is not bad for all $s$, $t$, $u$ and that by the equalities (2.8), (2.9), conjecture 1.5 holds in the degree $n_{s,u,t}$ for every $s \geqslant 1$, $u \neq 1$ and $t \geqslant 1$.

Because $\{h_s |\, s \geqslant 0\} \subset \operatorname{Im}(Tr_1(\mathbb{F}_2))$ (see Singer [**25**]) and $\{c_s |\, s \geqslant 0\} \subset \operatorname{Im}(Tr_3(\mathbb{F}_2))$ (see Boardman [**4**]) and the total transfer $\{Tr_q(\mathbb{F}_2)\}_{q\geqslant 0}$ is a homomorphism of algebras, by equalities (2.8) and (2.9), we may assert that $Tr_4(\mathbb{F}_2)$ is an isomorphism in the degrees $n_{s,u,t}$ whenever $s \geqslant 1$, $u \geqslant 1$, $t \geqslant 1$.

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We now prove the main theorem for items (ii) and (iii) with $t = 3$. Note that the method of calculation used is similar to our previous works in [**20**–**22**].

***Proof of main theorem.*** We first consider item (ii) with $t = 3$, i.e., degree $n := n_{s,\,3} = 2^{s+4} + 2^{s+1} - 3$ for all $s \geqslant 1$. Let us recall that due to Sum [**27**], the dimension of $(Q\mathcal{P}_4)_{n_{s,\,3}}$ is given by the following table:

| $n_{s,\,3}$ | $s = 1$ | $s = 2$ | $s \geqslant 3$ |
|---|---|---|---|
| $\dim(Q\mathcal{P}_4)_{n_{s,\,3}}$ | 136 | 180 | 195 |

A monomial basis of $(Q\mathcal{P}_4)_{n_{s,\,3}}$ is also given in the same paper [**27**]. Taking this basis, together with a computational technique similar to that of our works in [**21**, **22**], we obtain that the $GL(4)$-invariant space $(Q\mathcal{P}_4)^{GL(4)}_{n_{s,\,3}}$ is trivial if $s = 2$ and is 1-dimensional if $s \neq 2$. As it is known, $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,\,3}}(V^{\oplus 4}) \cong (Q\mathcal{P}_4)^{GL(4)}_{n_{s,\,3}}$, the dimensions of the domain of $Tr_4(\mathbb{F}_2)$ in degrees $n_{s,\,3}$ are determined by

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,\,3}}(V^{\oplus 4}) = \begin{cases} 0 & \text{if } s = 2, \\ 1 & \text{otherwise.} \end{cases} \tag{2.10}$$

On the other hand, from the result by Lin [**12**], it follows that

$$\mathrm{Ext}^{4,4+n_{s,\,3}}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle p_0 \rangle & \text{if } s = 1, \\ \langle p_0' \rangle & \text{if } s = 2, \\ \langle h_0 h_{s+1} h_{s+3}^2, h_0 h_s^2 h_{s+4} \rangle = \langle h_0 h_s^2 h_{s+4} \rangle & \text{if } s \geqslant 3. \end{cases} \tag{2.11}$$

So, by the equalities (2.10) and (2.11), we deduce that $n_{s,\,3}$ is bad for $s = 2$ and that conjecture 1.5 holds for the degrees $n_{s,\,3} = 2^{s+4} + 2^{s+1} - 3$ for $s \geqslant 1$.

Since $h_0 h_s^2 h_{s+4} \in \mathrm{Im}(Tr_4(\mathbb{F}_2))$, by theorems 1.3, 1.4 and the equalities (2.10), (2.11), the fourth transfer

$$Tr_4(\mathbb{F}_2) : \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,\,3}}(V^{\oplus 4}) \longrightarrow \mathrm{Ext}^{4,4+n_{s,\,3}}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism for $s \neq 2$, and that $Tr_4(\mathbb{F}_2)$ is a monomorphism, but not an epimorphism for $s = 2$.

Finally, we consider item (iii) with $t = 3$, i.e., degree $n := n_{s,\,3} = 2^{s+3} + 2^s - 2$ for all $s \geqslant 1$. Notice that $n_{s,\,3}$ is an even degree, and so, the Kameko map

$$(\overline{Sq}^0)_{n_{s,\,3}} := \overline{Sq}^0 : (Q\mathcal{P}_4)_{n_{s,\,3}} \to (Q\mathcal{P}_4)_{(n_{s,\,3}-4)/2} \equiv (Q\mathcal{P}_4)_{2^{s+2}+2^{s-1}-3}$$

is an epimorphism of $\mathbb{F}_2 GL(4)$-modules. This leads to an estimate

$$\begin{aligned} \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{s,\,3}}(V^{\oplus 4}) \ &\leqslant \dim[(\mathrm{Ker}((\overline{Sq}^0)_{n_{s,\,3}}))^{GL(4)}]^* \\ &\quad + \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{2^{s+2}+2^{s-1}-3}(V^{\oplus 4}). \end{aligned} \tag{2.12}$$

Here $[(\mathrm{Ker}((\overline{Sq}^0)_{n_{s,\,3}}))^{GL(4)}]^*$ denotes the dual of $(\mathrm{Ker}((\overline{Sq}^0)_{n_{s,\,3}}))^{GL(4)}$. Following Sum [**29**] and the equality (2.10), the coinvariant $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{2^{s+2}+2^{s-1}-3}(V^{\oplus 4})$ is trivial if $s = 1,\,4$ and is 1-dimensional otherwise. We need to compute the dimension of the $GL(4)$-invariant $(\mathrm{Ker}((\overline{Sq}^0)_{n_{s,\,3}}))^{GL(4)}$. By Sum [**27**, **28**], the dimension of the kenel of $(\overline{Sq}^0)_{n_{s,\,3}}$ is determined as follows:

| $n_{s,\,3}$ | $s=1$ | $s=2$ | $s\geqslant 3$ |
|---|---|---|---|
| $\dim \mathrm{Ker}((\overline{Sq}^0)_{n_{s,\,3}})$ | 49 | 90 | 105 |

Using this result and a similar computation as in [20, 22], we claim that $(\mathrm{Ker}((\overline{Sq}^0)_{n_{s,\,3}}))^{GL(4)}$ is trivial if $s=1,\,2$ and has dimension 1 if $s\geqslant 3$. From these data, the inequality (2.12) implies that

$$0\leqslant \dim \mathbb{F}_2\otimes_{GL(4)} P_{\mathscr{A}}H_{n_{s,\,3}}(V^{\oplus 4})\leqslant \begin{cases} 0 & \text{if } s=1,\\ 1 & \text{if } s=2,\,4,\\ 2 & \text{otherwise.}\end{cases} \tag{2.13}$$

On the other side, due to Lin [12], we find that

$$\mathrm{Ext}_{\mathscr{A}}^{4,4+n_{s,\,3}}(\mathbb{F}_2,\mathbb{F}_2)=\begin{cases} \langle h_1^2h_3^2\rangle =0 & \text{if } s=1,\\ \langle h_0^2h_2h_5, h_1h_2h_4^2\rangle = \langle h_1^3h_5\rangle & \text{if } s=2,\\ \langle h_0^2h_3h_6, p_1\rangle & \text{if } s=3,\\ \langle h_0^2h_4h_7, p_1'\rangle & \text{if } s=4,\\ \langle h_0^2h_sh_{s+3}, h_1h_{s-1}^2h_{s+3}\rangle & \text{if } s\geqslant 5.\end{cases} \tag{2.14}$$

It is known, the non-zero elements $h_1^3h_5$, $h_0^2h_sh_{s+3}$, for $s\geqslant 3$, and $h_1h_{s-1}^2h_{s+3}$ for $s\geqslant 5$ are detected by the fourth transfer. In fact, this could also be directly proved as our previous works [20–22] by using $E_1$-level of $Tr_4(\mathbb{F}_2)$. For instance, a direct computation shows that

$$\begin{aligned} Tr_4(\mathbb{F}_2)([a_1^{(1)}a_2^{(2^{s-1}-1)}a_3^{(2^{s-1}-1)}a_4^{(2^{s+3}-1)}]) &= [\psi_4(a_1^{(1)}a_2^{(2^{s-1}-1)}a_3^{(2^{s-1}-1)}a_4^{(2^{s+3}-1)})]\\ &= [\lambda_1\lambda_{2^{s-1}-1}^2\lambda_{2^{s+3}-1}] = h_1h_{s-1}^2h_{s+3},\end{aligned}$$

where the elements $a_1^{(1)}a_2^{(2^{s-1}-1)}a_3^{(2^{s-1}-1)}a_4^{(2^{s+3}-1)}$ belong to $P_{\mathscr{A}}H_{n_{s,\,3}}(V^{\oplus 4})$, while the linear transformation $\psi_4$ viewed as a representation in the lambda algebra of $Tr_4(\mathbb{F}_2)$ and determined as in [7]. The above equality implies that $h_1h_{s-1}^2h_{s+3}\in \mathrm{Im}(Tr_4(\mathbb{F}_2))$ for every $s\geqslant 5$. Hence, combining with (2.14) and theorems 1.3, 1.4 gives

$$\dim \mathbb{F}_2\otimes_{GL(4)} P_{\mathscr{A}}H_{n_{s,\,3}}(V^{\oplus 4})\geqslant \begin{cases} 0 & \text{if } s=1,\\ 1 & \text{if } s=2,\,4,\\ 2 & \text{otherwise.}\end{cases} \tag{2.15}$$

Noting that by (2.13) and (2.15), the coinvariant space $\mathbb{F}_2\otimes_{GL(4)} P_{\mathscr{A}}H_{n_{4,\,3}}(V^{\oplus 4})$ is 1-dimensional and has been computed by Hu'ng [9, Thm. 7.3] using other techniques. So, his result showed that conjecture 1.5 holds for the degrees $n_{4,\,3}=2^{4+3}+2^4-2$. It is easy to see that the only degree $n_{4,\,3}$ is bad. These data together with the inequalities (2.13), (2.14) and (2.15) imply that conjecture 1.5 is true for the degrees $n_{s,\,3}=2^{s+3}+2^s-2$ for every positive integer $s$.

It is well-known that by Hu'ng [9], the indecomposable element $p_1'$ is not in the image of $Tr_4(\mathbb{F}_2)$ (see also theorem 1.3) and that by Hu'ng and Quỳnh [10], the indecomposable element $p_1$ is in the image of $Tr_4(\mathbb{F}_2)$ (see also theorem 1.4(iii)). At the same time, as shown above, the decomposable elements $h_1^3h_5$, $h_0^2h_sh_{s+3}$, for

$s \geqslant 3$, and $h_1h_{s-1}^2h_{s+3}$ for $s \geqslant 5$ are in the image of $Tr_4(\mathbb{F}_2)$. So, combining with the inequalities (2.13), (2.14) and (2.15), we conclude that the fourth transfer

$$Tr_4(\mathbb{F}_2) : \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}}H_{n_{s,\,3}}(V^{\oplus 4}) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{4,4+n_{s,\,3}}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism for $s \neq 4$, and that $Tr_4(\mathbb{F}_2)$ is a monomorphism, but not an epimorphism for $s = 4$. The proof of the theorem is complete. □

Based on the main theorem and the results in Bruner, Hà and Hu'ng [**5**], Hu'ng [**9**], Hà [**8**], Nam [**15**], Hu'ng and Quỳnh [**10**], we obtain the following corollaries.

COROLLARY 2.1. *Let us consider the following generic degrees:*

$$\begin{aligned} n &:= n_s^{(1)} = 2^{s+4} + 2^{s+1} - 4,\\ n &:= n_s^{(2)} = 2^{s+4} + 2^{s+2} + 2^s - 4,\\ n &:= n_s^{(3)} = 2^{s+4} + 2^{s+2} + 2^{s+1} - 4,\\ n &:= n_s^{(4)} = 2^{s+5} + 2^{s+2} + 2^s - 4, \end{aligned}$$

*where $s$ is an arbitrary positive integer. Then, conjecture 1.5 holds true in these degrees and Singer's transfer is an isomorphism in the bidegree $(4,\,4 + n_s^{(j)})$ for $1 \leqslant j \leqslant 4$ and all $s > 0$.*

It is to be noted that in each degree $n_s^{(j)}$, we do not consider the case $s = 0$ since it has been discussed in the proof of the main theorem. Remark that by the previous works in Hà [**8**], Nam [**15**], Hu'ng and Quỳnh [**10**], Singer's transfer is an epimorphism in the bidegree $(4,\,4 + n_s^{(j)})$ for $1 \leqslant j \leqslant 4$ and all $s > 0$. Indeed, following theorem 1.7, for each positive integer $s$, we have

$$\mathrm{Ext}_{\mathscr{A}}^{4,4+n_s^{(j)}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle d_s\rangle & \text{if } j = 1,\\ \langle h_{s-1}^2h_{s+2}h_{s+4},\ e_s\rangle & \text{if } j = 2,\\ \langle h_{s-1}^2h_{s+1}h_{s+3} = h_s^3h_{s+3},\ f_s\rangle & \text{if } j = 3,\\ \langle h_{s-1}^2h_{s+2}h_{s+5},\ p_s\rangle & \text{if } j = 4. \end{cases} \tag{2.16}$$

By Singer [**25**], the decomposable elements $h_{s-1}^2h_{s+2}h_{s+4}$, $h_{s-1}^2h_{s+1}h_{s+3}$ and $h_{s-1}^2h_{s+2}h_{s+5}$ are detected by the fourth transfer, $Tr_4(\mathbb{F}_2)$ for any $s > 0$. On the other hand, by theorem 1.4, $\{d_s|\, s \geqslant 0\} \subset \mathrm{Im}(Tr_4(\mathbb{F}_2))$ (see Hà [**8**]), $\{e_s|\, s \geqslant 0\} \subset \mathrm{Im}(Tr_4(\mathbb{F}_2))$ (see Hà [**8**]), $\{f_s|\, s \geqslant 0\} \subset \mathrm{Im}(Tr_4(\mathbb{F}_2))$ (see Nam [**15**]) and $\{p_s|\, s \geqslant 0\} \subset \mathrm{Im}(Tr_4(\mathbb{F}_2))$ (see Hu'ng and Quỳnh [**10**]). So, the degrees $n_s^{(j)}$ are not bad and $Tr_4(\mathbb{F}_2)$ is an epimorphism in those degrees for $1 \leqslant j \leqslant 4$ and any $s > 0$. Thus, to prove that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_s^{(j)}$, we need only to show that conjecture 1.5 holds true in these degrees $n_s^{(j)}$. The proof is presented as follows.

*Proof.* It is straightforward to check that $\mu(n_s^{(3)}) = 4$ for every $s > 0$ and $\mu(n_s^{(j)}) = 4$ for every $s > 1$ and $j \neq 3$. Then, one has the following isomorphisms, which are special cases of a result in Hu'ng [9, Cor. 6.2]:

$$\begin{array}{ll}\mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_s^{(3)}}(V^{\oplus 4}) \cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_0^{(3)}}(V^{\oplus 4}), & \text{for all } s \geqslant 1,\\ \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_s^{(j)}}(V^{\oplus 4}) \cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_1^{(j)}}(V^{\oplus 4}), & \text{for all } s \geqslant 2 \text{ and } j \neq 3.\end{array} \tag{2.17}$$

Due to the equality (2.7) and the inequalities (2.13), (2.15) in the proof of the main theorem, we get the equalities of dimensions in (2.18) below (except the concrete values 1 or 2), which is also a special case of Hu'ng [9, Cor. 6.2]:

$$\begin{array}{l}\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_0^{(3)}}(V^{\oplus 4}) = \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{2,\,2}}(V^{\oplus 4}) = 2,\\ \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_1^{(j)}}(V^{\oplus 4}) = \begin{cases}\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{1,\,4}}(V^{\oplus 4}) = 1 & \text{if } j = 1,\\ \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{3,\,2}}(V^{\oplus 4}) = 2 & \text{if } j = 2,\\ \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_{3,\,3}}(V^{\oplus 4}) = 2 & \text{if } j = 4.\end{cases}\end{array} \tag{2.18}$$

Thus, the equalities (2.16), (2.17) and (2.18) indicated that conjecture 1.5 is true in degrees $n_s^{(j)}$ for $1 \leqslant j \leqslant 4$ and all $s > 0$. The corollary follows. □

COROLLARY 2.2. *Let us consider the following generic degrees:*

$$\begin{array}{l} n := n_s^{(5)} = 2^{s+3} + 2^{s+2} - 4,\\ n := n_s^{(6)} = 2^{s+6} + 2^{s+3} + 2^s - 4,\\ n := n_s^{(7)} = 2^{s+6} + 2^s - 4,\end{array}$$

*where $s$ is an arbitrary positive integer. Then, conjecture 1.5 also holds true in these degrees and Singer's transfer is a monomorphism, but it is not an epimorphism in the bidegree $(4,\, 4 + n_s^{(j)})$ for $5 \leqslant j \leqslant 7$ and arbitrary $s \geqslant 1$.*

We remark that by the previous works in [5] and [9], Singer's transfer is not an epimorphism in the bidegree $(4,\, 4 + n_s^{(j)})$ for $5 \leqslant j \leqslant 7$ and all $s > 0$. Indeed, due to theorem 1.7, for each positive integer $s$, one gets

$$\operatorname{Ext}_{\mathscr{A}}^{4,4+n_s^{(j)}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases}\langle g_s\rangle & \text{if } j = 5,\\ \langle h_{s-1}^2 h_{s+3} h_{s+6},\ p'_s\rangle & \text{if } j = 6,\\ \langle h_{s-1}^2 h_{s+5}^2,\ D_3(s)\rangle & \text{if } j = 7.\end{cases} \tag{2.19}$$

We see that for each integer $s > 0$, the decomposable elements $h_{s-1}^2 h_{s+3} h_{s+6}$ and $h_{s-1}^2 h_{s+5}^2$ are detected by the fourth transfer, $Tr_4(\mathbb{F}_2)$. However, by theorem 1.2, $g_s \notin \operatorname{Im}(Tr_4(\mathbb{F}_2))$ (see [5]), and by theorem 1.3, $p'_s \notin \operatorname{Im}(Tr_4(\mathbb{F}_2))$ (see [9]) and $D_3(s) \notin \operatorname{Im}(Tr_4(\mathbb{F}_2))$ (see [9]). So, the degrees $n_s^{(j)}$ are bad and $Tr_4(\mathbb{F}_2)$ is not an epimorphism in those degrees for $5 \leqslant j \leqslant 7$ and any $s > 0$. Thus, to prove that $Tr_4(\mathbb{F}_2)$ is a monomorphism in degrees $n_s^{(j)}$, we shall show that conjecture 1.5 holds true in these degrees $n_s^{(j)}$. Note that the case $j = 5$ was proved by Bruner, Hà and Hu'ng [5]. More explicitly, in [5], the authors show that the coinvariant space $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_s^{(5)}}(V^{\oplus 4})$ is trivial for any $s > 0$. This together with (2.19) imply that conjecture 1.5 is true in the degree $n_s^{(5)}$ for every positive integer $s$.

We now prove the corollary for the cases $j = 6$ and 7.

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*Proof.* It is easy to check that $\mu(n_s^{(6)}) = 4$, for every $s > 1$, and $\mu(n_s^{(7)}) = 4$, for every $s > 2$. So, we get the following isomorphisms, which are special cases of corollary 6.2 in Hu'ng [**9**]:

$$\begin{aligned} \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_s^{(6)}}(V^{\oplus 4}) &\cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_1^{(6)}}(V^{\oplus 4}), \text{ for any } s \geqslant 2, \\ \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_s^{(7)}}(V^{\oplus 4}) &\cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_2^{(7)}}(V^{\oplus 4}), \text{ for any } s \geqslant 3. \end{aligned} \tag{2.20}$$

On the other hand, from the proof of theorem 7.3 in Hu'ng [**9**], we have

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_1^{(6)}}(V^{\oplus 4}) = 1 = \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathscr{A}} H_{n_2^{(7)}}(V^{\oplus 4}). \tag{2.21}$$

Then, by the equalities (2.19), (2.20) and (2.21), it may be concluded that conjecture 1.5 is also true in the degree $n_s^{(6)}$ and $n_s^{(7)}$ for any $s > 0$. The corollary is proved. □

By the main theorem and corollaries 2.1, 2.2, we see that

Corollary 2.3. *Conjecture 1.5 is true for all $n$ and so is, conjecture 1.1 for $Tr_4(\mathbb{F}_2)$.*


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